THE EFFICIENT COMPUTATION AND THE SENSITIVITY ANALYSIS OF FINITE-TIME RUIN PROBABILITIES AND THE ESTIMATION OF RISK-BASED REGULATORY CAPITAL

BY

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ABSTRACT

Solvency regulations require financial institutions to hold initial capital so that ruin is a rare event. An important practical problem is to estimate the regulatory capital so the ruin probability is at the regulatory level, typically with less than 0.1% over a finite-time horizon. Estimating probabilities of rare events is challenging, since naive estimations via direct simulations of the surplus process is not feasible. In this paper, we present a stratified sampling algorithm for estimating finite-time ruin probabilities. We further introduce a sequence of measure changes to remove the pathwise discontinuities of the estimator, and compute unbiased first and second-order derivative estimates of the finite-time ruin probabilities with respect to both distributional and structural parameters. We then estimate the regulatory capital and its sensitivities. These estimates provide information to insurance companies for meeting prudential regulations as well as designing risk management strategies. Numerical examples are presented for the classical model, the Sparre Andersen model with interest and the periodic risk model with interest to demonstrate the speed and efficacy of our methodology.

KEYWORDS

Sensitivity analysis, ruin probabilities, Monte Carlo, risk-based capital.

1. INTRODUCTION

The standard theoretical approach underlying insurance regulations originated in risk theory. The aim of prudential regulations is to ensure that the probability of ruin for certain insurance portfolios is below some given "acceptable" level. To ensure this, regulators set a mandatory minimum amount of capital that may be used by the insurance companies as a buffer. The risk theory community aims to utilize sophisticated stochastic models to estimate

Astin Bulletin 46(2), 431-467. doi: 10.1017/asb.2016.5

- the probability of ruin within a finite-time horizon given the current level of capital,
- the sensitivities of the current ruin probability with respect to the underlying risk factors,
- the density of the time to ruin given the current portfolio condition,
- the minimum regulatory capital needed to obtain a survival probability of at least 99.9% (for example) over a given possible long time horizon,
- the sensitivities of the regulatory capital with respect to the underlying risk factors.

Traditionally, the risk theory community has been focusing on deriving analytical solutions of ruin probabilities with restrictions to some specific models. These analytical solutions provide helpful insights for the nature of insurance risks. However, due to the inherent complexity of insurance business, these models are insufficient to explain the underlying dynamics. While the analytical solutions of ruin probabilities under more realistic models are yet to be discovered, we address the above problems by efficient simulation algorithms, that can be used widely for practical purposes.

Monte Carlo simulation is the computational tool of this paper, which has long been used for estimating expected values of a function, that is, it produces an estimate of $\Upsilon(\phi, \eta)$ by computing the expectation of the pathwise estimator g with respect to a certain probability distribution,

$$\Upsilon(\phi, \eta) = \mathbb{E}_\eta[g(X, \phi)] = \int g(x, \phi) f(x, \eta) dx, \tag{1.1}$$

where f is the probability density function of the random variable X. Simulation techniques have been studied in a wide range of practical areas, including queues (Ross, 2006), financial engineering (Glasserman, 2004), computer networks (Peterson and Davie, 2007) and many more. Here, we apply some techniques of rare event simulation and sensitivity analysis to solving classical problems in risk theory. An extension of the work here is to consider other Monte Carlo methods for studying these ruin-related problems. We leave these to the future.

In order to study ruin-related problems of insurance portfolios, the risk theory community considers the excess of income over claims paid, i.e. the insurer's surplus process. In this paper, we use Monte Carlo simulation to generate repeated random samples of this process. The mathematical representation of the surplus process is

$$R_t = u + c(t) - \sum_{i=1}^{N_t} X_i, \tag{1.2}$$

where

- u is the initial surplus, or capital,
- N_t is the number of claims up to time t, i.e. a delayed renewal process generated by a sequence of inter-claim times T_i,

$$N_t = \inf\{j \geq 0 : T_0 + \cdots\ T_j \geq t\},$$

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

- X_i's are independent and identically distributed non-negative random variables,
- $c(t)$ is the amount of premiums collected by the insurer up to time t.

The classical risk model was the first model introduced to study the behaviour of the above process. It models the inter-claim times, T_i, as independent and identically distributed exponential random variables. Andersen (1957) extended the classical model by allowing the inter-claim times to have arbitrary distributions, such as Erlang, gamma and mixed exponential distributions. For both models, the premium collection is assumed to be

$$c(t) = (1+\theta)\mu\lambda t,$$

where $\lambda = 1/\mathbb{E}[T_i]$, $\mu = \mathbb{E}[X_i]$ and θ is the premium loading factor. Both the classical model and the Sparre Andersen model assume independent and identically distributed inter-claim times. This choice implies that it does not describe situations, such as motor insurances, where claim occurrence epochs depend on the time of the year. The non-homogeneous Poisson process is an alternative for this case, such that a typical periodic intensity function $\lambda(t) = a + b\cos(2\pi ct)$ can be used to describe the arrival rate of claims (Čížek *et al.*, 2011). For this model, the premium collection is assumed to be

$$c(t) = (1+\theta)\mu \int_0^t \lambda(s)ds.$$

In addition to the claim arrival process, various distributions can be used to describe the behaviour of X_i. While most of the literature has focused on light-tailed distributions, the nature of most insurance businesses implies more heavy-tailed claim size distributions. Since the aggregate claim of a portfolio is the sum of a random number of claims, its tail distribution depends heavily on the tail of the individual claim size distribution. It is then inadequate to model using distributions such as the exponential.

After specifying the theoretical basis, the time to ruin,

$$\tau(u) = \inf\{t > 0 : R_t < 0\},$$

is studied. In particular, the ruin probabilities are defined in terms of $\tau(u)$,

$$\psi(u) = \mathbb{P}(\tau(u) < \infty), \tag{1.3}$$

and

$$\psi(u,t) = \mathbb{P}(\tau(u) \le t). \tag{1.4}$$

One can use the Pollaczek–Khinchin formula (Pollaczek, 1930) and (Khinchin 1967) for calculating the ultimate ruin probabilities. However, it is only tractable when the Laplace transform of the ultimate-ruin probability is a rational function (Rolski *et al.*, 1999). From a practical point of view, the finite-time ruin probability, $\psi(u,t)$, may perhaps be regarded as more interesting than the infinite-time one. The finite-time, t, is related to the planning horizon of the company, and typically long term, for example 10 years. In recent years, there has been a surge of research into methods for computing finite-time ruin probabilities, especially for the classical model:

- Picard and Lefévre (1997) derived the finite-time ruin probabilities with discrete claim size distributions.
- Vylder (1999) provided numerical approximations of finite-time ruin probabilities for the continuous case by the Picard–Leféve Formula.
- Dickson and Willmot (2005b) derived an expression for the density of the time to ruin in the classical risk model by inverting its Laplace transform. They have shown that finite-time ruin probabilities can be calculated when the individual claim amount distribution is a mixed Erlang distribution.
- Dickson (2007) derived the density function of aggregate claims for joint density functions involving the time to ruin, the deficit at ruin and the surplus prior to ruin.

In general, ruin probabilities can only be calculated analytically for some special light-tailed claim size distributions under the classical risk model and the Sparre Andersen model. These analytical formulae can, however, require hours of computation. For more realistic models, there are no analytical results found in the literature. Albeit, one can always use Monte Carlo simulations to estimate these ruin probabilities, and results obtained can also be used in the future as benchmarks for the purposing analytical solutions.

In the first part of our paper, we develop an efficient simulation algorithm for estimating the finite-time ruin probabilities. For computing finite-time ruin probabilities, the direct simulation algorithm provides an unbiased pathwise estimator,

$$g = \mathbb{I}_{\{\tau(u)\leq t\}}, \tag{1.5}$$

and the choice of probability density function depends on the random variable of interest. This estimator, however, does not provide sensible answers in practical situations, because solvency regulations require insurance companies to hold capital so that ruin in a finite-time horizon is a rare event, i.e. $\psi(u, t)$ is a very small number. For a 99.9% sufficiency level, one can only expect to observe a non-zero pathwise estimator, i.e. $g = 1$, in just five paths of a 5,000 paths simulation. To provide reliable estimates via direct simulation would be a time-consuming exercise.

Rare event simulation has been studied extensively in the literature to address problems like this, standard techniques includes importance sampling (Blanchet and Lam, 2012), conditional Monte Carlo (Asmussen and Kroese, 2006), and multi-level splitting (Dean and Dupuis, 2011). However, none of these standard techniques have been applied to estimate finite-time ruin probabilities. Stratified sampling is another method, widely used as a variance reduction technique by the Monte Carlo community. As pointed out by Glasserman *et al.* (1999a), stratified sampling can be viewed as a special case of importance sampling in

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

that measure changes are performed on the standard uniforms for simulating the random variables.

Here, we introduce stratified sampling to the field of ruin theory and provide better estimates for finite-time ruin probabilities when they are small. The method is similar to the approach of Joshi and Kainth (2003) for studying credit derivative. In each claim time, we only require the program of simulating X_i to be monotonic in its first standard uniform. Then, given $N_t = n$, for $n = 1, 2, \ldots,$ we force the probability of ruin at the ith claim to be $\frac{1}{n+1-i}$ provided that ruin has not occurred through the first $i-1$ claims, by performing a measure change on the first standard uniform in the simulation algorithm for each X_i until ruin occurs. For paths with one or more claims, the probability of ruin becomes certain. The new pathwise estimator is

$$g = \mathbb{I}_{N_t>0} \prod_{i=1}^{n_r} W_i, \tag{1.6}$$

where n_r is the number of the claim at which the surplus level drops below zero and W_i is the likelihood ratio weight resulting from the measure change at the ith claim. The mild assumptions allow the algorithm to have wide applications, which provide an easy and efficient way to computing finite-time ruin probabilities. The numerical results show that, the stratified sampling algorithm produces better estimates of finite-time ruin probabilities when they are small.

The second practical problem we address in this paper is to compute derivative estimates of the finite-time ruin probabilities with respect to both distributional and structural parameters, i.e. the gradient and Hessian of $\psi(u, t)$. Sensitivity analysis is useful for discovering which risk factors are important and which ones are not. Further, the derivative of $\psi(u, t)$ with respect to t gives the density of the time to ruin, which is one important research focus of the risk theory community. Our techniques work under similar assumptions to those for stratified sampling.

Estimating the sensitivities of ruin probabilities is a difficult task. Due to the complexity of the underlying systems, the three traditional Monte Carlo methods of computing sensitivities, namely the finite-differencing method(FD), the pathwise method and the likelihood ratio method (LR), are not applicable for computing sensitivities of finite-time ruin probabilities:

- The FD method with a small bump size produces a biased estimator. Since a small bump in the parameter of interest may alter the number of claims at time t and R_t at each claim time, the resulting pathwise estimator of ruin from the bumped path might be materially different from the unbumped path, i.e. the pathwise estimators of ruin are discontinuous functions. Therefore, the resulting FD estimators of derivatives have extremely large variances if the bump size is small.
- The pathwise method can be viewed as the limit case of the FD method, which produces unbiased estimates when applicable. The computation of first-order

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

sensitivities under this approach requires the pathwise estimator, $g(x(\eta), \phi)$, to be Lipschitz continuous everywhere and differentiable almost surely as a function of η and ϕ. This is not the case for finite-time ruin pathwise estimators, which involves a sequence of indicator functions.

