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Uniqueness of renormalized solutions to nonlinear parabolic problems with lower-order terms

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We consider a general class of parabolic equations of the type

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(u, \nabla u)) + \operatorname{div}(K(u)) + H(\nabla u) = f - \operatorname{div} g$$

with Dirichlet boundary conditions and with a right-hand side belonging to $L^1 + L^{p'}(W^{-1,p'})$. Using the framework of renormalized solutions we prove uniqueness results under appropriate growth conditions and Lipschitz-type conditions on $a(u, \nabla u)$, $K(u)$ and $H(\nabla u)$.

1. Introduction

In this paper we investigate the uniqueness of the following class of nonlinear parabolic problems:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + \operatorname{div}(K(x,t,u)) + H(x,t,\nabla u) = f - \operatorname{div} g \quad &\text{in } Q_T, \\ u(x,t) = 0 \qquad &\text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad &\text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where Q_T is the cylinder $\Omega \times (0,T)$, Ω is a bounded open subset of $\mathbb{R}^N$, $T > 0$, $p > 1$ and $N \geqslant 2$. Moreover, $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator that is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u. The functions K and H are

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Carathéodory functions with suitable assumptions (see theorems 3.1–3.3). Finally, $f \in L^1(Q_T)$, $g \in (L^{p'}(Q_T))^N$ and $u_0 \in L^1(\Omega)$.

The difficulties connected to the existence and uniqueness of the solution to this problem are due to the L^1 data and to the presence of the two terms K and H, which can induce a lack of coercivity.

For L^1 data and $p > 2-1/(N+1)$ the existence of a weak solution to (1.1) (which belongs to $L^m((0,T);W_0^{1,m}(\Omega))$, with $m < (p(N+1)-N)/(N+1))$ was proved in [6] (see also [8]) when $K \equiv H \equiv 0$ and in [23] when $K \equiv 0$. It is well known that this weak solution is not, in general, unique (see [27] and [24] for a counter-example in the stationary case). In the present paper we use the framework of renormalized solutions, which provides uniqueness and stability properties.

The notion of renormalized solutions was introduced in [14, 15] for first-order equations and was adapted for elliptic problems with L^1 data in [18, 19], and for those with bounded measure data in [10]. This notion was also developed for parabolic equations with L^1 data in [2, 4] (see also [21] for measure data). Recall that the equivalent notion of an entropy solution for L^1 data was also developed for elliptic equations in [1] (see also [25] in the parabolic case).

In the case where $H \equiv 0$ and where the function $K(x,t,u)$ is independent of the (x,t) variable and continuous, the existence of a renormalized solution to (1.1) is proved in [4]. The case $H \equiv 0$, $g \equiv 0$ (and where K depends on (x,t) and u) is investigated in [11]. In [13] the authors prove the existence of a renormalized solution to (1.1) with the presence of both the terms $\operatorname{div}(K(x,t,u))$ and $H(x,t,\nabla u)$.

As far as the uniqueness of renormalized solutions to parabolic equations is concerned, we refer the reader mainly to [2, 4, 9], where, in short, the function K does not depend on the (x,t) variable, and where $H \equiv 0$ (see also [3] for the Stefan problem with L^1 data). In particular, when $H \equiv 0$ and under a local Lipschitz assumption on $a(x,t,r,\xi)$ and on $K(r)$ with respect to r, Blanchard *et al.* prove in [4] that the renormalized solution to (1.1) is unique. With respect to the aforementioned references, the main novelty of the present paper is that it presents uniqueness results to parabolic equations (1.1) with both the terms $\operatorname{div}(K(x,t,u))$ and $H(x,t,\nabla u)$. The first result (see theorem 3.1) deals with the case $H \equiv 0$ and establishes the uniqueness of the renormalized solution to (1.1) under a local Lipschitz condition on $a(x,t,r,\xi)$ and $K(x,t,r)$ with respect to r. The proof uses the techniques developed in [4], and the (x,t)-dependence of the function K leads to additional difficulties here. Such difficulties are overcome by a technical lemma (see lemma 4.1) that specifies the asymptotic behaviour of some terms that appear in the uniqueness process. The second result (see theorem 3.2 for $p \geqslant 2$ and theorem 3.3 for $2-1/(N+1) < p < 2$) addresses (1.1) with the presence of both the terms $\operatorname{div}(K(x,t,u))$ and $H(x,t,\nabla u)$. Under more restrictive assumptions on a and under a global Lipschitz-type condition on $K(x,t,s)$ with respect to s and $H(x,t,\xi)$ with respect to ξ, we show the uniqueness of the renormalized solution. The proof uses two technical lemmas (lemmas 4.1 and 4.2) and the techniques developed in [13] for the existence of a solution to (1.1) (see also [23]). We underline that we do not make any assumptions on the smallness of the coefficients. Indeed, for the analogous elliptic equation with two lower-order terms (see, for example, [12, 16]) it is necessary to assume that one of the terms K or H is small enough in order to obtain existence and uniqueness results.

