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The Weak Order on Weyl Posets

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Abstract. We define a natural lattice structure on all subsets of a finite root system that extends the weak order on the elements of the corresponding Coxeter group. For crystallographic root systems, we show that the subposet of this lattice induced by antisymmetric closed subsets of roots is again a lattice. We then study further subposets of this lattice that naturally correspond to the elements, the intervals, and the faces of the permutahedron and the generalized associahedra of the corresponding Weyl group. These results extend to arbitrary finite crystallographic root systems the recent results of G. Chatel, V. Pilaud, and V. Pons on the weak order on posets and its induced subposets.

1 Introduction

The weak order is a fundamental ordering of the elements of a Coxeter group. It can be defined as the prefix order in reduced expressions of the elements of the group, or more geometrically as the inclusion poset of the inversion sets of the elements of the group. For finite Coxeter groups, the weak order is known to be a lattice [Bjö84], and its Hasse diagram is the graph of the permutahedron of the group oriented in a linear direction. The rich theory of congruences of the weak order [Rea04] yield to the construction of Cambrian lattices [Rea06] with its connection to Coxeter Catalan combinatorics and finite type cluster algebras [FZ02, FZ03a]. This point of view was fundamental for the construction of generalized associahedra [HLT11]. We refer the reader to the survey papers [Rea12, Rea16, Hoh12] for details on these subjects.

More recently, some efforts were devoted to developing certain extensions of the weak order beyond the elements of the group. This led in particular to the notion of facial weak order of a finite Coxeter group, pioneered in type A in [KLN+01], defined for arbitrary finite Coxeter groups in [PR06], and proved to be a lattice in [DHP18]. This order is a lattice on the faces of the permutahedron that extends the weak order on the vertices.

In type A, an even more general notion of weak order on integer binary relations was recently introduced in [CPP17]. This order is defined by $\mathrm{R} \preccurlyeq \mathrm{S} \iff \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}}$ and $\mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$ for any two binary relations R, S on $[n]$, where $\mathrm{R}^{\mathsf{Inc}} := \{(a,b) \in \mathrm{R} \mid a < b\}$ and $\mathrm{R}^{\mathsf{Dec}} := \{(b,a) \in \mathrm{R} \mid a < b\}$, respectively, denote the increasing and decreasing subrelations of R. It turns out that the subposet of this weak order induced by posets on $[n]$ is a lattice. In fact, many relevant lattices can be recovered as subposets of the weak order on posets induced by certain families of posets. Such families

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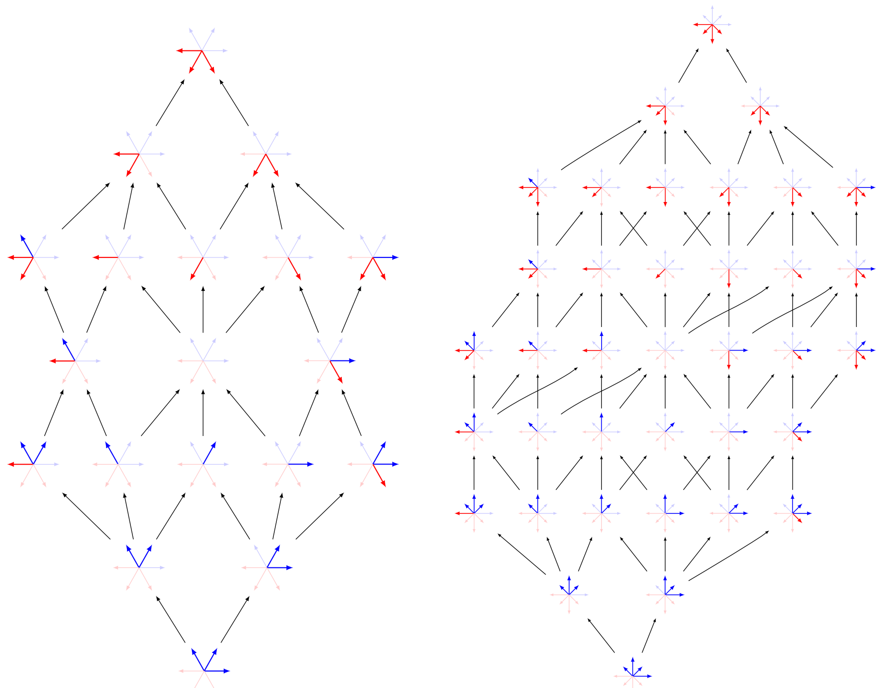

Figure 1: The weak order on A_2-posets (left) and on B_2-posets (right).

include the vertices, the intervals, and the faces of the permutahedron, associahedra [Lod04, HL07], permutreehedra [PP18], cube, etc. For the vertices, the corresponding lattices are the weak order on permutations, the Tamari lattice on binary trees, the type A Cambrian lattices, the permutree lattices [PP18], the boolean lattice on binary sequences, etc.

The goal of this paper is to extend these results beyond type A using subsets of root systems. We define the *weak order* on subsets R, S of a finite root system Φ by $\mathrm{R} \preccurlyeq \mathrm{S} \iff \mathrm{R}^+ \supseteq \mathrm{S}^+$ and $\mathrm{R}^- \subseteq \mathrm{S}^-$, where $\mathrm{R}^+ := \mathrm{R} \cap \Phi^+$ and $\mathrm{R}^- := \mathrm{R} \cap \Phi^-$. This order is a lattice on all subsets of Φ, which are the analogues of type A integer binary relations. In turn, the analogues of type A integer posets are Φ-*posets*, *i.e.*, subsets R of Φ that are both antisymmetric ($\alpha \in \mathrm{R}$ implies $-\alpha \notin \mathrm{R}$) and closed (in the sense of [Bou68], $\alpha, \beta \in \mathrm{R}$ and $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \mathrm{R}$). Our central result is that the subposet of the weak order induced by Φ-posets is also a lattice when the root system Φ is crystallographic. For example, the weak orders on A_2-, B_2- and G_2-posets are represented in Figures 1 and 2. Surprisingly, this property fails for non-crystallographic root systems, and the proof actually requires us to develop delicate properties on subsums of roots in crystallographic root systems.

We then switch our focus to our motivation to study the weak order on Φ-posets. We consider Φ-posets corresponding to the vertices, the intervals, and the faces of

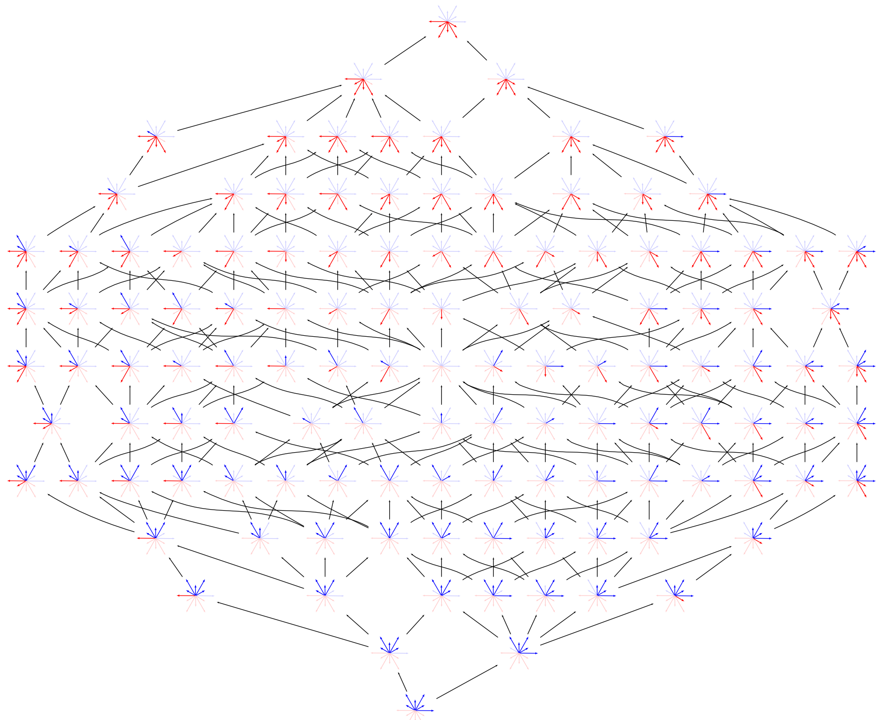

Figure 2: The weak order on G_2-posets.

the permutahedron, the associahedra, and the cube of type Φ. Considering the subposets of the weak order induced by these specific families of Φ-posets allows us to recover the classical weak order and the Cambrian lattices, their interval lattices, and their facial lattices. A roadmap presenting the different families of subsets of roots considered in this paper is given in Figure 3.

2 Root Systems

This section gathers some notions and properties of finite crystallographic root systems and Weyl groups. We refer the reader to the textbooks by J. Humphreys [Hum90], N. Bourbaki [Bou68], and A. Björner and F. Brenti [BB05] for further details on basic definitions and classical properties.

2.1 Root Systems

Let V be a real Euclidean space with scalar product $\langle \cdot \mid \cdot \rangle$. For $\alpha \in V \setminus \{0\}$, we define $\alpha^\vee := 2\alpha/\langle \alpha \mid \alpha \rangle$. We denote by s_α the reflection orthogonal to a non-zero vector $\alpha \in V$, defined by $s_\alpha(v) = v - \langle \alpha^\vee \mid v \rangle \alpha$. A *finite root system* Φ is a finite set of non-zero vectors in V such that $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ and $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

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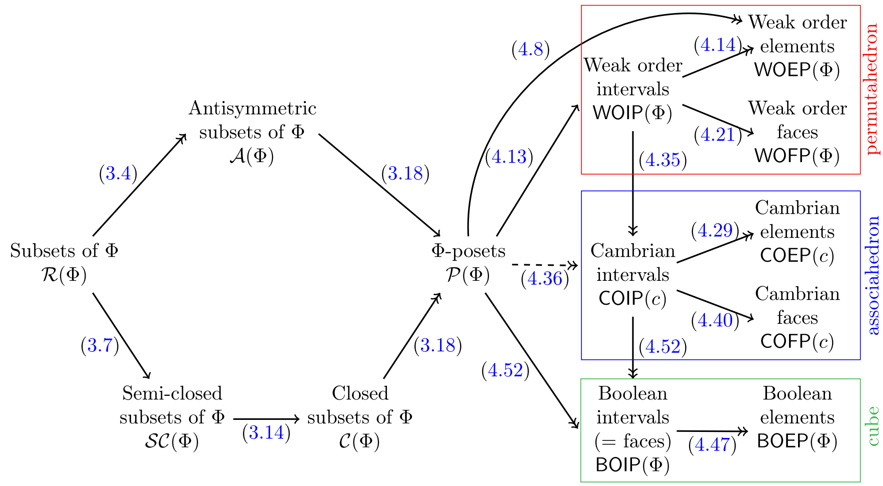

Figure 3: A roadmap through the different families of subsets of roots studied in this paper. An arrow → indicates a subposet, while an arrow ↠ indicates a sublattice. The dashed arrow is a conjectural sublattice relation. The label on each arrow refers to the corresponding statement in this paper.

We denote by W the *Coxeter group* generated by the reflections s_α for $\alpha \in \Phi$. Throughout this paper, we will denote by $\mathcal{R}(\Phi)$ the collection of all subsets of Φ.

We choose a generic linear functional f and denote the set of *positive roots* by $\Phi^+ := \{\alpha \in \Phi \mid f(\alpha) > 0\}$ and the set of *negative roots* by $\Phi^- := \{\alpha \in \Phi \mid f(\alpha) < 0\}$. We denote by Δ the *simple roots*. They are the roots of the rays of the cone $\mathbb{R}_{\geq 0}\Phi^+$ and form a linear basis, so that any positive root is a positive linear combination of simple roots. The *height* of a root $\alpha = \sum_{\delta \in \Delta} \alpha_\delta \delta$ is $h(\alpha) = \sum_{\delta\in\Delta} \alpha_\delta$. The *absolute height* of α is $|h|(\alpha) = |h(\alpha)|$.

The root system Φ is *crystallographic* if $\langle \alpha^\vee \mid \beta\rangle \in \mathbb{Z}$ for any $\alpha, \beta \in \Phi$. Equivalently, the Coxeter group W stabilizes the lattice $\mathbb{Z}\Phi$ and is called a *Weyl group*. In most of the paper, we restrict our attention to crystallographic root systems. Remarks 2.6, 2.7, 2.13, 3.16, 3.19, and 3.23 justify this restriction.

Example 2.1 (Type A) Let $(\mathbf{e}_i)_{i\in[n+1]}$ be the standard basis of $\mathbb{R}^{n+1}$. The symmetric group $\mathfrak{S}_{n+1}$ acts on $\mathbb{R}^{n+1}$ by permutation of coordinates. It is the Weyl group of type A_n. The roots are $\Phi_{A_n} = \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i \neq j \leq n+1\}$, the positive roots are $\Phi^+_{A_n} = \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i < j \leq n+1\}$ and the simple roots are $\Delta_{A_n} = \{\mathbf{e}_i - \mathbf{e}_{i+1} \mid i \in [n]\}$. A subset of Φ_{A_n} can thus be identified with a binary relation on $[n]$ via the bijection $(i,j) \in [n]^2 \longleftrightarrow \mathbf{e}_i - \mathbf{e}_j \in \Phi_A$. Note that the height of $\mathbf{e}_i - \mathbf{e}_j$ is $j - i$.

2.2 Sums of Roots in Crystallographic Root Systems

We now gather statements on sums of roots in crystallographic root systems that are needed throughout the paper and that we consider interesting for their own sake.

We start by a statement from [Bou68] providing sufficient conditions for the sum or difference of two roots to again be a root in a crystallographic root system Φ.

Theorem 2.2 ([Bou68, Chap. 6, 1.3, Thm. 1]) *For any α, β in a crystallographic root system Φ,*

(i) *if $\langle \alpha \mid \beta \rangle > 0$, then $\alpha - \beta \in \Phi$ or $\alpha = \beta$;*
(ii) *if $\langle \alpha \mid \beta \rangle < 0$, then $\alpha + \beta \in \Phi$ or $\alpha = -\beta$.*

We say that a (multi)set $X \subseteq \Phi$

- is *summable* if its sum ΣX is again a root of Φ,
- has *no vanishing subsum* if $\Sigma Y \neq 0$ for any $\varnothing \neq Y \subseteq X$.

Proposition 2.3 and Theorems 2.4 and 2.5 ensure that a summable set of roots with no vanishing subsum has many summable subsets. We start with sums of three roots.

Proposition 2.3 *Let Φ be a crystallographic root system. If $\alpha, \beta, \gamma \in \Phi$ are such that $\alpha + \beta + \gamma \in \Phi$ has no vanishing subsum, then at least two of the three subsums $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$ are in Φ.*

Proof Assume by means of contradiction that $\alpha+\beta \notin \Phi$ and $\alpha + \gamma \notin \Phi$. Since $\alpha+\beta+\gamma$ has no vanishing subsum, $\alpha \neq -\beta$ and $\alpha \neq -\gamma$. By contraposition of Theorem 2.2(ii), we obtain that $\langle \alpha \mid \beta \rangle \geq 0$ and $\langle \alpha \mid \gamma \rangle \geq 0$. Therefore,

$$\langle \alpha + \beta + \gamma \mid \beta + \gamma \rangle = \langle \alpha \mid \beta \rangle + \langle \alpha \mid \gamma \rangle + \langle \beta + \gamma \mid \beta + \gamma \rangle > 0,$$

since $\beta + \gamma \neq 0$. It follows that either $\langle \alpha + \beta + \gamma \mid \beta \rangle > 0$ or $\langle \alpha + \beta + \gamma \mid \gamma \rangle > 0$. Assume for instance $\langle \alpha + \beta + \gamma \mid \beta \rangle > 0$. Theorem 2.2(i) then implies that either $\alpha + \gamma \in \Phi$ or $\alpha + \gamma = 0$, contradicting either of our assumptions on $\alpha + \gamma$. ■

It is proved in [Bou68, Chap. 6, 1.6, Prop. 19] that any summable subset X of positive roots admits a filtration $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{|X|-1} \subsetneq X_{|X|} = X$ of summable subsets. We now use Proposition 2.3 to extend this property in two directions: first, we consider subsets of all roots (positive and negative); second, we show that we can additionally prescribe the initial set X_1 to be a chosen root of Φ. This latter improvement will be crucial throughout the paper.

Theorem 2.4 *Let Φ be a crystallographic root system. Any summable set $X \subseteq \Phi$ with no vanishing subsum admits a filtration of summable subsets*

$$\{\alpha\} = X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{|X|-1} \subsetneq X_{|X|} = X$$

for any $\alpha \in X$.

Proof The proof works by induction on $|X|$. It is clear for $|X| = 2$, so we consider $|X| > 2$. By induction, it suffices to find a summable subset $X_{|X|-1}$ of size $|X| - 1$ such that $\alpha \in X_{|X|-1} \subset X$. Since $\sum_{\beta \in X} \langle \beta \mid \Sigma X \rangle = \langle \Sigma X \mid \Sigma X \rangle > 0$, there exists $\beta \in X$ such that $\langle \beta \mid \Sigma X \rangle > 0$. Since X has no vanishing subsum, we have $\beta \neq \Sigma X$. Theorem 2.2(i) thus ensures that $X \smallsetminus \{\beta\}$ is summable. If $\alpha \neq \beta$, then we set $X_{|X|-1} := X \smallsetminus \{\beta\}$ and conclude by induction. Otherwise, we have proved that both $\{\alpha\}$ and $X \smallsetminus \{\alpha\}$ are summable. Let Y be inclusion maximal with $\alpha \in Y \subsetneq X$ such that both Y and $X \smallsetminus Y$ are summable. Assume that $|X \smallsetminus Y| \geq 2$. By induction hypothesis, there exists $Z \subset X \smallsetminus Y$ summable with $|Z| = |X \smallsetminus Y| - 1 \geq 1$. Let γ be the root in $(X \smallsetminus Y) \smallsetminus Z$. Since γ, ΣY, and ΣZ are roots and $\gamma + \Sigma Y + \Sigma Z = \Sigma X \in \Phi$, Proposition 2.3 affirms that either $\{\gamma\} \cup Y$ or $Y \cup Z$ is summable, contradicting the maximality of Y. We therefore obtained a summable subset Y with $\alpha \in Y \subseteq X$ with $|Y| = |X| - 1$. We set $X_{|X|-1} := Y$ and conclude by induction. ∎

Finally, we obtain the following generalization of Proposition 2.3.