- The LR method also produces unbiased estimates, but it is only applicable for the distributional parameters η when the underlying distributions have known and tractable probability density functions, $f(x, \eta)$. For structural parameters such as the initial surplus u, the premium loading θ and the finite-time t, the LR method is not applicable. In addition, for cases where the LR method is applicable, it has a tendency to produce derivative estimates with high variances.

Among these traditional methods, the pathwise method when applicable produces estimators with the smallest variances (Glasserman, 2004), but applying the pathwise method to compute sensitivities often requires additional endeavour. One approach is the Optimal Partial Proxy(OPP) algorithm by Chan and Joshi (2015); a measure change is performed at each pathwise discontinuity defined by the payoff function g, so the simulated payoff function $\hat{g}^{\text{OPP}}$ is Lipschitz continuous everywhere and differentiable almost surely, the pathwise method is then applied to $\hat{g}^{\text{OPP}}$ to compute the pathwise estimators of first-order sensitivities. Joshi and Zhu (2014a) extended the OPP method to computing Hessians of financial products with angular or discontinuities payoffs(HOPP); their measure change removes the pathwise discontinuities of both the payoff function g and its first order derivatives Δ_g. The resulting simulated payoff function $\hat{g}^{\text{HOPP}}$ has first order derivatives which are Lipschitz continuous everywhere and differentiable almost surely (we shall say a function with such properties is $\hat{C}^2$), the pathwise method is then applied to $\hat{g}^{\text{HOPP}}$ to compute the pathwise estimator of the Hessian. Joshi and Zhu (2014b) further introduced the Optimal Sensitivities of Rejection Sampling (OSRS) algorithm to compute first- and second-order sensitivities of systems simulated by rejection techniques; they use a sequence of measure changes to remove the pathwise discontinuities resulted from the acceptance–rejection decisions.

While computing sensitivities of performance measures has been studied extensively in the Monte Carlo literature, there has been little progress made for estimating derivatives of ruin probabilities due to practical difficulties of the problem. Asmussen and Rubinstein (1999) estimated the ultimate ruin probability $\psi(u)$ and its sensitivities, under various distributions of the arrival processes and the claim sizes by simulation. For distributional parameters, they used the LR method; for the structural parameters, they used the combination of the LR and the push-out method. The idea of the push-out method is to rewrite the model such that the parameter of interest appears in the usual way as the parameter of the density, and the LR method is then applicable. Privault and Wei (2004) introduced the Malliavin calculus approach to compute sensitivities of the ultimate ruin probability for a classical risk model with a constant interest rate and an

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

arbitrary claim size distribution. Another approach by Vazquez-Abad (2000) is rare perturbation analysis (RPA), which assumes that there exists a σ-field such that the interchange of differentiation and integration is valid. However, for the sensitivities of the finite-time ruin probabilities, only the Malliavin calculus approach was applied to compute $\frac{\partial\psi(u,t)}{\partial u}$ in the classical model, where the claim sizes were stricted to fixed, exponential and Pareto random variables (Loisel and Privault, 2009).

In this paper, we adapt HOPP to compute first and second-order sensitivities of the finite-time ruin probabilities. The pathwise estimator of ruin by direct simulation is shown in equation (1.5), it is clearly a discontinuous function of both the distributional and the structural parameters. To remove the pathwise discontinuities, we perform measure changes to ensure

- the bumped paths and the unbumped paths have the same number of claims, i.e. $N_t(\eta) = N_t(\eta_0)$, where $N_t(\eta)$ is the number of claims up to time t given the distributional parameters η,
- the surplus level $R_t(\eta, \phi) < 0$ if and only if $R_t(\eta_0, \phi_0) < 0$, where $R_t(\eta, \phi)$ is the surplus level at time t given the distributional and the structural parameters.

We shall call the resulting simulation algorithm, the sensitivity of finite-time ruin by direct simulation (SFRDS). The pathwise estimator by the stratified sampling algorithm as shown in equation (1.6) is $\hat{C}^2$ as a function of the structural parameters as well as the parameters of the claim size distribution. This is because the LR weights, W_i, are continuously differentiable functions of the structural parameters as well as the parameters of the claim size distribution. The product of them is then also continuously differentiable. However, the function is still discontinuous with respect to the parameters of the inter-claim time random variables. That is, a small bump in them may change the value of the indicator function as well as the value of n_r. We perform changes of measure to ensure such pathwise discontinuities are removed. We shall call the resulting simulation algorithm, the sensitivity of the finite-time ruin by stratified sampling (SFRSS). Numerical experiments are performed for the three models discussed, the results show that the sensitivities computed by the two method agree. However, when finite-time ruin is rare, the SFRSS method outperforms.

After obtaining efficient algorithms for computing the finite-time ruin probabilities and their sensitivities, we can then address the problem of estimating the regulatory capital. Since the stratified sampling algorithm and the SFRSS method produce better estimates when finite-time ruin is rare, they are used to approximate the critical initial surplus level, i.e. the regulatory capital u^* such that the finite-time ruin probabilities calculated is at the solvency level α,

$$\psi(u^*, t) = \alpha. \tag{1.7}$$

We use the Newton–Raphson method to provide an approximate, $\hat{u}$, of the regulatory capital given $\alpha = 0.001$ in our numerical examples. Since the value of

$\frac{\partial\psi(u,t)}{\partial u}$ is always negative, the initial surplus obtained by the process is guaranteed to approach the critical value.

In addition to regulatory capital, insurance companies are also interested in its sensitivities to risk factors. For instance, if the claims arrival rate increases, how much more capital h_u, they should raise in order to maintain the solvency level, i.e. $\psi(u + h_u, t) = \alpha$. Sensitivity analysis is required for insurance companies in their financial condition reports in Australia, the UK and in Canada. For meeting the prudential regulations, Cocozza and Lorenzo (2006) presented various methodologies for solvency assessment of life insurance businesses, but their results only focused on the investment risk; Hardy (1993) has shown that a stochastic simulation method exceeds the traditional deterministic sensitivity test approach in estimating the relative probabilities of insolvency of different investment strategies and the timing of potential solvency problems. We compute sensitivities of the regulatory capital with respect to other parameters. These sensitivities, $\frac{\partial\hat{u}}{\partial\eta}$, provide an approach to risk management for insurance companies in terms of meeting solvency requirements.

To demonstrate the validity and efficiency of our simulation algorithm, numerical experiments are performed under the classical risk model, the Sparre Andersen model with interest and the periodic risk model with interest. All numerical examples are computed using single-threaded Monte Carlo C^{++} programs. The use of multi-core *central processing unit* (CPUs) and graphics cards to speed up computer programs has received considerable attention. Aldrich *et al.* (2011) demonstrated the effectiveness of **Graphics Processing Unit** (GPUs) for solving dynamic equilibrium problems in economics using iterative methods; Joshi (2014) demonstrated that over one hundred times speed up can be achieved in a realistic case for the pricing of cancellable swaps using the displaced diffusion LIBOR market model using a multi-core graphics card. The algorithms we develop in the paper are naturally parallel in nature and so multiple processing cores could be similarly employed for massive speed ups.

The remaining sections of the report are organized as follows. The basic idea of the stratified sampling algorithm is presented in Section 2. In Section 3, we compute first- and second-order sensitivities of finite-time ruin probabilities by both SFRSS and SFRDS. In Section 4, we apply the Newton–Raphson method to approximate the regulatory capital and compute its sensitivities.

2. Simulating the surplus process

2.1. Overview on rare event simulation

Estimating $\psi(u, t)$ via direct simulation of the surplus process R_t would require estimating $\mathbb{P}(E)$, where

$$E = \{\tau(u) \leq t\}.$$

When the initial surplus level u is large, $\alpha = \mathbb{P}(E)$ is small. The direct Monte Carlo estimator of α is

$$\alpha_n = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}_E^j,$$

where $\mathbb{I}_E^j$'s are independent pathwise estimators. By the Central Limit Theorem, the relative error in the estimator is described by the approximation,

$$\frac{\alpha_n}{\alpha} - 1 \approx \sqrt{\frac{(1-\alpha)}{n\alpha}}\, Z,$$

where Z is the standard normal random variable. When α is small, there are only a few non-zero paths. It is unlikely to produce reliable estimation unless the sample is large. Consequently, naïve rare event simulations are prohibitively expensive. Due to the stringent solvency requirements of prudential regulations, estimating $\psi(u, T)$ by direct simulation is clearly not sensible.

Due to its importance across different fields of endeavour, fast simulation techniques have been introduced to study the occurrence and impact of rare situations. These methods efficiently reduced relative errors of the estimates in practical situations where crude Monte Carlo is insufficient to provide a sensible answer. Rubino and Tuffin (2009) presented a detailed account of the theoretical basis for modern rare event simulation techniques. They also devoted a significant part of the book to applications of these techniques, such as performance measure simulation in queues, nuclear particle transport simulation, biological system simulation, etc. While these advanced techniques have many interesting applications, none of them have been applied to estimate finite-time ruin probabilities. In this paper, we apply one of these methods, i.e. stratified sampling, to estimate finite-time ruin probabilities. The idea of stratified sampling is very similar to that of importance sampling. The sample space of the target distribution is divided into K regions, called strata, and then one can compute the sample estimate, g_i, in each stratum. The resulting pathwise estimate is

$$g = \sum_{i=1}^{K} p_i g_i,$$

where p_i is the allocation of the ith stratum. The idea of our approach is originated from Joshi and Kainth (2003), and is explained in the next section. Here, we briefly summarize other rare event simulation techniques and leave the application of these methods to the future.

The most commonly used technique for rare event simulation is importance sampling, especially when the underlying state variables are light-tailed. It modifies the direct Monte Carlo method, and generates random weighted samples from an equivalent probability distribution rather than the distribution of interest. The basic idea is to introduce a probability measure $\mathbb{Q}$ such that the Radon–Nikodym derivative between the original probability measure, $\mathbb{P}$, and $\mathbb{Q}$ is well defined on the event of interest. Then, the pathwise unbiased estimator becomes

$$\mathbb{I}_{\omega\in E}\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega).$$

Here, ω is the underlying random outcome simulated according to the probability measure $\mathbb{Q}$. Asmussen and Glynn (2007) showed that the optimal choice of measure change is $\mathbb{P}(\cdot|E)$. Thus, we wish to use a sampling distribution $\mathbb{Q}$ that resembles as closely as possible to the conditional distribution of $\mathbb{P}$ given E. This approach is widely used for estimating performance measures in queuing systems. A typical example is to consider a random variable $S_n = \sum_{i=1}^{n} x_i$, where x_i is identically distributed independent random variables with $\mathbb{E}[x_i] < 0$, and estimate the rare event probability

$$\mathbb{P}\Big(\sup_{n\in\mathcal{N}} S_n \geq b\Big),$$

for a deterministic b that is large (Blanchet and Glynn, 2008). This is different from our problem of estimating finite-time ruin probabilities, the value of b in our context is stochastic and depends on the claim arrival process.

