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# THE SMOOTHNESS OF ORBITAL MEASURES ON NONCOMPACT SYMMETRIC SPACES

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**Abstract**

Let $G/K$ be an irreducible symmetric space, where $G$ is a noncompact, connected Lie group and $K$ is a compact, connected subgroup. We use decay properties of the spherical functions to show that the convolution product of any $r = r(G/K)$ continuous orbital measures has its density function in $L^2(G)$ and hence is an absolutely continuous measure with respect to the Haar measure. The number $r$ is approximately the rank of $G/K$. For the special case of the orbital measures, $\nu_{a_i}$, supported on the double cosets $Ka_iK$, where $a_i$ belongs to the dense set of regular elements, we prove the sharp result that $\nu_{a_1} * \nu_{a_2} \in L^2$, except for the symmetric space of Cartan class $AI$ when the convolution of three orbital measures is needed (even though $\nu_{a_1} * \nu_{a_2}$ is absolutely continuous).

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## 1. Introduction

Let $G$ be a real, connected, noncompact, semisimple Lie group with finite center, and $K$ a maximal compact subgroup of $G$. The quotient space, $G/K$, is a symmetric space of noncompact type, which we also assume to be irreducible. For $a \in G \diagdown N_G(K)$, we let $\nu_a$ denote the $K$-bi-invariant, orbital, singular measure supported on the compact double coset $KaK$ in $G$. The smoothness properties of convolution products of these orbital measures has been of interest for many years and is related to questions about products of double cosets and spherical functions. Ragozin, in [21], proved that for $r \geq \dim G/K$, the convolution product measure, $\nu_{a_1} * \cdots * \nu_{a_r}$, is absolutely continuous with respect to any Haar measure on $G$; equivalently, its density function is a compactly supported function in $L^1(G)$. This was improved in a series of papers, culminating with

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[8, 13], where $r$ was reduced to either rank $G/K$ or rank $G/K + 1$ depending on the Lie type. See [11] for a good history of this problem.

For the special case of regular elements, $a_j$, it was shown in [2] that the density function of $\nu_{a_1} * \cdots * \nu_{a_r}$ belongs to the smaller space of compactly supported functions in $L^2(G)$ for $r \geq \dim G/K + 1$. The decay properties of spherical functions and the Plancherel theorem were used to prove this. In this paper, we develop a more refined analysis of the decay properties of spherical functions, using the classification of these symmetric spaces in terms of their restricted root systems, to significantly improve this result. This analysis allows us to both extend the $L^2$ result to convolutions of *all* orbital measures $\nu_a$ for $a \notin N_G(K)$, as well as to reduce the number of convolution products to approximately rank $G/K$; the precise values are given in Section 4 and depend only on the Lie and Cartan types of the symmetric space. In the special case of convolution products of orbital measures at regular elements, we prove that $r = 2$ suffices, except for one symmetric space (Cartan class $AI$ of rank one), where $r = 3$ is both necessary and sufficient. This latter fact shows that, unlike the situation for the analogous problem in compact Lie groups and algebras, it is not true that $\nu_a^k$ belongs to $L^2$ if and only if $\nu_a^k$ is absolutely continuous (where the exponent means convolution powers). The decay properties are also applied to study the differentiability of orbital measures.

## 2. Notation and basic facts

**2.1. Lie algebra setup.** Let $G$ be a real, connected, noncompact, semisimple Lie group with finite center and let $K$ be a maximal compact subgroup of $G$ fixed by the Cartan involution $\theta$. We assume that $G/K$ is irreducible. The quotient space, $G/K$, is a symmetric space of noncompact type III in Helgason's terminology [19]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$, where $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. We fix a maximal abelian (as a subalgebra of $\mathfrak{g}$) subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\mathfrak{a}^*$ denote its dual. The rank of $G/K$ is the dimension of $\mathfrak{a}$. If we put $A = \exp \mathfrak{a}$, where $\exp : \mathfrak{g} \to G$ is the exponential function, then $G = KAK$.

The set of restricted roots, $\Phi$, is defined by

$$\Phi = \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq 0\},$$

where $\mathfrak{g}_\alpha$ are the root spaces. The multiplicity of the restricted root $\alpha$ is denoted

$$m_\alpha = \dim \mathfrak{g}_\alpha.$$

The subset of positive restricted roots is denoted $\Phi^+$. The set $\Phi$ is a root system, although not necessarily reduced, as it is possible for both $\alpha$ and $2\alpha$ to be in $\Phi$.

Take a basis $\mathcal{B}$ for $\mathfrak{a}^*$ consisting of positive simple roots and let $\mathfrak{a}^+$ be the elements $H \in \mathfrak{a}$ with $\alpha(H) > 0$ for all $\alpha \in \mathcal{B}$. Similarly, let $\mathcal{D} \subseteq \mathfrak{a}$ be the dual basis to $\mathcal{B}$ and let

$$\mathfrak{a}^{*+} = \{\lambda \in \mathfrak{a}^* : \lambda(H) > 0 \text{ for all } H \in \mathcal{D}\}.$$

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We have $\mathfrak{a}^* = \bigcup_{w \in W} w(\overline{\mathfrak{a}^{*+}})$ for $W$ equal to the Weyl group, with a similar statement holding for $\mathfrak{a}$.

Consequently, $G = K\overline{\exp \mathfrak{a}^+}K$. Indeed, given any $g \in G$, there are a pair $k_1, k_2 \in K$ and a unique $X_g \in \overline{\mathfrak{a}^+}$ such that $g = k_1(\exp X_g)k_2$. We can thus view $\lambda \in \mathfrak{a}^*$ as also acting on $g \in A$ by setting $\lambda(g) = \lambda(X_g)$.

The symmetric spaces can be classified by their Cartan class and the Lie type of their restricted root system, these being one of types $A_n, B_n, C_n, BC_n, D_n$ (the classical types) or $G_2, F_4, E_6, E_7, E_8$ (the exceptional types), the subscript in all cases being the rank of the symmetric space. We remark that for types $B_n$ and $C_n$, we may assume that $n \geq 2$, as the symmetric spaces of Lie types $B_1$ and $C_1$ are isomorphic to type $A_1$. Similarly, with $D_n$, we may assume that $n \geq 4$. For more details, please see the appendix.

For further background on this material and proofs of the facts stated above we refer the reader to [18–20].

**2.2. Orbital measures.** Next, we introduce the orbital measures of interest in this paper. We let $dm$ denote normalized Haar measure on $K$.

DEFINITION 2.1. Let $a \in A$. By an *orbital measure on* $G$, we mean the measure denoted $\nu_a$ defined by the rule

$$\int_G f(g)\, d\nu_a(g) = \int_K \int_K f(k_1 a k_2)\, dm(k_1)\, dm(k_2)$$

for all continuous, compactly supported functions $f$ on $G$.

The orbital measure $\nu_a$ is the $K$-bi-invariant, probability measure supported on the compact, double coset $KaK \subseteq G$. Orbital measures are always singular with respect to Haar measure on $G$ and they are continuous measures (that is, have no atoms) when $a \notin N_G(K)$, the normalizer of $K$ in $G$.

It is a classical problem to study the smoothness of convolution products of continuous orbital measures. Some of the earliest work was done by Ragozin in [21], who showed that $\nu_{a_1} * \cdots * \nu_{a_r}$ is absolutely continuous if and only if the product of double cosets, $Ka_1Ka_2 \cdots Ka_rK$, has nonempty interior in $G$. He, then, used geometric arguments to prove that the latter statement was true whenever $r \geq \dim G/K$. Using algebraic methods, this was subsequently improved to $r \geq \operatorname{rank} G/K + 1$ by Graczyk and Sawyer in [8], who also showed that this was sharp in the case of noncompact symmetric spaces with restricted root systems of type $A_n$. Inspired by Graczyk and Sawyer's work in [9, 10], in [13] the authors proved that $r \geq \operatorname{rank} G/K$ is the sharp $L^1$ result for all the classical noncompact symmetric spaces except those of type $A_n$, and characterized precisely which convolution products are absolutely continuous for the classical types.