Theorem 2.5 *Let Φ be a crystallographic root system. Any summable set $X \subseteq \Phi$ with no vanishing subsum admits at least p distinct summable subsets of size $|X| - p + 1$, for any $1 \leq p \leq |X|$.*

Proof Note that it holds for $p = 1$ and $p = |X|$. We now proceed by induction on $|X|$ to prove the result for $1 < p < |X|$. By Theorem 2.4, X admits a summable subset Y of size $|X| - 1$. Since $1 < p$, we can apply the induction hypothesis to find $p - 1$ distinct summable subsets $Z_1, \dots, Z_{p-1}$ of Y of size $|Y| - p + 2 = |X| - p + 1$. Moreover, by Theorem 2.4 there exists at least one summable subset Z_p of X of size $|X| - p + 1$ containing the root α in $X \smallsetminus Y$. This subset Z_p is distinct from all the subsets $Z_1, \dots, Z_{p-1}$ of Y, since it contains α. This concludes the proof. ∎

Remark 2.6 All results presented in this section fail for non-crystallographic root systems. For example, consider the Coxeter group of type H_3 with Dynkin diagram $1 \overset{5}{\text{——}} 2 \text{——} 3$ and the positive roots $\alpha := \alpha_1$, $\beta := \alpha_2$ and $\gamma := s_1 s_2 s_3(\alpha_2) = \psi(\alpha_1 + \alpha_2 + \alpha_3)$, where $\psi = -2\cos(4\pi/5)$. Then

- $\langle \alpha \mid \beta \rangle < 0$ when $\alpha + \beta \notin \Phi$ and $\alpha \neq -\beta$;
- $\alpha + \beta + \gamma \in \Phi$ when $\alpha + \beta \notin \Phi$ and $\beta + \gamma \notin \Phi$ (although $\alpha + \gamma \in \Phi$).

Remark 2.7 For later purposes, we need an even stronger counter-example to Theorem 2.4 in non-crystallographic root systems. Consider the Coxeter group of type $H_2 = I_2(5)$ and the roots $\alpha := \alpha_1$, $\beta := \alpha_2$, $\gamma := \psi\alpha_1 + \psi\alpha_2$ and $\delta := -\alpha_1 - \psi\alpha_2$, where $\psi = -2\cos(4\pi/5)$. It is not difficult to check that

$$\Phi \cap \{a\alpha + b\beta + c\gamma + d\delta \mid a, b, c, d \in \mathbb{N}\} = \{\alpha, \beta, \gamma, \delta, \alpha + \beta + \gamma + \delta\}.$$

In particular, there is not even a single flag $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \{\alpha, \beta, \gamma, \delta\}$ of summable subsets of $\{\alpha, \beta, \gamma, \delta\}$, even though $\{\alpha, \beta, \gamma, \delta\}$ is itself summable.

2.3 Φ-posets

In Section 3, we will consider certain specific families of collections of roots. We start with the simple definition of symmetric and antisymmetric subsets of roots.

Definition 2.8 A subset $R \subseteq \Phi$ is *symmetric* if $-R = R$ and *antisymmetric* if $R \cap -R = \varnothing$. We denote by $\mathcal{S}(\Phi)$ (resp. $\mathcal{A}(\Phi)$) the set of symmetric (resp. antisymmetric) subsets of roots of Φ.

We now want to define closed sets of roots. The next statement is proved by A. Pilkington [Pil06, Sect. 2] for subsets of positive roots. We extend it to subsets of all roots using Theorem 2.4.

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Lemma 2.9 *In a crystallographic root system Φ, the following conditions are equivalent for $\mathrm{R} \subseteq \Phi$:*

(i) $\alpha + \beta \in \mathrm{R}$ *for any* $\alpha, \beta \in \mathrm{R}$ *such that* $\alpha + \beta \in \Phi$*;*
(ii) $m\alpha + n\beta \in \mathrm{R}$ *for any* $\alpha, \beta \in \mathrm{R}$ *and* $m, n \in \mathbb{N}$ *such that* $m\alpha + n\beta \in \Phi$*;*
(iii) $\alpha_1 + \cdots + \alpha_p \in \mathrm{R}$ *for any* $\alpha_1, \dots, \alpha_p \in \mathrm{R}$ *such that* $\alpha_1 + \cdots + \alpha_p \in \Phi$.

Proof The proof follows [Pil06, Sect. 2]. The implications (iii) ⇒ (ii) ⇒ (i) are clear. Assume now (i) and let $\alpha_1, \dots, \alpha_p \in \mathrm{R}$ such that $\alpha_1 + \cdots + \alpha_p \in \Phi$. By Theorem 2.4, there exists a flag $\mathrm{X}_1 \subsetneq \mathrm{X}_2 \subsetneq \cdots \subsetneq \mathrm{X}_p = \{\alpha_1, \dots, \alpha_p\}$ of summable subsets of Φ. Applying (i) inductively, we get that $\Sigma \mathrm{X}_i \in \mathrm{R}$ for all $i \in [p]$, and thus $\alpha_1 + \cdots + \alpha_p \in \mathrm{R}$. ∎

Definition 2.10 In a crystallographic root system Φ, a subset $\mathrm{R} \subseteq \Phi$ is *closed* if it satisfies the equivalent conditions of Lemma 2.9. We denote by $\mathcal{C}(\Phi)$ the set of closed subsets of roots of Φ.

Definition 2.11 In a crystallographic root system Φ, the *closure* of $\mathrm{R} \subseteq \Phi$ is the set $\mathrm{R}^{\mathrm{cl}} := \mathbb{N}\mathrm{R} \cap \Phi$.

Remark 2.12 The map $\mathrm{R} \mapsto \mathrm{R}^{\mathrm{cl}}$ is a closure operator on Φ, meaning that

$$\varnothing^{\mathrm{cl}} = \varnothing, \quad \Phi \subseteq \Phi^{\mathrm{cl}}, \quad \mathrm{R} \subseteq \mathrm{S} \Longrightarrow \mathrm{R}^{\mathrm{cl}} \subseteq \mathrm{S}^{\mathrm{cl}}, \quad \text{and} \quad (\mathrm{R}^{\mathrm{cl}})^{\mathrm{cl}} = \mathrm{R}^{\mathrm{cl}}$$

for all $\mathrm{R}, \mathrm{S} \subseteq \Phi$. Moreover R^{cl} is closed and R is closed if and only if $\mathrm{R} = \mathrm{R}^{\mathrm{cl}}$.

Remark 2.13 Lemma 2.9 fails for non-crystallographic root systems. For example, consider the roots $\alpha, \beta, \gamma, \delta$ of Remark 2.7. Then the set $\mathrm{R} := \{\alpha, \beta, \gamma, \delta\}$ satisfies (i) but not (iii).

Remark 2.14 As studied in detail by A. Pilkington in [Pil06], even in crystallographic root systems, there are other possible notions of closed sets of roots. Namely, one says that $\mathrm{R} \subseteq \Phi$ is

(i) $\mathbb{N}$-*closed* if $m\alpha + n\beta \in \mathrm{R}$ for $\alpha, \beta \in \mathrm{R}$ and $m, n \in \mathbb{N}$ with $m\alpha + n\beta \in \Phi$;
(ii) $\mathbb{R}$-*closed* if $x\alpha + y\beta \in \mathrm{R}$ for $\alpha, \beta \in \mathrm{R}$ and $x, y \in \mathbb{R}$ with $x\alpha + y\beta \in \Phi$;
(iii) *convex* if $\mathrm{R} = \Phi \cap C$ for a convex cone C in V.

Note that convex implies $\mathbb{R}$-closed, which implies $\mathbb{N}$-closed, but that the converse statements are wrong even for finite root systems [Pil06, p. 3192]. In this paper, we will only work with the notion of $\mathbb{N}$-closedness in crystallographic root systems, as it is discussed in [Bou68]. Remarks 3.17 and 3.20 justify this restriction.

Example 2.15 (Type A) Following Example 2.1, identify subsets of roots with integer binary relations via the bijection $(i, j) \in [n]^2 \longmapsto \mathbf{e}_i - \mathbf{e}_j \in \Phi_A$. A subset of roots is symmetric (resp. antisymmetric, resp. closed) if the corresponding integer binary relation is symmetric (resp. antisymmetric, resp. transitive). (Note that here the three notions of closed sets of roots mentioned in Remark 2.14 coincide in type A.)

This example motivates the definition of the central object of this paper.

Definition 2.16 In a crystallographic root system Φ, a Φ-*poset* is an antisymmetric and $\mathbb{N}$-closed subset of roots of Φ. We denote by $\mathcal{P}(\Phi)$ the set of all Φ-posets.

We speak of Weyl posets when we do not want to specify the root system. We will introduce in Section 3.4 a natural lattice structure on Φ-posets. We will see in Section 4 various subfamilies of Φ-posets arising from classical Coxeter and Coxeter Catalan combinatorics.

To conclude this preliminary section on Φ-posets, we gather simple observations on their subsums and their extensions.

Lemma 2.17 *For any Φ-poset* R *and any roots $\alpha_1, \dots, \alpha_p \in \mathrm{R}$, we have $\alpha_1 + \cdots + \alpha_p \neq 0$.*

Proof Assume that R is a Φ-poset and there are $\alpha_1, \dots, \alpha_p \in \mathrm{R}$ such that $\alpha_1 + \cdots + \alpha_p = 0$. Then $\alpha_2 + \cdots + \alpha_p = -\alpha_1$ is a root, so Lemma 2.9(iii) ensures that $\alpha_2 + \cdots + \alpha_p \in \mathrm{R}$, since R is closed. We obtain that $\alpha_1 \in \mathrm{R}$ and $-\alpha_1 \in \mathrm{R}$, contradicting the antisymmetry of R. ∎

Finally, we need Φ-poset extensions. The subsets of Φ are naturally ordered by inclusion, and we consider the restriction of this inclusion order on Φ-posets. For $\mathrm{R} \in \mathcal{P}(\Phi)$, we call *extensions* of R the Φ-posets S containing R, and we let $\mathcal{E}(\mathrm{R}) := \{\mathrm{S} \in \mathcal{P}(\Phi) \mid \mathrm{R} \subseteq \mathrm{S}\}$. Note that $\mathrm{R} \in \mathcal{E}(\mathrm{R})$ and $\mathrm{R} \subseteq \mathrm{S}$ for all $\mathrm{S} \in \mathcal{E}(\mathrm{R})$ so that $\mathrm{R} = \bigcap \mathcal{E}(\mathrm{R})$. For later purposes, we are interested in maximal Φ-posets in the extension order.

Proposition 2.18 *For $\mathrm{R} \in \mathcal{P}(\Phi)$, we have $\mathcal{E}(\mathrm{R}) = \{\mathrm{R}\}$ if and only if $\{\alpha, -\alpha\} \cap \mathrm{R} \neq \varnothing$ for all $\alpha \in \Phi$.*

Proof Clearly, if $\{\alpha, -\alpha\} \cap \mathrm{R} \neq \varnothing$ for all $\alpha \in \Phi$, then adding any root to R breaks the antisymmetry, so that $\mathcal{E}(\mathrm{R}) = \{\mathrm{R}\}$. Reciprocally, assume that there exists $\alpha \in \Phi$ such that $\{\alpha, -\alpha\} \cap \Phi = \varnothing$. Let $\mathrm{S} := (\mathrm{R} \cup \{\alpha\})^{\mathsf{cl}}$ and $\mathrm{T} := (\mathrm{R} \cup \{-\alpha\})^{\mathsf{cl}}$. By definition, both S and T are closed, and we claim that at least one of them is antisymmetric, thus proving that R admits a non-trivial extension. Assume by means of contradiction that neither S nor T are antisymmetric. Let $\beta \in \mathrm{S} \cap -\mathrm{S}$ and $\gamma \in \mathrm{T} \cap -\mathrm{T}$. By definition of the closure, we can write

$$\beta = \sum_{\delta \in \mathrm{R}} \lambda_\delta \delta + \lambda_\alpha \alpha = -\sum_{\delta \in \mathrm{R}} \kappa_\delta \delta - \kappa_\alpha \alpha,$$
$$\gamma = \sum_{\delta \in \mathrm{R}} \mu_\delta \delta - \mu_\alpha \alpha = -\sum_{\delta \in \mathrm{R}} \nu_\delta \delta + \nu_\alpha \alpha,$$

where $\lambda_\delta, \kappa_\delta, \mu_\delta, \nu_\delta$ are non-negative integer coefficients for all $\delta \in \mathrm{R} \cup \{\alpha\}$. Moreover, we have $\lambda_\alpha + \kappa_\alpha \neq 0 \neq \mu_\alpha + \nu_\alpha$, since R is antisymmetric and closed. This implies that

$$\sum_{\delta \in \mathrm{R}} \big((\lambda_\alpha + \kappa_\alpha)(\mu_\delta + \nu_\delta) + (\mu_\alpha + \nu_\alpha)(\lambda_\delta + \kappa_\delta) \big)\, \delta = 0.$$

Lemma 2.17 thus ensures that $(\lambda_\alpha + \kappa_\alpha)(\mu_\delta + \nu_\delta) + (\mu_\alpha + \nu_\alpha)(\lambda_\delta + \kappa_\delta) = 0$, which in turns implies that $\lambda_\delta = \kappa_\delta = \mu_\delta = \nu_\delta = 0$ for all $\delta \in \mathrm{R}$, a contradiction. ∎

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3 Weak Order on Φ-posets

3.1 Weak Order on All Subsets

Let Φ be a finite root system (not necessarily crystallographic for the moment) with positive roots Φ^+ and negative roots Φ^-. We denote by $\mathcal{R}(\Phi)$ the set of all subsets of Φ. For $R \in \mathcal{R}(\Phi)$, we denote by $R^+ := R \cap \Phi^+$ its positive part and $R^- := R \cap \Phi^-$ its negative part. The following order was considered in type A in [CPP17].

Definition 3.1 The *weak order* on $\mathcal{R}(\Phi)$ is defined by

$$R \preccurlyeq S \quad \Longleftrightarrow \quad R^+ \supseteq S^+ \text{ and } R^- \subseteq S^-.$$

Remark 3.2 The name for this order relation will be transparent in Section 4. Note that there is an arbitrary choice of orientation in Definition 3.1. The choice we have made here may seem unusual, as the apparent contradiction in Proposition 4.5 suggests. However, it is more coherent with the case of type A as treated in [CPP17], and it simplifies the presentation of Section 4.1.3.

Proposition 3.3 *The weak order on $\mathcal{R}(\Phi)$ is a lattice with meet and join*

$$R \wedge_{\mathcal{R}} S = (R^+ \cup S^+) \sqcup (R^- \cap S^-) \quad \textit{and} \quad R \vee_{\mathcal{R}} S = (R^+ \cap S^+) \sqcup (R^- \cup S^-).$$

Furthermore, it is graded by $R \mapsto |R^-| - |R^+|$, *and its cover relations are given by*

$$R \preccurlyeq R \smallsetminus \{\alpha\} \textit{ for } \alpha \in R^+ \quad \textit{and} \quad R \smallsetminus \{\beta\} \preccurlyeq R \textit{ for } \beta \in R^-.$$

Proof It is the Cartesian product of two boolean lattices (the reverse inclusion poset on the positive roots and the inclusion poset on the negative roots). ∎

This section is devoted to showing that the restriction of the weak order to certain families of subsets of roots (antisymmetric, closed and Φ-posets — see Figure 3 for a roadmap) still defines a lattice structure when Φ is crystallographic, and expressing its meet and join operations. For example, the weak orders on A_2-, B_2- and G_2-posets are represented in Figures 1 and 2.

3.2 Weak Order on Antisymmetric Subsets

We start with the antisymmetry condition.

Proposition 3.4 *The meet $\wedge_{\mathcal{R}}$ and the join $\vee_{\mathcal{R}}$ both preserve antisymmetry. Thus, the set $\mathcal{A}(\Phi)$ of antisymmetric subsets of Φ induces a sublattice of the weak order on $\mathcal{R}(\Phi)$.*

Proof Consider two antisymmetric subsets $R, S \in \mathcal{R}(\Phi)$ and let $\alpha \in (R \wedge_{\mathcal{R}} S)^+ = R^+ \cup S^+$. Assume, for instance, that $\alpha \in R^+$. Since R is antisymmetric, $-\alpha \notin R^-$, so that $-\alpha \notin R^- \cap S^- = (R \wedge_{\mathcal{R}} S)^-$. We conclude that $R \wedge_{\mathcal{R}} S$ is antisymmetric. The proof for $R \vee_{\mathcal{R}} S$ is similar. ∎

Proposition 3.5 *All cover relations in the weak order on $\mathcal{A}(\Phi)$ are cover relations in the weak order on $\mathcal{R}(\Phi)$. In particular, the weak order on $\mathcal{A}(\Phi)$ is still graded by* $R \mapsto |R^+| - |R^-|$.