Importance sampling has limited applications, when the problem dimensionality is high or when the optimal importance sampling density is too complex to obtain. For computing finite-time ruin probabilities, one can apply importance sampling, i.e. perform measure changes on either the claim arrival process and the individual claim size random variable. However, the surplus process R_t depends critically on the timing of the claims as well as other factors such as investment returns. Consequently, it is hard to determine the optimal measure change. Thus, it is challenging to implement importance sampling efficiently, especially in a general framework that will allow solutions for many classes of problems.

Conditional Monte Carlo is another method for rare event simulation (Asmussen and Glynn, 2007). The idea of the method is that given an auxiliary vector of random variables, Z, we have

$$\mathbb{E}[\mathbb{P}(E|Z)] = \mathbb{P}(E),$$

so $\mathbb{P}(E|Z)$ is an unbiased estimate of $\mathbb{P}(E)$. More importantly, it has a smaller variance than the crude Monte Carlo estimator. Based on this idea, efficient method was introduced in Asmussen and Binswanger (1997) to estimate ultimate ruin probabilities with heavy-tailed random claim sizes. Using the Pollaczeck–Khinchine formula, they transformed the problem of computing equation (1.3) to computing

$$\mathbb{P}\Big(\sum_{i=1}^{K} X_i > u\Big),$$

where K follows a geometric distribution. It was then estimated using an efficient conditional Monte Carlo algorithm. This method was then improved in the random K case by incorporating control variates and stratification techniques (Asmussen and Kroese, 2006). Their estimators of ultimate ruin probabilities have bounded relative errors. However, we do not have the luxury of the Pollaczeck–Khinchine formula in the finite case. Thus, it is difficult to apply the existing conditional Monte Carlo method to computing finite-time ruin probabilities.

Another method called "Splitting" was introduced to deal with the problem of estimating rare event probabilities (Villen-Altamirano and Villen-Atamirano, 1991). Basically, it partitions the space-state of the system into a series of nested subsets and considers the rare event as the intersection of a nested sequence of events. That is, consider

$$E_1 \subset E_2 \ldots E_m = E,$$

the small probability of event E can be decomposed into

$$\mathbb{P}(E) = \mathbb{P}(E_1)\mathbb{P}(E_2|E_1)\ldots\mathbb{P}(E_m|E_{m-1}),$$

with each conditional event being "not rare". When a given sub-set is entered by a sample path, random sub-simulations are generated from the initial state corresponding to the state of the system at the entry point. Thus, the system has been split into a number of new sub-simulations. The final estimate is the product of individual estimates. Splitting is typically used for estimating the probability that a Markov process first enters an unlikely set B before another likely set A, after starting in neither A nor B (Dean and Dupuis, 2009). This is different from estimating finite-time ruin probabilities.

Glasserman *et al.* (1999b) analysed the splitting method, and showed that choosing the degree of splitting correctly produces asymptotically optimal estimates when the state space satisfies some dimensionality conditions. Practically, it implies that too much splitting results in explosive computational requirements, and too little splitting eliminates any reduction in variance. To address the issue, Dean and Dupuis (2011) formulated multi-level splitting algorithms for simulate rare probabilities, where the underlying state variable is discrete. In case of computing finite-time ruin probabilities, the distribution of the underlying surplus process is often continuous and typically difficult to work with. A carefully designed sequence of splitting is required for applying the method, moreover, an efficient algorithm of estimating these conditional probability is crucially important.

2.2. The idea of the stratified sampling algorithm

While the rare event simulation techniques in the previous section are not yet applied to computing finite-time ruin probabilities, we use a simple stratified sampling to deal with the problem. Stratified sampling is widely used for variance reduction purposes by the Monte Carlo community. The idea in this paper is similar to the simulation algorithm in Joshi and Kainth (2003) for pricing credit derivatives. As opposed to direct simulation, it produces non-zero pathwise estimates for all paths with one or more claims.

The conditions for the stratified sampling to apply are

1. we can observe the number of claims $N_t = n$, for $n = 0, 1, \ldots$, and the exact values of T_i's before the observation of the individual claim sizes;
2. given $R_i^* > 0$, the surplus level just before the ith claim, there exists a twice differentiable function $A_{X_i}(R_i^*(\eta, \phi), \eta)$ such that

$$V_{X_i} < A_{X_i} \Leftrightarrow R_i < 0,$$

where V_{X_i} is one simulated random uniform for generating X_i.

This is the actual probability of ruin at the ith claim, X_i, given the surplus level just before the claim. Here, we shall only present this case, a similar approach is valid for the opposite situation, i.e. $V_{X_i} > A_{X_i} \Leftrightarrow R_i < 0$.

The first step of the algorithm is to simulate the number of claims, n, within a finite-time horizon. If $n = 0$, the pathwise estimator is zero. For other paths, the surplus level is forced to drop below zero at the ith claim with a probability,

$$a_i = \frac{1}{n + 1 - i},$$

provided that ruin has not occurred through the first $i - 1$ claims. This number a_i only depends on the total number of claims within the time horizon, not the surplus level just before the claim. This ensures that ruin always occurs and is equidistributed between the n claims. To achieve this, a change of measure is performed on one simulated standard uniform random variable V_{X_i} of X_i until ruin occurs. The standard uniform V_{X_i} is replaced by a change of variable function $U_i(V_{X_i}, A_{X_i})$ of it, i.e.

$$U_i(V_{X_i}, A_{X_i}) = \begin{cases} \frac{A_{X_i}}{a_i} V_{X_i}, & V_{X_i} < a_i, \\ \frac{1 - A_{X_i}}{1 - a_i}(V_{X_i} - a_i) + A_{X_i}, & V_{X_i} \geq a_i. \end{cases} \tag{2.1}$$

The resulting LR weight from the measure change is

$$W_i(V_{X_i}, A_{X_i}) = \begin{cases} \frac{A_{X_i}}{a_i}, & V_{X_i} < a_i, \\ \frac{1 - A_{X_i}}{1 - a_i}, & V_{X_i} \geq a_i. \end{cases} \tag{2.2}$$

The pathwise estimator of the finite-time ruin probability under the stratified sampling algorithm is

$$\hat{g}^{SS}(\eta) = \mathbb{I}_{N_t > 0} \prod_{i=1}^{n_r} W_i, \tag{2.3}$$

where $n_r = 1, 2, \ldots$ is the number of the claim at which the surplus level drops below zero.

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

TABLE 1

THE CLASSICAL RISK MODEL: SIMULATED FINITE-TIME RUIN PROBABILITIES BY THE DIRECT METHOD AND THE STRATIFIED SAMPLING ALGORITHM WITH 50,000 PATHS SAMPLE.

t	Analytical	DM mean(S.E.)	SS mean (S.E.)
2	0.0013	0.0013(0.000158)	0.0012(0.00013)
4	0.0059	0.0056(0.00333)	0.0065(0.00046)
6	0.0131	0.0140(0.00052)	0.0138(0.00071)
8	0.0220	0.0216(0.00065)	0.0226(0.00099)
10	0.0319	0.0307(0.00077)	0.0322(0.00131)
20	0.0822	0.0801(0.00121)	0.0807(0.00256)
40	0.1573	0.1523(0.00161)	0.1506(0.00408)

2.3. Numerical examples

We consider three models in our numerical experiment for comparing the results from direct simulation and the stratified sampling algorithm in estimating the finite-time ruin probabilities. The numerical experiments in this section are conducted by 50,000 paths samples.

2.3.1. *The classical risk model.* We choose the exponential inter-claim time random variable and the exponential claim size random variable for benchmark purposes, see Dickson and Willmot (2005a) for the analytical results.

For the stratified sampling algorithm, we need to compute the critical value functions A_i. Given V_{X_i}, the exponential claim size random variable X_i is computed as

$$X_i = -\mu \log(V_{X_i}),$$

where $\mu = \mathbb{E}[X_i]$. The critical value function is

$$A_{X_i}(\mu, R_i^*) = \exp\left(-\frac{R_i^*}{\mu}\right).$$

We first set the parameters $\mu = 1$, $\lambda = 1$, $\theta = 0.1$, $u = 10$, and vary the finite-time t. The finite-time ruin probabilities computed by both methods are consistent with the ones in Dickson and Willmot (2005a), the numerical results are shown in Table 1.

- The stratified sampling algorithm underperforms the direct simulation when $\psi(u, t)$ is large, this is because the change of measure at each X_i introduces excessive LR weights.
- The stratified sampling algorithm outperforms the direct simulation when ruin is rare and the change of measure induces a faster convergence.

For the classical model, the numerical results suggest that the stratified sampling algorithm is the preferred method when simulating rare finite-time ruin probabilities.

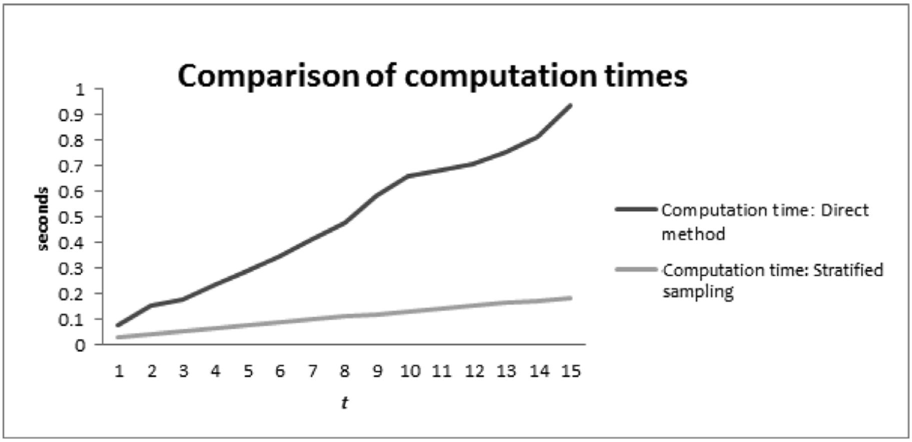

FIGURE 1: The classical model: the computation times of the direct method and the stratified sampling method based on 50,000 samples.

We further consider the computational times of the two methods for estimating the finite-time ruin probabilities. Let $u = 20$ and vary $t = 1, 2, \ldots 15$, in figure 1 we plot the time taken to estimate $\psi(u, t)$ by the two method with the same set of parameters as above. The plot indicates that the stratified sampling algorithm reduces the computational effort when the probability of ruin is small, because

- the algorithm stops on each path when ruin occurs,
- we have forced ruin to occur at a faster rate, i.e. $a_i > A_i$.

2.3.2. *The Sparre Andersen model with a deterministic investment return on surplus.* The traditional surplus process in risk theory assumes that the surplus receives no interest over time, but a large portion of the surplus of insurance companies comes from investment income. In recent years, we have seen an increasing interest in risk models with interest incomes. Infinite time ruin probabilities with a constant interest rate were estimated in Sundt and Teugels (1995) for the classical risk model. In particular, they considered the case with zero initial reserve, and the case with exponential claim sizes. Konstantinides *et al.* (2002) extended the results to allow heavy-tail claim sizes. These results are restricted to the infinite-time ruin probabilities under the classical model. Despite the practical importance of with-interest models, there has been no progress made in literature on solving the finite-time ruin probabilities with interest, nor the Sparre Andersen model with interest.