**2.3. $L^1$–$L^2$ dichotomy.** Similar smoothness questions have been explored in a number of related settings, including $K$-bi-invariant measures supported on double cosets in compact symmetric spaces $G/K$, invariant measures supported on conjugacy

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classes of compact Lie groups or $Ad$-invariant measures supported on adjoint orbits of compact Lie algebras. In the case of compact Lie groups and algebras, the authors in [12, 14] used a combination of harmonic analysis and geometric arguments to show that convolution powers of such measures belong to $L^1$ if and only they belong to $L^2$, and determined the sharp exponent for each such measure. In contrast, in [3], it was shown that this dichotomy fails to hold in the compact symmetric space SU(2)/SO(2).

The harmonic analysis approach to the $L^2$ problem for compact Lie groups involved studying the rate of decay of the characters of the group and applying the Plancherel theorem. For symmetric spaces, the analogous approach is to study, instead, the decay of the spherical transform. We recall the definitions of the spherical function and spherical transform.

DEFINITION 2.2. The *spherical transform* of a compactly supported measure $\nu$ on the noncompact Lie group $G$ is defined by

$$\widehat{\nu}(\lambda) = \int_G \phi_\lambda(g^{-1})\, d\nu(g),$$

where $\phi_\lambda$ is the *spherical function* corresponding to $\lambda \in \mathfrak{a}^*$ given by the expression

$$\phi_\lambda(g) = \int_K \exp((i\lambda - \rho)\mathcal{H}(gk))\, dm(k).$$

Here $\rho$ is half the sum of the positive roots and $\mathcal{H}$ is the Iwasawa projection; that is, $\mathcal{H}(gk)$ is the unique element in $\mathfrak{a}$ such that $gk \in K \exp \mathcal{H}(gk)N$, where $N$ is a Lie subgroup of $G$ with Lie algebra $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

This formula for the spherical function can be found in [19, Ch. IV, Theorem 4.3], where it is also seen that $\phi_\lambda = \phi_{w(\lambda)}$ for all $w \in W$ and $\lambda \in \mathfrak{a}^*$.

From the definition of orbital measures it is easy to see that $\widehat{\nu_a}(\lambda) = \phi_\lambda(a^{-1})$, while in [2] it is shown that

$$(\nu_{a_1} * \cdots * \nu_{a_r})\widehat{\ }(\lambda) = \prod_{i=1}^{r} \phi_\lambda(a_i^{-1}).$$

A version of Plancherel's theorem holds in this setting. For the remainder of the paper, $c = c(\lambda)$ is the Harish-Chandra $c$ function and $d\lambda$ denotes Lebesgue measure on $\mathfrak{a}^*$.

THEOREM 2.3 (Plancherel, see [19, Ch. IV, Theorem 9.1]). *The $K$-bi-invariant measure $\mu$ belongs to $L^2(G)$ if and only if*

$$\|\mu\|^2_{L^2(G)} = \int_{\mathfrak{a}^*} |\widehat{\mu}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda < \infty.$$

COROLLARY 2.4. *The k-fold convolution product of the orbital measure $\nu_a$ belongs to $L^2(G)$ if and only if $|\phi_\lambda(a)|^k |c(\lambda)|^{-1} \in L^2(\mathfrak{a}^*)$.*

It is known that the spherical functions have good decay properties. To explain, it is helpful to introduce further terminology and notation.

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DEFINITION 2.5.

(i) Given $a \in A$ (or $a \in \mathfrak{a}$), by the *set of annihilating roots* of $a$ we mean the set

$$\Phi(a) = \{\alpha \in \Phi : \alpha(a) = 0\}.$$

Put $\Phi^+(a) = \Phi(a) \cap \Phi^+$. By $(\Phi^+(a))^c$ we mean the complement of $\Phi(a)$ in $\Phi^+$, that is, $(\Phi^+(a))^c = \{\alpha \in \Phi^+ : \alpha(a) \neq 0\}$.

(ii) If $\Phi(a)$ is empty, the element $a$ is called *regular*. If $a$ is regular, we call $\nu_a$ a *regular orbital measure*.

We let

$$A_0 = \{g \in A : g \notin N_G(K)\}.$$

The set $N_G(K)$ can be characterized as the set of elements $g \in G$ such that $\alpha(g) = 0$ for all roots $\alpha$; and hence the set of annihilating roots of an element in $A_0$ is a proper root subsystem. The set of regular elements is dense in $A$ and in the special case of a rank-one symmetric space all the elements of $A_0$ are regular.

Here is the decay result that we use.

PROPOSITION 2.6 ([6, Theorem 11.1], see also [2, Proposition 4.1]). *For each $a \in A_0$, there is a constant $C_a$ such that for all $\lambda \in \mathfrak{a}^*$,*

$$|\phi_\lambda(a)| \leq C_a \sum_{w \in W} \prod_{\alpha \in (\Phi^+(w(a)))^c} (1 + |\langle \lambda, \alpha \rangle|)^{-m_\alpha/2}. \tag{2-1}$$

It is well known (see [19, Ch. IV, Proposition 7.2]) that there is a constant $C$ such that

$$|c(\lambda)|^{-1} \leq C \prod_{\alpha \in \Phi^+} (1 + |\langle \lambda, \alpha \rangle|)^{m_\alpha/2};$$

thus,

$$\begin{aligned}(|\phi_\lambda(a)|^k |c(\lambda)|^{-1})^2 \\ \leq C_a \max_{w \in W} \prod_{\alpha \in (\Phi^+(w(a)))^c} |1 + |\langle \lambda, \alpha \rangle||^{-m_\alpha k} \prod_{\alpha \in \Phi^+} |1 + |\langle \lambda, \alpha \rangle||^{m_\alpha}\end{aligned} \tag{2-2}$$

for a new constant $C_a$. Combined with Plancherel's theorem, this implies that $\nu_a^k$ belongs to $L^2(G)$ provided

$$\int_{\overline{\mathfrak{a}^{*+}}} \max_{w \in W} \prod_{\alpha \in (\Phi^+(w(a)))^c} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\alpha \in \Phi^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha} d\lambda < \infty. \tag{2-3}$$

## 3. $L^2$ results for convolutions of orbital measures at regular elements

In [2], bounds were found for the right-hand side of (2-2) that were sufficient to show that any convolution product of more than $\dim G/K$ regular orbital measures was in $L^2(G)$. We begin by improving this result, in fact, obtaining sharp $L^2$ results for convolution products of regular orbital measures.

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THEOREM 3.1. *Suppose that $a \in A_0$ is a regular element. The convolution products, $\nu_a^k$, belong to $L^2(G)$ if and only if $k \geq 2$, except if the symmetric space $G/K$ has a restricted root system of type $A_1$ and is of Cartan class AI, in which case $k \geq 3$ is both necessary and sufficient.*

REMARK 3.2. We remark that $k \geq 2$ is necessary since $\nu_a$ is always a singular measure.

We first obtain bounds for $|\phi_\lambda(a)|^k |c(\lambda)|^{-1}$ for the symmetric spaces of classical Lie types. Let $\eta_0$ denote the multiplicity of the standard roots $e_i \pm e_j$, $\eta_1$ the multiplicity of the short roots $e_i$ and $\eta_2$ the multiplicity of the long roots $2e_i$ (should there be roots of these forms). The reader can find the values of $\eta_j$ for each type in the appendix.

LEMMA 3.3. *Suppose that the restricted root system of $G/K$ is one of the Lie types $A_n$, $B_n$, $C_n$, $BC_n$ or $D_n$ and that $a \in A_0$ is a regular element. There is a positive constant $C$, depending only on $G/K$ and $a$, such that*

$$(|\phi_\lambda(a)|^k |c(\lambda)|^{-1})^2 \leq C \min(1, \|\lambda\|^{(1-k)\varrho}) \quad \text{for all } \lambda \in \mathfrak{a}^* \text{ and } k \geq 1,$$

*where*

$$\varrho = \varrho(G/K) = \begin{cases} \eta_0 n & \text{for Lie type } A_n, \\ \max(\eta_0(2n-3) + \eta_1 + \eta_2, 3) & \text{for Lie types } B_n, C_n, BC_n \text{ with } n \geq 2, \\ \eta_1 + \eta_2 & \text{for Lie type } BC_1, \\ \eta_0 2(n-1) & \text{for Lie type } D_n. \end{cases}$$

PROOF. Throughout the proof, the constant $C$ may vary from one occurrence to another. We assume that $G/K$ has rank $n$; and there is no loss of generality in assuming that $\lambda \in \overline{\mathfrak{a}^{*+}}$.