The paper has the following structure. In § 2, we present the assumptions on the data and we recall the definition of a renormalized solution to (1.1). In § 3, we state the main results of the paper. In § 4, we give the proof of the uniqueness results.

2. Assumptions and definitions

We recall the definition of a renormalized solution to nonlinear parabolic problems with lower-order terms and $L^1(\Omega \times (0,T)) + L^{p'}((0,T); W^{-1,p'}(\Omega))$ data in this section.

More precisely, we consider the problem

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + \operatorname{div}(K(x,t,u)) + H(x,t,\nabla u) = f - \operatorname{div} g \quad \text{in } Q_T,\\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T),\\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{aligned}\right\} \tag{2.1}$$

where Q_T is the cylinder $\Omega \times (0,T)$, Ω is a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$, $T > 0$, $p > 1$ and $N \geqslant 2$.

The following assumptions hold true.

- $a\colon Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$a(x,t,s,\xi)\xi \geqslant \alpha_0|\xi|^p, \quad \alpha_0 > 0, \tag{2.2}$$

and

$$(a(x,t,s,\xi) - a(x,t,s,\bar{\xi}) \cdot \xi - \bar{\xi}) > 0 \tag{2.3}$$

for almost everywhere (a.e.) $(x,t) \in Q_T$, for any $s \in \mathbb{R}$ and any $\xi, \bar{\xi} \in \mathbb{R}^N$, with $\xi \neq \bar{\xi}$.

Moreover, for any $k > 0$ there exists $\beta_k > 0$ and $h_k \in L^{p'}(Q_T)$ such that

$$|a(x,t,s,\xi)| \leqslant h_k + \beta_k|\xi|^{p-1} \quad \text{for every } s \text{ such that } |s| \leqslant k, \tag{2.4}$$

for a.e. $(x,t) \in Q_T$ and any $\xi \in \mathbb{R}^N$.

- $K\colon Q_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$|K(x,t,s)| \leqslant c(x,t)(|s|^\gamma + 1), \tag{2.5}$$

with

$$\gamma = \frac{N+2}{N+p}(p-1) \quad \text{and} \quad c \in L^r(Q_T) \quad \text{with } r \geqslant \frac{N+p}{p-1}, \tag{2.6}$$

for a.e. $(x,t) \in Q_T$, for every $s \in \mathbb{R}$.

- $H\colon Q_T \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|H(x,t,\xi)| \leqslant b(x,t)(|\xi|^\delta + 1), \tag{2.7}$$

with

$$\delta = \frac{N(p-1)+p}{N+2} \quad \text{and} \quad b \in L^{N+2,1}(Q_T) \tag{2.8}$$

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for a.e. $(x,t) \in Q_T$, for every $\xi \in \mathbb{R}^N$ and where the Lorentz space $L^{N+2,1}(Q_T)$ is defined later.

Moreover, we assume that

$$f \in L^1(Q_T), \tag{2.9}$$

$$g \in (L^{p'}(Q_T))^N, \tag{2.10}$$

$$u_0 \in L^1(\Omega). \tag{2.11}$$

Under these assumptions, problem (2.1) does not admit, in general, a solution in the sense of distribution, since we cannot expect to have the field $a(x,t,u,\nabla u)$ in $L^1_{\mathrm{loc}}(Q_T)$. For this reason, in the present paper we consider the framework of renormalized solutions.

For any $k > 0$ we denote by T_k the truncation function at height $\pm k$, $T_k(s) = \max(-k, \min(k,s))$ for any $s \in \mathbb{R}$.

We recall the definition of a renormalized solution (see [2, 4]) to (2.1).

Definition 2.1. A real function u defined in Q_T is a renormalized solution of (2.1) if it satisfies

$$u \in L^\infty((0,T); L^1(\Omega)), \tag{2.12}$$

$$T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega)) \quad \text{for any } k > 0, \tag{2.13}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{(x,t)\in Q_T\colon |u(x,t)|\leqslant n\}} a(x,t,u,\nabla u)\nabla u \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.14}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ that is piecewise C^1 and such that S' has a compact support,

$$\begin{aligned} \frac{\partial S(u)}{\partial t} &- \operatorname{div}(a(x,t,u,\nabla u)S'(u)) \\ &+ S''(u)a(x,t,u,\nabla u)\nabla u + \operatorname{div}(K(x,t,u)S'(u)) \\ &- S''(u)K(x,t,u)\nabla u + H(x,t,\nabla u)S'(u) \\ &\qquad = fS'(u) - (\operatorname{div} g)S'(u) \quad \text{in } \mathcal{D}'(Q_T) \end{aligned} \tag{2.15}$$