Here, we consider the Sparre Andersen model with interest, that is,

$$dR_t = r\,R_t dt + (1+\theta)\mu\lambda dt - d\sum_{i=1}^{N_t} X_i,$$

TABLE 2

THE SPARRE ANDERSEN WITH INTEREST RISK MODEL: SIMULATED FINITE-TIME RUIN PROBABILITIES BY THE DIRECT METHOD AND THE STRATIFIED SAMPLING ALGORITHM WITH 50,000 PATHS SAMPLE.

u	DM mean	DM S.E.	SS mean	SS S.E.
1	0.42712	0.00221	0.42236	0.00263
5	0.09768	0.001328	0.09822	0.000973
10	0.02076	0.000638	0.02150	0.000369
25	0.00174	0.000186	0.00135	2.2011E-05
30	0.00106	0.000146	0.00078	1.2905E-05

where $\mu = \mathbb{E}[X_i]$ and $\lambda = \frac{1}{\mathbb{E}[T_i]}$. While obtaining the analytical solution of the ruin probabilities via Laplace transform is intractable, Monte Carlo simulation provides an alternative solution to the problem. For the numerical experiments,

- the inter-claim times are distributed $Erlang(m, \beta)$, we set $m = 2$ and $\beta = 2$, so $\mathbb{E}[T_i] = 1$,
- the claim size random variables are distributed $Pareto(a, b)$, we set $a = 3$ and $b = 2$, so $\mathbb{E}[X_i] = 1$,
- the finite-time horizon is 10,
- the premium loading is 0.1 and the constant force of interest is 0.1.

We set the initial surplus $u = 1, 5, 10, 25$ and 30, and compare the results simulated by the direct method and the stratified sampling algorithm, see table 2. The numerical results show that the stratified sampling algorithm outperforms the direct method significantly for the Sparre Andersen model with interest for most cases, except when the ruin probability is significantly large. The time taken to compute the finite-time ruin probability with $u = 30$ under the direct method is 0.5208 seconds and the stratified sampling is 0.168 seconds. This indicates a reduction of computational effort due to a faster rate of ruin occurrence for the stratified sampling method.

2.3.3. *The periodic risk model with a deterministic investment return on surplus.* Another important property of insurer's risk business is that, claims are sometimes caused by periodic phenomena. The previous two models have a constant claim arrival intensity over time, which makes them crude models for insurance portfolios under periodic environments. We allow a periodic intensity for the compound Poisson process in the next experiment, that is the number of claims up to time t, N_t follows a non-homogeneous Poisson process. The general theory for the periodic case has been derived in Asmussen and Rolski (1994), whose discussion relied on the properties of the martingale non-homogeneous Poisson. A practical simulation algorithm for the surplus process was introduced by Morales (2004). The ultimate ruin probabilities computed using their method were compared with the classical model results, it demonstrated a significant fluctuation depending on the current state of the cycle. However, their

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discussions of ruin probabilities were restricted to the ultimate cases and exponential claim sizes.

For the numerical experiment, we compute the finite-time ruin probabilities with lognormal claim size random variables. This example is of more practical interest to the insurance industry. The following intensity function is considered:

$$\lambda(t) = a + b\cos(2\pi ct), \text{ for } t > 0, \tag{2.4}$$

where $a > 0$, $b > 0$ and $c > 0$ are parameters of the model. This intensity function has a maximum value of $a + b$ and a minimum value of $a - b$. The period of the seasonal behaviour can be modified through the value c in the function. To simulate the time of the next arrival conditional on a claim just arrived at time s, the thinning algorithm is used (Ross, 2006). In particular, the process is simulated by a majoring Poisson process with an intensity function $\lambda^*(t) = a + b$, as follows:

1. *Let* $t_s = s$,
2. *generate* $V_{T_i} = U(0, 1)$*, set* $t_s = t_s - \frac{1}{a+b} log(V_{T_i})$;
3. *generate* V_D,
4. *if* $V_D \leq \frac{a+b\cos(2\pi ct_s)}{a+b}$*, set* $T_s = t_s$ *and accept, else go back to step 2.*

We also modify the model to incorporate a constant force of interest r, so that

$$dR_t = rR_t dt + (1+\theta)\mu\lambda(t)dt - d\sum_{i=1}^{N_t} X_i,$$

where N_t is a non-homogeneous Poisson process with intensity $\lambda(t)$.

For the numerical experiments, we choose

- the intensity parameters, $a = 1$, $b = 1$ and $c = 0.1$,
- the claim sizes lognormally distributed with $\nu = -0.5$ and $\sigma = 1$, so that $\mathbb{E}[X_i] = 1$,
- the force of interest $r = 0.1$ and the loading $\theta = 0.1$,
- the finite-time horizon $t = 10$.

We set the initial surplus $u = 1, 5, 10, 25$ and 30 to show that the stratified sampling algorithm results and the direct method results agree, and the stratified sampling algorithm outperforms the direct method when the probability of ruin is small, see table 3. The time taken to compute the finite-time ruin probability with $u = 30$ under the direct method is 0.643 seconds and the stratified sampling method is 0.372 seconds.

The above numerical results suggest that, the stratified sampling algorithm outperforms the direct method when the initial surplus is large, i.e. the probability of ruin is small. Our ultimate objective is to find the regulatory capital so the finite-time ruin probability over a 10-year period is very small, typically less than 0.1%, thus it is preferable to use the stratified sampling algorithm.

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TABLE 3

THE PERIODIC RISK WITH INTEREST RISK MODEL: SIMULATED FINITE-TIME RUIN PROBABILITIES BY THE DIRECT METHOD AND THE STRATIFIED SAMPLING ALGORITHM WITH 50,000 PATHS SAMPLE.

u	DM mean	DM S.E.	SS mean	SS S.E.
1	0.50542	0.002236	0.50379	0.003752
5	0.12144	0.001461	0.12254	0.001715
10	0.02076	0.000638	0.02304	0.000750
25	0.00038	8.716E-05	0.000473	3.391E-05
30	0.00028	7.482E-05	0.000181	6.422E-06

3. SENSITIVITY ANALYSIS ON FINITE-TIME RUIN PROBABILITIES AND DENSITY ESTIMATION

Sensitivity analysis in risk theory is concerned with estimating derivatives of the ruin probabilities with respect to parameters of interest. Derivative estimates of the finite-time ruin probabilities serves the following purposes:

- it unravels the underlying risk process, and identifies the most significant operational parameters;
- one can make use of sensitivities to find the optimal solution with respect to the parameters of interest;
- if the parameter is only partially known, they can be used to estimate the parameter from data.

The pathwise method, or *infinitesimal perturbation analysis* (*IPA*), has been introduced as an efficient way to compute parameter sensitivities of discrete-event systems. Glasserman (1991) provided a general formulation of IPA for a broad class of discrete-event systems, and stated sufficient conditions for these estimates to be unbiased. Pathwise estimators of the finite-time ruin probabilities often fail to fall into this restricted class.

3.1. Sensitivities with respect to the inter-claim time distribution: remove pathwise discontinuities of N_t

For both the direct method and the stratified sampling algorithm, we need to simulate the inter-claim time random variables. To compute the sensitivities of the finite-time ruin probabilities with respect to the distributional parameters of the inter-claim time random variable via IPA, we need to ensure that the pathwise estimator of finite-time ruin is a $\hat{C}^2$ function of these parameters. Define

$$t_{i-1} = \sum_{j=0}^{i-1} T_j,$$

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the arrival time of the ith claim. There is a pathwise discontinuity for each T_i when

$$T_i = t - t_{i-1}, \text{ provided that } t_{i-1} < t, \text{ for } i = 1, 2, \ldots,$$

and at the first claim arrival time when

$$T_0 = t.$$

That is, a small bump in the distributional parameters of T_i will change of the number of claims in the finite-time horizon, i.e. $N_t(\eta) \neq N_t(\eta_0)$. We remove such pathwise discontinuities by HOPP in Joshi and Zhu (2014a), to ensure the unbumped path and the bumped path have the same number of claims. A change of measure performed at each T_i to force the bumped path to finish on the same side of discontinuities as the unbumped paths, that is,

$$T_i > t - t_{i-1} \text{ if and only if } T_i^0 > t^0 - t_{i-1}^0. \tag{3.1}$$

We assume that, there exists a twice differentiable function $A_{T_i}(\eta)$ such that

$$V_{T_i} < A_{T_i} \Leftrightarrow T_i > t - t_{i-1},$$

where V_{T_i} is one simulated standard uniform for generating T_i. This A_{T_i} is the probability of having i claims conditional on $i-1$ claims already arrived, which is different from A_{X_i} in the previous section. We replace V_{T_i} by a change of variable function, $U_{T_i}(V_{T_i}, \eta)$, of it, such that

$$U_{T_i}(V_{T_i}, \eta) = \begin{cases} \frac{1-A_{T_i}(\eta)}{1-A_{T_i}(\eta_0)}\left(V_{T_i} - A_{T_i}(\eta_0)\right) + A_{T_i}(\eta), & V_{T_i} \geq A_{T_i}(\eta_0), \\ \frac{A_{T_i}(\eta)}{A_{T_i}(\eta_0)} V_{T_i}, & V_{T_i} < A_{T_i}(\eta_0). \end{cases} \tag{3.2}$$

The corresponding LR weight is

$$W_{T_i}(V_{T_i}, \eta) = \begin{cases} \frac{1-A_{T_i}(\eta)}{1-A_{T_i}(\eta_0)}, & V_{T_i} \geq A_{T_i}(\eta_0), \\ \frac{A_{T_i}(\eta)}{A_{T_i}(\eta_0)}, & V_{T_i} < A_{T_i}(\eta_0). \end{cases} \tag{3.3}$$

Example: $T_i \sim \exp(\lambda)$
Given a standard random uniform V_{T_i},

$$T_i = -\frac{1}{\lambda}\log(V_{T_i}),$$

where $\lambda = 1/\mathbb{E}[T_i]$. The critical value function for T_i is

$$A_{T_i}(t_{i-1}, t, \lambda) = \exp -\lambda(t - t_{i-1}), \text{ for } t_{i-1} < t \text{ and } i = 1, 2, \ldots,$$

the critical value function for T_0 is

$$A_{T_0}(t, \lambda) = \exp\left(-\lambda t\right).$$

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3.2. Sensitivities with respect to claim size distribution and structural parameters: remove the pathwise discontinuities when R_s $\mathcal{D}$ 0

For the stratified sampling, when the number of claims $N_t > 0$, simulated ruin is certain. As shown in Section 2, the decision of ruin is determined by the set of probabilities $a_i = \frac{1}{n+1-i}$. The changes of measure in Section 3.1 ensures that the bumped path has the same number of claims as the unbumped path, i.e. the value of n is the same. The decision of ruin is then determined by the same value of a_i for both the bumped and unbumped paths, thus the pathwise estimator of the stratified sampling algorithm has no pathwise discontinuities as R_s passing through zero.