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Proof Consider a cover relation $R \lessdot S$ in the weak order on $\mathcal{A}(\Phi)$. We have $R^+ \supseteq S^+$ and $R^- \subseteq S^-$ where at least one of the inclusions is strict. Suppose first that $R^+ \neq S^+$. Let $\alpha \in R^+ \smallsetminus S^+$ and $T := R \smallsetminus \{\alpha\}$. Note that $T \in \mathcal{A}(\Phi)$ and $R < T \leqslant S$. Since S covers R, we get $S = T = R \smallsetminus \{\alpha\}$. Similarly, if $S^- \neq R^-$, let $\alpha \in S^- \smallsetminus R^-$ and $T := S^- \smallsetminus \{\alpha\}$. Then $T \in \mathcal{A}(\Phi)$, and $R \leqslant T < S$ implies that $T = R = S \smallsetminus \{\alpha\}$. In both cases, $R \lessdot S$ is a cover relation of the weak order on $\mathcal{R}(\Phi)$. ∎

Corollary 3.6 *In the weak order on $\mathcal{A}(\Phi)$, the antisymmetric subsets that cover a given antisymmetric subset $R \in \mathcal{A}(\Phi)$ are precisely the following relations:*

- $R \smallsetminus \{\alpha\}$ *for any* $\alpha \in R^+$,
- $R \cup \{\beta\}$ *for any* $\beta \in \Phi^- \smallsetminus R^-$ *such that* $-\beta \notin R^+$.

3.3 Weak Order on Closed Subsets

We want to prove that the weak order on closed subsets of Φ is also a lattice. Contrary to Propositions 3.3 and 3.4, we now need to assume that the root system Φ is crystallographic (see Remarks 2.13, 3.16, and 3.19). Unfortunately, as $\mathcal{C}(\Phi)$ is stable by intersection but not by union, it is not preserved by the meet $\wedge_{\mathcal{R}}$ and the join $\vee_{\mathcal{R}}$, so that it does not induce a sublattice of the weak order on $\mathcal{R}(\Phi)$. Proving that it is still a lattice requires more work. Following [CPP17], we start with a weaker notion of closedness. We say that a subset $R = R^+ \sqcup R^-$ is *semiclosed* if both R^+ and R^- are closed. We denote by $\mathcal{SC}(\Phi)$ the set of semiclosed subsets of Φ. Note that $\mathcal{C}(\Phi) \subseteq \mathcal{SC}(\Phi)$ but that the reverse inclusion does not hold in general.

Proposition 3.7 *The weak order on $\mathcal{SC}(\Phi)$ is a lattice with meet and join*

$$R \wedge_{\mathcal{SC}} S = (R^+ \cup S^+)^{\mathrm{cl}} \sqcup (R^- \cap S^-) \quad \textit{and} \quad R \vee_{\mathcal{SC}} S = (R^+ \cap S^+) \sqcup (R^- \cup S^-)^{\mathrm{cl}}.$$

Proof Observe first that $R \wedge_{\mathcal{SC}} S$ is indeed semiclosed (T^{cl} is always closed and $\mathcal{C}(\Phi)$ is stable by intersection). Moreover, $R \wedge_{\mathcal{SC}} S \leqslant R$ and $R \wedge_{\mathcal{SC}} S \leqslant S$. Assume now that $T \subseteq \Phi$ is semiclosed such that $T \leqslant R$ and $T \leqslant S$. Then $T^+ \supseteq R^+ \cup S^+$ and $T^- \subseteq R^- \cap S^-$. Moreover, since T^+ is closed, we get that $T^+ \supseteq (R^+ \cup S^+)^{\mathrm{cl}}$ so that $T \leqslant R \wedge_{\mathcal{SC}} S$. We conclude that $R \wedge_{\mathcal{SC}} S$ is indeed the meet of R and S. The proof is similar for the join. ∎

Proposition 3.8 *All cover relations in the weak order on $\mathcal{SC}(\Phi)$ are cover relations in the weak order on $\mathcal{R}(\Phi)$. In particular, the weak order on $\mathcal{SC}(\Phi)$ is still graded by $R \mapsto |R^-| - |R^+|$.*

Proof Consider a cover relation $R \lessdot S$ in the weak order on $\mathcal{SC}(\Phi)$. We have $R^+ \supseteq S^+$ and $R^- \subseteq S^-$ where at least one of the inclusions is strict. We distinguish two cases.

Suppose first that $R^+ \neq S^+$, and consider $\alpha \in R^+ \smallsetminus S^+$ of minimal height in $R^+ \smallsetminus S^+$. Observe that α cannot be decomposed in R^+: if $\alpha = \gamma + \delta$ with $\gamma, \delta \in R^+$, then $h(\gamma), h(\delta) < h(\alpha)$, so $\gamma, \delta \in S^+$ by minimality of $h(\alpha)$, which contradicts the closedness of S^+. Consider now $T := R \smallsetminus \{\alpha\}$. Let $\gamma, \delta \in T^+$ with $\gamma + \delta \in \Phi$. Then $\gamma, \delta \in R^+$ so that $\gamma + \delta \in R^+$, since R^+ is closed. Since $\gamma + \delta \neq \alpha$, this implies that $\gamma + \delta \in T^+$. This shows that T^+ is closed. Since $T^- = R^-$ is also closed, we obtain that T is semiclosed. Since $R \neq T$ and $R \leqslant T \leqslant S$, this proves that $T = S = R \smallsetminus \{\alpha\}$.

Assume now that $R^- \neq S^-$, and let $\beta \in S^- \smallsetminus R^-$ of minimal height (or equivalently maximal absolute height). Consider $T := R \cup \{\beta\}$. Let $\gamma, \delta \in T^-$ with $\gamma + \delta \in \Phi$. If $\gamma, \delta \in R^-$, then $\gamma + \delta \in R^-$, since R^- is closed. Assume now that $\delta = \beta$. Then $\gamma, \beta \in S^-$ and S^- is closed, we have $\gamma + \beta \in S^-$ and $h(\gamma + \beta) < h(\beta)$, which ensures that $\gamma + \beta \in R^-$ by minimality of $h(\beta)$. This shows that T^- is closed. Since $T^+ = R^+$ is also closed, we obtain that T is semiclosed. Since $R \neq T$ and $R \preccurlyeq T \preccurlyeq S$, this proves that $T = S = R \cup \{\beta\}$. ∎

Corollary 3.9 *In the weak order on $\mathcal{SC}(\Phi)$, the semiclosed subsets of Φ that cover a given semiclosed subset $R \in \mathcal{SC}(\Phi)$ are precisely the relations:*

- $R \smallsetminus \{\alpha\}$ *for any* $\alpha \in R^+$ *such that there is no* $\gamma, \delta \in R^+$ *with* $\alpha = \gamma + \delta$,
- $R \cup \{\beta\}$ *for any* $\beta \in \Phi^- \smallsetminus R^-$ *such that* $\beta + \gamma \in \Phi \implies \beta + \gamma \in R$ *for all* $\gamma \in R^-$.

We now come back to closed subsets of Φ introduced in Definition 2.16. Unfortunately, $\mathcal{C}(\Phi)$ still does not induce a sublattice of $\mathcal{SC}(\Phi)$. We thus need a transformation similar to the closure $R \mapsto R^{\mathsf{cl}}$ to transform a semiclosed subset of Φ into a closed one.

Definition 3.10 For $R \in \mathcal{R}(\Phi)$, we define the *negative closure deletion* R^{ncd} and the *positive closure deletion* R^{pcd} by

$$R^{\mathsf{ncd}} := R \smallsetminus \left\{\alpha \in R^- \;\middle|\; \exists X \subseteq R^+ \text{such that } \alpha + \Sigma X \in \Phi \smallsetminus R\right\},$$
$$R^{\mathsf{pcd}} := R \smallsetminus \left\{\alpha \in R^+ \;\middle|\; \exists X \subseteq R^- \text{such that } \alpha + \Sigma X \in \Phi \smallsetminus R\right\}.$$

Remark 3.11 Note that in the formulas for R^{ncd} and R^{pcd} in Definition 3.10, the following hold:

(i) The notation ΣX denotes the sum of all roots in X, as in Section 2.2.
(ii) We do not assume that X is summable, just that $X \cup \{\alpha\}$ is.
(iii) In the case where R is semiclosed, we can assume that the set X is such that the $\alpha + \Sigma X$ has no vanishing subsum. Observe first that no vanishing subsum can contain α. Indeed, if $Y \subseteq X$ is such that $\alpha + \Sigma Y = 0$, then $X \smallsetminus Y \subseteq R^-$ and R^- closed implies that $\alpha + \Sigma X = \Sigma(X \smallsetminus Y) \in R$. Now if $Y \subseteq X$ is such that $\Sigma Y = 0$, then $\alpha + \Sigma(X \smallsetminus Y) = \alpha + \Sigma X \notin R$, so that we can replace X by $X \smallsetminus Y$.

Lemma 3.12 *For any* $R \in \mathcal{R}(\Phi)$, *we have* $R^{\mathsf{ncd}} \preccurlyeq R \preccurlyeq R^{\mathsf{pcd}}$.

Proof As R^{ncd} (resp. R^{pcd}) is obtained from R deleting negative (resp. positive) roots, we get $(R^{\mathsf{ncd}})^+ = R^+ \supseteq (R^{\mathsf{pcd}})^+$ and $(R^{\mathsf{ncd}})^- \subseteq R^- = (R^{\mathsf{pcd}})^-$, so that $R^{\mathsf{ncd}} \preccurlyeq R \preccurlyeq R^{\mathsf{pcd}}$. ∎

Lemma 3.13 *If Φ is crystallographic and $R \subseteq \Phi$ is semiclosed, then both R^{ncd} and R^{pcd} are closed.*

Proof Assume by means of contradiction that R is semiclosed and R^{ncd} is not closed. Then there are roots $\alpha, \beta \in R^{\mathsf{ncd}}$ such that $\alpha + \beta \in \Phi \smallsetminus R^{\mathsf{ncd}}$. Consider two such roots such that $\alpha + \beta$ has minimal absolute height. We distinguish four cases:

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(1) If $\alpha, \beta \in \Phi^+$, then $\alpha, \beta \in (\mathrm{R}^{\mathrm{ncd}})^+ = \mathrm{R}^+$, which is closed, so that $\alpha + \beta \in \mathrm{R}^+ = (\mathrm{R}^{\mathrm{ncd}})^+$. Contradiction.
(2) If $\alpha \in \Phi^-$ and $\beta \in \Phi^+$, we distinguish again two cases:
- If $\alpha + \beta \notin \mathrm{R}$, then the set $\{\beta\}$ ensures $\alpha \notin \mathrm{R}^{\mathrm{ncd}}$. Contradiction.
- If $\alpha + \beta \in \mathrm{R}$, then since $\alpha + \beta \in \mathrm{R} \smallsetminus \mathrm{R}^{\mathrm{ncd}}$, there exists $\mathrm{X} \subseteq \mathrm{R}^+$ such that $\alpha + \beta + \Sigma\mathrm{X} \in \Phi \smallsetminus \mathrm{R}$. Since $\beta \in \mathrm{R}^+$, the set $\{\beta\} \cup \mathrm{X}$ ensures $\alpha \notin \mathrm{R}^{\mathrm{ncd}}$. Contradiction.

(3) If $\alpha \in \Phi^+$ and $\beta \in \Phi^-$, the argument is symmetric.
(4) If $\alpha, \beta \in \Phi^-$, then $\alpha + \beta \in \mathrm{R}^-$, since R^- is closed. As $\alpha + \beta \in \mathrm{R} \smallsetminus \mathrm{R}^{\mathrm{ncd}}$, there exists $\mathrm{X} \subseteq \mathrm{R}^+$ such that $(\alpha + \beta) + \Sigma\mathrm{X} \in \Phi \smallsetminus \mathrm{R}$. By Remark 3.11 (iii), we can assume that $(\alpha + \beta) + \Sigma\mathrm{X}$ has no vanishing subsum. By Theorem 2.4, there exists $\gamma \in \mathrm{X}$ such that $\alpha + \beta + \gamma \in \Phi$. By Proposition 2.3, we can assume without loss of generality that $\beta + \gamma \in \Phi$. We now distinguish four cases:
- If $\beta + \gamma \notin \mathrm{R}$, then the set $\{\gamma\}$ ensures $\beta \notin \mathrm{R}^{\mathrm{ncd}}$. Contradiction.
- If $\beta + \gamma \in \mathrm{R}^+$, then $\mathrm{T} = \{\beta + \gamma\} \cup (\mathrm{X} \smallsetminus \{\gamma\}) \subseteq \mathrm{R}^+$ satisfies $\alpha + \Sigma\mathrm{T} = \alpha + \beta + \Sigma\mathrm{X} \in \Phi \smallsetminus \mathrm{R}$ so that $\alpha \notin \mathrm{R}^{\mathrm{ncd}}$. Contradiction.
- If $\beta + \gamma \in \mathrm{R}^- \smallsetminus \mathrm{R}^{\mathrm{ncd}}$, then there exists a subset $\mathrm{T} \subseteq \mathrm{R}^+$ such that $\beta + \gamma + \Sigma\mathrm{T} \in \Phi \smallsetminus \mathrm{R}$. Since $\gamma \in \mathrm{R}^+$, the set $\{\gamma\} \cup \mathrm{T}$ ensures that $\beta \notin \mathrm{R}^{\mathrm{ncd}}$. Contradiction.
- If $\beta + \gamma \in (\mathrm{R}^{\mathrm{ncd}})^-$, then we have $\alpha \in \mathrm{R}^{\mathrm{ncd}}$ and $\beta + \gamma \in \mathrm{R}^{\mathrm{ncd}}$ with $\alpha + \beta + \gamma \in \Phi$. Moreover, $h(\alpha + \beta + \gamma) < h(\alpha + \beta)$, since $\alpha + \beta \in \Phi^-$ while $\gamma \in \Phi^+$ and $\beta + \gamma \in \Phi^-$. By minimality in the choice of $\alpha + \beta$, we obtain that $\alpha + \beta + \gamma \in \mathrm{R}^{\mathrm{ncd}}$. Observe now that $\mathrm{X} \smallsetminus \{\gamma\} \subseteq \mathrm{R}^+$ and $\alpha + \beta + \gamma + \Sigma(\mathrm{X} \smallsetminus \{\gamma\}) = \alpha + \beta + \Sigma\mathrm{X} \in \Phi \smallsetminus \mathrm{R}$. Therefore, the following hold:
  - if $\alpha + \beta + \gamma$ is negative, the set $\mathrm{X} \smallsetminus \{\gamma\}$ ensures $\alpha + \beta + \gamma \notin \mathrm{R}^{\mathrm{ncd}}$. Contradiction.
  - if $\alpha + \beta + \gamma$ is positive, then R^+ is not closed. Contradiction.

In all cases, we have reached a contradiction. We conclude that if R is semiclosed, then $\mathrm{R}^{\mathrm{ncd}}$ is closed. The proof is symmetric for $\mathrm{R}^{\mathrm{pcd}}$. ∎

Proposition 3.14 *When Φ is crystallographic, the weak order on $\mathcal{C}(\Phi)$ is a lattice with meet and join*

$$\mathrm{R} \wedge_{\mathcal{C}} \mathrm{S} = \left((\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathrm{cl}} \sqcup (\mathrm{R}^- \cap \mathrm{S}^-)\right)^{\mathrm{ncd}} \textit{ and } \mathrm{R} \vee_{\mathcal{C}} \mathrm{S} = \left((\mathrm{R}^+ \cap \mathrm{S}^+) \sqcup (\mathrm{R}^- \cup \mathrm{S}^-)^{\mathrm{cl}}\right)^{\mathrm{pcd}}.$$

Proof First, the weak order $\preccurlyeq$ on $\mathcal{C}(\Phi)$ is a subposet of the weak order $\preccurlyeq$ on $\mathcal{R}(\Phi)$, and it is bounded below by Φ^+ and above by Φ^-. We therefore just need to show that there is a meet and a join and that they are given by the above formulas.

Let $\mathrm{R}, \mathrm{S} \in \mathcal{C}(\Phi)$ and $\mathrm{M} = \mathrm{R} \wedge_{\mathcal{SC}} \mathrm{S}$ so that $\mathrm{M}^{\mathrm{ncd}} = \mathrm{R} \wedge_{\mathcal{C}} \mathrm{S}$. Observe that we have $\mathrm{M}^{\mathrm{ncd}} \preccurlyeq \mathrm{M} \preccurlyeq \mathrm{R}$ and $\mathrm{M}^{\mathrm{ncd}} \preccurlyeq \mathrm{M} \preccurlyeq \mathrm{S}$ by Lemma 3.12. Moreover, since M is semiclosed, $\mathrm{M}^{\mathrm{ncd}}$ is closed by Lemma 3.13. Therefore, $\mathrm{M}^{\mathrm{ncd}}$ is closed and below both R and S.

Consider now $\mathrm{T} \in \mathcal{C}(\Phi)$ such that $\mathrm{T} \preccurlyeq \mathrm{R}$ and $\mathrm{T} \preccurlyeq \mathrm{S}$. Since $\mathrm{T} \in \mathcal{SC}(\Phi)$ and $\mathrm{M} = \mathrm{R} \wedge_{\mathcal{SC}} \mathrm{S}$, we have $\mathrm{T} \preccurlyeq \mathrm{M}$. Therefore, $\mathrm{T}^+ \supseteq \mathrm{M}^+ = (\mathrm{M}^{\mathrm{ncd}})^+$ and $\mathrm{T}^- \subseteq \mathrm{M}^-$. Assume by means of contradiction that $\mathrm{T} \not\preccurlyeq \mathrm{M}^{\mathrm{ncd}}$. Then we have $\mathrm{T}^- \nsubseteq (\mathrm{M}^{\mathrm{ncd}})^-$. Consider $\alpha \in \mathrm{T}^- \smallsetminus (\mathrm{M}^{\mathrm{ncd}})^-$ of minimal absolute height. By definition of $\mathrm{M}^{\mathrm{ncd}}$, there exists $\mathrm{X} \subseteq \mathrm{M}^+$ such that $\alpha + \Sigma\mathrm{X} \in \Phi \smallsetminus \mathrm{M}$. Since $\mathrm{M}^+ = (\mathrm{R} \wedge_{\mathcal{SC}} \mathrm{S})^+ = (\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathrm{cl}}$, we can rewrite each root of X as a sum of roots in $\mathrm{R}^+ \cup \mathrm{S}^+$, and thus we can assume without loss of generality that $\mathrm{X} \subseteq (\mathrm{R}^+ \cup \mathrm{S}^+)$. By Remark 3.11 (iii), we can moreover assume

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that $\alpha + \Sigma X$ has no vanishing subsum. By Theorem 2.4, there exists $\beta \in X$ such that $\alpha + \beta \in \Phi$.