and

$$S(u)(t=0) = S(u_0) \quad \text{in } \Omega. \tag{2.16}$$

Remark 2.2. It is well known that (2.12) and (2.13) allow us to define ∇u almost everywhere in Q_T: for any $k > 0$ we have that $\nabla T_k(u) = \chi_{\{|u|<k\}}\nabla u$ almost everywhere in Q_T, where $\chi_{\{|u|<k\}}$ denotes the characteristic function of the set $\{(x,t)\colon |u(x,t)| < k\}$. We note that (2.15) can be formally obtained through pointwise multiplication of (2.1) by $S'(u)$, and all terms except $S(u)_t$ in (2.15) belong to $L^1(Q_T) + L^{p'}((0,T); W^{-1,p'}(\Omega))$ since $T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega))$, for any $k > 0$, and S' has a compact support. It follows that (2.15) has a meaning in $\mathcal{D}'(Q_T)$ and that the initial condition (2.16) makes sense (see [22, theorem 1.1]). Finally, (2.14) gives additional information on ∇u for large values of $|u|$.

We use, in the present paper, the two Lorentz spaces $L^{q,1}(Q_T)$ and $L^{q,\infty}(Q_T)$ (for information about Lorentz spaces $L^{q,s}$, see, for example, [17, 20] and the references therein). If f^* denotes the decreasing rearrangement of a measurable function f,

$$f^*(r) = \inf\{s \geqslant 0\colon\ \operatorname{meas}\{(x,t) \in Q_T\colon |f(x,t)| > s\} < r\},$$

with $r \in [0, \operatorname{meas}(Q_T)]$, then $L^{q,1}(Q_T)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,1}(Q_T)} = \left(\int_0^{\operatorname{meas}(Q_T)} f^*(r) r^{1/q} \frac{\mathrm{d}r}{r}\right) < +\infty,$$

while $L^{q,\infty}(Q_T)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,\infty}(Q_T)} = \sup_{r>0} r[\operatorname{meas}\{(x,t) \in Q_T\colon |f(x,t)| > r\}]^{1/q} < +\infty.$$

If $1 < q < +\infty$, we have the generalized Hölder inequality,

$$\forall f \in L^{q,\infty}(Q_T), \quad \forall g \in L^{q',1}(Q_T), \quad \int_{Q_T} |fg| \leqslant \|f\|_{L^{q,\infty}(Q_T)} \|g\|_{L^{q',1}(Q_T)}. \tag{2.17}$$

Under (2.2)–(2.11) the existence of a renormalized solution to (2.1) is established in [13] and it is well known that (2.12)–(2.14) lead to

$$|\nabla u| \in L^{(N(p-1)+p)/(N+1),\infty}(Q_T) \tag{2.18}$$

and

$$u \in L^{(N(p-1)+p)/N,\infty}(Q_T). \tag{2.19}$$

Moreover, the growth assumptions (2.5) and (2.7) on K and H, the regularities (2.6) and (2.8) of c and b together with (2.12) and (2.14) allow us to prove (see [13]) that any renormalized solution to (2.1) verifies that

$$H(x,t,\nabla u) \in L^1(Q_T) \tag{2.20}$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{(x,t)\in Q_T;\, |u(x,t)|<n\}} |K(x,t,u)||\nabla u| \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.21}$$

Properties (2.20) and (2.21) are crucial to obtain uniqueness results.

Notation. Throughout the paper, for the sake of brevity, if u is a measurable function defined on Q_T, we denote by $\{|u| \leqslant k\}$ (respectively, $\{|u| < k\}$) the measurable subset $\{(x,t) \in Q_T;\ |u(x,t)| \leqslant k\}$ (respectively, $\{(x,t) \in Q_T;\ |u(x,t)| < k\}$). Moreover, the explicit dependence in x and t of the functions a, K and H will be omitted, so $a(x,t,u,\nabla u) = a(u,\nabla u)$, $K(u) = K(x,t,u)$ and $H(\nabla u) = H(x,t,\nabla u)$.

3. Statement of the results

3.1. First case: $H \equiv 0$

In order to prove the uniqueness result in the case $H(x,t,\xi) = 0$, we further assume that $a(x,t,s,\xi)$ and $K(x,t,s)$ are locally Lipschitz continuous with respect to s: for

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any compact set C of $\mathbb{R}$, there exists L_C belonging to $L^{p'}(Q_T)$ and $\gamma_C > 0$ such that, for all $s, \bar{s} \in C$,

$$|a(x,t,s,\xi) - a(x,t,\bar{s},\xi)| \leqslant (L_C(x,t) + \gamma_C|\xi|^{p-1})|s-\bar{s}|, \tag{3.1}$$
$$|K(x,t,s) - K(x,t,\bar{s})| \leqslant L_C(x,t)|s-\bar{s}| \tag{3.2}$$

for a.e. $(x,t) \in Q_T$ and for every $\xi \in \mathbb{R}^N$.