However, we need to remove the pathwise discontinuities for the direct method, as small bumps of parameters may cause the surplus level of the bumped path to finish on different sides of zero from the unbumped path at each claim. Adopting the same idea as Joshi and Zhu (2014a), given $R_i^*(\eta, \phi) > 0$, assume there exists a twice differentiable function $A_{X_i}(R_i^*(\eta, \phi), \eta)$ for $i = 1, 2, \ldots,$ such that

$$V_{X_i} < A_{X_i} \Leftrightarrow R_i < 0,$$

where V_{X_i} is one simulated random uniform for generating X_i. Notice, this function is exactly the same as the critical value function in Section 2. We replace V_{X_i} by a change of measure function of it, $U_{X_i}(V_{X_i}, \eta, \phi)$, such that

$$U_{X_i}(V_{X_i}, \eta, \phi) = \begin{cases} \frac{1-A_{X_i}(R_i^*(\eta,\phi),\eta)}{1-A_{X_i}(R_i^*(\eta_0,\phi_0),\eta_0)}(V_{X_i} - A_{X_i}(R_i^*(\eta_0, \phi_0), \eta_0)) + A_{X_i}(R_i^*(\eta, \phi), \eta), \\ \qquad\qquad\qquad\qquad V_{X_i} \geq A_{X_i}(R_i^*(\eta_0, \phi_0), \eta_0), \\ \frac{A_{X_i}(R_i^*(\eta,\phi),\eta)}{A_{X_i}(R_i^*(\eta_0,\phi_0),\eta_0)} V_{T_i}, \quad V_{X_i} < A_{X_i}(R_i^*(\eta_0, \phi_0), \eta_0). \end{cases} \tag{3.4}$$

The corresponding LR weight is

$$W_{X_i}(V_{X_i}, \eta, \phi) = \begin{cases} \frac{1-A_{X_i}(R_i^*(\eta,\phi),\eta)}{1-A_{X_i}(R_i^*(\eta_0,\phi_0),\eta_0)}, & V_{X_i} \geq A_{X_i}(R_i^*(\eta_0, \phi_0), \eta_0), \\ \frac{A_{X_i}(R_i^*(\eta,\phi),\eta)}{A_{X_i}(R_i^*(\eta_0,\phi_0),\eta_0)}, & V_{X_i} < A_{X_i}(R_i^*(\eta_0, \phi_0), \eta_0). \end{cases} \tag{3.5}$$

3.3. The SFRSS method and the SFRDS method

After the sequence of measure changes, we obtain two new pathwise estimators for finite-time ruin probabilities. For the direct simulation, changes of measure are performed at each T_i and X_i, the pathwise estimator of the finite-time ruin probabilities under SFRDS is

$$\hat{g}^{\mathrm{SFRDS}}(\eta) = \left(\prod_{i=1}^{n_r-1} \mathbb{I}_{t_{i-1}<t} W_{T_{i-1}} \mathbb{I}_{X_i < R_i^*} W_{X_i}\right)\left(\mathbb{I}_{t_{n_r-1}<t} W_{T_{n_r-1}} \mathbb{I}_{X_{n_r} > R_{n_r}^*} W_{X_{n_r}}\right), \tag{3.6}$$

where $n_r = 1, 2, \cdots$ is the number of the claim at which the surplus level drop below zero. Similarly, we derive the pathwise estimator for the finite-time ruin

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probabilities under SFRSS,

$$\hat{g}^{SFRSS}(\eta) = \left(\prod_{i=1}^{N_t} \mathbb{I}_{t_{i-1}<t} W_{T_{i-1}}\right) \mathbb{I}_{t_{N_t}>t} W_{T_{N_t}} \times \left(\prod_{i=1}^{n_r-1} \mathbb{I}_{V_{X_i}<R_i^*} W_i\right) I_{V_{X_{n_r}}>R_{n_r}^*} W_{n_r}. \quad (3.7)$$

The SFRDS and SFRSS pathwise estimators for the finite-time ruin probabilities have the following properties:

1. $\hat{g}^{\mathrm{SFRDS}}(\eta)$ and $\hat{g}^{\mathrm{SFRSS}}(\eta)$ are $\hat{C}^2$ since the composite of differentiable functions are differentiable;
2. the FD estimator of first- and second-order derivatives Glasserman (2004) under the new schemes are uniformly integrable.

Now, we can apply the pathwise method to $\hat{g}^{\mathrm{SFRDS}}(\eta)$ and $\hat{g}^{\mathrm{SFRSS}}(\eta)$ to construct unbiased estimators of the gradient and Hessian of the finite-ruin probabilities.

3.4. Automatic differentiation for multiple parameters of interest

Insurance companies and prudential regulators are interested in the sensitivities of the finite-time ruin probabilities with respect to more than one parameter, as well as the impact of their interactions on the probability of ruin. Calculating sensitivities in such multi-dimensional settings is often computationally demanding.

For stochastic systems computed by Monte Carlo simulations, automatic differentiation can be applied to compute sensitivities of functions (Griewank and Walther, 2008). It differentiates computer program functions, rather than the actual formulae of the expected performance measure. It is based on the fact that all computer program functions can be decomposed into a string of simple operations; the derivatives with respect to the input variables are then computed by a chain-rule-based technique. The adjoint version of automatic differentiation was introduced by Giles and Glasserman (2006) to derivative pricing to speed up the computation of Greeks for cases where there are a small number of outputs and a large number of inputs. Joshi and Pitt (2010) applied the adjoint method to sensitivity analysis of the valuation results for pension funds. They have shown that such simulation approach quickly produces a large number of sensitivities which can be combined to assess the effect of a wide range of possible scenarios.

The computation of Hessians has long been explored by the research community, especially by algorithmic means. One important method is the Backwards Algorithm Hessian introduced by Joshi and Yang (2010), their method computes all first- and second-order derivatives simultaneously in adjoint fashion. Their key result is that, given the Hessian H_G of a function, $G : \mathbb{R}^M \to \mathbb{R}$, if l is a elementary operation which is the identity mapping in all coordinate except one and in that coordinate depends on only one or two coordinates then the Hessian of $G \circ l$, $H_{G\circ l}$ can be computed by overwriting H_G with $AM + B$

additional operations for some constants A and B depending only on the class of elementary operations. However, their method is only applicable when the integrand is $\hat{C}^2$.

For financial products with discontinuous or angular payoffs, they solved the problem by smoothing the points of discontinuity or angularity, so the resulting discounted payoff function are continuously twice-differentiable. However, the pathwise estimators of the finite-time ruin probability simulated by the SFRSS method and the SFRDS method are $\hat{C}^2$ by construction, so we can apply the Backwards Algorithmic Hessian approach to them directly without smoothing to compute pathwise sensitivities of performance measure functions. Essentially, we decompose the algorithm into L elementary operations and apply the method in a backward fashion to compute the Hessian estimator with $L(AM + B)$ additional operations.

3.5. Numerical results for the gradient and Hessian of the finite-time ruin probabilities

We use the same set of parameters as to the numerical experiments in Section 2, and the sample size is 50,000 paths for each simulation.

3.5.1. *The classical model.* We consider $t = 10$ for $u = 10$ and $u = 20$, see Tables A1–A4. The results show that the gradient and Hessian computed by the two methods agree. When $u = 10$, $\psi(10, 10) = 0.0319$, for a sample of 50,000, the event ruin occurred in 1,533 paths by the direct method. The SFRDS method is better algorithm for estimating the sensitivities with the sum of standard errors equals to 2.302782 as opposed to 64.0927 by SFRSS for all first- and second-order sensitivities. When $u = 20$, the probability of ruin is 0.0002776 (by the stratified sampling method), ruin occurred in 17 paths by the direct method. The SFRSS method is better in this case since ruin is a rare event, the sum of standard errors produced by the SFRDS method is 0.695968 as opposed to 0.092013 by SFRSS for all first- and second-order sensitivities.

We further consider the computational times of the two methods for estimating first- and second-order sensitivities of the finite-time ruin probabilities. We set $t = 1, 2 \ldots, 15$ and $u = 20$ and plot the time taken by the two methods with the same set of parameters as above. The plot in figure 2 suggests that

- the SFRSS method introduces more computational efforts when the time is small and the probability of ruin is extremely small, since the number of non-zero paths is significantly more than the direct method;
- the SFRSS method reduces the computational efforts when the time is big and it forces ruin to occur at a faster rate.

3.5.2. *The Sparre Andersen model with interest.* We set $u = 30$ and $t = 10$, the probability of ruin is small in this case (ruin occurred in 53 paths by direct simulation), see Tables A5 and A6 for the results. The SFRSS method outper-

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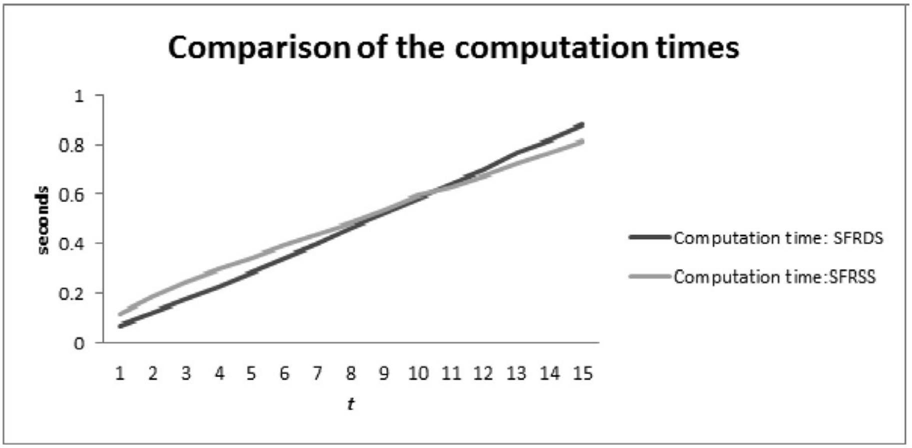

FIGURE 2: The classical model: the computation times of the SFRDS method and the SFRSS based on 50,000 samples.

forms this case with the sum of standard errors equals to 0.017551 as opposed to 0.1505458 by the SFRDS method. The time taken in this case under the SFRDS method is 0.473 seconds and the SFRSS method is 1.1415 seconds.

3.5.3. *The Periodic risk model with interest.* One important subtlety to point out about the thinning algorithm in Section 2.3 for generating the non-homogeneous Poisson process is its inherent discontinuities at each acceptance–rejection point, i.e. if

$$V_D \leq \frac{a + b\cos(2\pi c t_s)}{a+b},$$

set $T_s = t_s$ and accept, else reject. A small bump in the parameters of interest may alter the acceptance–rejection decision, and consequently results in different final accepted outcomes from the bumped path and the unbumped path. To remove such pathwise discontinuities, we use the OSRS algorithm introduced by Joshi and Zhu (2014b). A changed of measure is performed on each V_D to ensure the bumped path makes the same acceptance–rejection decision as the unbumped path.

We set $t = 10$ and $u = 25$, the probability of ruin is small in this case, see Tables A7 and A8 for the results. There are only 24 paths of the direct simulation with non-zero estimates. The SFRSS method outperforms in this case with the sum of standard errors equals to 0.287805 as opposed to 1.192388 by the SFRDS method. The time taken in this case under the SFRDS method is 0.707 seconds and the SFRSS method is 1.416 seconds.

The numerical results show that the SFRSS method is more appropriate for cases where ruin is rare, because it produces unbiased non-zero pathwise estimators for all paths with at least one claim. Unlike the the SFRSS method, the SFRDS method only produces non-zero estimates when ruin occurs, thus it suffers the following deficits when the event ruin is rare:

- it produces estimates with large sample standard errors;
- the estimates produced by a few significant paths are unlikely to be reliable.