As $a$ is regular, $\Phi(w(a))$ is empty for all $w \in W$ and thus

$$(|\phi_\lambda(a)|^k |c(\lambda)|^{-1})^2 \leq C \prod_{\alpha \in \Phi^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha(1-k)}. \tag{3-1}$$

Of course, if $\|\lambda\| \leq 1$, then $\prod_{\alpha \in \Phi^+} |1 + \langle \lambda, \alpha \rangle| \leq C$, so our interest is in $\|\lambda\| \geq 1$.

We let

$$T_\lambda = \{\alpha \in \Phi^+ : \langle \alpha, \lambda \rangle \geq c_G \|\lambda\|\} \tag{3-2}$$

for $c_G = 1$ for types $A_n$ and $D_n$, and $c_G = 1/2$ otherwise. Set

$$S_0 = \{e_i \pm e_j : 1 \leq i < j \leq n\}, \quad S_1 = \{e_i : 1 \leq i \leq n\} \quad \text{and} \quad S_2 = \{2e_i : 1 \leq i \leq n\}$$

(should they exist). For example, in type $A_n$, $S_0 = \Phi^+$ and $S_1, S_2$ do not exist. Notice that $m_\alpha = \eta_j$ if $\alpha \in S_j$. Put

$$U_{\lambda,j} = T_\lambda \cap S_j \tag{3-3}$$

and write $|U_{\lambda,j}|$ for the cardinality of this set.

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With this notation,

$$(|\phi_\lambda(a)|^k|c(\lambda)|^{-1})^2 \le C\min(1, \|\lambda\|^{(1-k)\sum_j \eta_j|U_{\lambda,j}|}). \tag{3-4}$$

We find lower bounds on $|U_{\lambda,j}|$ by analyzing on a type-by-type basis.

Type $A_n$: We can write $\lambda = \sum_{i=1}^n a_i\lambda_i$, where $\lambda_i$ are the fundamental dominant weights (the dual basis to the basis of simple roots) and $a_i \ge 0$. Since all norms are equivalent on a finite-dimensional normed space, we can take $\|\lambda\| = \max_i a_i = a_m$ (say). It suffices to determine which positive roots $\alpha = \sum_{i=1}^n b_i\alpha_i$ have $b_m > 0$ (and hence $b_m \ge 1$), for then $\langle\alpha, \lambda\rangle = \sum_i a_ib_i \ge a_mb_m \ge \|\lambda\|$ and $U_{\lambda,0}$ will contain that set of roots. These will be the roots $\alpha = e_i - e_j$, where $1 \le i \le m$ and $m < j \le n+1$; thus, the minimum value of $|U_{\lambda,0}|$ is $n$.

Type $B_n$, $C_n$, $BC_n$: We leave the very easy case of $BC_1$ to the reader and assume that $n \ge 2$. Here we can write $\lambda = \sum_{i=1}^n a_ie_i$, where $a_i \ge 0$ are nonincreasing and $e_i$ are the standard basis vectors for $\mathbb{R}^n$. Taking the Euclidean norm, we have $a_1 \le \|\lambda\| \le na_1$. We have $\langle\alpha, \lambda\rangle \ge a_1$ if $\alpha = e_1 + e_j$ for $j = 2, \dots, n$ or $\alpha = (2)e_1$. In particular, we have $|U_{\lambda,j}| \ge 1$ for $j = 1$ for type $B_n$, for $j = 2$ for type $C_n$ and for both $j = 1, 2$ for type $BC_n$. If $a_2 \le a_1/2$, then we also have $\langle\alpha, \lambda\rangle \ge a_1/2$ if $\alpha = e_1 - e_j$ for $j = 2, \dots, n$. In this case, $|U_{\lambda,0}| \ge 2(n-1)$ and hence $\sum \eta_j|U_{\lambda,j}| \ge 2\eta_0(n-1) + \eta_1 + \eta_2$.

Otherwise, $a_2 > a_1/2$ and then $\langle\alpha, \lambda\rangle \ge a_1/2$ if $a = e_2 + e_j, j = 3, \dots, n$ or $\alpha = (2)e_2$. In this case, we have $|U_{\lambda,0}| \ge 2n-3$ and $|U_{\lambda,1}| \ge 2$ for type $B_n$, with similar statements for $C_n$ and $BC_n$, and then $\sum \eta_j|U_{\lambda,j}| \ge \eta_0(2n-3) + 2(\eta_1 + \eta_2)$.

In either situation, $\sum \eta_j|U_{\lambda,j}|$ is both at least 3 and at least $\eta_0(2n-3) + (\eta_1 + \eta_2)$.

Type $D_n$: As with type $A_n$, we write $\lambda = \sum_{i=1}^n a_i\lambda_i$, where $\lambda_i$ are the fundamental dominant weights and $a_i \ge 0$. It suffices to determine which $\alpha = \sum b_i\alpha_i$ have $b_m > 0$, where $a_m = \max_i a_i$. If $m \ne n-1, n$, these will be the roots $\alpha = e_i + e_j$ for $i \le m$ and $j > i$ and for $\alpha = e_i - e_j$ for $i \le m < j$. There are at least $2(n-1)$ of these roots. If $m = n$, all the roots $e_i + e_j$ have the desired property, while, if $m = n-1$, the positive roots $e_i - e_n$, $i < n$, and $e_i + e_j$, $i < j < n$, all work. Thus, for all $\lambda$, $|U_{\lambda,0}| \ge \min(2(n-1), \binom{n}{2}) = 2(n-1)$, as we may assume that $n \ge 4$ for this type. □

PROOF OF THEOREM 3.1. We begin by proving the sufficiency of the choice of $k$. As in the lemma, the constant $C > 0$, depending on $G/K$ and $a$, which appears throughout may change from one occurrence to another. We again assume that $G/K$ has rank $n$.

When $G/K$ has a restricted root space of classical Lie type, the previous lemma shows that

$$\|\nu_a^k\|_2^2 \le C\int_{\overline{\mathfrak{a}^{*+}}} \min(1, \|\lambda\|^{(1-k)\varrho})\,d\lambda \le C\int_1^\infty t^{(1-k)\varrho}t^{n-1}dt \tag{3-5}$$

and this will be finite if $(1-k)\varrho + n - 1 < -1$. It is a routine exercise, using the values of $\varrho$ given in the lemma, to see that if $k \ge 2$, then this is true for all these classical types, except if $G/K$ is of Lie type $A_n$ and Cartan class $AI$. In this latter case, $\eta_0 = 1$ and we have that the integral above is finite provided $k \ge 3$.

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However, the argument can be improved for the Lie type $A_n$, Cartan class $AI$, when $n \geq 2$. Let

$$\Lambda_0 = \Big\{\lambda = \sum_{i=1}^{n} a_i\lambda_i \in \overline{\mathfrak{a}^{*+}} : a_j = \max a_i \text{ for some } j \neq 1, n\Big\},$$

$$\Lambda_1 = \Big\{\lambda = \sum_{i=1}^{n} a_i\lambda_i \in \overline{\mathfrak{a}^{*+}} : a_1 = a_n = \max a_i\Big\}$$

and let $\Lambda_2$ be the rest of $\overline{\mathfrak{a}^{*+}}$. Note that

$$\|\nu_a^2\|_2^2 \leq C \sum_{j=0}^{2} \int_{\Lambda_j} (|\phi_\lambda(a)|^2 |c(\lambda)|^{-1})^2 d\lambda.$$

Let $U_{\lambda,0}$ be as in the lemma. Note that $|U_{\lambda,0}| \geq n+1$ if $\lambda \in \Lambda_0 \cup \Lambda_1$, whence one can see that $\int_{\Lambda_j} (|\phi_\lambda(a)|^2 |c(\lambda)|^{-1})^2 d\lambda < \infty$ for $j = 0, 1$.