Since $\beta \in X \subseteq (R^+ \cup S^+)$, we can assume that $\beta \in R^+$. Since $\alpha \in T^- \subseteq R^-$, $\beta \in R^+ \subseteq T^+$ and both R and T are closed, we obtain that $\alpha + \beta \in R \cap T$. We now distinguish two cases:

- If $\alpha + \beta$ is positive, then $\alpha + \beta \in R^+ \subseteq M^+$. Since $X \smallsetminus \{\beta\} \subseteq M^+$ and M^+ is closed, we obtain that $\alpha + \Sigma X = (\alpha + \beta) + \Sigma(X \smallsetminus \{\beta\}) \in M^+$. Contradicion.
- If $\alpha + \beta$ is negative, we have $\alpha + \beta \in T^-$. Moreover, $\alpha + \beta$ has smaller absolute height than α, since $\alpha \in \Phi^-$, $\beta \in \Phi^+$ and $\alpha + \beta \in \Phi^-$. By minimality in the choice of α, we obtain that $\alpha + \beta \in M^{\mathrm{ncd}}$. Since $X \smallsetminus \{\beta\} \subseteq M^+$, this implies that

$$\alpha + \Sigma X = (\alpha + \beta) + \Sigma(X \smallsetminus \{\beta\}) \in M.$$

  Contradiction.

Since we reached a contradiction in both cases, we obtain that $T \preccurlyeq M^{\mathrm{ncd}}$. Hence, M^{ncd} is indeed the meet of R and S for the weak order on $\mathcal{C}(\Phi)$. The proof is similar for the join. ∎

Remark 3.15 In contrast to Propositions 3.5 and 3.8 and Corollaries 3.6 and 3.9, the cover relations in the weak order on $\mathcal{C}(\Phi)$ are more intricate, and the weak order on $\mathcal{C}(\Phi)$ is not graded in general.

Remark 3.16 All results presented in this section fail for non-crystallographic root systems. In view of Remark 2.13, it might a priori depend of the notion of $\mathbb{N}$-closed subsets considered. However, the following example works for any of the notions (i), (ii), and (iii) of Lemma 2.9.

Consider the Coxeter group of type H_3 with Dynkin diagram $1 \overset{5}{\text{———}} 2 \text{———} 3$. Consider $\alpha := \alpha_1 \in \Phi^+$, $\beta := -\alpha_1 - \psi\alpha_2 \in \Phi^-$ and $\gamma := -\psi\alpha_1 - \alpha_2 - \alpha_3 \in \Phi^-$, where $\psi = -2\cos(4\pi/5)$. Note that $\beta + \gamma \in \Phi^-$ and $\alpha + \beta + \gamma \in \Phi^-$, while $\alpha + \beta \notin \Phi$ and $\alpha + \gamma \notin \Phi$. Consider the sets $R := \{\alpha, \beta, \gamma, \beta + \gamma, \alpha + \beta + \gamma\}$, $S := \{\beta, \gamma, \beta + \gamma\}$, $U := \{\alpha, \beta\}$, and $V := \{\alpha, \gamma\}$. Note that R, S, U, and V are closed, and that both U and V are weak order smaller than both R and S. Moreover, we claim that there is no closed subset T that is weak order larger than both U and V and weak order smaller than both R and S. Indeed, such a set T should contain α, β, γ and thus $\beta + \gamma$ and $\alpha + \beta + \gamma$ by closedness, which would contradict $T \preccurlyeq S$. This implies that R and S have no meet and that U and V have no join in the weak order on closed subsets of Φ, thus contradicting the result of Proposition 3.14 in the non-crystallographic type H_3. In fact, even Lemma 3.13 fails in type H_3 since $\{\alpha, \beta, \gamma, \beta + \gamma\}^{\mathrm{ncd}} = \{\alpha, \beta, \gamma\}$ is not closed.

Remark 3.17 As mentioned in Remark 2.14, even for crystallographic root systems there are different possible notions of closed subsets (which all coincide in type A). Unfortunately, it turns out that Proposition 3.14 fails for the other notions of closed sets. The smallest counter-example is in type B_3. Consider the sets

$$R := \{-\alpha_1, -\alpha_1 - \alpha_2, -\alpha_1 - \alpha_2 - \alpha_3, -\alpha_1 - 2\alpha_2 - 2\alpha_3, \alpha_3\},$$
$$S := \{-\alpha_1, -\alpha_1 - \alpha_2 - \alpha_3, -\alpha_1 - 2\alpha_2 - 2\alpha_3\},$$
$$U := \{-\alpha_1 - 2\alpha_2 - 2\alpha_3, \alpha_3\}, \quad \text{and} \quad V := \{-\alpha_1, \alpha_3\}.$$

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Note that R, S, U, and V are convex, and that both U and V are weak order smaller than both R and S. Moreover, we have $U \vee_{\mathcal{C}} V = R \wedge_{\mathcal{C}} S = \{-\alpha_1, -\alpha_1 - 2\alpha_2 - 2\alpha_3, \alpha_3\}$, but this set is not convex. In fact, we claim that there is no convex subset T that is weak order larger than both U and V and weak order smaller than both R and S. Indeed, such a set T should contain $\{-\alpha_1, -\alpha_1 - 2\alpha_2 - 2\alpha_3, \alpha_3\}$ and thus also the root

$$-\alpha_1 - \alpha_2 = (-\alpha_1)/2 + (-\alpha_1 - 2\alpha_2 - 2\alpha_3)/2 + \alpha_3,$$

contradicting $T \preccurlyeq S$. This implies that R and S have no meet and that U and V have no join in the weak order on convex subsets of Φ.

3.4 Weak Order on Φ-posets

Recall from Definition 2.16 that $\mathcal{P}(\Phi)$ denotes the set of Φ-posets, *i.e.*, of antisymmetric closed subsets of Φ. We finally show that the restriction of the weak order to the Φ-posets still defines a lattice structure. The weak orders on A_2-, B_2-, and G_2-posets are represented in Figures 1 and 2.

Theorem 3.18 *The meet $\wedge_{\mathcal{C}}$ and the join $\vee_{\mathcal{C}}$ both preserve antisymmetry. Thus, when Φ is crystallographic, the set $\mathcal{P}(\Phi)$ of Φ-posets induces a sublattice of the weak order on $\mathcal{C}(\Phi)$.*

Proof Let $R, S \in \mathcal{P}(\Phi)$ and $M = R \wedge_{\mathcal{SC}} S$ so that $M^{ncd} = R \wedge_{\mathcal{C}} S$. Assume M^{ncd} is not antisymmetric, and let $\alpha \in (M^{ncd})^+$ such that $-\alpha \in (M^{ncd})^-$. Since we have $(M^{ncd})^- \subseteq M^- = R^- \cap S^-$ and both R and S are antisymmetric, we get $\alpha \notin R^+ \cup S^+$. Since we have $\alpha \in (M^{ncd})^+ = (R^+ \cup S^+)^{cl}$, there exists $X \subseteq R^+ \cup S^+$ such that $|X| \geq 2$ and $\alpha = \Sigma X$. By Theorem 2.4, there exists $\beta \in X$ such that $\Sigma(X \smallsetminus \{\beta\}) \in \Phi$. Since $X \smallsetminus \{\beta\} \subseteq M^+ \subseteq M^{ncd}$, $-\alpha \in M^{ncd}$, and M^{ncd} is closed, we get $\Sigma(X \smallsetminus \{\beta\}) + (-\alpha) = -\beta \in (M^{ncd})^- \subseteq R^- \cap S^-$. As $\beta \in R^+ \cup S^+$, this contradicts the antisymmetry of either R or S. ∎

Remark 3.19 Theorem 3.18 fails for non-crystallographic types. An example in type H_3 is given in Remark 3.16 (since the sets R, S, U, and V are all antisymmetric and thus Φ-posets).

Remark 3.20 Even for crystallographic root systems, Theorem 3.18 fails for the other notions of closed sets. An example in type B_3 is given in Remark 3.17 (since the sets R, S, U and V are all antisymmetric and thus Φ-posets).

Proposition 3.21 *All cover relations in the weak order on $\mathcal{P}(\Phi)$ are cover relations in the weak order on $\mathcal{R}(\Phi)$. In particular, the weak order on $\mathcal{P}(\Phi)$ is still graded by $R \mapsto |R^-| - |R^+|$.*

Proof Consider a cover relation $R \preccurlyeq S$ in the weak order on $\mathcal{P}(\Phi)$. We have $R^+ \supseteq S^+$ and $R^- \subseteq S^-$ where at least one of the inclusions is strict. Suppose first that $R^+ \neq S^+$ and let $X := \{\alpha \in R^+ \smallsetminus S^+ \mid \nexists\, \beta, \gamma \in R^+ \text{ with } \alpha = \beta + \gamma\}$. Thisset X is nonempty, as it contains any α in $R^+ \smallsetminus S^+$ with $|h|(\alpha)$ minimal. Consider now $\alpha \in X$ with $|h|(\alpha)$

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type	A	B/C	D $(n \geq 4)$
#antisym.	$3^1, 3^3, 3^6, 3^{10}$ [A047656]	$3^1, 3^4, 3^9$ [A060722]	3^{12} [A053764]
#semiclosed	$2^2, 7^2, 40^2, 357^2$ [A006455]	$2^2, 12^2, 172^2, 5310^2 / 5318^2$	888^2
#closed	4, 29, 355, 6942 [A000798]	4, 55, 1785 / 1803	18291
#Φ-posets	3, 19, 219, 4231 [A001035]	3, 37, 1235 / 1225	219
#WOEP	2, 6, 24, 120 [A000142]	2, 8, 48, 384 [A000165]	192 [A002866]
#WOIP	3, 17, 151, 1899 [A007767]	3, 27, 457	3959
#WOFP	3, 13, 75, 541 [A000670]	3, 17, 147, 1697 [A080253]	865 [A080254]
#COEP	2, 5, 14, 42 [A000108]	2, 6, 20, 70 [A000984]	50 [A051924]
#COIP(bip)	3, 13, 70, 433	3, 18, 138, 1185	622
#COIP(lin)	3, 13, 68, 399 [A000260]	3, 18, 132, 1069	578
#COFP	3, 11, 45, 197 [A001003]	3, 13, 63, 321 [A001850]	233
#BOEP	2, 4, 8, 16, 32 [A000079]	2, 4, 8, 16, 32 [A000079]	16 [A000079]
#BOIP	3, 9, 27, 81 [A000244]	3, 9, 27, 81 [A000244]	81 [A000244]
#BOFP	3, 9, 27, 81 [A000244]	3, 9, 27, 81 [A000244]	81 [A000244]

Table 1: Numerology in types A_n, B_n, C_n and D_n for small values of n. Further values can be found using the given references to [OEI10].

maximal and let $\mathrm{T} := \mathrm{R} \smallsetminus \{\alpha\}$. We claim that T is still a Φ-poset. It is clearly still antisymmetric. For closedness, assume by means of contradiction that there are β, $\gamma \in \mathrm{T}$ such that $\alpha = \beta + \gamma$. Since $\alpha \in \mathrm{X} \subseteq \Phi^+$, we can assume that $\beta \in \mathrm{R}^-$ and $\gamma \in \mathrm{R}^+$, and we choose β so that $|h|(\beta)$ is minimal. We claim that there are no $\delta, \varepsilon \in \mathrm{R}^+$ such that $\gamma = \delta + \varepsilon$. Otherwise, since $\alpha = \beta + \gamma = \beta + \delta + \varepsilon \in \Phi$, we can assume by Proposition 2.3 that $\beta + \delta \in \Phi \cup \{0\}$. If $\beta + \delta \in \Phi^-$, then $\beta + \delta \in \mathrm{R}^-$ (since R is closed), which contradicts the minimality of β. If $\beta + \delta \in \Phi^+$, then $\beta + \delta \in \mathrm{R}^+$ (since R is closed), which together with $\gamma \in \mathrm{R}^+$ and $(\beta + \delta) + \gamma = \alpha$ contradicts $\alpha \in X$. Finally, if $\beta + \delta = 0$, then $\beta = -\delta$, which contradicts the antisymmetry of R. This proves that there is no $\delta, \varepsilon \in \mathrm{R}^+$ such that $\gamma = \delta + \varepsilon$. By maximality of $h(|\alpha|)$ in our choice of α this implies that $\gamma \in \mathrm{S}$. Since $\beta \in \mathrm{R}^- \subseteq \mathrm{S}^-$, we therefore obtain that $\beta + \gamma = \alpha \notin \mathrm{S}$, while $\beta, \gamma \in \mathrm{S}$, contradicting the closedness of S. This proves that T is closed and thus it is a Φ-poset. Moreover, we have $\mathrm{R} \neq \mathrm{T}$ and $\mathrm{R} \preccurlyeq \mathrm{T} \preccurlyeq \mathrm{S}$ where S covers R, which implies that $\mathrm{S} = \mathrm{T} = \mathrm{R} \smallsetminus \{\alpha\}$. We prove similarly that if $\mathrm{R}^- \neq \mathrm{S}^-$, there exists $\alpha \in \Phi^-$ such that $\mathrm{S} = \mathrm{R} \cup \{\alpha\}$. In both cases, $\mathrm{R} \preccurlyeq \mathrm{S}$ is a cover relation in the weak order on $\mathcal{R}(\Phi)$. ∎

Corollary 3.22 *In the weak order on $\mathcal{P}(\Phi)$, the Φ-posets that cover a given Φ-poset* $\mathrm{R} \in \mathcal{SC}(\Phi)$ *are precisely the relations*

- $\mathrm{R} \smallsetminus \{\alpha\}$ *for any* $\alpha \in \mathrm{R}^+$ *such that there is no* $\gamma, \delta \in \mathrm{R}^+$ *with* $\alpha = \gamma + \delta$,
- $\mathrm{R} \cup \{\beta\}$, *for any* $\beta \in \Phi^- \smallsetminus \mathrm{R}^-$ *such that* $-\beta \notin \mathrm{R}^+$ *and* $\beta + \gamma \in \Phi \Rightarrow \beta + \gamma \in \mathrm{R}$ *for all* $\gamma \in \mathrm{R}$.

Remark 3.23 We have gathered in Table 1 the number of Φ-posets for the root systems of type A_n, B_n, C_n, and D_n for small values of n (the other lines of the table will be explained in the next section). Note that the number of semiclosed sets, closed sets, and posets differ in types B_4 and C_4. This should not come as a surprise, since the notion of closed sets used in this paper (Definition 2.10) is not preserved when

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passing from roots to coroots. This is just one more hint that crystallographic root systems are the right notion for this paper rather than finite Coxeter groups.

4 Some Relevant Subposets

In this section, we consider certain specific families of Φ-posets corresponding to the vertices, the intervals, and the faces in the permutahedron (Section 4.1), the generalized associahedra (Section 4.2), and the cube (Section 4.3). A roadmap through the different families of Φ-posets considered in this paper is given in Figure 3.

4.1 Permutahedron

The *W-permutahedron* $\mathsf{Perm}^p(W)$ is the convex hull of the orbit under W of a point p in the interior of the fundamental chamber of W. It has one vertex $w(p)$ for each element $w \in W$ and its graph is the Cayley graph of the set S of simple reflections of W. Moreover, when oriented in the linear direction $w_\circ(p) - p$, its graph is the Hasse diagram of the *weak order* on W. Recall that the weak order is defined equivalently for any $v, w \in W$ by $v \preccurlyeq w$ if and only if the following hold:

- $\ell(v) + \ell(v^{-1}w) = \ell(w)$, where $\ell(w)$ is the *length* of w, *i.e.,* the minimal length of an expression of the form $\ell = s_1 \cdots s_k$ with $s_1, \dots, s_k \in S$;
- v is a prefix of w, *i.e.,* there exists $u \in W$ such that $w = vu$ and $\ell(w) = \ell(v) + \ell(u)$;
- $\mathrm{inv}(v) \subseteq \mathrm{inv}(w)$, where inv is the *inversion set* $\mathrm{inv}(w) := \Phi^+ \cap w(\Phi^-)$;
- there is an oriented path from $v(p)$ to $w(p)$ in the graph of the permutahedron oriented in the linear direction $w_\circ(p) - p$.

In the sequel, we will often drop p from the notation $\mathsf{Perm}^p(W)$ as the combinatorics of $\mathsf{Perm}^p(W)$ is independent of p as long as this point is generic.

4.1.1 Elements

For an element $w \in W$, we consider the Φ-poset

$$\mathrm{R}(w) := w(\Phi^+).$$

We say that $\mathrm{R}(w)$ is a *weak order element poset* and let

$$\mathsf{WOEP}(\Phi) := \{\mathrm{R}(w) \mid w \in W\}$$

denote the collection of all such Φ-posets.

Remark 4.1 Table 1 reports the cardinality of $\mathsf{WOEP}(\Phi)$ in type A_n, B_n, C_n, and D_n for small values of n. It is just the order of W, which is known as the product formula

$$|\mathsf{WOEP}(\Phi)| = |W| = \prod_{i \in [n]} d_i,$$

where $(d_1, \dots, d_n)$ are the degrees of W.