3.6. The density of the time to ruin

In recent years, the Risk theory community has focussed on the actual distribution of the time to ruin. So far, its density only can be derived analytically for a few special cases, and often requires an enormous amount of computational effort to compute.

In this section, we estimate the density function of the time to ruin for the three models we considered via Monte Carlo simulation. Since the finite-time ruin probability,

$$\psi(u, t) = \mathbb{P}(\tau(u) \leq t),$$

is the cumulative density function of the finite-time ruin probabilities, the probability density function of $\tau(u)$ is then

$$f_{\tau(u)}(t) = \frac{\partial \psi(u, t)}{\partial t}.$$

Both the SFRSS method and the SFRDS method provide unbiased estimates of the above density.

3.6.1. *The classical model.* We first compare the density estimated by the SFRSS method and the SFRDS method against the analytic results. The density of the time to ruin can be computed explicitly for the classical model with exponential claim size by the following formula:

$$f_{\tau(u)}(t) = \lambda \exp\left(-\mu u - (\lambda + \mu c)t\right) \Bigg(I_0\left((4\mu\lambda t(u + ct))^{0.5}\right) - \frac{ct}{ct + u} I_2\left((4\mu\lambda t(u + ct))^{0.5}\right) \Bigg),$$

where I_i is the modified Bessel function of the ith kind (Dickson, 2007). The numerical experiments in this section is performed by setting $u = 5$ and $\theta = 0.25$, the other parameters are the same as Section 2. In order to create a smooth graph, we use 500,000 paths for each simulation and 60 equally spaced-time points over a period of 15 years.

The density estimated by the SFRDS method (the unsmooth dark line) is plotted against the analytic formula (the smooth dotted line), we also included the line representing the density estimated plus one standard error and the line representing the density estimated minus one standard error to show the convergence of the estimated density function, see Figure 3. The maximum obtained is

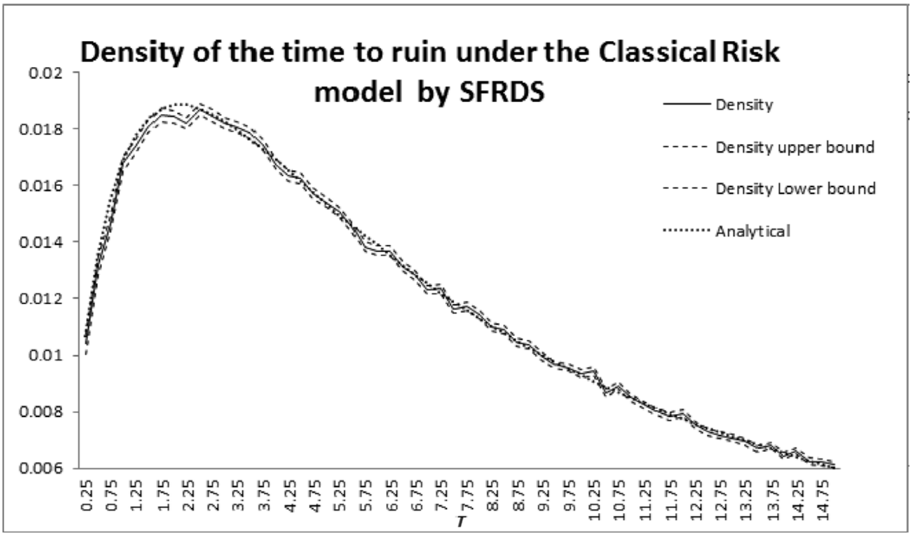

FIGURE 3: The classical risk model: the density of the time to ruin estimated by setting $u = 5$ and $\theta = 0.25$ with 500,000 paths sample using SFRDS.

0.01869 with 2.087E-04 standard error when $t = 2.5$, and the corresponding analytically result is 0.01873. The minimum obtained is 0.00613 when $t = 15$ with 1.110E-04 standard error and the corresponding analytical result is 0.00603. The maximum absolute difference between the density estimated by SFRDS and the analytical density is 6.452E-04 when $t = 2.25$, the corresponding Monte Carlo standard error is 4.105E-04. The minimum absolute difference between the density estimated by SFRSS and the analytical density is 4.275E-06 when $t = 13.25$, the corresponding Monte Carlo standard error is 1.119E-04.

In Figure 4, we plot the density estimated by SFRSS against the analytical results. The maximum obtained is 0.01909 when $t = 2.5$ with 3.407E-04 standard error, and the corresponding analytically result is 0.01873. The minimum obtained is 0.00618 when $t = 15$ with 7.480E-04 standard error and the corresponding analytical result is 0.00603. The maximum absolute difference between the density estimated by SFRSS and the analytical density is 9.116E-04 when $t = 14.25$, the corresponding Monte Carlo standard error is 7.320E-04. The minimum absolute difference between the density estimated by SFRSS and the analytical density is 2.065E-05 when $t = 1.25$, the corresponding Monte Carlo standard error is 3.008E-04. The numerical results demonstrates

- both the SFRSS and the SFRDS methods produce unbiased estimates of the density function,
- when t is small, it is preferable to use the SFRSS method since the corresponding finite-time ruin probability is small,
- when t is big, it is preferable to use the SFRDS method since the corresponding finite-time ruin probability is big.

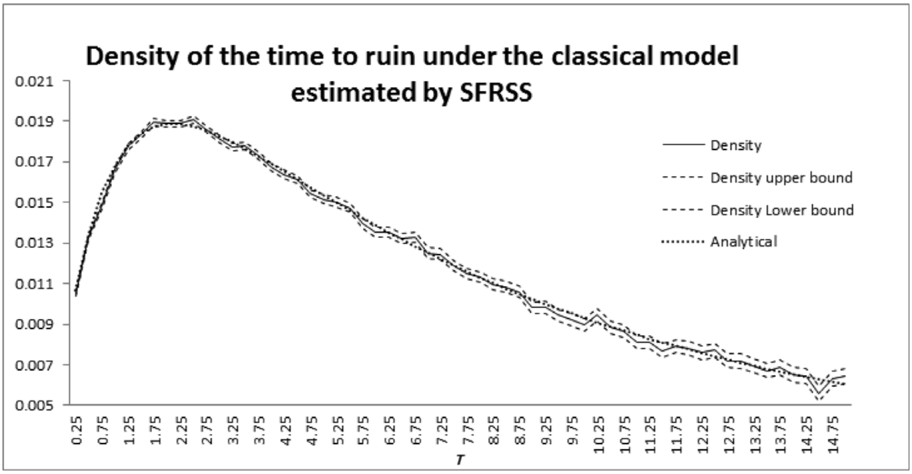

FIGURE 4: The classical risk model: the density of the time to ruin estimated by setting $u = 5$ and $\theta = 0.25$ with 500,000 paths sample using SFRSS.

3.6.2. *The Sparre Andersen risk model with interest.* Insurance companies typically hold large initial capital for meeting the prudential requirements. The density of the time to ruin sheds important insights into the risk profile over a long term horizon. In figure 5 the density of the time to ruin over the period [0, 10] under the Sparre Andersen model with interest is plotted. For the plot, we assume $u = 30, \theta = 0.1$, and the other parameters the same as Section 2. The finite-time probability $\psi(30, 10)$ is 0.00078 by the stratified sampling algorithm, therefore, it is preferable to use the SFRSS method. The graph plotted demonstrates a low probability of ruin at the beginning, follows by a period of higher ruin probabilities. The ruin probability gradually decreases due to the premium collection with a loading factor at 10% as well as the interest accumulation at 10%. Based on the risk profile illustrated by the density function, insurance companies can design their pricing as well as risk management strategies to reflect such pattern. For instance, companies need to hold large capitals during the early years of the risk business, and some portion of the capital may be released later if significant surpluses have been accumulated.

In order to create a smooth graph, we use 5,000,000 paths for each simulation and 40 equally spaced-time points over a period of 10 years. The maximum obtained is 1.841E-04 when $t = 1$ with 2.7025E-06 standard error. The minimum obtained is 2.609E-05 when $t = 10$ with 2.835E-06 standard error. The time taken to produce the density estimated is 53 minutes and 6.75 seconds.

3.6.3. *The periodic risk model with interest.* Regulators are interested in the density of the time to ruin after the surplus level reaches zero. It provides information on the possibility of the surplus recovery. In particular, the seasonal pattern of insurance claims plays an important role in the performance of

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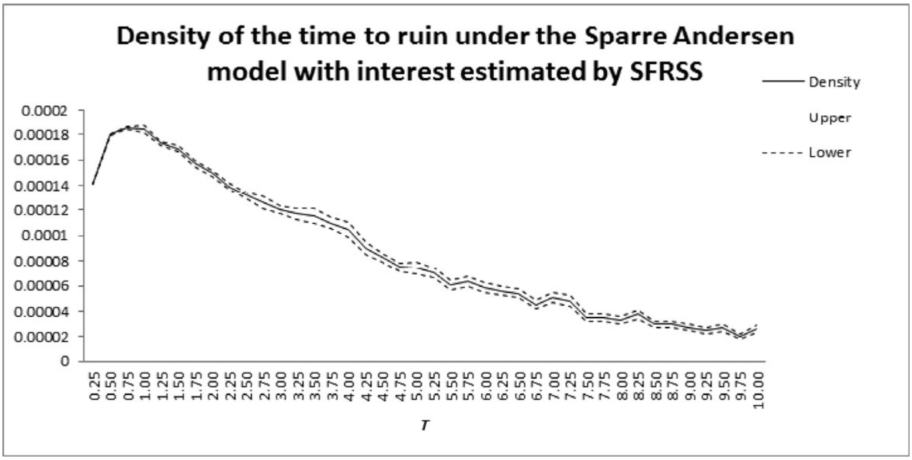

FIGURE 5: The Sparre Andersen with interest risk model with Erlang(2,2) inter-claim times and Pareto(3,2)claim sizes: the density of the time to ruin estimated by setting $u = 30$ and $\theta = 0.1$ with 5,000,000 paths sample using SFRSS.

the insurance companies, it allows companies to recover after significant losses over the period with low claim arrival intensity. The periodic risk model captures such phenomena.

In figure 6 we plot the density of the time to ruin over the period [0, 10] under the periodic risk model with interest by setting $u = 0$, $b = 0.9$ and $c = 0.25$, the other parameters are in Section 2. The density is produced by the SFRDS method, since the probability of ruin in this case is large, even for $t = 0.25$. The graph plotted demonstrates that once the insurance portfolio survives through the initial period of high claim arrival intensity, it will recover its surplus level through the premium collection with a loading factor at 10% as well as the interest accumulation at 10%. The density of the time to ruin has a cyclical pattern due to the periodic risk structure, however the probability of ruin is generally low after the initial couple of cycles (4 years in this case).

In order to create a smooth graph, we use 50,000 paths for each simulation and 40 equally spaced-time points over a period of 10 years. The maximum obtained is 0.72433 when $t = 0.25$ with 0.00629 standard error. The minimum obtained is 5.660E-04 when $t = 10$ with 1.350E-04 standard error. The time taken to produce the density estimated is 17.483 seconds.