If, instead, $\lambda \in \Lambda_2$ (so either $a_1$ or $a_n$ is the unique maximal coordinate), then $|U_{\lambda,0}| = n$. However, there will also be at least $n-1$ positive roots $\alpha \notin U_{\lambda,0}$ such that $\langle \alpha, \lambda \rangle \geq a_J$, where $a_J$ is the second largest coefficient. Using this fact, we obtain the bound

$$\begin{aligned}
&\int_{\Lambda_2} (|\phi_\lambda(a)|^2 |c(\lambda)|^{-1})^2 d\lambda \\
&\quad \leq C \int_0^\infty (1+t_1)^{-n} \Big( \int_0^{t_1} (1+t_2)^{-(n-1)} t_2^{n-2}\, dt_2 \Big) dt_1 \\
&\quad \leq C\Big(1 + \int_1^\infty t_1^{-n} \int_1^{t_1} t_2^{-1} dt_1\Big) \\
&\quad = C\Big(1 + \int_1^\infty t_1^{-n} \log t_1 dt_1\Big)
\end{aligned}$$

and this is finite since we are assuming that $n \geq 2$.

Thus, even when the symmetric space is of Cartan class $AI$, we have $\nu_a^2 \in L^2$ provided the rank of $G/K$ is at least $n = 2$. That completes the proof of sufficiency of the choice of $k$ for the classical Lie types.

For the symmetric spaces with restricted root spaces of exceptional Lie types, we argue in a similar fashion. We define $T_\lambda$ as in (3-2) and decompose the set of positive restricted roots into maximal disjoint sets $S_j$, consisting of the positive roots of a given multiplicity. Again, put $U_{\lambda,j} = T_\lambda \cap S_j$ and observe that again (3-4) holds.

If the restricted root space is of Lie type $G_2$, $E_6$, $E_7$ or $E_8$, then all the roots have the same multiplicity, so we take $S_0 = \Phi^+$. It is shown in [15] (see, for example, Tables 2–4) that the minimum cardinality of $U_{\lambda,0}$ is at least 5, 16, 27 and 57, respectively.

If the restricted root space is of Lie type $F_4$ and all the roots have the same multiplicity, again $S_0 = \Phi^+$ and the minimum cardinality of $U_{\lambda,0}$ is shown in [15] to be 15. Otherwise, there are two distinct multiplicities and we define $S_0, S_1$ accordingly.

As can be seen from [15], $|U_{\lambda,0}| \geq 9$ and $|U_{\lambda,1}| \geq 6$. Using (3-5) again, it is easy to check that $k \geq 2$ suffices.

We turn now to proving the necessity of the choice of $k$. Since $\nu_a$ is a singular measure with respect to Haar measure, $k \geq 2$ is certainly necessary (in all cases). Thus, we need only consider the symmetric space $G/K$ of Lie type $A_1$ and Cartan class $AI$ and show that $\nu_a^2$ does not belong to $L^2$.

For this symmetric space, the spherical functions can be expressed in terms of the hypergeometric functions ${}_2F_1$ as follows. Denote by $\alpha$ the (single) positive root and choose $H_0 \in \mathfrak{a}$ such that $\alpha(H_0) = 1$. For any $t \neq 0$, it is known [22, 11.5.15] that

$$\phi_\lambda(\exp tH_0) = {}_2F_1\Big(\frac{1+i\lambda}{4}, \frac{1-i\lambda}{4}, 1, -\sinh^2 t\Big).$$

Next, we use the relationship between the hypergeometric functions and the Jacobi and Bessel functions (cf. [7, Section 6.4]):

$$J_u^{(0,b)}(t) = {}_2F_1\Big(\frac{b+1+iu}{2}, \frac{b+1-iu}{2}, 1, -\sinh^2 t\Big),$$

while

$$J_u^{(0,b)}(t) = cJ_0(ut) + O(u^{-3/2}),$$

where $J_0(\cdot)$ is the Bessel function and $c$ is a nonzero constant depending on $t$. It is well known [1, 9.2.1] that for $z > 0$,

$$J_0(z) = \frac{C}{\sqrt{z}}(\cos(z - \pi/4) + O(z^{-1}))$$

for some $C \neq 0$. Thus, for all $\lambda > 0$,

$$\phi_\lambda(\exp tH_0) = \frac{C}{\sqrt{\lambda}}\cos(\lambda t/2 - \pi/4) + O(|\lambda|^{-3/2}), \tag{3-6}$$

where the nonzero constant $C$ depends only on $t$.

If $\lambda$ is chosen from an interval of the form

$$I_j = \frac{2}{t}\Big[2j\pi + \frac{\pi}{8}, 2j\pi + \frac{3\pi}{8}\Big]$$

for an integer $j$, then $\cos(\lambda t/2 - \pi/4) \geq \cos\pi/8 = \varepsilon_0 > 0$. It follows from (3-6) that we can choose $\lambda_1$ sufficiently large so that if $\lambda \in I_j$ and $\lambda \geq \lambda_1$, then

$$|\phi_\lambda(\exp tH_0)| \geq \frac{\varepsilon_0}{2\sqrt{\lambda}}.$$

It is shown in the proof of [19, Ch. IV, Proposition 7.2] that for the Harish-Chandra $c$ function, $\lim_{\lambda\to\infty} c(\lambda)^{-1}\lambda^{-1/2} = 2\sqrt{\pi}$. Thus, $c(\lambda)^{-1} \geq \sqrt{\pi\lambda}$ for all $\lambda \geq \lambda_2$, say. Choose

$j_0$ so large that if $\lambda \in \bigcup_{j=j_0}^{\infty} I_j$, then $\lambda \geq \max(\lambda_1, \lambda_2)$. Since the intervals $I_j$ are disjoint,

$$\int |\phi_\lambda^2(\exp tH_0)c(\lambda)^{-1}|^2 d\lambda \geq \sum_{j=j_0}^{\infty} \int_{I_j} |\phi_\lambda^2(\exp tH_0)c(\lambda)^{-1}|^2 d\lambda$$
$$\geq \sum_{j=j_0}^{\infty} \int \Big(\frac{\varepsilon_0}{2\sqrt{\lambda}}\Big)^4 (\sqrt{\pi\lambda})^2 d\lambda \geq C \sum_{j=j_0}^{\infty} \int_{I_j} \frac{d\lambda}{\lambda}.$$

We deduce that for a new constant $C = C(t) > 0$,

$$\int |\phi_\lambda^2(\exp tH_0)c(\lambda)^{-1}|^2 d\lambda \geq C \sum_{j=j_0}^{\infty} \frac{\text{length}(I_j)}{j} = \infty.$$

Consequently, $\phi_\lambda^2(\exp tH_0)c(\lambda)^{-1} \notin L^2$ and this proves that $\nu_a^2 \notin L^2$ for any $a = \exp tH_0, t \neq 0$, and hence for any regular $a$. □

REMARK 3.4. It is known that for any noncompact, rank-one symmetric space, $\nu_a * \nu_a$ belongs to $L^1$ for all $a \in A_0$ [8]. Thus, the $L^1$–$L^2$ dichotomy fails for the symmetric space of Lie type $A_1$ and Cartan class $AI$. Interestingly, the $L^1$–$L^2$ dichotomy holds for all the regular orbital measures in all the other symmetric spaces since we obviously have $\nu_a^k \in L^1$ only if $k \geq 2$.