Remark 4.2 Geometrically, $\mathrm{R}(w)$ is the set of roots of Φ not contained in the cone of $\mathsf{Perm}^p(W)$ at the vertex $w(p)$, *i.e.,*

$$\mathrm{R}(w) = \Phi \smallsetminus \mathrm{cone}\,\{w'(p) - w(p) \mid w' \in W\}.$$

See Figure 4.

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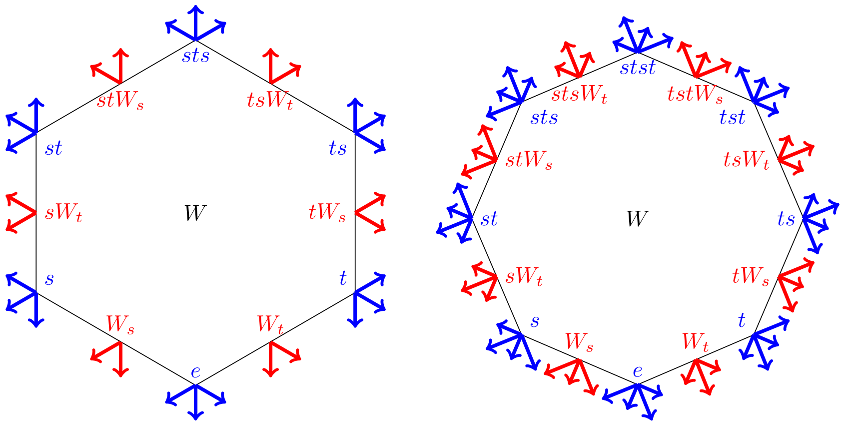

Figure 4: The sets $\mathrm{R}(xW_I)$ of the standard parabolic cosets xW_I in type A_2 (left) and B_2 (right). Note that positive roots point downwards.

We now characterize the Φ-posets of $\mathsf{WOEP}(\Phi)$.

Proposition 4.3 *A Φ-poset $\mathrm{R} \in \mathcal{P}(\Phi)$ is in $\mathsf{WOEP}(\Phi)$ if and only if $\alpha \in \mathrm{R}$ or $-\alpha \in \mathrm{R}$ for all $\alpha \in \Phi$.*

Proof This is folklore. See, for instance, [Bou68, Chap. 6, 1.7, Coro. 1]. ∎

Remark 4.4 We have already encountered these Φ-posets in Proposition 2.18: a poset is in $\mathsf{WOEP}(\Phi)$ if and only if it is its unique extension. In other words, the maximal extensions of a Φ-poset R are all in $\mathsf{WOEP}(\Phi)$, and it is thus natural to consider $\mathcal{L}(\mathrm{R}) := \{w \in W \mid \mathrm{R} \subseteq \mathrm{R}(w)\}$. For example, in type A, the set $\mathcal{L}(\mathrm{R})$ is the set of linear extensions of the poset R. Note however that we have $\mathrm{R} = \bigcap \mathcal{E}(\mathrm{R})$, but sometimes $\mathrm{R} \neq \bigcap_{w \in \mathcal{L}(\mathrm{R})} \mathrm{R}(w)$, in contrast to the type A situation (consider, for example, $\mathrm{R} = \{\alpha_1 + \alpha_2, \alpha_2\}$ in type B_2).

The following statement connects the subposet of the weak order induced by $\mathsf{WOEP}(\Phi)$ with the classical weak order on W, and thus justifies the name in Definition 3.1.

Proposition 4.5 *For $w \in W$, we have*

$$\mathrm{inv}(w) = \Phi^+ \cap -\mathrm{R}(w) \quad \textit{and} \quad \mathrm{R}(w) = (\Phi^+ \smallsetminus \mathrm{inv}(w)) \sqcup -\mathrm{inv}(w).$$

In particular, for $v, w \in W$, we have $\mathrm{R}(v) \preccurlyeq \mathrm{R}(w)$ in the weak order on $\mathsf{WOEP}(\Phi)$ if and only if $v \preccurlyeq w$ in the weak order on W.

Proof The first equality is just the definition of $\mathrm{inv}(w)$ and the second comes from the fact that $|\{\alpha, -\alpha\} \cap \mathrm{R}(w)| = 1$, which in turn implies that

$$\mathrm{R}(w)^- = \Phi^- \smallsetminus -\mathrm{R}(w)^+ = \Phi^- \smallsetminus -\mathrm{inv}(w).$$

Finally, $v \preccurlyeq w$ in the weak order on W if and only if $\mathrm{inv}(v) \subseteq \mathrm{inv}(w)$ if and only if $\Phi^+ \smallsetminus \mathrm{inv}(v) \supseteq \Phi^+ \smallsetminus \mathrm{inv}(w)$. This shows the equivalence with $\mathrm{R}(v) \preccurlyeq \mathrm{R}(w)$. ∎

Remark 4.6 In fact,

$$\mathrm{R}(v) \preccurlyeq \mathrm{R}(w) \iff \mathrm{R}(v)^+ \supseteq \mathrm{R}(w)^+ \iff \mathrm{R}(v)^- \subseteq \mathrm{R}(w)^- \iff v \preccurlyeq w.$$

Corollary 4.7 *The weak order on* $\mathsf{WOEP}(\Phi)$ *is a lattice with meet and join*

$$\mathrm{R}(v) \wedge_{\mathsf{WOEP}} \mathrm{R}(w) = \mathrm{R}(v \wedge_W w) \quad \textit{and} \quad \mathrm{R}\,(v) \vee_{\mathsf{WOEP}} \mathrm{R}(w) = \mathrm{R}(v \vee_W w).$$

The following statement connects this lattice structure on $\mathsf{WOEP}(\Phi)$ with that on $\mathcal{P}(\Phi)$, and is our original motivation to study the weak order on $\mathcal{P}(\Phi)$.

Proposition 4.8 *The set* $\mathsf{WOEP}(\Phi)$ *induces a sublattice of the weak order on* $\mathcal{P}(\Phi)$.

Proof Let $\mathrm{R}, \mathrm{S} \in \mathsf{WOEP}(\Phi)$ and $\mathrm{M} = \mathrm{R} \wedge_{\mathcal{SC}} \mathrm{S} = (\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathrm{cl}} \sqcup (\mathrm{R}^- \cap \mathrm{S}^-)$ so that $\mathrm{M}^{\mathrm{ncd}} = \mathrm{R} \wedge_{\mathcal{C}} \mathrm{S}$. Assume by means of contradiction that $\mathrm{M}^{\mathrm{ncd}}$ is not in $\mathsf{WOEP}(\Phi)$, and consider a root $\alpha \in \Phi^+$ with $|h|(\alpha)$ minimal such that $\{\alpha, -\alpha\} \cap \mathrm{M}^{\mathrm{ncd}} = \varnothing$.

Since $(\mathrm{M}^{\mathrm{ncd}})^+ = \mathrm{M}^+ = (\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathrm{cl}}$, we have $\alpha \notin \mathrm{R}^+$ and $\alpha \notin \mathrm{S}^+$. Since we have $\mathrm{R}, \mathrm{S} \in \mathsf{WOEP}(\Phi)$, we get $-\alpha \in \mathrm{R}^-$ and $-\alpha \in \mathrm{S}^-$, so that $-\alpha \in \mathrm{M}^-$. Therefore, $-\alpha \in \mathrm{M} \smallsetminus \mathrm{M}^{\mathrm{ncd}}$, so that there exists $\mathrm{X} \subseteq \mathrm{M}^+$ such that $\Sigma\mathrm{X} - \alpha \in \Phi \smallsetminus \mathrm{M}$. Since $\mathrm{M}^+ = (\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathrm{cl}}$, we can rewrite each root of X as a sum of roots in $\mathrm{R}^+ \cup \mathrm{S}^+$, and we can assume without loss of generality that $\mathrm{X} \subseteq (\mathrm{R}^+ \cup \mathrm{S}^+)$. As usual, we can assume moreover that X has no vanishing subsum. We finally choose an inclusion minimal such subset X of $\mathrm{R}^+ \cup \mathrm{S}^+$.

Assume first that $\mathrm{X} = \{\beta\}$. We have $\beta \in \mathrm{R}^+ \cup \mathrm{S}^+$, say, for instance, $\beta \in \mathrm{R}^+$. Since $-\alpha \in \mathrm{M}^- = \mathrm{R}^- \cap \mathrm{S}^-$, $\beta \in \mathrm{R}$ and R is closed, we have $\beta - \alpha \in \mathrm{R}$. Since $\beta - \alpha \notin \mathrm{M}^+$, we obtain that $\beta - \alpha \in \Phi^-$. Therefore, as $\beta \in \Phi^+$, we have $|h|(\beta - \alpha) < |h|(\alpha)$. By minimality of $|h|(\alpha)$, we obtain that $\alpha - \beta \in \mathrm{M}^{\mathrm{ncd}}$. We conclude that $\alpha - \beta \in \mathrm{M}^{\mathrm{ncd}}$ and $\beta \in \mathrm{R}^+ \subseteq \mathrm{M}^{\mathrm{ncd}}$, while $\alpha \notin \mathrm{M}^{\mathrm{ncd}}$, contradicting the closedness of $\mathrm{M}^{\mathrm{ncd}}$.

Assume now that $|\mathrm{X}| \geq 2$. Since $\alpha \notin \mathrm{M}^+ = (\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathrm{cl}}$ and $\mathrm{X} \subseteq \mathrm{R}^+ \cup \mathrm{S}^+$, we obtain that $\mathrm{X} \cup \{-\alpha\}$ has no vanishing subsums. Therefore, Proposition 2.5 ensures that $\mathrm{X} \cup \{-\alpha\}$ has at least two strict summable subsets. In particular, there is $\mathrm{Y} \subsetneq \mathrm{X}$ such that $\Sigma\mathrm{Y} - \alpha \in \Phi$. By minimality of X, we obtain that $\Sigma\mathrm{Y} - \alpha \in \mathrm{M}$. We distinguish two cases:

- If $\Sigma\mathrm{Y} - \alpha \in \mathrm{M}^+$, then $\Sigma\mathrm{X} - \alpha \notin \mathrm{M}^+$, while $\Sigma\mathrm{Y} - \alpha \in \mathrm{M}^+$ and $\mathrm{X} \smallsetminus \mathrm{Y} \subseteq \mathrm{M}^+$ contradicts the closedness of M^+.
- If $\Sigma\mathrm{Y} - \alpha \in \mathrm{M}^-$, then $|h|(\Sigma\mathrm{Y} - \alpha) < |h|(\alpha)$. By minimality of $|h|(\alpha)$, we obtain that
  - either $\Sigma\mathrm{Y} - \alpha \in \mathrm{M}^{\mathrm{ncd}}$, which implies that $\Sigma\mathrm{X} - \alpha = (\Sigma\mathrm{Y} - \alpha) + (\Sigma(\mathrm{X} \smallsetminus \mathrm{Y})) \in \mathrm{M}$, a contradiction;
  - or $\alpha - \Sigma\mathrm{Y} \in \mathrm{M}^{\mathrm{ncd}}$, which implies that $\alpha = (\alpha - \Sigma\mathrm{Y}) + \Sigma\mathrm{Y} \in \mathrm{M}^{\mathrm{ncd}}$, which contradicts our assumption on α.

As we reached a contradiction in all cases, we conclude that $\mathrm{M}^{\mathrm{ncd}} \in \mathsf{WOEP}(\Phi)$. The proof is similar for the join. ∎

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4.1.2 Intervals

For $w, w' \in W$ with $w \preccurlyeq w'$, we denote by $[w, w'] := \{v \in W \mid w \preccurlyeq v \preccurlyeq w'\}$ the *weak order interval* between w and w'. We associate with each weak order interval $[w, w']$ the Φ-poset

$$\mathrm{R}(w, w') := \bigcap_{v \in [w, w']} \mathrm{R}(v) = \mathrm{R}(w) \cap \mathrm{R}(w') = \mathrm{R}(w)^- \sqcup \mathrm{R}(w')^+.$$

Say that $\mathrm{R}(w, w')$ is a *weak order interval poset* and let

$$\mathsf{WOIP}(\Phi) := \{\mathrm{R}(w, w') \mid w, w' \in W, w \preccurlyeq w'\}$$

denote the collection of all such Φ-posets. Table 1 reports the cardinality of $\mathsf{WOIP}(\Phi)$ in type A_n, B_n, C_n, and D_n for small values of n.

Recall from Remark 4.4 that we denote by $\mathcal{L}(\mathrm{R}) := \{w \in W \mid \mathrm{R} \subseteq \mathrm{R}(w)\}$ the set of maximal extensions of a Φ-poset R. We will use the following observation to characterize these Φ-posets.

Lemma 4.9 *A Φ-poset $\mathrm{R} \in \mathcal{P}(\Phi)$ is in $\mathsf{WOIP}(\Phi)$ if and only if $\mathcal{L}(\mathrm{R})$ has a unique weak order minimum w (resp. maximum w') that satisfies $\mathrm{R}(w)^- = \mathrm{R}^-$ (resp. $\mathrm{R}(w')^+ = \mathrm{R}^+$).*

Proof Observe first that Remark 4.6 implies that $\mathrm{R}(w, w') \subseteq \mathrm{R}(v)$ if and only if $\mathrm{R}(w)^- \subseteq \mathrm{R}(v)^-$ and $\mathrm{R}(w')^+ \subseteq \mathrm{R}(v)^+$ if and only if $v \in [w, w']$. Therefore, $\mathcal{L}(\mathrm{R}(w, w'))$ has a unique weak order minimum w and a unique weak order maximum w' and $\mathrm{R}(w)^- = \mathrm{R}(w, w')^-$ while $\mathrm{R}(w')^+ = \mathrm{R}(w, w')^+$.

Conversely, if $\mathcal{L}(\mathrm{R})$ has a unique weak order minimum w and a unique weak order maximum w' and $\mathrm{R}(w)^- = \mathrm{R}^-$ while $\mathrm{R}(w')^+ = \mathrm{R}^+$, then we have, by definition, $\mathrm{R} = \mathrm{R}(w)^- \sqcup \mathrm{R}(w')^+ = \mathrm{R}(w, w')$. ∎

Remark 4.10 In Lemma 4.9, the final hypothesis is crucial as it may happen that $\mathrm{R} \neq \bigcap_{w \in \mathcal{L}} \mathrm{R}(w)$ (consider for example $\mathrm{R} = \{\alpha_1 + \alpha_2, \alpha_2\}$ in type B_2).

We can now characterize the Φ-posets of $\mathsf{WOIP}(\Phi)$.

Proposition 4.11 *A Φ-poset $\mathrm{R} \in \mathcal{P}(\Phi)$ is in $\mathsf{WOIP}(\Phi)$ if and only if $\alpha + \beta \in \mathrm{R}$ implies $\alpha \in \mathrm{R}$ or $\beta \in \mathrm{R}$ for all $\alpha, \beta \in \Phi^-$ and all $\alpha, \beta \in \Phi^+$.*

Proof By Lemma 4.9, this boils down to showing that the following assertions are equivalent:

(i) $\mathcal{L}(\mathrm{R})$ has a unique weak order minimum w (resp. maximum w') that moreover satisfies $\mathrm{R}(w)^- = \mathrm{R}^-$ (resp. $\mathrm{R}(w')^+ = \mathrm{R}^+$).
(ii) $\alpha + \beta \in \mathrm{R}$ implies $\alpha \in \mathrm{R}$ or $\beta \in \mathrm{R}$ for all $\alpha, \beta \in \Phi^-$ (resp. for all $\alpha, \beta \in \Phi^+$).

We prove the result for the maximum and $\alpha, \beta \in \Phi^+$. The result for the minimum and $\alpha, \beta \in \Phi^-$ follows by symmetry.

Assume first that (ii) holds. Consider the set $\mathrm{S} := \mathrm{R}^+ \cup (\Phi^- \smallsetminus -\mathrm{R}^+)$. Note that $\mathrm{R} \subseteq \mathrm{S}$ (since R is antisymmetric), that S is antisymmetric, and that $\mathrm{T} \preccurlyeq \mathrm{S}$ for any antisymmetric T such that $\mathrm{R} \subseteq \mathrm{T}$ (as R has been completed with all possible negative roots to obtain S). We claim, moreover, that S is closed. Indeed, consider $\alpha, \beta \in \mathrm{S}$ such that $\alpha + \beta \in \Phi$. We distinguish four cases:

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- If $\alpha \in \mathrm{R}$ and $\beta \in \mathrm{R}$, then $\alpha + \beta \in \mathrm{R} \subseteq \mathrm{S}$.
- If $\alpha \notin \mathrm{R}$ and $\beta \in \mathrm{R}$, then $\alpha \in \mathrm{S} \smallsetminus \mathrm{R} \subseteq \Phi^-$ so that $-\alpha \in \Phi^+ \smallsetminus \mathrm{R}^+$. Then
  - if $\alpha+\beta \in \Phi^+$, then we have $-\alpha \in \Phi^+ \smallsetminus \mathrm{R}^+$ and $\alpha+\beta \in \Phi^+$ with $-\alpha+(\alpha+\beta) = \beta \in \mathrm{R}$ so that condition (ii) ensures that $\alpha + \beta \in \mathrm{R}$,
  - if $\alpha + \beta \in \Phi^-$, then $-(\alpha + \beta) \notin \mathrm{R}$ (otherwise $-\alpha = -(\alpha + \beta) + \beta \in \mathrm{R}$, a contradiction). Therefore, $\alpha + \beta \in \Phi^- \smallsetminus -\mathrm{R}^+ \subseteq \mathrm{S}$.
- If $\alpha \in \mathrm{R}$ and $\beta \notin \mathrm{R}$, the argument is symmetric.
- If $\alpha \notin \mathrm{R}$ and $\beta \notin \mathrm{R}$, then $\alpha, \beta \in \mathrm{S} \smallsetminus \mathrm{R} \subseteq \Phi^-$ and $-\alpha, -\beta \in \Phi^+ \smallsetminus \mathrm{R}$. By condition (ii), this implies that $-\alpha - \beta \in \Phi^+ \smallsetminus \mathrm{R}^+$. Therefore, $\alpha + \beta \in \Phi^- \smallsetminus -\mathrm{R}^+ \subseteq \mathrm{S}$.