4. REGULATORY CAPITAL AND ITS SENSITIVITIES

Regulatory capital requirements for insurers are the focus of the current development of a global framework for insurer solvency assessment. Risk theory is well known for its theoretic approach to the macro-analysis of underwriting portfolio risk, that assists insurance companies to answer the question "how

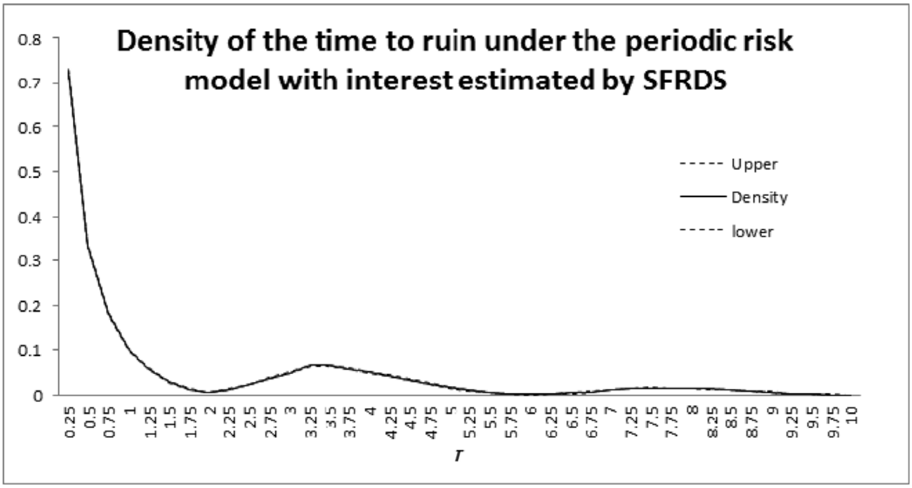

FIGURE 6: The periodic with interest risk model with $LN(-0.5, 1)$ claim sizes: the density of the time to ruin estimated by setting $u = 0$ and $\theta = 0.1$ with 50,000 paths sample using SFRDS.

much capital to hold, so the finite-time probabilities is smaller than $\alpha\%$". To solve this question, we need to invert equation (1.7), i.e. solve

$$\psi^{-1}(\alpha, t) = u^*.$$

4.1. Numerically approximating u^* by the Newton–Raphson method

The Newton–Raphson method gives a numerical solution to this question. With an initial guess $\hat{u}_0$, the algorithm is performed iteratively as

$$\hat{u}_{i+1} = \hat{u}_i + \frac{\alpha - \psi(\hat{u}_i, t)}{\frac{\partial\psi}{\partial u}(\hat{u}_i, t)}, \tag{4.1}$$

until $|\psi(\hat{u}, t) - \alpha| < \epsilon$. Since the finite-time ruin probability ψ and its derivative with respect to u need to be computed repetitively, efficient algorithms for computing them are critical for the iterative process to converge. Generally, we do not have the luxury of analytical solutions for ψ and $\frac{\partial\psi}{\partial u}$; even for cases where they do exist, using them for such iterative calculation is not practically due to the computational cost. For the following numerical experiments, we use the stratified sampling algorithm for ψ and the SFRSS method for $\frac{\partial\psi}{\partial u}$, since the value of α is typically small in practice.

We perform experiments to numerically search for the regulatory capital under the three models we considered. The parameters are in Section 2, for a finite time horizon of $t = 10$ and $\alpha = 0.1\%$. The results are summarized in Tables 4–6. The numerical experiments show that we can approximate the regulatory capital u^* using less than five iterations to reduce $|\psi(\hat{u}, t) - 0.001| < 10^{-5}$ for the three models considered.

TABLE 4

THE CLASSICAL RISK MODEL: THE ESTIMATED REGULATORY CAPITAL FOR $\alpha = 0.1\%$ AND ITS SENSITIVITIES WITH 50,000 PATHS SAMPLE.

	u^*	$\frac{\partial u^*}{\partial \theta}$	$\frac{\partial u^*}{\partial t}$	$\frac{\partial u^*}{\partial \lambda}$	$\frac{\partial u^*}{\partial \mu}$
Mean	18.1647	−8.1679	0.7280	7.2805	18.1648
S.E.	0.1878	0.5962	0.1550	1.5499	1.3445

TABLE 5

THE SPARRE ANDERSEN RISK MODEL WITH INTEREST: THE ESTIMATED REGULATORY CAPITAL FOR $\alpha = 0.1\%$ AND ITS SENSITIVITIES WITH 50,000 PATHS SAMPLE.

	u^*	$\frac{\partial u^*}{\partial \theta}$	$\frac{\partial u^*}{\partial t}$	$\frac{\partial u^*}{\partial \beta}$	$\frac{\partial u^*}{\partial r}$	$\frac{\partial u^*}{\partial a}$	$\frac{\partial u^*}{\partial b}$
Mean	27.5705	−3.7089	0.4329	6.8424	−93.5544	13.6638	−27.6281
S.E.	0.1179	0.2525	0.0787	0.5301	4.9944	0.6309	1.1929

TABLE 6

THE PERIOD RISK MODEL WITH INTEREST: THE ESTIMATED REGULATORY CAPITAL FOR $\alpha = 0.1\%$ AND ITS SENSITIVITIES WITH 50,000 PATHS SAMPLE.

	u^*	$\frac{\partial u^*}{\partial \theta}$	$\frac{\partial u^*}{\partial t}$	$\frac{\partial u^*}{\partial a}$	$\frac{\partial u^*}{\partial b}$	$\frac{\partial u^*}{\partial c}$	$\frac{\partial u^*}{\partial r}$	$\frac{\partial u^*}{\partial \nu}$	$\frac{\partial u^*}{\partial \sigma}$
Mean	21.3467	−3.5519	0.1087	6.9633	0.4363	−4.0693	−52.4456	21.3467	67.79286
S.E.	0.2548	0.21683	0.0539	2.1794	0.0355	0.7668	2.16133	0.8590	2.0081

4.2. Sensitivities of u^*

Our next objective is to perform sensitivity analysis of the regulatory capital. Since the value u^* is estimated based on the current assumptions made about the underlying risk process, insurance companies are also interested in how they should adjust the capital, u^*, in response to changes in the risk process. The derivatives of the value u^* with respect to the parameters of the underlying risk model provide a solution to this question.

To compute $\frac{\partial u^*}{\partial \eta}$, we rely on the following relationship,

$$\alpha = \psi(u^*, t, \eta),$$

where $\psi(u^*, t, \eta)$ is the finite-time probability of ruin given the current distributional and structural parameters. Since $u^* = \psi^{-1}(\alpha, t, \eta)$, we have

$$\alpha = \psi(\psi^{-1}(\alpha, t, \eta), t, \eta).$$

Differentiate the above expression with respect to η, and obtain

$$0 = \frac{\partial \psi}{\partial \eta}(u^*, t) + \frac{\partial \psi}{\partial u}(u^*, t)\frac{\partial u^*}{\partial \eta},$$

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

so that

$$\frac{\partial u^*}{\partial \eta} = -\frac{\partial \psi}{\partial \eta}(u^*, t) / \frac{\partial \psi}{\partial u}(u^*, t). \tag{4.2}$$

Since we have already obtained the regulatory capital by the Newton–Raphson method and its sensitivities by the SFRSS method, the sensitivities of u^* is then easily computable. It allows the practitioners to focus more on selecting an appropriate valuation basis, than on computations of the sensitivities.

Numerical results are summarized in Tables 4–6 for the three models. Standard errors are for 50,000 paths and these could be reduced by sampling more paths. Based on the numerical results, the most important factor influencing the regulatory capital is the distributional parameters of the claim size. For the Sparre Andersen with interest and the periodic risk model with interest, another critical parameter is the deterministic interest rate. Insurance companies should constantly monitor their underlying insurance portfolios to obtain accurate estimates of the claim size distribution, especially for businesses with heavy tail nature. In addition, adequate estimations of the business cycle are crucial for insurance companies which rely heavily on their investment portfolio returns.

5. Conclusion

We have introduced a stratified sampling method for computing finite-time ruin probabilities, which outperforms the direct simulation method when ruin is rare. We perform changes of measure to remove discontinuities of the pathwise estimators of ruin, so the pathwise method is applicable to provide unbiased estimates of first- and second-order sensitivities of the finite-time ruin probabilities. The derivative with respect to the finite-time horizon, t, provides a way of density estimation for the time to ruin random variable, especially when analytical approximations are not feasible. We further use the Newton–Raphson method to estimate the regulatory capital, and as well as its sensitivities. Numerical experiments on the classical, the Sparre Andersen with interest and the periodic risk model with interest, are performed to demonstrate the validity and the efficiency of the methods suggested,

Whilst our methods are fast, we have not explored other methods of acceleration such as the use of parallel processing and further methodologies for variance reduction such as randomized quasi-Monte Carlo. We would like to emphasis on the applicability of the methods introduced in this paper for cases where analytical results do not exist, and its ability to often produce unbiased results within seconds.

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APPENDIX A. TABLES OF NUMERICAL RESULTS

TABLE A1

THE CLASSICAL MODEL: THE MEAN ×1000 OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN u = 10.

SFRSS vs SFRDS	u	θ	t	λ	μ
u	3.913 vs 4.114	26.319 vs 26.152	−1.99 vs −1.35	−19.898 vs −13.505	−26.304 vs −29.425
θ	26.319 vs 26.152	217.779 vs 209.952	−21.937 vs −17.709	−219.369 vs −177.095	−263.195 vs −261.516
t	−1.99 vs −1.35	−21.937 vs −17.709	0.039 vs 0.063	4.686 vs 5.702	19.898 vs 13.505
λ	−19.898 vs −13.505	−219.369 vs −177.095	4.686 vs 5.702	−138764.653 vs −9327.25	198.978 vs 135.05
μ	−26.304 vs −29.425	−263.195 vs −261.516	19.898 vs 13.505	198.978 vs 135.05	134.79 vs 177.059
First Orders	−12.825 vs −11.719	−90.858 vs −83.461	5.074 vs 5.074	50.741 vs 50.738	128.254 vs 117.187

TABLE A2

THE CLASSICAL MODEL: THE STANDARD ERRORS ×1000 OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN u = 10.

SFRSS vs SFRDS	u	θ	t	λ	μ
u	0.349 vs 0.357	2.206 vs 2.244	0.411 vs 0.148	4.111 vs 1.48	3.411 vs 3.256
θ	2.206 vs 2.244	17.235 vs 17.836	3.319 vs 1.535	33.19 vs 15.35	22.056 vs 22.44
t	0.411 vs 0.148	3.319 vs 1.535	1.533 vs 0.388	16.078 vs 4.077	4.111 vs 1.48
λ	4.111 vs 1.48	33.19 vs 15.35	16.078 vs 4.077	63757.311 vs 2120.802	41.113 vs 14.804
μ	3.411 vs 3.256	22.056 vs 22.44	4.111 vs 1.48	41.113 vs 14.804	33.989 vs 29.768
First Orders	0.491 vs 0.462	3.38 vs 3.099	1.196 vs 0.322	11.962 vs 3.217	4.915 vs 4.623

TABLE A3

THE CLASSICAL MODEL: THE MEAN ×1000 OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN u = 20.