The main result of this subsection is the following theorem.

Theorem 3.1. *Under the assumptions (2.2)–(2.6), (2.9)–(2.11), (3.1) and (3.2), the renormalized solution to (2.1) is unique.*

3.2. Second case: general operator

In order to prove the uniqueness result for (2.1) with the term $H(x,t,\nabla u)$, we assume in this subsection that the function a is independent of r and is strongly monotone (see (3.5) and (3.7)).

Moreover, the function $K(x,t,s)$ (respectively, $H(x,t,\xi)$) is locally Lipschitz continuous with respect to s (respectively, ξ) with a global control of the Lipschitz coefficient:

$$|K(x,t,s) - K(x,t,\bar{s})| \leqslant c(x,t)(1+|s|+|\bar{s}|)^{\tau}|s-\bar{s}|, \quad \tau \geqslant 0, \tag{3.3}$$

and

$$|H(x,t,\xi) - H(x,t,\bar{\xi})| \leqslant b(x,t)(1+|\xi|+|\bar{\xi}|)^{\sigma}|\xi-\bar{\xi}|, \quad \sigma \geqslant 0, \tag{3.4}$$

for a.e. $(x,t) \in Q_T$, for every $s, \bar{s} \in \mathbb{R}$, for every $\xi, \bar{\xi} \in \mathbb{R}^N$ with $c \in L^{r,1}(Q_T)$ and $b \in L^{\lambda,1}(Q_T)$, and where the parameters τ, σ, r and λ belong to suitable intervals (see theorems 3.2 and 3.3).

We investigate the case $p \geqslant 2$ and the case $2 - 1/(N+1) < p < 2$ in two different results.

Theorem 3.2. *Let $p \geqslant 2$. We assume that (2.2)–(2.11) hold and that the function a is independent of r and satisfies*

$$(a(x,t,\xi) - a(x,t,\bar{\xi})) \cdot (\xi - \bar{\xi}) \geqslant \beta(1+|\xi|+|\bar{\xi}|)^{p-2}|\xi-\bar{\xi}|^2 \tag{3.5}$$

for a.e. $(x,t) \in Q_T$, for every $\xi, \bar{\xi} \in \mathbb{R}^N$ with $\xi \neq \bar{\xi}$ and $\beta > 0$.

Moreover, we assume that (3.3) and (3.4) are satisfied, with

$$\left.\begin{aligned} r \geqslant N+2, \quad 0 \leqslant \tau \leqslant \frac{N(p-1)+p}{N}\left(\frac{1}{N+2} - \frac{1}{r}\right), \\ \lambda \geqslant N+2, \quad 0 \leqslant \sigma \leqslant \frac{N(p-1)+p}{N+1}\left(\frac{1}{N+2} - \frac{1}{\lambda}\right). \end{aligned}\right\} \tag{3.6}$$

Then, the renormalized solution to (2.1) is unique.

Theorem 3.3. *Let $2 - 1/(N+1) < p < 2$. We assume that (2.2)–(2.4), (2.9)–(2.11) hold and that the function a is independent of r and satisfies*

$$(a(x,t,\xi) - a(x,t,\bar{\xi})) \cdot (\xi - \bar{\xi}) \geqslant \beta\frac{|\xi-\bar{\xi}|^2}{(|\xi|+|\bar{\xi}|)^{2-p}} \tag{3.7}$$

for a.e. $(x,t) \in Q_T$*, for every* $\xi, \bar\xi \in \mathbb{R}^N$ *with* $\xi \neq \bar\xi$ *and* $\beta > 0$*. Moreover, we assume that (3.3) and (3.4) are satisfied, with*

$$\left.\begin{aligned} & r > \frac{p(N+1)-N}{(p-1)(N+1)-N}, \\ & 0 \leqslant \tau < \frac{N(p-1)+p}{N}\left(\frac{(p-1)(N+1)-N}{p(N+1)-N} - \frac{1}{r}\right), \\ & \lambda > \frac{p(N+1)-N}{(p-1)(N+1)-N}, \\ & 0 \leqslant \sigma < \frac{N(p-1)+p}{N+1}\left(\frac{(p-1)(N+1)-N}{p(N+1)-N} - \frac{1}{\lambda}\right). \end{aligned}\right\} \tag{3.8}$$

Then, the renormalized solution to (2.1) is unique.