COROLLARY 3.5. *Let $a_1, a_2, a_3$ be regular elements in $A$. If $G/K$ is of Lie type $A_1$ and Cartan class $AI$, then $\nu_{a_1} * \nu_{a_2} * \nu_{a_3} \in L^2$. Otherwise, $\nu_{a_1} * \nu_{a_2} \in L^2$.*

PROOF. We prove the first statement, as the second is even easier. Let $\mu = \nu_{a_1} * \nu_{a_2} * \nu_{a_3}$. By the Plancherel formula,

$$\|\mu\|_2^2 = \int_{\mathfrak{a}^*} |\widehat{\mu}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda = \int_{\mathfrak{a}^*} \Big|\prod_{i=1}^{3} \phi_\lambda(a_i^{-1})\Big|^2 |c(\lambda)|^{-2} d\lambda.$$

Applying the generalized Holder's inequality gives

$$\|\mu\|_2^2 \leq \prod_{i=1}^{3} \Big(\int_{\mathfrak{a}^*} |\phi_\lambda(a_i^{-1})|^6 |c(\lambda)|^{-2} d\lambda\Big)^{1/3} = \prod_{i=1}^{3} \|\nu_{a_i}^3\|_2^{2/3}$$

and the latter product is finite according to the theorem. □

## 4. Smoothness of convolutions of arbitrary orbital measures

**4.1. $L^2$ results.** The goal of this section is to show that for all $a \in A_0$ (not just regular $a$) there is an index $k$ such that $\nu_a^k \in L^2(G)$. As in the proof of Theorem 3.1, we continue to use the notation $\eta_0$ to denote the multiplicity of the roots $e_i \pm e_j$, $\eta_1$ for the multiplicity of the short roots $e_i$ and $\eta_2$ for the multiplicity of the long roots $2e_i$ when the symmetric space is of classical Lie type $A_n$, $B_n$, $C_n$, $BC_n$ or $D_n$. We recall

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that the values of $\eta_j$ depend on the Lie type and Cartan class and can be found in the appendix.

THEOREM 4.1. *Let $G/K$ be a noncompact symmetric space of type $A_n$, $B_n$, $C_n$, $D_n$ or $BC_n$. If $\nu_{a_1}, \ldots, \nu_{a_k}$ are any orbital measures on $G$ with $a_i \in A_0$, then $\nu_{a_1} * \cdots * \nu_{a_k} \in L^2(G)$ provided $k > k_G$, where*

$$k_G = \begin{cases} n + n/\eta_0 & \text{for type } A_n, \\ n - 1 + n/(2\eta_0) & \text{for type } D_n, \\ 2(n-1) + (n + \eta_1 + \eta_2)/\eta_0 & \text{for types } B_n, C_n, BC_n, n \geq 3, \\ \max(4, 2 + (\eta_1 + \eta_2)/(2\eta_0)) & \text{for } B_2, C_2, BC_2. \end{cases}$$

REMARK 4.2. We remark that the symmetric spaces of Lie type $A_n$, $(B)C_n$ or $D_n$ have rank $n$ and dimension comparable to $n^2 + n(\eta_1 + \eta_2)$. Note that for type $(B)C_n$ we can assume that $n \geq 2$, as the regular orbital measure case has already been done.

The key to the proof of this theorem is finding bounds for the products

$$P^w_{G/K}(\lambda, k, a) = \prod_{\alpha \in \Phi^+(w(a))^c} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\alpha \in \Phi^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha} \tag{4-1}$$

and

$$P_{G/K}(\lambda, k, a) = \max_{w \in W} P^w_{G/K}(\lambda, k, a)$$

for $\lambda \in \overline{\mathfrak{a}^{*+}}$ since we have already seen in (2-2) that

$$(|\phi_\lambda(a)|^k |c(\lambda)|^{-1})^2 \leq C_a P_{G/K}(\lambda, k, a).$$

This will be mainly accomplished in two lemmas. We again write $C$ for a positive constant (depending only on $G/K$ and $a$) that may change throughout the proof. We begin with the symmetric spaces of Lie type $A_n$ or $D_n$. These are easier, as all roots have the same multiplicity.

LEMMA 4.3. *Suppose that $G/K$ is of Lie type $A_{n-1}$ or $D_n$ and $a \in A_0$. There is a constant $C$ such that*

$$P_{G/K}(\lambda, k, a) \leq C \min(1, \|\lambda\|^{-\eta_0 p_k})$$

*for all integers $k \geq n - 1$ and $\lambda \in \overline{\mathfrak{a}^{*+}}$, where*

$$p_k = p_k(G/K) = \begin{cases} k - n + 1 & \text{for } G/K \text{ type } A_{n-1}, \\ 2(k - n + 1) & \text{for } G/K \text{ type } D_n. \end{cases}$$

PROOF. Obviously, there is a constant $C$ such that if $\|\lambda\| \leq 1$, then $P^w_{G/K}(\lambda, k, a) \leq C \min(1, \|\lambda\|^{-\eta_0 p_k})$. Thus, our interest is with $\|\lambda\| \geq 1$.

In [16], the analogous problem was studied for the invariant measures supported on conjugacy classes in the classical simple compact Lie groups. Specifically, in (3.1) of [16], it was shown that there is a constant $C = C(\mathcal{G})$ such that if $\mathcal{G}$ is a compact Lie

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group of type $A_{n-1}$ or $D_n$, $X^+$ is the set of positive roots for the Lie algebra associated with $\mathcal{G}$, $Y^+$ is the set of positive roots of some maximal root subsystem (such as $\Phi^+(w(a))$) and $\rho$ is half the sum of the positive roots, then, for all representations $\lambda$ of $\mathcal{G}$,

$$\prod_{\alpha\in Y^+} |\langle\rho+\lambda,\alpha\rangle|^s \prod_{\alpha\in X^+\setminus Y^+} |\langle\rho+\lambda,\alpha\rangle|^{s-1} \le C \tag{4-2}$$

when $s = 1/(n-1)$. As $\langle\rho+\lambda,\alpha\rangle \sim 1+\langle\lambda,\alpha\rangle$, this is equivalent to the statement that

$$\prod_{\alpha\in(Y^+)^c} |1+\langle\lambda,\alpha\rangle|^{-1} \le C \prod_{\alpha\in X^+} |1+\langle\lambda,\alpha\rangle|^{-s} \le C. \tag{4-3}$$

The arguments of [16] were based on the combinatorial structure of root systems, properties of roots and the fact that representations of a compact group belong to $\overline{\mathfrak{a}^{*+}}$. They did not rely upon the fact that group representations of a compact group belong to the integer lattice of $\overline{\mathfrak{a}^{*+}}$; thus, the same reasoning applies to all $\lambda \in \overline{\mathfrak{a}^{*+}}$.

Now consider the compact Lie group $\mathcal{G}$ with the same root system $\Phi$ as the restricted root system of $G/K$ (although, with all roots having multiplicity two, rather than $\eta_0$). For any $a \in A_0$ and $w \in W$, the set of positive annihilating roots of $w(a)$ is contained in the set of positive roots of a maximal root subsystem of $\Phi$, say $\Psi^+$. Appealing to (4-3), we deduce that

$$\begin{aligned} P^w_{G/K}(\lambda,k,a) &\le \Bigg(\prod_{\alpha\in(\Psi^+)^c} |1+\langle\lambda,\alpha\rangle|^{-k} \prod_{\alpha\in\Phi^+} |1+\langle\lambda,\alpha\rangle|\Bigg)^{\eta_0} \\ &\le C \prod_{\alpha\in\Phi^+} |1+\langle\lambda,\alpha\rangle|^{(1-ks)\eta_0} \end{aligned}$$

(for the appropriate choice of $s$). Hence, if we let $q$ be the minimal number of positive roots $\alpha$ (not counting multiplicity) such that $\langle\lambda,\alpha\rangle \ge \|\lambda\|$, then $P_{G/K}(\lambda,k,a) \le C\|\lambda\|^{(1-ks)\eta_0 q}$. Of course, $\|\lambda\|^{(1-ks)\eta_0 q} \le 1$ if $\|\lambda\| \ge 1$. In the notation of (3-3), $q = \min_\lambda |U_{\lambda,0}|$. Thus, $q(A_{n-1}) = n-1$ and $q(D_n) = 2(n-1)$. Inputting the values for $s$ and $q$ gives the desired result. □

LEMMA 4.4. *Suppose that $G/K$ is of Lie type $B_n$, $C_n$ or $BC_n$, $\lambda \in \overline{\mathfrak{a}^{*+}}$ and $a \in A_0$.*