We thus obtained in all cases that $\alpha + \beta \in \mathrm{S}$ so that S is closed. We conclude that S is a Φ-poset and that $\mathrm{T} \preccurlyeq \mathrm{S}$ for any antisymmetric T such that $\mathrm{R} \subseteq \mathrm{T}$. In particular, S is the unique maximum of the set $\mathcal{E}(\mathrm{R})$ of extensions of R. Moreover, $\mathrm{S}^+ = \mathrm{R}^+$. Using Propositions 2.18 and 4.3, we obtain that there exist $w' \in W$ such that $\mathrm{S} = \mathrm{R}(w')$. This concludes the proof that (ii)$\Rightarrow$(i).

Conversely, assume by means of contradiction that (i) holds but not (ii). Let w' denote the weak order maximal element of $\mathcal{L}(\mathrm{R})$, and let $\alpha, \beta \in \Phi^+ \smallsetminus \mathrm{R}$ be such that $\alpha + \beta \in \mathrm{R}$. We then distinguish two cases:

- If $\alpha \in \mathrm{R}(v)$ for all $v \in \mathcal{L}(\mathrm{R})$, then $\alpha \in \mathrm{R}(w')^+ = \mathrm{R}^+$. Contradiction.
- Otherwise, there exists $v \in \mathcal{L}(\mathrm{R})$ such that $-\alpha \in \mathrm{R}(v)$. Since $v \preccurlyeq w'$, this gives $-\alpha \in \mathrm{R}(w')$. Since $\alpha + \beta \in \mathrm{R} \subseteq \mathrm{R}(w')$ and $\mathrm{R}(w')$ is closed, we get $\beta \in \mathrm{R}(w')^+ = \mathrm{R}^+$. Contradiction. ∎

We now describe the weak order on $\mathsf{WOIP}(\Phi)$. It corresponds to the Cartesian product order on intervals of the weak order.

Proposition 4.12 *For any two weak order intervals $v \preccurlyeq v'$ and $w \preccurlyeq w'$, we have $\mathrm{R}(v, v') \preccurlyeq \mathrm{R}(w, w')$ in the weak order on $\mathsf{WOIP}(\Phi)$ if and only if $v \preccurlyeq w$ and $v' \preccurlyeq w'$.*

Proof From the definition of $\mathrm{R}(w, w')$ and Remark 4.6, we have

$$\begin{aligned} &\mathrm{R}(v, v') \preccurlyeq \mathrm{R}(w, w') \\ &\iff \mathrm{R}(v, v')^+ \supseteq \mathrm{R}(w, w')^+ \text{ and } \mathrm{R}(v, v')^- \subseteq \mathrm{R}(w, w')^- \\ &\iff \mathrm{R}(v')^+ \supseteq \mathrm{R}(w')^+ \quad \text{and} \quad \mathrm{R}(v)^- \subseteq \mathrm{R}(w)^- \\ &\iff v' \preccurlyeq w' \quad \text{and} \quad v \preccurlyeq w. \end{aligned}$$ ∎

Corollary 4.13 *The weak order on $\mathsf{WOIP}(\Phi)$ is a lattice with meet and join*

$$\mathrm{R}(v, v') \wedge_{\mathsf{WOIP}} \mathrm{R}(w, w') = \mathrm{R}(v \wedge_W w, v' \wedge_W w'),$$
$$\textit{and} \quad \mathrm{R}(v, v') \vee_{\mathsf{WOIP}} \mathrm{R}(w, w') = \mathrm{R}(v \vee_W w, v' \vee_W w').$$

Remark 4.14 It follows from the expressions of $\wedge_{\mathsf{WOIP}}$ and $\vee_{\mathsf{WOIP}}$ that $\mathsf{WOEP}(\Phi)$ also induces a sublattice of $\mathsf{WOIP}(\Phi)$.

Remark 4.15 To conclude on intervals, however, we observe that the weak order on $\mathsf{WOIP}(\Phi)$ is not a sublattice of the weak order on Φ-posets. For example, in type A_2, $\{\alpha_1, \alpha_1+\alpha_2\} \vee_{\mathcal{C}} \{\alpha_2, \alpha_1+\alpha_2\} = \{\alpha_1+\alpha_2\}$, while $\{\alpha_1, \alpha_1+\alpha_2\} \vee_{\mathsf{WOIP}} \{\alpha_2, \alpha_1+\alpha_2\} = \varnothing$.

4.1.3 Faces

The faces of the permutohedron $\mathsf{Perm}^p(W)$ correspond to the cosets of the standard parabolic subgroups of W. Recall that a *standard parabolic subgroup* of W is a subgroup W_I generated by a subset I of the simple reflections of W. Its simple roots are the simple roots Δ_I of Δ corresponding to I, its root system is $\Phi_I = W_I(\Delta_I) = \Phi \cap \mathbb{R}\Delta_I$ and its longest element is denoted by $w_{\circ,I}$. A *standard parabolic coset* is a coset under the action of a standard parabolic subgroup W_I. Such a standard parabolic coset can be written as xW_I where x is its minimal length coset representative (thus x has no descent in I, see Section 4.3). Each standard parabolic coset xW_I (with $I \subseteq S$ disjoint from the descent set $\operatorname{des}(x)$ of x) corresponds to a face

$$\mathrm{F}(xW_I) = x\big(\mathsf{Perm}^p(W_I)\big) = \mathsf{Perm}^{x(p)}\big(xW_Ix^{-1}\big).$$

See Figure 4 for an illustration in type A_2 and B_2.

In [DHP18], A. Dermenjian, C. Hohlweg, and V. Pilaud also associated with each standard parabolic coset xW_I the set of roots $\overline{\mathrm{R}}(xW_I) := x(\Phi^- \cup \Phi_I^+)$. These Φ-posets were characterized in [DHP18] as follows.

Proposition 4.16 ([DHP18, Coro. 3.9]) *The following assertions are equivalent for a subset of roots* $\mathrm{R} \in \mathcal{R}(\Phi)$:

(i) $\mathrm{R} = \overline{\mathrm{R}}(xW_I)$ *for some parabolic coset* xW_I *of* W.
(ii) $\mathrm{R} = \{\alpha \in \Phi \mid \psi(\alpha) \geq 0\}$ *for some linear function* $\psi : V \to \mathbb{R}$.
(iii) $\mathrm{R} = \Phi \cap \operatorname{cone}(\mathrm{R})$ *is convex closed and* $|\mathrm{R} \cap \{\alpha, -\alpha\}| \geq 1$ *for all* $\alpha \in \Phi$.

Moreover, they used this definition to recover the following order on faces of the permutahedron, defined initially in type A in [KLN⁺01] and later for arbitrary finite Coxeter groups in [PR06].

Proposition 4.17 ([DHP18]) *The following assertions are equivalent for two standard parabolic cosets* $xW_I = [x, xw_{\circ,I}]$ *and* $yW_J = [y, yw_{\circ,J}]$ *of* W:

- $x \preccurlyeq y$ *and* $xw_{\circ,I} \preccurlyeq yw_{\circ,J}$.
- $\overline{\mathrm{R}}(xW_I)^+ \subseteq \overline{\mathrm{R}}(yW_J)^+$ *and* $\overline{\mathrm{R}}(xW_I)^- \supseteq \overline{\mathrm{R}}(yW_J)^-$.
- $xW_I \preccurlyeq yW_J$ *for the transitive closure* $\preccurlyeq$ *of the two cover relations* $xW_I \prec xW_{I\cup\{s\}}$ *for* $s \notin I \cup \operatorname{des}(x)$ *and* $xW_I \prec (xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}})W_{I\smallsetminus\{s\}}$ *for* $s \in I$.

The resulting order on standard parabolic cosets is the facial weak order *defined in* [KLN⁺01, PR06, DHP18]. *This order extends the weak order on* W, *since* $xW_\varnothing \preccurlyeq yW_\varnothing$ *if and only if* $x \preccurlyeq y$ *for any* $x, y \in W$. *Moreover, it defines a lattice on standard parabolic cosets of* W *with meet and join*

$$xW_I \wedge_{FW} yW_J = z_\wedge W_{K_\wedge} \quad \text{and} \quad xW_I \vee_{FW} yW_J = z_\vee W_{K_\vee}$$

where

$$z_\wedge = x \wedge_W y \qquad\qquad z_\vee = xw_{\circ,I} \vee_W yw_{\circ,J},$$
$$K_\wedge = \operatorname{des}(z_\wedge^{-1}(xw_{\circ,I} \wedge_W yw_{\circ,J})) \quad \text{and} \quad K_\vee = \operatorname{des}(z_\vee^{-1}(x \vee_W y)).$$

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Note that $\overline{\mathrm{R}}(xW_I)$ is not a Φ-poset as it is not antisymmetric when $I \neq \varnothing$. Here, we will therefore associate with xW_I the set of roots

$$\mathrm{R}(xW_I) := \Phi \smallsetminus \overline{\mathrm{R}}(xW_I) = x(\Phi^+ \smallsetminus \Phi_I^+).$$

Note that $\mathrm{R}(xW_I)$ coincides with the weak order interval poset $\mathrm{R}(x, xw_{\circ,I})$. We say that $\mathrm{R}(xW_I)$ is a *weak order face poset* and we let

$$\mathsf{WOFP}(\Phi) := \{\mathrm{R}(xW_I) \mid xW_I \text{ standard parabolic coset of } W\}$$

denote the collection of all such Φ-posets. Table 1 reports the cardinality of $\mathsf{WOFP}(\Phi)$ in type A_n, B_n, C_n, and D_n for small values of n.

Remark 4.18 Geometrically, $\mathrm{R}(xW_I)$ is the set of roots of Φ not contained in the cone of $\mathsf{Perm}^p(W)$ at the face $\mathrm{F}(xW_I)$, *i.e.*,

$$\mathrm{R}(xW_I) = \Phi \smallsetminus \mathrm{cone}\{w'(p) - w(p) \mid w \in xW_I, w' \in W\}.$$

See Figure 4.

Proposition 4.16 yields the following characterization of the Φ-posets in $\mathsf{WOFP}(\Phi)$.

Proposition 4.19 *The following assertions are equivalent for a subset of roots* $\mathrm{R} \in \mathcal{R}(\Phi)$*:*

(i) R *is a weak order face poset of* $\mathsf{WOFP}(\Phi)$.
(ii) $\mathrm{R} = \{\alpha \in \Phi \mid \psi(\alpha) < 0\}$ *for some linear function* $\psi : V \to \mathbb{R}$.
(iii) $\mathrm{R} = \Phi \cap \mathrm{cone}(\mathrm{R})$ *is convex closed and* $|\mathrm{R} \cap \{\alpha, -\alpha\}| \leq 1$ *for all* $\alpha \in \Phi$.

Proof This immediately follows from the characterization of $\overline{\mathrm{R}}(xW_I)$ in Proposition 4.16 and the definition $\mathrm{R}(xW_I) := \Phi \smallsetminus \overline{\mathrm{R}}(xW_I)$. ∎

We now observe that the weak order induced by $\mathsf{WOFP}(\Phi)$ corresponds to the facial weak order of [PR06, DHP18].

Proposition 4.20 *For any standard parabolic cosets* xW_I *and* yW_J*, we have that* $\mathrm{R}(xW_I) \preccurlyeq \mathrm{R}(yW_J)$ *in the weak order on* $\mathsf{WOFP}(\Phi)$ *if and only if* $xW_I \preccurlyeq yW_J$ *in the facial weak order.*

Proof By definition of $\mathrm{R}(xW_I)$ and Proposition 4.17, we have

$$\begin{aligned}
&\mathrm{R}(xW_I) \preccurlyeq \mathrm{R}(yW_J) \\
&\qquad \iff \quad \mathrm{R}(xW_I)^+ \supseteq \mathrm{R}(yW_J)^+ \text{ and } \mathrm{R}(xW_I)^- \subseteq \mathrm{R}(yW_J)^- \\
&\qquad \iff \quad \overline{\mathrm{R}}(xW_I)^+ \subseteq \overline{\mathrm{R}}(yW_J)^+ \text{ and } \overline{\mathrm{R}}(xW_I)^- \supseteq \overline{\mathrm{R}}(yW_J)^- \\
&\qquad \iff \quad xW_I \preccurlyeq yW_J.
\end{aligned}$$

∎

Corollary 4.21 *The weak order on* $\mathsf{WOFP}(\Phi)$ *is a lattice with meet and join*

$$\mathrm{R}(xW_I) \wedge_{\mathsf{WOFP}} \mathrm{R}(yW_J) = \mathrm{R}(xW_I \wedge_{FW} yW_J)$$
$$\textit{and} \quad \mathrm{R}(xW_I) \vee_{\mathsf{WOFP}} \mathrm{R}(yW_J) = \mathrm{R}(xW_I \vee_{FW} yW_J).$$

Remark 4.22 To conclude, note that the weak order on $\mathsf{WOFP}(\Phi)$ is a lattice but not a sublattice of the weak order on $\mathcal{P}(\Phi)$, nor on $\mathsf{WOIP}(\Phi)$. This was observed in [CPP17, Rem. 31] in type A. For example, already in type A_2, we have

$$\{-\alpha_1, \alpha_2\} \vee_{\mathcal{C}} \varnothing = \{-\alpha_1, \alpha_2\} \vee_{\mathsf{WOIP}} \varnothing = \{\alpha_2\},$$

while $\{-\alpha_1, \alpha_2\} \vee_{\mathsf{WOIP}} \varnothing = \{\alpha_2, \alpha_1 + \alpha_2\}$.

4.2 Generalized Associahedra

We now consider Φ-posets corresponding to the vertices, the intervals, and the faces of the generalized associahedra of type Φ. These polytopes provide geometric realizations of the type Φ cluster complex, in connection to the type Φ cluster algebra of S. Fomin and A. Zelevinsky [FZ02, FZ03a]. A first realization was constructed by F. Chapoton, S. Fomin, and A. Zelevinsky in [CFZ02] based on the compatibility fan of [FZ03b, FZ03a]. An alternative realization was constructed later by C. Hohlweg, C. Lange, and H. Thomas in [HLT11] based on the Cambrian fan of N. Reading and D. Speyer [RS09].

Although the sets of roots that we consider in this section have a strong connection to these geometric realizations (see Remarks 4.24 and 4.38), for our purposes we do not really need the precise definition of the geometry of these associahedra or of these Cambrian fans. We rather need a combinatorial description of their vertices and faces. The combinatorial model behind these constructions is the Cambrian lattice on sortable elements as developed by N. Reading [Rea06, Rea07a, Rea07b], which we briefly recall now.

Let c be a Coxeter element, *i.e.,* the product of the simple reflections of W in an arbitrary order. The *c-sorting word* of an element $w \in W$ is the lexicographically smallest reduced expression for w in the word $c^\infty := ccccc\cdots$. We write this word as $w = c_{I_1} \cdots c_{I_k}$ where c_I is the subword of c consisting only of the simple reflections in I. An element $w \in W$ is *c-sortable* when these subsets are nested: $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$. An element $w \in W$ is *c-antisortable* when $ww_\circ$ is (c^{-1})-sortable. See [Rea07a] for details on Coxeter sortable elements and their connections to other Coxeter-Catalan families.

For an element $w \in W$, we denote by $\pi_\downarrow^c(w)$ the maximal c-sortable element below w in weak order and by $\pi_c^\uparrow(w)$ the minimal c-antisortable element above w in weak order. The projection maps $\pi_\downarrow^c$ and $\pi_c^\uparrow$ can also be defined inductively, see [Rea07b]. Here, we only need that these maps are order preserving projections from W to sortable (resp. antisortable) elements, and that their fibers are intervals of the weak order of the form $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$. Therefore, they define a lattice congruence $\equiv_c$ of the weak order, called the *c-Cambrian congruence*. The quotient of the weak order by this congruence $\equiv_c$ is the *c-Cambrian lattice*. It is isomorphic to the sublattice of the weak order induced by c-sortable (or c-antisortable) elements. In particular, for two c-Cambrian classes X, Y, we have $X \preccurlyeq Y$ in the c-Cambrian lattice $\Leftrightarrow$ there exists $x \in X$ and $y \in Y$ such that $x \preccurlyeq y$ in the weak order on

$$W \iff \pi_\downarrow^c(X) \preccurlyeq \pi_\downarrow^c(Y) \iff \pi_c^\uparrow(X) \preccurlyeq \pi_c^\uparrow(Y).$$

We denote by $X \wedge_c Y$ and $X \vee_c Y$ the meet and join of the two c-Cambrian classes X, Y.

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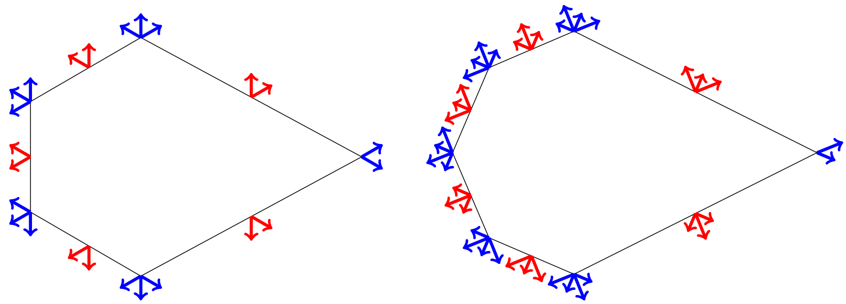

Figure 5: The sets $\mathrm{R}(F)$ for the faces F of the c-associahedron in type A_2 (left) and B_2 (right). Note that positive roots point downwards.