Remark 3.4. We compare assumptions (2.5) and (2.7) on the growth condition and assumptions (3.3) and (3.4) on the locally Lipschitz continuity made on $K(x,t,s)$ and $H(x,t,\xi)$, respectively. Observe that (3.3) ((3.4), respectively) implies a growth condition on $K(x,t,s)$ (on $H(x,t,\xi)$, respectively) that can be more restrictive than (2.5) ((2.7), respectively), depending on the value of τ (σ, respectively).

The model function $a(x,t,\xi)$ that satisfies (2.4), (3.5) or (3.7) is

$$a(x,t,\xi) = \begin{cases} a(x,t)|\xi|^{p-2}\xi & \text{if } 2 - \dfrac{1}{N+1} < p < 2, \\ a(x,t)(1+|\xi|^2)^{(p-2)/2}\xi & \text{if } p \geqslant 2, \end{cases}$$

where $a(x,t) \in L^\infty(Q_T)$ and $a(x,t) \geqslant \beta > 0$.

Examples of functions $K(x,t,s)$ and $H(x,t,\xi)$ are given by

$$\begin{aligned} K(x,t,s) &= c(x,t)(1+|s|)^{\bar\gamma}, \quad \text{with } \bar\gamma = \min\{\gamma, \tau+1\}, \\ H(x,t,\xi) &= b(x,t)(1+|\xi|)^{\bar\delta}, \quad \text{with } \bar\delta = \min\{\delta, \sigma+1\}, \end{aligned}$$

where $c(x,t) \in L^{r,1}(Q_T)$ and $b(x,t) \in L^{\lambda,1}(Q_T)$, with

$$\begin{aligned} & r > \frac{p(N+1)-N}{(p-1)(N+1)-N} && \text{if } 2 - \frac{1}{N+1} < p < 2, \\ & r \geqslant N+2 && \text{if } p \geqslant 2 \end{aligned}$$

and

$$\begin{aligned} & \lambda > \frac{p(N+1)-N}{(p-1)(N+1)-N} && \text{if } 2 - \frac{1}{N+1} < p < 2, \\ & \lambda \geqslant N+2 && \text{if } p \geqslant 2. \end{aligned}$$

4. Proof of the results

This section is devoted to proving theorems 3.1–3.3. We start with a technical lemma, similar to [4, lemma 6], for a different parabolic equation with L^1 data. It

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allows us to control the behaviour of some quantities that appear in the uniqueness process. We stress that our proof is different to the one in [4] and uses only the fact that two renormalized solutions of (2.1) verify (2.14) and (2.21) (note that (2.21) is a consequence of (2.14) and the growth assumption of K). See also [5] for such a generalization on parabolic equations of the kind

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = f + \operatorname{div} g.$$

Lemma 4.1. *Under the assumptions (2.2)–(2.11), let u and v be two renormalized solutions to (2.1). We define, for any $0 < k < s$,*

$$\begin{aligned} \Gamma(u,v,s,k) = {} & \int_{\{s-k<|u|<s+k\}} (a(u,\nabla u)\nabla u + |K(u)||\nabla u| + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\{s-k<|v|<s+k\}} (a(v,\nabla v)\nabla v + |K(v)||\nabla v| + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.1}$$

and, for any $0 < r < s$,

$$\begin{aligned} \Theta(u,v,s,r) = {} & \int_{\{s-r<|u|<s\}} (a(u,\nabla u)\nabla u + |K(u)||\nabla u|)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\{s-r<|v|<s\}} (a(v,\nabla v)\nabla v + |K(v)||\nabla v|)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.2}$$

Then, we have, for any $r > 0$, that

$$\liminf_{s\to\infty}\left(\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k) + \Theta(u,v,s,r)\right) = 0. \tag{4.3}$$

Proof. We argue by contradiction. Let r be a positive real number. If the thesis of lemma 4.1 is not true, let $\varepsilon_0 > 0$ and let $n_0 > r$ be an integer such that for every real number $s \geqslant n_0$ we have that

$$\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k) + \Theta(u,v,s,r) \geqslant \varepsilon_0. \tag{4.4}$$

We consider the function

$$\begin{aligned} F(s) = {} & \int_{\{|u|<s\}} (a(u,\nabla u)\nabla u + |K(u)||\nabla u| + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\{|v|<s\}} (a(v,\nabla v)\nabla v + |K(v)||\nabla v| + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Due to (2.2) the function F is monotone increasing. It follows (see, for example, [26]) that F is derivable almost everywhere, with F' measurable, and that we have, for any $s > \eta > 0$, that

$$F(s) - F(\eta) \geqslant \int_\eta^s F'(w)\,\mathrm{d}w \tag{4.5}$$

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and, for almost any $s > 0$,

$$F'(s) = \frac{1}{2}\limsup_{k\to 0}\frac{1}{k}\biggl[\int_{\{s-k\leqslant|u|<s+k\}}(a(u,\nabla u)\nabla u + |K(u)||\nabla u| + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}t + \int_{\{s-k\leqslant|v|<s+k\}}(a(v,\nabla v)\nabla v + |K(v)||\nabla v| + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}t\biggr].$$