(i) *If $n \ge 3$, there is a constant $C_n$ such that if an integer*

$$k \ge \kappa_n := 2(n-1) + (\eta_1+\eta_2)/\eta_0,$$

*then*

$$P_{G/K}(\lambda,k,a) \le C_n \min(1, \|\lambda\|^{\eta_0(2(n-1)-k)+\eta_1+\eta_2}). \tag{4-4}$$

(ii) *Suppose that $n = 2$, $m = \min(\eta_0, \eta_1+\eta_2)$ and $M = \max(\eta_0, \eta_1+\eta_2)$. Then, if an integer $k \ge \kappa_2 = 1 + M/2m$,*

$$P_{G/K}(\lambda,k,a) \le C_2 \min(1, \|\lambda\|^{2m(1-k)+M}). \tag{4-5}$$

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PROOF. As noted previously, we obviously have $P_{G/K}(\lambda, k, a)$ uniformly bounded when $\|\lambda\| \le 1$. Moreover, when $n \ge 3$ and an integer $k \ge \kappa_n$, then $\eta_0(2(n-1)-k)+\eta_1+\eta_2 \le 0$ and, when $k \ge \kappa_2$, $2m(1-k)+M \le 0$. Thus, the task is to check that $P_{G/K}(\lambda, k, a) \le C_n\|\lambda\|^{\eta_0(2(n-1)-k)+\eta_1+\eta_2}$ when $n \ge 3$ and the corresponding statement of (ii) when $n = 2$.

Our proof of (i) proceeds by induction on $n$. We leave the arguments for the base case until the end when it will be done in conjunction with the proof of (ii).

We give the proof for type $BC_n$, but the modifications for the other types are essentially notational. For the induction argument, it is natural to write $P_n(\lambda, k, a)$ rather than $P_{G/K}(\lambda, k, a)$ when the rank of $G/K$ is $n$.

Let $a \in A_0$. Since $\Phi^+(w(a))$ is a proper root subsystem, in bounding $P_n(\lambda, k, a)$ we may as well assume that $\Phi^+(w(a)) = \Psi^+$, where $\Psi$ is one of the finitely many maximal root subsystems and that $w = \mathrm{id}$.

The maximal root subsystems of a symmetric space of Lie type $BC_n$ are: (a) Lie type $BC_{n-1}$, (b) Lie type $A_{n-1}$ and (c) Lie types $BC_{n-j} \times A_{j-1}$ with $n-j \ge 1, j \ge 2$.

Any spherical representation in $BC_n$ can be written as $\lambda = \sum_{i=1}^{n} \lambda_i e_i$, where $\lambda_i$ are nonincreasing, nonnegative integers. Thus, $\lambda_1 \le \|\lambda\| \le n\lambda_1$ and, consequently,

$$\prod_{\alpha\in\Phi^+} |1 + \langle\lambda, \alpha\rangle|^{m_\alpha} \le C\lambda_1^{2\binom{n}{2}\eta_0+n(\eta_1+\eta_2)}. \tag{4-6}$$

We now consider the three cases of maximal annihilating root subsystems separately.

Case (a): $\Psi$ is of type $BC_{n-1}$. This means that there is some index $n_0 \in \{1, \dots, n\}$ such that

$$\Psi^+ = \{e_i \pm e_j, e_k, 2e_k : 1 \le i < j \le n,\ i, j, k \ne n_0\}$$

and hence

$$(\Psi^+)^c = \{e_{n_0} \pm e_j, e_{n_0}, 2e_{n_0} : j \ne n_0\}$$

(where $e_{n_0} - e_j$ should be replaced by $e_j - e_{n_0}$ if $j < n_0$).

If $n_0 = 1$, then, as $1 + \langle\lambda, e_1 + e_j\rangle \ge \lambda_1$ for all $j = 2, \dots, n$ and $1 + \langle\lambda, (2)e_1\rangle \ge \lambda_1$, we see that

$$\prod_{\alpha\in(\Psi^+)^c} |1 + \langle\lambda, \alpha\rangle|^{m_\alpha} \ge \lambda_1^{(n-1)\eta_0+\eta_1+\eta_2}.$$

Thus, for such $a$,

$$P_n(\lambda, k, a) \le \lambda_1^{(n-1)\eta_0(n-k)+(\eta_1+\eta_2)(n-k)} \tag{4-7}$$

and that is dominated by the right-hand side of (4-4) when $k \ge \kappa_n$.

So, assume that $n_0 \ne 1$. Here we use an induction argument assuming that the statement holds for $n-1$. (Actually, all we need to inductively assume is that $P_{n-1}(\lambda, k, a)$ is uniformly bounded for $k \ge \kappa_n$ and the claims of the lemma certainly ensure this.)

We consider the root subsystem

$$\Phi' = \{e_i \pm e_j, e_k, 2e_k : 2 \le i \ne j \le n, 2 \le k \le n\} \subseteq \Phi,$$

with the same multiplicities. This can be viewed as the restricted root system of the same Cartan class as $G/K$, but with rank $n-1$. For instance, if $G/K$ is of Cartan class *AIII*, so that

$$G/K = \mathrm{SU}(p,n)/\mathrm{SU}(p) \times \mathrm{SU}(n)$$

for some $p > n$, then $\Phi'$ is the restricted root system of the symmetric space

$$\mathrm{SU}(p-1,n-1)/\mathrm{SU}(p-1) \times \mathrm{SU}(n-1)$$

of Cartan class *AIII* and Lie type $BC_{n-1}$. For the purposes of this proof, we call this the 'reduced symmetric space'. We remark that the reduced symmetric space has rank $n-1$ and that the multiplicities of the roots are unchanged.

By identifying $a \in A_0$ with $X_a \in \mathfrak{a}$, we can assume that $a = \sum_{i=1}^n a_i e_i$. We let $a' = \sum_{i=2}^n a_i e_i$ (understood, appropriately, as an element in the reduced symmetric space) and observe that the annihilating root system of $a'$ is of type $BC_{n-2}$.

Put $\lambda' = \sum_{i=2}^n \lambda_i e_i$, so that for $\alpha \in \Phi'$, $\langle \alpha, \lambda' \rangle = \langle \alpha, \lambda \rangle$. An elementary, but useful, observation is that $\Phi^+(a)^c$ consists of the union of the nonannihilating positive roots of $a$ that belong to $\Phi'$ together with those nonannihilating positive roots that do not belong to $\Phi'$, namely $e_1 \pm e_{n_0}$. Moreover, the nonannihilating roots that are in $\Phi'$ are precisely the nonannihilating roots of $a'$. Thus,

$$P_{n-1}(\lambda', k, a') = \prod_{\alpha \in (\Psi^+)^c \cap \Phi'^+} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\alpha \in \Phi'^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha}.$$

Since $\langle \lambda, e_1 + e_{n_0} \rangle \geq c\lambda_1$ and the induction assumption ensures that $P_{n-1}(\lambda', k, a')$ is bounded independently of $\lambda'$ and $k$, we see that

$$\begin{aligned}
&P_n(\lambda, k, a) \\
&\quad = P_{n-1}(\lambda', k, a') \prod_{\alpha \in (\Psi^+)^c \setminus \Phi'^+} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\alpha \in \Phi^+ \setminus \Phi'^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha} \\
&\quad \leq P_{n-1}(\lambda', k, a') \prod_{\alpha = e_1 \pm e_{n_0}} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\substack{\alpha = e_1 \pm e_j, j = 2, \ldots, n \\ e_1, 2e_1}} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha} \\
&\quad \leq P_{n-1}(\lambda', k, a') \lambda_1^{\eta_0(2(n-1)-k)+\eta_1+\eta_2} \leq C\lambda_1^{\eta_0(2(n-1)-k)+\eta_1+\eta_2}.
\end{aligned} \tag{4-8}$$