Let $w_\circ(c) = q_1 \cdots q_N$ denote the c-sorting word for the longest element $w_\circ$. It orders Φ^+ by $\alpha_{q_1} <_c q_1(\alpha_{q_2}) <_c q_1q_2(\alpha_{q_3}) <_c \cdots <_c q_1 \cdots q_{N-1}(\alpha_{q_N})$. A subset R of positive roots is called *c-aligned* if for any $\alpha <_c \beta$ such that $\alpha + \beta \in \mathrm{R}$, we have $\alpha \in \mathrm{R}$. It is known that $w \in W$ is c-sortable if and only if its inversion set $\mathrm{inv}(w)$ is c-aligned [Rea07b].

4.2.1 Elements

For a c-Cambrian class X, we consider the Φ-poset

$$\mathrm{R}(X) := \bigcap_{w \in X} \mathrm{R}(w) = \mathrm{R}\big(\pi_\downarrow^c(X)\big) \cap \mathrm{R}\left(\pi_c^\uparrow(X)\right) = \mathrm{R}\big(\pi_\downarrow^c(X)\big)^- \sqcup \mathrm{R}\big(\pi_c^\uparrow(X)\big)^+.$$

Note that by definition, $\mathrm{R}(X)$ coincides with the weak order interval poset $\mathrm{R}(\pi_\downarrow^c(X), \pi_c^\uparrow(X))$. We say that $\mathrm{R}(X)$ is a *c-Cambrian order element poset*, and we denote the collection of all such Φ-posets by

$$\mathsf{COEP}(c) := \{\mathrm{R}(X) \mid X \ c\text{-Cambrian class}\}.$$

Remark 4.23 Table 1 reports the cardinality of $\mathsf{COEP}(c)$ in type A_n, B_n, C_n and D_n for small values of n. Observe that this cardinality is independent of the choice of the Coxeter element c, and is the Coxeter–Catalan number (counting many related objects from clusters of type Φ to non-crossing partitions of W)

$$|\mathsf{COEP}(c)| = \mathrm{Cat}(W) = \prod_{i \in [n]} \frac{1 + d_i}{d_i},$$

where $(d_1, \ldots, d_n)$ still denote the degrees of W.

Remark 4.24 Geometrically, $\mathrm{R}(X)$ is the set of roots of Φ not contained in the cone of the vertex corresponding to X in the generalized associahedron $\mathsf{Asso}(c)$ of C. Hohlweg, C. Lange, and H. Thomas in [HLT11]. See Figure 5.

Let us now take a little detour to comment on a conjectured characterization of these Φ-posets, inspired by a similar characterization in type A proved in [CPP17, Prop. 60]. Note that it uses the c-Cambrian order interval posets formally defined in the next section and characterized in Proposition 4.32. It also requires the notion of c-snakes. A *c-snake* in a Φ-poset R is a sequence of roots $\alpha_1, \dots, \alpha_p \in \mathrm{R}$ such that

- either $\alpha_{2i} \in \Phi^-$, $\alpha_{2i+1} \in \Phi^+$ and $\alpha_1 <_c -\alpha_2 >_c \alpha_3 <_c -\alpha_4 >_c \cdots$,
- or $\alpha_{2i} \in \Phi^+$, $\alpha_{2i+1} \in \Phi^-$ and $-\alpha_1 >_c \alpha_2 <_c -\alpha_3 >_c \alpha_4 <_c \cdots$.

A *c-snake decomposition* of a root α in R is a decomposition $\alpha = \sum_{i \in [p]} \lambda_i \alpha_i$, where $\lambda_i \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_p$ is a c-snake of R. The following conjectural characterization of c-Cambrian order element posets was proved in type A in [CPP17, Prop. 60] and has been checked computationally for small Coxeter types using [Sd16].

Conjecture 4.25 *A Φ-poset $\mathrm{R} \in \mathcal{P}(\Phi)$ is in $\mathsf{COEP}(c)$ if and only if it is in $\mathsf{COIP}(c)$ (characterized in Proposition 4.32) and any root $\alpha \in \Phi$ admits a c-snake decomposition in* R.

Even without this characterization, we can at least describe the weak order on these posets.

Proposition 4.26 *For any two c-Cambrian classes X and Y, we have $\mathrm{R}(X) \preccurlyeq \mathrm{R}(Y)$ in the weak order on $\mathsf{COEP}(c)$ if and only if $X \preccurlyeq Y$ in the c-Cambrian lattice.*

Proof By definition, a c-Cambrian class X admits both a minimal element $\pi_\downarrow^c(X)$ and a maximal element $\pi_c^\uparrow(X)$. Therefore,

$$\mathrm{R}(X) = \mathrm{R}\big(\pi_\downarrow^c(X), \pi_c^\uparrow(X)\big) \in \mathsf{WOIP}(\Phi).$$

Moreover, for two c-Cambrian classes X, Y, Proposition 4.12 implies that $\mathrm{R}(X) \preccurlyeq \mathrm{R}\,(Y)$ in the weak order on $\mathsf{WOIP}(\Phi)$ if and only if $\pi_\downarrow^c(X) \preccurlyeq \pi_\downarrow^c(Y)$ and $\pi_c^\uparrow(X) \preccurlyeq \pi_c^\uparrow(Y)$ in weak order on W. But this is equivalent to $X \preccurlyeq Y$ in the c-Cambrian lattice as mentioned above. ∎

Remark 4.27 In fact,

$$\mathrm{R}(X) \preccurlyeq \mathrm{R}(Y) \iff \mathrm{R}(X)^+ \supseteq \mathrm{R}(Y)^+ \iff \mathrm{R}(X)^- \subseteq \mathrm{R}(Y)^- \iff X \preccurlyeq Y.$$

Corollary 4.28 *For any Coxeter element c, the weak order on $\mathsf{COEP}(c)$ is a lattice with meet and join*

$$\mathrm{R}(X) \wedge_{\mathsf{COEP}(c)} \mathrm{R}(Y) = \mathrm{R}(X \wedge_c Y) \quad \textit{and} \quad \mathrm{R}(X) \vee_{\mathsf{COEP}(c)} \mathrm{R}(Y) = \mathrm{R}(X \vee_c Y).$$

Although it anticipates the c-Cambrian order interval posets studied in the next section, let us state the following result that will be a direct consequence of Corollary 4.34 and Proposition 4.35.

Proposition 4.29 *For any Coxeter element c, the set $\mathsf{COEP}(c)$ induces a sublattice of the weak order on $\mathsf{COIP}(c)$ and thus also a sublattice of the weak order on $\mathsf{WOIP}(\Phi)$.*

We conclude our discussion on $\mathsf{COEP}(c)$ with one more conjecture, which was proved in type A in [CPP17, Coro. 88] and checked computationally for small Coxeter types using [Sd16]. Note that there is little hope to attack this conjecture before proving either Conjecture 4.25 or Conjecture 4.36.

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Conjecture 4.30 *For any Coxeter element c, the set* $\mathsf{COEP}(c)$ *induces a sublattice of the weak order on* $\mathcal{P}(\Phi)$.

4.2.2 Intervals

For two c-Cambrian classes X, X' with $X \preccurlyeq X'$ in the c-Cambrian order, we denote by $[X, X'] := \{Y \text{ } c\text{-Cambrian class} \mid X \preccurlyeq Y \preccurlyeq X'\}$ the *c-Cambrian order interval* between X and X'. We associate with each c-Cambrian order interval $[X, X']$ the Φ-poset

$$\mathrm{R}(X, X') := \bigcap_{Y \in [X, X']} \mathrm{R}(Y) = \mathrm{R}(X) \cap \mathrm{R}(X') = \mathrm{R}(X)^{-} \cup \mathrm{R}(X')^{+}.$$

Note that by definition, $\mathrm{R}(X, X')$ coincides with the weak order interval poset $\mathrm{R}(\pi_{\downarrow}^{c}(X), \pi_{c}^{\uparrow}(X'))$. We say that $\mathrm{R}(X, X')$ is a *c-Cambrian order interval poset*, and we denote the collection of all such Φ-posets by

$$\mathsf{COIP}(c) := \{\mathrm{R}(X, X') \mid X, X' \text{ } c\text{-Cambrian classes}, X \preccurlyeq X'\}.$$

Remark 4.31 Table 1 reports the cardinality of $\mathsf{COIP}(c)$ in type A_n, B_n, C_n, and D_n for small values of n and different choices of the Coxeter element c. We have denoted by bip the bipartite Coxeter element, and by lin the linear one (with the special vertex first in type B/C and the two special vertices first in type D). Note that in contrast to $\mathsf{COEP}(c)$, the cardinality of $\mathsf{COIP}(c)$ depends on the choice of the Coxeter element c (this comes from the fact that the c-Cambrian lattices for different choices of Coxeter element c are not isomorphic and have distinct intervals, although they have the same number of elements).

We now characterize the Φ-posets in $\mathsf{COIP}(c)$.

Proposition 4.32 *A Φ-poset $\mathrm{R} \in \mathcal{P}(\Phi)$ is in* $\mathsf{COIP}(c)$ *if and only if $\alpha + \beta \in \mathrm{R}$ and $\alpha <_c \beta$ implies $\beta \in \mathrm{R}$ for all $\alpha, \beta \in \Phi^{+}$ (resp. $\alpha \in \mathrm{R}$ for all $\alpha, \beta \in \Phi^{-}$).*

Proof Consider a Φ-poset $\mathrm{R} \in \mathcal{P}(\Phi)$. By definition, R is in $\mathsf{COIP}(c)$ if and only if $\mathrm{R} = \mathrm{R}(w, w')$ is in $\mathsf{WOIP}(\Phi)$ where w is c-sortable while w' is c-antisortable. However, w is c-sortable if and only if its inversion set

$$\mathrm{inv}(w) = \Phi^{+} \cap -\mathrm{R}(w) = -\mathrm{R}(w)^{-} = -\mathrm{R}(w, w')^{-} = -\mathrm{R}^{-}$$

is c-aligned, *i.e.,* if and only if $\alpha + \beta \in \mathrm{R}^{-} \Rightarrow \alpha \in \mathrm{R}^{-}$ for any $\alpha <_c \beta$. Similarly, w' is c-antisortable if and only if $\alpha + \beta \in \mathrm{R}^{+} \Rightarrow \beta \in \mathrm{R}^{+}$ for any $\alpha <_c \beta$. ■

Proposition 4.33 *For two c-Cambrian intervals $X \preccurlyeq X'$ and $Y \preccurlyeq Y'$, we have $\mathrm{R}(X, X') \preccurlyeq \mathrm{R}(Y, Y')$ in the weak order on* $\mathsf{COIP}(c)$ *if and only if $X \preccurlyeq Y$ and $X' \preccurlyeq Y'$ in the c-Cambrian order.*

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Proof By definition of $\mathrm{R}(X, X')$ and Remark 4.27, we obtain

$$\begin{aligned} &\mathrm{R}(X, X') \preccurlyeq \mathrm{R}(Y, Y') \\ &\iff \quad \mathrm{R}(X, X')^{+} \supseteq \mathrm{R}(Y, Y')^{+} \quad \text{and} \quad \mathrm{R}(X, X')^{-} \subseteq \mathrm{R}(Y, Y')^{-} \\ &\iff \quad \mathrm{R}(X')^{+} \supseteq \mathrm{R}(Y')^{+} \quad \text{and} \quad \mathrm{R}(X)^{-} \subseteq \mathrm{R}(Y)^{-} \\ &\iff \quad X' \preccurlyeq Y' \quad \text{and} \quad X \preccurlyeq Y. \end{aligned}$$

■

Corollary 4.34 *For any Coxeter element c, the weak order on* $\mathsf{COIP}(c)$ *is a lattice with meet and join*

$$\mathrm{R}(X, X') \wedge_{\mathsf{COIP}(c)} \mathrm{R}(Y, Y') = \mathrm{R}(X \wedge_c Y, X' \wedge_c Y')$$
$$\text{and} \quad \mathrm{R}(X, X') \vee_{\mathsf{COIP}(c)} \mathrm{R}(Y, Y') = \mathrm{R}(X \vee_c Y, X' \vee_c Y').$$

The following statement connects this lattice structure on $\mathsf{COIP}(c)$ with that on $\mathsf{WOIP}(\Phi)$.

Proposition 4.35 *For any Coxeter element c, the set* $\mathsf{COIP}(c)$ *induces a sublattice of the weak order on* $\mathsf{WOIP}(\Phi)$.

Proof Consider two c-Cambrian intervals $X \preccurlyeq X'$ and $Y \preccurlyeq Y'$. By Corollary 4.13, we have

$$\begin{aligned} \mathrm{R}(X, X') \wedge_{\mathsf{WOIP}} \mathrm{R}(Y, Y') &= \mathrm{R}\big(\pi_{\downarrow}^{c}(X), \pi_{c}^{\uparrow}(X')\big) \wedge_{\mathsf{WOIP}} \mathrm{R}\big(\pi_{\downarrow}^{c}(Y), \pi_{c}^{\uparrow}(Y')\big) \\ &= \mathrm{R}\big(\pi_{\downarrow}^{c}(X) \wedge_W \pi_{\downarrow}^{c}(Y), \pi_{\downarrow}^{c}(X') \wedge_W \pi_{\downarrow}^{c}(Y')\big) \\ &= \mathrm{R}\big(\pi_{\downarrow}^{c}(X \wedge_c Y), \pi_{\downarrow}^{c}(X' \wedge_c Y')\big), \end{aligned}$$

where the last equality follows from the fact that c-sortable elements (resp. c-antisortable elements) induce a sublattice of the weak order. ■

The following conjecture indicates that $\mathsf{COIP}(c)$ behaves much better than $\mathsf{WOIP}(\Phi)$ as subposet of $\mathcal{P}(\Phi)$. This conjecture unfortunately remains open for now but was proved in type A in [CPP17, Coro. 82] and verified for small Coxeter types using [Sd16]. Note that it is not implied by Proposition 4.35, since $\mathsf{WOIP}(\Phi)$ is not a sublattice of $\mathcal{P}(\Phi)$. Observe also that it would imply Conjecture 4.30.

Conjecture 4.36 *For any Coxeter element c, the set* $\mathsf{COIP}(c)$ *induces a sublattice of the weak order on* $\mathcal{P}(\Phi)$.

4.2.3 Faces

To remain at a combinatorial level and avoid geometric descriptions (see Remark 4.38), we consider a combinatorial model for the faces of the associahedron $\mathsf{Asso}(c)$ that rely on results of [DHP18, Sec. 4]. The c-Cambrian congruence $\equiv_c$ extends to the *c-Cambrian facial congruence* on all faces of the permutahedron $\mathsf{Perm}(W)$ defined by $xW_I \equiv_c yW_J \iff x \equiv_c y$ and $xw_{\circ,I} \equiv_c yw_{\circ,J}$. This relation is a lattice congruence of the facial weak order on faces of the permutahedron $\mathsf{Perm}(W)$

[DHP18, Prop. 4.12], and we denote by $\Pi_{\downarrow}^{c}$ and $\Pi_{c}^{\uparrow}$ its down and up projections. Moreover, the c-Cambrian facial congruence classes precisely correspond to the faces of the associahedron $\mathsf{Asso}(c)$ of [HLT11].

For a c-Cambrian facial congruence class F, we consider the Φ-poset

$$\mathrm{R}(F) := \bigcap_{xW_I \in F} \mathrm{R}(xW_I) = \mathrm{R}\big(\Pi_{\downarrow}^{c}(F)\big)^{-} \cap \mathrm{R}\left(\Pi_{c}^{\uparrow}(F)\right)^{+}.$$

Note that if $\Pi_{\downarrow}^{c}(F) = xW_I$ and $\Pi_{c}^{\uparrow}(F) = yW_J$, then $\mathrm{R}(F)$ coincides with the weak order interval poset $\mathrm{R}(x, yw_{\circ,J})$. We say that $\mathrm{R}(F)$ is a *c-Cambrian order face poset* and denote the set of such Φ-posets by

$$\mathsf{COFP}(c) := \{\mathrm{R}(F) \mid F \text{ } c\text{-Cambrian facial congruence class}\}.$$

Remark 4.37 Table 1 reports the cardinality of $\mathsf{COFP}(c)$ in type A_n, B_n, C_n, and D_n for small values of n. Note that this cardinality is again independent of the choice of the Coxeter element c (it is the number of faces in the generalized associahedron, *i.e.,* the number of partial clusters in the corresponding cluster algebra).

Remark 4.38 Geometrically, $\mathrm{R}(F)$ is the set of roots of Φ not contained in the cone of the face F in the generalized associahedron $\mathsf{Asso}(c)$ of C. Hohlweg, C. Lange and H. Thomas in [HLT11]. See Figure 5.

It would be particularly interesting to have a characterization of the Φ-posets in $\mathsf{COFP}(c)$ similar to that given in [CPP17] in type A (see [CPP17, Prop. 46] for the Tamari faces and [CPP17, Prop. 63] for the type A Cambrian faces in general).

Here, we just connect the weak order on $\mathsf{COFP}(c)$ with the facial weak order on the associahedron $\mathsf{Asso}(c)$ considered in [DHP18, Sec. 4.7.2]. This order is the quotient of the facial weak order on the faces of the permutahedron $\mathsf{Perm}(W)$ by the c-Cambrian facial congruence $\equiv_c$.