Moreover, due to (2.14) and (2.21) and since g belongs to $(L^{p'}(Q_T))^N$ we have that

$$\lim_{s\to+\infty}\frac{F(s)}{s} = 0. \tag{4.6}$$

Due to the definition of $\Gamma(u,v,s,k)$, (4.4) leads to

$$F'(w) + \tfrac{1}{2}\Theta(u,v,w,r) \geqslant \tfrac{1}{2}\varepsilon_0$$

for almost every $w \geqslant n_0$. From (4.5) it follows that

$$\frac{1}{s-n_0}\biggl(F(s) + \frac{1}{2}\int_{n_0}^{s}\Theta(u,v,w,r)\,\mathrm{d}w\biggr) \geqslant \frac{\varepsilon_0}{2} + \frac{F(n_0)}{s-n_0} \quad \text{for } s > n_0. \tag{4.7}$$

Writing, for any $s > n_0$,

$$\begin{aligned}\int_{n_0}^{s}\int_{\{w-r<|u|<w\}} & a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w \\ &= \int_{n_0}^{s}\int_{\{|u|<w\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w - \int_{n_0}^{s}\int_{\{|u|\leqslant w-r\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w \\ &= \int_{n_0}^{s}\int_{\{|u|<w\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w - \int_{n_0-r}^{s-r}\int_{\{|u|\leqslant w\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w \\ &= \int_{s-r}^{s}\int_{\{|u|<w\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w - \int_{n_0-r}^{n_0}\int_{\{|u|<w\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w,\end{aligned}$$

and since $a(u,\nabla u)\nabla u \geqslant 0$ almost everywhere in Q_T, we obtain that

$$\int_{n_0}^{s}\int_{\{w-r<|u|<w\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w \leqslant r\int_{\{|u|<s\}} a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t.$$

In view of the definition of Θ and using similar arguments we deduce that

$$\int_{n_0}^{s}\Theta(u,v,\xi,r)\,\mathrm{d}\xi \leqslant r\biggl(\int_{\{|u|<s\}}(a(u,\nabla u)\nabla u + |K(u)||\nabla u|)\,\mathrm{d}x\,\mathrm{d}t + \int_{\{|v|<s\}}(a(v,\nabla v)\nabla v + |K(v)||\nabla v|)\,\mathrm{d}x\,\mathrm{d}t\biggr).$$

From (4.7) and the above inequality it follows that

$$\frac{1}{s-n_0}\biggl(F(s) + \frac{r}{2}\biggl(\int_{\{|u|<s\}}(a(u,\nabla u)\nabla u + |K(u)||\nabla u|)\,\mathrm{d}x\,\mathrm{d}t + \int_{\{|v|<s\}}(a(v,\nabla v)\nabla v + |K(v)||\nabla v|)\,\mathrm{d}x\,\mathrm{d}t\biggr)\biggr) \geqslant \frac{\varepsilon_0}{2} + \frac{F(n_0)}{s-n_0}$$

for $s > n_0$. The last inequality contradicts (2.14) and (4.6). □

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Proof of theorem 3.1. The strategy is similar to that in the proof of [4, theorem 2]. It consists of defining a smooth approximation T_s^σ of the truncation T_s and considering two renormalized solutions u and v to (2.1) for the same data f, g and u_0. In step 1 we plug the test function $T_k(T_s^\sigma(u)-T_s^\sigma(v))/k$ into the difference of equations (2.15) for u and v, in which we have taken $S = T_s^\sigma$. This process then leads to (4.9). In step 2 we study the behaviour of the terms of (4.9) with respect to σ, k and s, with the help of lemma 4.1. In step 3 we then pass to the limit when $\sigma \to 0$, $k \to 0$ and $s \to +\infty$.