Case (b): $\Psi$ is of type $A_{n-1}$. Hence, $\Psi^+ = \{s_i e_i - s_j e_j : 1 \leq i < j \leq n\}$ for some choice of $s_i = \pm 1$. We define $\Phi', a', \lambda'$ as above, so that $\Phi'$ is of type $BC_{n-1}$ and the subset of annihilating roots of $a'$ is of type $A_{n-2}$. Again we factor $P_n(\lambda, k, a)$ and use the fact that $P_{n-1}(\lambda', k, a')$ is bounded to see that

$$\begin{aligned}
&P_n(\lambda, k, a) \\
&\quad = P_{n-1}(\lambda', k, a') \prod_{\alpha \in (\Psi^+)^c \setminus \Phi'^+} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\alpha \in \Phi^+ \setminus \Phi'^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha}
\end{aligned}$$

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$$\le P_{n-1}(\lambda', k, a') \prod_{\substack{\alpha = s_1 e_1 + s_j e_j, \\ (2)e_1}} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \prod_{\substack{\alpha = \varepsilon_1 \pm e_j, \\ (2)e_1}} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha}$$

$$\le C \prod_{\alpha = s_1 e_1 + s_j e_j, (2)e_1} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \cdot \lambda_1^{2(n-1)\eta_0 + \eta_1 + \eta_2}.$$

There is no loss in assuming that $s_1 = 1$; thus,

$$\prod_{\alpha = s_1 e_1 + s_j e_j, (2)e_1} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \le \lambda_1^{-\#\{j>1: s_j = 1\}\eta_0 k} \lambda_1^{-(\eta_1 + \eta_2)k}.$$

If there is at least one $j > 1$ such that $s_j = 1$, then

$$P_n(\lambda, k, a) \le C \lambda_1^{\eta_0(2(n-1)-k) + (\eta_1+\eta_2)(1-k)}, \tag{4-9}$$

agreeing with (4-4).

So, assume that $s_j = -1$ for all $j > 1$. We note that if $\alpha = e_1 - e_j$, then $|1 + \langle \lambda, \alpha \rangle| = 1 + \lambda_1 - \lambda_j$, so, if there is some $j$ with $\lambda_j \le \lambda_1/2$, then $|1 + \langle \lambda, \alpha \rangle| \ge \lambda_1/2$. Hence,

$$\prod_{\alpha \in s_1 e_1 + s_j e_j, (2)e_1} |1 + \langle \lambda, \alpha \rangle|^{-m_\alpha k} \le C \lambda_1^{-\eta_0 k - (\eta_1 + \eta_2)k}$$

and we can obtain the same bound on $P_n(k, \lambda, a)$ as in (4-9) (with a different choice of constant).

Thus, we can also assume that $\lambda_j \ge \lambda_1/2$ for all $j > 1$. Then we give a direct argument, rather than appealing to induction. The choice of $s_1 = 1$ and $s_j = -1$ for all $j \ne 1$ means that

$$\Phi^+(a)^c = \{e_1 - e_k, e_i + e_j, (2)e_t : 2 \le i < j \le n,\ k \ge 2, t \ge 1\}.$$

Furthermore, $|1 + \langle \lambda, e_i + e_j \rangle| \ge \lambda_i + \lambda_j \ge \lambda_1$ for all $i, j \ge 2$ and similarly we have $|1 + \langle \lambda, (2)e_t \rangle| \ge \lambda_1/2$ for all $t \ge 1$. Thus,

$$\prod_{\alpha \in (\Psi^+)^c} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha} \ge \lambda_1^{\binom{n-1}{2}\eta_0 + n(\eta_1 + \eta_2)}.$$

Coupled with (4-6), this gives

$$P_n(\lambda, k, a) \le C \lambda_1^{\eta_0 (2\binom{n}{2} - k\binom{n-1}{2}) + n(\eta_1 + \eta_2)(1-k)}. \tag{4-10}$$

It is routine to check that this implies that the claim of the lemma holds.

Case (c): $\Psi$ is of type $BC_{n-j} \times A_{j-1}$ with $2 \le j \le n-1$. In this situation, there are disjoint sets of indices, $I, J \subseteq \{1, \ldots, n\}$, where $|I| = n - j$, $|J| = j \ge 2$, and a choice $s_t = \pm 1$ for $t \in J$ such that

$$\Psi^+ = \{e_i \pm e_j, (2)e_t : i < j, t \in I\} \cup \{s_i e_i - s_j e_j : i < j \in J\}.$$

We set up the usual induction/factoring argument. If $1 \in I$, then the set of annihilating roots of $a'$ is of type $BC_{n-j-1} \times A_{j-1}$ in the reduced symmetric space of type $BC_{n-1}$

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(or type $A_{n-2}$ in $BC_{n-1}$ if $j = n-1$). Under this assumption, $(\Psi^+)^c \diagdown \Phi'$ contains all the roots $a = e_1 + e_j$ for $j \in J$ and, for such $\alpha$, we have $\langle \alpha, \lambda \rangle \geq \lambda_1$. As $|J| \geq 2$,

$$P_n(\lambda, k, a) \leq C\lambda_1^{-k\eta_0|J|+2(n-1)\eta_0+\eta_1+\eta_2} \leq C\lambda_1^{\eta_0(2(n-1)-2k)+\eta_1+\eta_2}, \tag{4-11}$$

a better bound than (4-4).

Otherwise, $1 \in J$, so the set of annihilating roots of $a'$ is of type $BC_{n-j} \times A_{j-2}$ in type $BC_{n-1}$ (or $BC_{n-2}$ in $BC_{n-1}$ if $|J| = 2$). Then $(\Psi^+)^c \diagdown \Phi'$ contains the roots $a = e_1 + e_i$ for $i \in I$ and $(2)e_1$; hence, the usual arguments give

$$\begin{aligned} P_n(\lambda, k, a) &\leq C\lambda_1^{-k\eta_0|I|-k(\eta_1+\eta_2)+2(n-1)\eta_0+\eta_1+\eta_2} \\ &\leq C\lambda_1^{\eta_0(2(n-1)-k)+(\eta_1+\eta_2)(1-k)}. \end{aligned} \tag{4-12}$$

This completes the induction step.

We have seen that to start the induction argument we need only prove that $P_2(\lambda, k, a)$ is uniformly bounded for $k \geq \kappa_3$. Since $k \geq \kappa_3$ ensures that $2m(1-k) + M \leq 0$, we need only prove (4-5) to see this. For $BC_2$, we have $\Phi^+ = \{e_1 \pm e_2, (2)e_1, (2)e_2\}$; thus,

$$\prod_{\alpha \in \Phi^+} |1 + \langle \lambda, \alpha \rangle|^{m_\alpha} \leq C\lambda_1^{\eta_0}(1 + \lambda_1 - \lambda_2)^{\eta_0} \lambda_1^{\eta_1+\eta_2} \lambda_2^{\eta_1+\eta_2}.$$

The maximal root subsystems of type $BC_2$ are of type $BC_1$ with positive root being either $(2)e_1$ or $(2)e_2$, or of type $A_1$ with the positive root being either $e_1 - e_2$ or $e_1 + e_2$. We can analyze each of these cases separately, using the fact that $\lambda_1 - \lambda_2 \sim \lambda_1$ if $\lambda_2 \leq \lambda_1/2$ and $\lambda_2 \sim \lambda_1$ if $\lambda_2 \geq \lambda_1/2$. It is these different cases that lead to the definitions of $m$ and $M$. The details are left for the reader. □

PROOF OF THEOREM 4.1. First, suppose that $G/K$ is of Lie type $A_{n-1}$ or $D_n$. In the notation of Lemma 4.3,

$$\begin{aligned} \|\nu_a^k\|_2^2 &\leq C \int_{\overline{\mathfrak{a}^{*+}}} (|\phi_\lambda(a)|^k |c(\lambda)|^{-1})^2 d\lambda \leq C \int \min(1, \|\lambda\|^{-\eta_0 p_k})\, d\lambda \\ &\leq C \int_1^\infty t^{-\eta_0 p_k} t^{n-1} dt. \end{aligned}$$

Of course, the last integral is finite if $k$ is chosen so that $\eta_0 p_k > n$. Using the values obtained for $p_k$ in the lemma gives the specified choice of $k_G$.