SFRSS vs SFRDS	u	θ	t	λ	μ
u	0.13 vs 0.114	1.018 vs 0.892	−0.058 vs −0.065	−0.582 vs −0.648	−2.409 vs −2.072
θ	1.018 vs 0.892	8.299 vs 7.284	−0.527 vs −0.683	−5.271 vs −6.835	−20.358 vs −17.831
t	−0.058 vs −0.065	−0.527 vs −0.683	−0.094 vs −0.068	0.872 vs 0.792	1.163 vs 1.296
λ	−0.582 vs −0.648	−5.271 vs −6.835	0.872 vs 0.792	−62.847 vs −628.937	11.63 vs 12.959
μ	−2.409 vs −2.072	−20.358 vs −17.831	1.163 vs 1.296	11.63 vs 12.959	44.365 vs 37.378
First Orders	−0.191 vs −0.203	−1.46 vs −1.597	0.071 vs 0.116	0.706 vs 1.159	3.825 vs 4.064

TABLE A4

THE CLASSICAL MODEL: THE STANDARD ERRORS ×1000 OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN $u = 20$.

SFRSS vs SFRDS	u	θ	t	λ	μ
u	0.032 vs 0.047	0.254 vs 0.353	0.033 vs 0.041	0.328 vs 0.406	0.588 vs 0.885
θ	0.254 vs 0.353	2.236 vs 2.831	0.275 vs 0.416	2.75 vs 4.159	5.073 vs 7.065
t	0.033 vs 0.041	0.275 vs 0.416	0.035 vs 0.114	0.319 vs 1.188	0.656 vs 0.812
λ	0.328 vs 0.406	2.75 vs 4.159	0.319 vs 1.188	42.908 vs 629.389	6.564 vs 8.119
μ	0.588 vs 0.885	5.073 vs 7.065	0.656 vs 0.812	6.564 vs 8.119	10.92 vs 16.7
First Orders	0.06 vs 0.06	0.435 vs 0.467	0.046 vs 0.059	0.464 vs 0.589	1.196 vs 1.194

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

TABLE A5

THE SPARRE ANDERSEN MODEL WITH INTEREST: THE MEAN ×10000 OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN u = 30.

SFRSS vs SFRDS	u	θ	t	β	r	a	b
u	0.111 vs 0.114	0.431 vs 0.379	−0.079 vs –0.065	−0.815 vs –0.699	8.442 vs 7.81	–1.234 vs –1.297	2.324 vs 2.386
θ	0.431 vs 0.379	3.228 vs 2.323	−0.469 vs –0.062	−5.042 vs –1.763	53.926 vs 41.43	−6.461 vs –5.691	10.667 vs 9.384
t	−0.079 vs –0.065	−0.469 vs –0.062	1.312 vs 0.512	8.019 vs 2.44	−22.246 vs –12.239	1.18 vs 0.75	−2.67 vs –1.004
β	−0.815 vs –0.699	−5.042 vs –1.763	8.019 vs 2.44	43.101 vs 12.762	−120.136 vs –127.828	12.23 vs 8.107	−25.79 vs –26.365
r	8.442 vs 7.81	53.926 vs 41.43	−22.246 vs –12.239	−120.136 vs –127.828	1365.632 vs 1093.043	−126.633 vs –117.15	248.791 vs 227.662
a	−1.234 vs –1.297	−6.461 vs –5.691	1.18 vs 0.75	12.23 vs 8.107	−126.633 vs –117.15	12.009 vs 13.151	−34.866 vs –35.79
b	2.324 vs 2.386	10.667 vs 9.384	−2.67 vs –1.004	−25.79 vs –26.365	248.791 vs 227.662	−34.866 vs –35.79	82.175 vs 81.925
First Orders	−0.867 vs –0.84	−2.897 vs –2.757	0.694 vs 0.411	7.705 vs 5.865	−84.746 vs –76.04	12.998 vs 12.595	−26.993 vs –25.942

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

TABLE A6

THE SPARRE ANDERSEN MODEL WITH INTEREST: THE STANDARD ERRORS $\times 10,000$ OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN $u = 30$.

SFRSS vs SFRDS	u	θ	t	β	r	a	b
u	0.019 vs 0.005	0.142 vs 0.038	0.076 vs 0.039	0.425 vs 0.193	1.968 vs 0.291	0.223 vs 0.065	0.407 vs 0.101
θ	0.142 vs 0.038	1.743 vs 0.688	0.448 vs 0.254	2.616 vs 1.261	16.362 vs 3.329	2.125 vs 0.574	3.331 vs 0.842
t	0.076 vs 0.039	0.448 vs 0.254	1.392 vs 1.332	8.353 vs 6.717	21.263 vs 2.268	1.133 vs 0.585	2.552 vs 0.944
β	0.425 vs 0.193	2.616 vs 1.261	8.353 vs 6.717	45.608 vs 33.291	113.819 vs 11.85	6.382 vs 2.902	14.164 vs 4.692
r	1.968 vs 0.291	16.362 vs 3.329	21.263 vs 2.268	113.819 vs 11.85	382.783 vs 31.282	29.524 vs 4.37	57.501 vs 7.091
a	0.223 vs 0.065	2.125 vs 0.574	1.133 vs 0.585	6.382 vs 2.902	29.524 vs 4.37	2.412 vs 0.89	6.109 vs 1.518
b	0.407 vs 0.101	3.331 vs 0.842	2.552 vs 0.944	14.164 vs 4.692	57.501 vs 7.091	6.109 vs 1.518	13.778 vs 2.549
First Orders	0.138 vs 0.031	0.741 vs 0.384	0.664 vs 0.344	3.734 vs 1.69	17.435 vs 1.986	2.065 vs 0.459	4.221 vs 0.722

TABLE A7

THE PERIODIC RISK MODEL WITH INTEREST: THE MEAN $\times 1,000$ OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN $u = 25$.

SFRSS vs SFRDS	u	θ	t	a	b	c	r	ν	σ
u	0.016 vs 0.036	0.039 vs 0.045	−0.014 vs −0.001	−0.093 vs −0.706	0.123 vs 0.556	0.135 vs 0.547	0.818 vs 2.056	−0.34 vs −0.82	−1.083 vs −2.026
θ	0.039 vs 0.045	0.196 vs 0.65	−0.049 vs −0.007	−0.987 vs −0.839	−0.062 vs −0.464	1.148 vs 2.784	3.397 vs 5.344	−0.974 vs −1.617	−2.749 vs −3.587
t	−0.014 vs −0.001	−0.049 vs −0.007	0.008 vs 0.003	0.022 vs 0.013	−0.002 vs −0.028	0.096 vs 0.115	−0.348 vs −0.874	0.13 vs 0.128	0.335 vs 0.415
a	−0.093 vs −0.706	−0.987 vs −0.839	0.022 vs 0.013	−25.136 vs −25.223	−23.01 vs −21.567	26.237 vs 10.117	−6.097 vs −6.213	2.335 vs 1.661	7.069 vs 6.528
b	0.123 vs 0.556	−0.062 vs −0.464	−0.002 vs −0.028	−23.01 vs −21.567	−21.107 vs −21.391	22.213 vs 13.298	7.23 vs 6.081	−3.065 vs −3.906	−9.876 vs −7.021
c	0.135 vs 0.547	1.148 vs 2.784	0.096 vs 0.115	26.237 vs 10.117	22.213 vs 13.298	−46.792 vs −34.892	11.39 vs 14.173	−3.363 vs −3.667	−10.182 vs −6.42
r	0.818 vs 2.056	3.397 vs 5.344	−0.348 vs −0.874	−6.097 vs −6.213	7.23 vs 6.081	11.39 vs 14.173	84.873 vs 69.498	−20.45 vs −31.393	−69.441 vs −80.81
ν	−0.34 vs −0.82	−0.974 vs −1.617	0.13 vs 0.128	2.335 vs 1.661	−3.065 vs −3.906	−3.363 vs −3.667	−20.45 vs −31.393	8.491 vs 20.506	27.077 vs 19.922
σ	−1.083 vs −2.026	−2.749 vs −3.587	0.335 vs 0.415	7.069 vs 6.528	−9.876 vs −7.021	−10.182 vs −6.42	−69.441 vs −80.81	27.077 vs 19.922	88.752 vs 127.542
First Orders	−0.065 vs −0.086	−0.197 vs −0.178	0.046 vs 0.042	0.587 vs 0.756	−0.099 vs −0.081	−0.633 vs −0.844	−4.038 vs −4.769	1.625 vs 2.159	5.831 vs 5.956

https://doi.org/10.1017/asb.2016.5 Published online by Cambridge University Press

TABLE A8

THE PERIODIC RISK MODEL WITH INTEREST: THE STANDARD ERRORS ×1, 000 OF FINITE-TIME RUIN PROBABILITIES DERIVATIVES BY THE SFRSS METHOD AND THE SFRDS WITH 50,000 PATHS SAMPLE WHEN $u = 25$.

SFRSS vs SFRDS	u	θ	t	a	b	c	r	ν	σ
u	0.003 vs 0.015	0.014 vs 0.072	0.006 vs 0.001	0.063 vs 0.47	0.06 vs 0.61	0.095 vs 0.413	0.186 vs 1.133	0.067 vs 0.353	0.183 vs 0.869
θ	0.014 vs 0.072	0.067 vs 0.389	0.025 vs 0.006	0.401 vs 2.644	0.404 vs 3.427	0.597 vs 2.247	1.06 vs 6.813	0.34 vs 1.804	0.939 vs 4.352
t	0.006 vs 0.001	0.025 vs 0.006	0.173 vs 0.003	0.361 vs 0.012	0.398 vs 0.003	1.933 vs 0.054	0.695 vs 0.12	0.146 vs 0.022	0.521 vs 0.067
a	0.063 vs 0.47	0.401 vs 2.644	0.361 vs 0.012	10.135 vs 9.795	8.872 vs 9.499	11.813 vs 13.015	3.967 vs 48.91	1.573 vs 11.741	4.562 vs 31.791
b	0.06 vs 0.61	0.404 vs 3.427	0.398 vs 0.003	8.872 vs 9.499	8.173 vs 12.708	11.15 vs 11.429	4.234 vs 66.002	1.498 vs 15.251	4.57 vs 41.41
c	0.095 vs 0.413	0.597 vs 2.247	1.933 vs 0.054	11.813 vs 13.015	11.15 vs 11.429	26.246 vs 16.415	8.417 vs 44.125	2.373 vs 10.32	7.494 vs 28.225
r	0.186 vs 1.133	1.06 vs 6.813	0.695 vs 0.12	3.967 vs 48.91	4.234 vs 66.002	8.417 vs 44.125	20.676 vs 120.206	4.659 vs 28.33	14.533 vs 73.6
ν	0.067 vs 0.353	0.34 vs 1.804	0.146 vs 0.022	1.573 vs 11.741	1.498 vs 15.251	2.373 vs 10.32	4.659 vs 28.33	1.678 vs 8.83	4.584 vs 21.733
σ	0.183 vs 0.869	0.939 vs 4.352	0.521 vs 0.067	4.562 vs 31.791	4.57 vs 41.41	7.494 vs 28.225	14.533 vs 73.6	4.584 vs 21.733	12.982 vs 55.241
First Orders	0.007 vs 0.027	0.033 vs 0.109	0.019 vs 0.001	0.208 vs 0.775	0.225 vs 0.993	0.304 vs 0.683	0.59 vs 1.956	0.167 vs 0.681	0.525 vs 1.814