STEP 1. Let u and v be two renormalized solutions to (2.1) for the same data f, g and u_0. For every real number $s > 0$ and $\sigma > 0$, let T_s^σ be the function defined by

$$\left.\begin{aligned} &T_s^\sigma(0) = 0,\\ &(T_s^\sigma)'(r) = \begin{cases} 1 & \text{for } |r| < s,\\ \dfrac{1}{\sigma}(s+\sigma-|r|) & \text{for } s \leqslant |r| \leqslant s+\sigma,\\ 0 & \text{for } |r| > s+\sigma. \end{cases}\end{aligned}\right\} \tag{4.8}$$

We take $S = T_s^\sigma$ in (2.15) for u and v. Subtracting these two equations and plugging in the test function $T_k(T_s^\sigma(u)-T_s^\sigma(v))/k$, we obtain, upon integration on $(0,t)$, that, for every $k > 0$, $s > 0$, $\sigma > 0$,

$$\begin{aligned} \frac{1}{k}\int_0^t \left\langle \frac{\partial}{\partial t}[T_s^\sigma(u) - T_s^\sigma(v)], T_k(T_s^\sigma(u) - T_s^\sigma(v))\right\rangle \mathrm{d}\tau + \frac{1}{k}(A_{s,k}^\sigma(t) + \tilde{A}_{s,k}^\sigma(t))&\\ = \frac{1}{k}(C_{s,k}^\sigma(t) + \tilde{C}_{s,k}^\sigma(t) + F_{s,k}^\sigma(t) + G_{s,k}^\sigma(t) + \tilde{G}_{s,k}^\sigma(t))& \end{aligned} \tag{4.9}$$

for almost any $t \in (0,T)$, where $\langle\cdot,\cdot\rangle$ denotes the duality between $L^1(\Omega)+W^{-1,p'}(\Omega)$ and $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$, and where

$$\begin{aligned} A_{s,k}^\sigma(t) &= \int_0^t\int_\Omega [(T_s^\sigma)'(u)a(u,\nabla u) - (T_s^\sigma)'(v)a(v,\nabla v)]\\ &\qquad\qquad \times \nabla T_k(T_s^\sigma(u) - T_s^\sigma(v))\,\mathrm{d}x\,\mathrm{d}\tau,\\ \tilde{A}_{s,k}^\sigma(t) &= \int_0^t\int_\Omega (T_s^\sigma)''(u)a(u,\nabla u)\nabla u T_k(T_s^\sigma(u) - T_s^\sigma(v))\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad - \int_0^t\int_\Omega (T_s^\sigma)''(v)a(v,\nabla v)\nabla v T_k(T_s^\sigma(u) - T_s^\sigma(v))\,\mathrm{d}x\,\mathrm{d}\tau,\\ C_{s,k}^\sigma(t) &= \int_0^t\int_\Omega [(T_s^\sigma)'(u)K(u) - (T_s^\sigma)'(v)K(v)]\\ &\qquad\qquad \times \nabla T_k(T_s^\sigma(u) - T_s^\sigma(v))\,\mathrm{d}x\,\mathrm{d}\tau,\\ \tilde{C}_{s,k}^\sigma(t) &= \int_0^t\int_\Omega [(T_s^\sigma)''(u)K(u)\nabla u - (T_s^\sigma)''(v)K(v)\nabla v]\\ &\qquad\qquad \times T_k(T_s^\sigma(u) - T_s^\sigma(v))\,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned}$$

$$F^\sigma_{s,k}(t) = \int_0^t \int_\Omega f[(T^\sigma_s)'(u) - (T^\sigma_s)'(v)] T_k(T^\sigma_s(u) - T^\sigma_s(v))\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$G^\sigma_{s,k}(t) = \int_0^t \int_\Omega g[(T^\sigma_s)'(u) - (T^\sigma_s)'(v)] \nabla T_k(T^\sigma_s(u) - T^\sigma_s(v))\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$\tilde{G}^\sigma_{s,k}(t) = \int_0^t \int_\Omega g\nabla[(T^\sigma_s)'(u) - (T^\sigma_s)'(v)] T_k(T^\sigma_s(u) - T^\sigma_s(v))\,\mathrm{d}x\,\mathrm{d}\tau.$$

In order to pass to the limit in (4.9) when $\sigma \to 0$, $k \to 0$ and $s \to +\infty$, we observe that by (4.8) we have, for almost any $t \in (0,T)$, that

$$T^\sigma_s(u) \to T_s(u) \qquad \text{in } L^p((0,t); W^{1,p}_0(\Omega)) \text{ and almost everywhere in } \Omega \times (0,t) \tag{4.10}$$

and

$$(T^\sigma_s)'(u) \to \chi_{\{|u|\leqslant s\}} \qquad \text{in } L^q(\Omega \times (0,t)) \text{ and almost everywhere in } \Omega \times (0,t) \tag{4.11}$$

for every $1 < q < +\infty$, for fixed $s > 0$ when σ tends to zero.