A similar argument can be applied for types $B_n, C_n$ or $BC_n$, using the claims of Lemma 4.4.

To prove the statement about the convolution of $k$ (possibly distinct) orbital measures $\nu_{a_i}$, with $a_i \in A_0$, we use the fact that $\nu_{a_i}^k \in L^2$ for the specified choices of $k$ and apply the generalized Holder's inequality in the same manner as in the proof of Corollary 3.5. □

REMARK 4.5. The technique of Lemma 4.3 could be applied to the symmetric spaces of type $B_n$ or $C_n$ that have the additional property that all restricted roots have the same multiplicity. But the results are no better than can be obtained by Lemma 4.4.

The induction technique of Lemma 4.4 could also be applied to types $A_n$ and $D_n$. This takes much more work than Lemma 4.3 and gives only modest improvements.

Similar techniques can also be applied to the symmetric spaces with root systems of exceptional types.

PROPOSITION 4.6. *Suppose that $G/K$ is a symmetric space with restricted root system of exceptional Lie type $G_2$, $F_4$, $E_6$, $E_7$ or $E_8$. Then $\nu_{a_1} * \cdots * \nu_{a_k} \in L^2$ if $k \geq k_G$ as stated in the chart.*

| *Lie type* | $k_G$ | $F_4$ – *Cartan class* | $k_G$ |
|---|---|---|---|
| $E_7, E_8$ | 8 | *EII* | 7 |
| $G_2$ | 4 | *EVI* | 11 |
| $E_6, F_4$ *all same mult.* | 7 | *EIX* | 19 |

REMARK 4.7. For comparison, the dimension of $G/K$ is 40 for *EII*, 64 for *EVI* and 112 for *EIX*.

PROOF. When all the restricted roots of the symmetric space have the same multiplicity, we reason as in the proof of Lemma 4.3, using the fact (with the notation of that lemma) shown in [17] that $s = 1/(n-1)$ if the Lie type is $E_n$, $s = 1/5$ for type $F_4$ and $s = 2/5$ for type $G_2$.

For the final three cases (Lie type $F_4$, Cartan classes *EII*, *EVI* or *EIX*), we note that the maximal annihilating root systems are types $A_1 \times A_2$, $A_1 \times B_2$, $A_1 \times C_2$, $A_1 \times A_1 \times A_1$, $B_3$ and $C_3$, all of which have cardinality at most nine, and do a counting argument similar to that done in the proof of Theorem 3.1. □

REMARK 4.8. It would be interesting to know the sharp $L^2$ results and whether the $L^1$–$L^2$ dichotomy only fails for the symmetric space of Lie type $A_1$ and Cartan class *AI*.

**4.2. Differentiability properties.** If $\nu^k \in L^2$, then $\nu^{2k} = \nu^k * \nu^k$ has a continuous density function. However, more can be said about the smoothness of these measures using the following fact shown in [4, Proposition 3 (vi)]:

$$\left|\frac{d^m}{dt^m}(\phi_\lambda(g\exp(tX)))|_{t=0}\right| \leq C_1(1+\|\lambda\|)^m.$$

In proving Theorems 3.1 and 4.1, we have seen that there are constants $C$ and $q(k)$ such that $|(\phi_\lambda(a))^k c(\lambda)^{-1}|^2 \leq C\min(1, \|\lambda\|^{q(k)})$ for all $\lambda$. Thus, with $n = \operatorname{rank} G/K$,

$$\begin{aligned}
&\int_{\mathfrak{a}^*}\left|\phi_\lambda(a)^k\frac{d}{dt}(\phi_\lambda(g\exp(tX)))|_{t=0}|c(\lambda)|^{-2}\right| d\lambda\\
\leq& C\int_{\overline{\mathfrak{a}^{*+}}}|\phi_\lambda(a)^{k/2}c(\lambda)^{-1}|^2(1+\|\lambda\|)\,d\lambda\\
\leq& C\int_{\overline{\mathfrak{a}^{*+}}}\|\lambda\|^{q(k/2)}(1+\|\lambda\|)\,d\lambda\\
\leq& C\int_1^\infty t^{n-1+q(k/2)+1}\,dt
\end{aligned}$$

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and this is finite provided $n + q(k/2) < -1$. If $k$ is chosen sufficiently large that this inequality holds, then Leibniz's rule applied to the inversion formula [19, Ch. IV, Theorem 7.5] shows that

$$X\nu_a^k(g) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \phi_\lambda(a)^k \frac{d}{dt}(\phi_\lambda(g\exp(tX)))|_{t=0}|c(\lambda)|^{-2}\,d\lambda$$

is well defined and hence $\nu_a^k$ is differentiable. More generally, $\nu_a^k$ is $r$-times differentiable if $n - 1 + q(k/2) + r < -1$.

For example, if $G/K$ is of Lie type $A_n$, then Lemma 4.3 yields $q(k/2) \leq \eta_0(n - k/2)$. Thus, we have that $\nu_a^k$ is differentiable for all $a \in A_0$ if $k > 2n + 2(n+1)/\eta_0$. If $a$ is a regular element and $G/K$ is not of Lie type $A_1$ and Cartan class $AI$, then one can similarly use Lemma 3.3 to check that $\nu_a^k$ is differentiable if $k > 4$. Similar statements can be made about higher orders of differentiability. These observations improve upon [2], where it was shown that if $a$ is a regular element, then $\nu_a^k$ is differentiable for $k > \dim G/K + 1$.

## 5. Appendix

In the charts below we summarize some of the important facts about these symmetric spaces. These are taken from [5] and [18, Ch. X].

| Restricted root space | Cartan class | $G/K$ | $\dim G/K$ | Multiplicities $\eta_0; \eta_1; \eta_2$ |
|---|---|---|---|---|
| $A_{n-1}$ | $AI$ | $\frac{SL(n,R)}{SO(n)}$ | $\frac{1}{2}(n-1)(n+2)$ | $1; 0; 0$ |
| $A_{n-1}$ | $AII$ | $\frac{SL(n,H)}{Sp(n)}$ | $(n-1)(2n+1)$ | $4; 0; 0$ |
| $BC_n, p > n$<br>$C_n, p = n$ | $AIII$ | $\frac{SU(p,n)}{S(U(p)\times U(n))}$ | $2pn$ | $2; 1; 2(p-n)$ |
| $C_n$ | $CI$ | $\frac{Sp(n,R)}{U(n)}$ | $n(n+1)$ | $1; 1; 0$ |
| $BC_n, p > n$<br>$C_n, p = n$ | $CII$ | $\frac{Sp(p,n)}{Sp(p)\times Sp(n)}$ | $4pn$ | $4; 3; 4(p-n)$ |
| $C_n$ | $DIII$ (even) | $\frac{SO^*(4n)}{U(2n)}$ | $2n(2n-1)$ | $4; 1; 0$ |
| $BC_n$ | $DIII$ (odd) | $\frac{SO^*(4n+2)}{U(2n+1)}$ | $2n(2n+1)$ | $4; 1; 4$ |
| $B_n, p > n$<br>$D_n, p = n$ | $BDI$ | $\frac{SO_0(p,n)}{SO(p)\times SO(n)}$ | $pn$ | $1; 0; p-n$ |

| Restricted root space | Cartan class | $\dim G/K$ | Multiplicities |
|---|---|---|---|
| $BC_2$ | $EIII$ | 32 | $8; 6; 1$ |
| $A_2$ | $EIV$ | 26 | 8 |
| $C_3$ | $EVII$ | 54 | $8; 1$ |
| $BC_1$ | $FII$ | 16 | $8; 7$ |

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| Restricted root space | Cartan class | dim $G/K$ | Multiplicities |
|---|---|---|---|
| $G_2$ | $G$ | 8 | 1 |
| $F_4$ | $EII$ | 40 | 1; 2 |
| | $EVI$ | 64 | 1; 4 |
| | $EIX$ | 112 | 8; 1 |
| | $FI$ | 28 | 1 |
| $E_6$ | $EI$ | 42 | 1 |
| $E_7$ | $EV$ | 70 | 1 |
| $E_8$ | $EVIII$ | 128 | 1 |

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