Proposition 4.39 *For any two c-Cambrian facial congruence classes F and G, we have $\mathrm{R}(F) \preccurlyeq \mathrm{R}(G)$ in the weak order on $\mathsf{COFP}(c)$ if and only if $F \preccurlyeq G$ in the c-Cambrian facial lattice.*

Proof This is immediate from the definitions:

$$\begin{aligned}
&\mathrm{R}(F) \preccurlyeq \mathrm{R}(G) \\
&\quad\iff \quad \mathrm{R}\big(\Pi_{c}^{\uparrow}(F)\big)^{+} \supseteq \mathrm{R}\big(\Pi_{c}^{\uparrow}(G)\big)^{+} \text{ and } \mathrm{R}\big(\Pi_{\downarrow}^{c}(F)\big)^{-} \subseteq \mathrm{R}\big(\Pi_{\downarrow}^{c}(G)\big)^{-} \\
&\quad\iff \quad \Pi_{c}^{\uparrow}(F) \preccurlyeq \Pi_{c}^{\uparrow}(G) \quad \text{and} \quad \Pi_{\downarrow}^{c}(F) \preccurlyeq \Pi_{\downarrow}^{c}(G) \\
&\quad\iff \quad F \preccurlyeq G.
\end{aligned}$$

■

Corollary 4.40 *For any Coxeter element c, the weak order on $\mathsf{COFP}(c)$ is a lattice.*

Remark 4.41 To conclude, note that the weak order on $\mathsf{COFP}(c)$ is a lattice but not a sublattice of the weak order on $\mathcal{P}(\Phi)$, nor on $\mathsf{WOIP}(\Phi)$, nor on $\mathsf{COIP}(c)$. This was already observed in [CPP17, Rem. 47] in type A. For example, consider the example of Remark 4.22 for the Coxeter element $s_1 s_2$ in type A_2.

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4.3 Cube

To conclude this paper, we consider Φ-posets corresponding to the vertices, the intervals, and the faces of the cube (see Remarks 4.42 and 4.48), corresponding to the descent congruence on W. Recall that a (left) *descent* of $w \in W$ is a simple root $\alpha \in \Delta$ such that $s_\alpha w \preccurlyeq w$, or equivalently $\alpha \in \mathrm{inv}(w)$. The *descent set* of w is $\mathrm{des}(w) := \mathrm{inv}(w) \cap \Delta$. The *descent class* of w is the set of elements of W that have precisely the same descent set as w. Note that descent classes correspond to subsets of Δ: for $A \subseteq \Delta$, we denote by Z_A the descent class of elements of W with A as descent set. These classes define the *descent congruence* on W, whose down and up projections we denote by $\pi_\downarrow^d$ and $\pi_d^\uparrow$.

4.3.1 Elements

For a subset $A \subseteq \Delta$ corresponding to the descent class Z_A, we consider the Φ-poset

$$\begin{aligned}\mathrm{R}(A) &:= (-A \sqcup (\Delta \smallsetminus A))^{\mathsf{cl}} = \Phi \cap \mathbb{N}(-A \sqcup (\Delta \smallsetminus A)) \\ &= \bigcap_{w \in Z_A} \mathrm{R}(w) = \mathrm{R}\big(\pi_\downarrow^d(Z_A)\big) \cap \mathrm{R}\left(\pi_d^\uparrow(Z_A)\right) = \mathrm{R}\big(\pi_\downarrow^d(Z_A)\big)^- \sqcup \mathrm{R}\big(\pi_d^\uparrow(Z_A)\big)^+.\end{aligned}$$

Note that by definition, $\mathrm{R}(A)$ coincides with the weak order interval poset $\mathrm{R}(\pi_\downarrow^c(Z_A), \pi_c^\uparrow(Z_A))$. We say that $\mathrm{R}(A)$ is a *boolean order element poset* and we denote the collection of all such Φ-posets by $\mathsf{BOEP}(\Phi) := \{\mathrm{R}(A) \mid A \subseteq \Delta\}$. Note that there are 2^n many Φ-posets in $\mathsf{BOEP}(\Phi)$, see Table 1.

Remark 4.42 Geometrically, $\mathrm{R}(A)$ is the set of roots of Φ not contained in the cone of the vertex corresponding to A in the parallelepiped generated by the simple roots Δ. See Figure 6.

These Φ-posets are characterized in the next statement. Its proof is delayed to Section 4.3.2, as it requires the characterization of the boolean order interval posets.

Proposition 4.43 *A Φ-poset* $\mathrm{R} \in \mathcal{P}(\Phi)$ *is in* $\mathsf{BOEP}(\Phi)$ *if and only if*

(i) $\alpha + \beta \in \mathrm{R} \Rightarrow \alpha \in \mathrm{R}$ *and* $\beta \in \mathrm{R}$ *for all* $\alpha, \beta \in \Phi^+$ *and all* $\alpha, \beta \in \Phi^-$,
(ii) $\alpha \in \mathrm{R}$ *or* $-\alpha \in \mathrm{R}$ *for any simple root* $\alpha \in \Delta$.

The following statement characterizes the weak order induced by $\mathsf{BOEP}(\Phi)$.

Proposition 4.44 *For any subsets* $A, B \subseteq \Delta$, *we have* $\mathrm{R}(A) \preccurlyeq \mathrm{R}(B)$ *in the weak order on* $\mathsf{BOEP}(\Phi)$ *if and only if* $A \subseteq B$ *in boolean order.*

Proof From the definition $\mathrm{R}(A) = \Phi \cap \mathbb{N}(-A \sqcup (\Delta \smallsetminus A))$, we obtain that

$$\begin{aligned}\mathrm{R}(A) \preccurlyeq \mathrm{R}(B) \quad &\Longleftrightarrow \quad \mathrm{R}(A)^+ \supseteq \mathrm{R}(B)^+ \quad \text{and} \quad \mathrm{R}(A)^- \subseteq \mathrm{R}(B)^- \\ &\Longleftrightarrow \quad \Delta \smallsetminus A \supseteq \Delta \smallsetminus B \quad \text{and} \quad A \subseteq B.\end{aligned}$$
■

Remark 4.45 In fact,

$$\mathrm{R}(A) \preccurlyeq \mathrm{R}(B) \iff \mathrm{R}(A)^+ \supseteq \mathrm{R}(B)^+ \iff \mathrm{R}(A)^- \subseteq \mathrm{R}(B)^- \iff A \subseteq B.$$

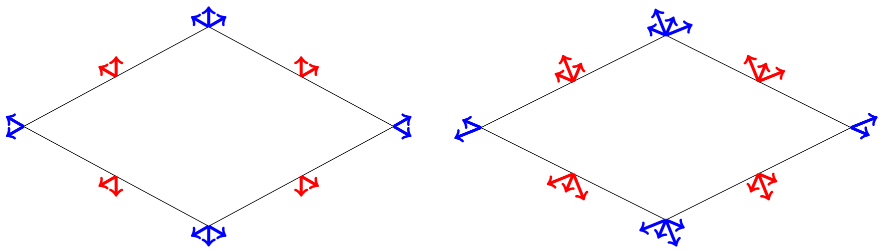

Figure 6: The sets $\mathrm{R}(F)$ for the faces F of the cube in type A_2 (left) and B_2 (right). Note that positive roots point downwards.

Corollary 4.46 *The weak order on* $\mathsf{BOEP}(\Phi)$ *is a lattice with meet and join*

$$\mathrm{R}(A) \wedge_{\mathsf{BOEP}} \mathrm{R}(B) = \mathrm{R}(A \cap B) \quad \textit{and} \quad \mathrm{R}(A) \vee_{\mathsf{BOEP}} \mathrm{R}(B) = \mathrm{R}(A \cup B).$$

Although it anticipates the boolean order interval posets studied in the next section, let us state the following result, which will be a direct consequence of Corollary 4.51 and Proposition 4.52.

Proposition 4.47 *The set* $\mathsf{BOEP}(\Phi)$ *induces a sublattice of the weak order on* $\mathsf{BOIP}(\Phi)$ *and therefore on the weak orders on* $\mathcal{P}(\Phi)$, *on* $\mathsf{WOIP}(\Phi)$ *and on* $\mathsf{COIP}(c)$ *for all Coxeter element* c.

4.3.2 Intervals and Faces

We finally consider intervals in the boolean order or, equivalently, faces of the cube (see Remark 4.48). For two subsets $A \subseteq A'$ of Δ, we consider

$$\mathrm{R}(A, A') \coloneqq \bigcap_{A \subseteq B \subseteq A'} \mathrm{R}(B) = \mathrm{R}(A) \cap \mathrm{R}(A') = \mathrm{R}(A)^- \sqcup \mathrm{R}(A')^+.$$

Note that by definition, $\mathrm{R}(A, A')$ coincides with the weak order interval poset $\mathrm{R}(\pi_\downarrow^c(Z_A), \pi_c^\uparrow(Z_{A'}))$. Observe also that $\mathsf{BOIP}(\Phi) \subseteq \mathsf{COIP}(c)$ for any Coxeter element c, since the descent congruence coarsens the c-Cambrian congruence. We say that $\mathrm{R}(A, A')$ is a *boolean order interval poset* and we denote the set of such Φ-posets by $\mathsf{BOIP}(\Phi) \coloneqq \{\mathrm{R}(A, A') \mid A \subseteq A' \subseteq \Delta\}$.

Remark 4.48 Geometrically, $\mathrm{R}(A, A')$ is the set of roots of Φ not contained in the cone of the face corresponding to $A \subseteq A'$ in the parallelepiped generated by the simple roots Δ. See Figure 6.

These Φ-posets are characterized as follows.

Proposition 4.49 *A* Φ*-poset* $\mathrm{R} \in \mathcal{P}(\Phi)$ *is in* $\mathsf{BOIP}(\Phi)$ *if and only if* $\alpha + \beta \in \mathrm{R}$ *implies* $\alpha \in \mathrm{R}$ *and* $\beta \in \mathrm{R}$ *for all* $\alpha, \beta \in \Phi^+$ *and all* $\alpha, \beta \in \Phi^-$.

Proof Consider first $\mathrm{R}(A, A') \in \mathsf{BOIP}(\Phi)$ and $\alpha + \beta \in \mathrm{R}(A, A')$ with $\alpha, \beta \in \Phi^-$. For $\gamma \in \Delta$, denote by $[\alpha : \gamma]$ the coefficient of γ in the decomposition of α on the simple root basis. If $[\alpha : \gamma] \neq 0$, then $[\alpha + \beta : \gamma] \neq 0$, which implies that $\gamma \in A$, since $\alpha + \beta \in \mathrm{R}(A, A')^- = \mathrm{R}(A)^- \subseteq \mathbb{N}(-A)$. We get $\alpha \in \Phi \cap \mathbb{N}(-A) = \mathrm{R}(A)^- \subseteq \mathrm{R}(A, A')$. By symmetry, we conclude that $\alpha \in \mathrm{R}(A, A')$ and $\beta \in \mathrm{R}(A, A')$ for any $\alpha, \beta \in \Phi^-$ such that $\alpha + \beta \in \mathrm{R}(A, A')$. The proof is similar when $\alpha, \beta \in \Phi^+$.

Conversely, consider $\mathrm{R} \in \mathcal{P}(\Phi)$ such that $\alpha + \beta \in \mathrm{R} \Rightarrow \alpha \in \mathrm{R}$ and $\beta \in \mathrm{R}$ for all $\alpha, \beta \in \Phi^+$ and all $\alpha, \beta \in \Phi^-$. Define $A := -(\mathrm{R} \cap -\Delta)$ and $A' := \Phi \smallsetminus (\mathrm{R} \cap \Delta)$. We claim that $\mathrm{R} = \mathrm{R}(A, A')$, *i.e.*, that $\mathrm{R}^- = \mathrm{R}(A)^-$ and $\mathrm{R}^+ = \mathrm{R}(A')^+$. We prove the latter; the former is similar. Observe first that $\Delta \smallsetminus A' \subseteq \mathrm{R}$, so that $\mathrm{R}(A')^+ = \Phi \cap \mathbb{N}(\Delta \smallsetminus A') \subseteq \mathrm{R}$, since R is closed. Conversely, we prove by induction on $|\gamma|$ that any $\gamma \in \mathrm{R}^+$ belongs to $\mathrm{R}(A')^+$. Consider $\gamma \in \mathrm{R}^+$, and let X be the multiset of simple roots such that $\gamma = \Sigma \mathrm{X}$. By Theorem 2.4, there exists $\alpha \in \mathrm{X}$ such that $\beta = \Sigma(\mathrm{X} \smallsetminus \{\alpha\}) \in \Phi$. Since $\alpha + \beta = \gamma \in \mathrm{R}$, we get that $\alpha \in \mathrm{R}$ and $\beta \in \mathrm{R}$. We have $\alpha \in \Delta \cap \mathrm{R} = \Phi \smallsetminus A' \subseteq \mathrm{R}(A')^+$ and $\beta \in \mathrm{R}(A')^+$ (by induction hypothesis). Since $\mathrm{R}(A')^+$ is closed, this shows that $\gamma = \alpha + \beta \in \mathrm{R}(A')^+$. ∎

We are now in position to provide the proof of Proposition 4.43 that we postponed in Section 4.3.1.

Proof of Proposition 4.43 Observe first that for $A \subseteq \Delta$, the boolean order element poset $\mathrm{R}(A)$ satisfies (i) by Proposition 4.49, and (ii) since $\alpha \in \mathrm{R}(A)$ if $\alpha \in \Delta \smallsetminus A$ and $-\alpha \in \mathrm{R}(A)$ if $\alpha \in A$.

Conversely, consider a Φ-poset R satisfying (i) and (ii). The proof of Proposition 4.49 ensures that $\mathrm{R} = \mathrm{R}(A, A')$ where $A := -(\mathrm{R} \cap -\Delta)$ and $A' := \Phi \smallsetminus (\mathrm{R} \cap \Delta)$. Condition (ii) ensures that $A = A'$ so that we obtain $\mathrm{R} = \mathrm{R}(A, A) = \mathrm{R}(A) \in \mathsf{BOEP}(\Phi)$. ∎

The following statement characterizes the weak order induced by $\mathsf{BOIP}(\Phi)$.

Proposition 4.50 *For two boolean intervals $A \subseteq A'$ and $B \subseteq B'$, we have that $\mathrm{R}(A, A') \preccurlyeq \mathrm{R}(B, B')$ in the weak order on $\mathsf{BOIP}(\Phi)$ if and only if $A \subseteq B$ and $A' \subseteq B'$ in boolean order.*

Proof Using Remark 4.45, we obtain that

$$\begin{aligned} &\mathrm{R}(A, A') \preccurlyeq \mathrm{R}(B, B') \\ &\iff \mathrm{R}(A, A')^+ \supseteq \mathrm{R}(B, B')^+ \quad \text{and} \quad \mathrm{R}(A, A')^- \subseteq \mathrm{R}(B, B')^- \\ &\iff \mathrm{R}(A')^+ \supseteq \mathrm{R}(B')^+ \quad \text{and} \quad \mathrm{R}(A)^- \subseteq \mathrm{R}(B)^- \\ &\iff \Delta \smallsetminus A' \supseteq \Delta \smallsetminus B' \quad \text{and} \quad A \subseteq B \\ &\iff A' \subseteq B' \quad \text{and} \quad A \subseteq B. \end{aligned}$$
∎

Corollary 4.51 *The weak order on $\mathsf{BOIP}(\Phi)$ is a lattice with meet and join*

$$\mathrm{R}(A, A') \wedge_{\mathsf{BOIP}} \mathrm{R}(B, B') = \mathrm{R}(A \cap B, A' \cap B')$$
$$\text{and} \quad \mathrm{R}(A, A') \vee_{\mathsf{BOIP}} \mathrm{R}(B, B') = \mathrm{R}(A \cup B, A' \cup B').$$

We conclude with a connection between the lattice structure of the weak orders on $\mathsf{BOIP}(\Phi)$ with that on $\mathcal{P}(\Phi)$, $\mathsf{WOIP}(\Phi)$, and $\mathsf{COIP}(c)$.

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Proposition 4.52 *The set* $\mathsf{BOIP}(\Phi)$ *induces a sublattice of the weak order on* $\mathcal{P}(\Phi)$*, on* $\mathsf{WOIP}(\Phi)$ *and on* $\mathsf{COIP}(c)$ *for every Coxeter element* c.

Proof Let $\mathrm{R} = \mathrm{R}(A, A')$ and $\mathrm{S} = \mathrm{R}(B, B')$ be two boolean order interval posets, and consider $\mathrm{M} = \mathrm{R} \wedge_{\mathcal{SC}} \mathrm{S}$. Observe that

$$\mathrm{M}^- = \mathrm{R}^- \cap \mathrm{S}^- = -A^{\mathsf{cl}} \cap -B^{\mathsf{cl}} = -(A \cap B)^{\mathsf{cl}}$$

$$\text{and} \quad \mathrm{M}^+ = (\mathrm{R}^+ \cup \mathrm{S}^+)^{\mathsf{cl}} = \big((\Delta \smallsetminus A')^{\mathsf{cl}} \cup (\Delta \smallsetminus B')^{\mathsf{cl}}\big)^{\mathsf{cl}} = (\Delta \smallsetminus (A' \cap B'))^{\mathsf{cl}}.$$

In other words, we obtain that $\mathrm{M} = \mathrm{R} \wedge_{\mathsf{BOIP}} \mathrm{S}$ is already in $\mathsf{BOIP}(\Phi)$, and consequently,

$$\mathrm{R} \wedge_{\mathcal{C}} \mathrm{S} = \mathrm{M}^{\mathsf{ncd}} = \mathrm{M} = \mathrm{R} \wedge_{\mathsf{BOIP}} \mathrm{S} \in \mathsf{BOIP}(\Phi).$$

As $\mathsf{BOIP}(\Phi) \subseteq \mathsf{COIP}(c) \subseteq \mathsf{WOIP}(\Phi) \subseteq \mathcal{P}(\Phi)$, we have

$$\mathrm{R} \wedge_{\mathsf{BOIP}} \mathrm{S} \preccurlyeq \mathrm{R} \wedge_{\mathsf{COIP}(c)} \mathrm{S} \preccurlyeq \mathrm{R} \wedge_{\mathsf{WOIP}} \mathrm{S} \preccurlyeq \mathrm{R} \wedge_{\mathsf{BOIP}} \mathrm{S}$$

so that all these meets coincide. The proof is similar for the join. ∎

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