By defining

$$\Psi_k(r) = \int_0^r T_k(s)\,\mathrm{d}s,$$

an integration by parts (see [7]) gives that, for almost any $t \in (0,T)$,

$$\frac{1}{k}\int_0^t \left\langle \frac{\partial}{\partial t}[T^\sigma_s(u) - T^\sigma_s(v)], T_k(T^\sigma_s(u) - T^\sigma_s(v)) \right\rangle \mathrm{d}\tau = \frac{1}{k}\int_\Omega \Psi_k(T^\sigma_s(u)(t) - T^\sigma_s(v)(t))\,\mathrm{d}x. \tag{4.12}$$

We deduce from the above equality that, for almost any $t \in (0,T)$,

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_0^t \left\langle \frac{\partial}{\partial t}[T^\sigma_s(u) - T^\sigma_s(v)], T_k(T^\sigma_s(u) - T^\sigma_s(v)) \right\rangle \mathrm{d}\tau = \int_\Omega |T_s(u)(t) - T_s(v)(t)|\,\mathrm{d}x. \tag{4.13}$$

Step 2. Reasoning as in [4] we have that

$$\liminf_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}A^\sigma_{s,k}(t) \geqslant 0 \quad \text{for every } s > 0 \tag{4.14}$$

and

$$\lim_{s\to+\infty}\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}F^\sigma_{s,k}(t) = 0 \quad \text{for almost any } t \in (0,T). \tag{4.15}$$

We give the argument here for completeness. Due to (4.11) and (4.10) and with the help of (2.2) we have, for almost any $t \in (0,T)$, that

$$\lim_{\sigma\to 0}\frac{1}{k}A^\sigma_{s,k}(t) = \frac{1}{k}\int_0^t\int_\Omega [a(T_s(u), DT_s(u)) - a(T_s(v), DT_s(v))] \times DT_k(T_s(u) - T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau,$$

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which can written as

$$\lim_{\sigma\to 0}\frac{1}{k}A^{\sigma}_{s,k}(t)=\frac{1}{k}\int_0^t\int_{\Omega}[a(T_s(u),DT_s(u))-a(T_s(u),DT_s(v))]\times DT_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau+\frac{1}{k}\int_0^t\int_{\Omega}[a(T_s(u),DT_s(v))-a(T_s(v),DT_s(v))]\times DT_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau. \quad (4.16)$$

Since the operator a is monotone (see (2.3)), the first term of the right-hand side of (4.16) is non-negative. It remains to prove that the second term goes to zero as k goes to zero. Indeed, using the local Lipschitz condition (3.1) on a we get that

$$\begin{aligned}\frac{1}{k}\bigg|\int_0^t\int_{\Omega}&[a(T_s(u),DT_s(v))-a(T_s(v),DT_s(v))]DT_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau\bigg|\\&\leqslant\frac{1}{k}\int_0^t\int_{\Omega}\chi_{\{|T_s(u)-T_s(v)|<k\}}|T_s(u)-T_s(v)|(L_s(x,t)+\gamma_s|DT_s(v)|^{p-1})\\&\qquad\times|DT_s(u)+DT_s(v)|\,\mathrm{d}x\,\mathrm{d}\tau\\&\leqslant\int_{\{0<|T_s(u)-T_s(v)|<k\}}(L_s(x,t)+\gamma_s|DT_s(v)|^{p-1})|DT_s(u)+DT_s(v)|\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Due to the regularity of $T_s(u)$, $T_s(v)$ and L_s we have that

$$(L_s(x,t)+\gamma_s|DT_s(v)|^{p-1})|DT_s(u)+DT_s(v)|\in L^1(Q_T).$$

Since $\chi_{\{|T_s(u)-T_s(v)|<k\}}$ tends to zero almost everywhere in Q_T as k goes to zero, the Lebesgue dominated convergence allows us to conclude that (4.14) holds.

As far as (4.15) is concerned, we have, for almost any $t\in(0,T)$, that

$$\lim_{\sigma\to 0}\frac{1}{k}F^{\sigma}_{s,k}(t)=\frac{1}{k}\int_0^t\int_{\Omega}f(\chi_{\{|u|\leqslant s\}}-\chi_{\{|v|\leqslant s\}})T_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau,$$

so, for almost any $t\in(0,T)$,

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}F^{\sigma}_{s,k}(t)=\int_0^t\int_{\Omega}f\times(\chi_{\{|u|\leqslant s\}}-\chi_{\{|v|\leqslant s\}})\operatorname{sgn}(u-v)\,\mathrm{d}x\,\mathrm{d}\tau,$$

where $\operatorname{sgn}(r)=r/|r|$ for any $r\neq 0$ and $\operatorname{sgn}(0)=0$. Since u and v are finite almost everywhere in $\Omega\times(0,T)$ and since f belongs to $L^1(Q_T)$, the Lebesgue dominated convergence theorem implies (4.15).

We now claim that, for almost any $t\in(0,T)$,

$$\frac{1}{k}(|\tilde{A}^{\sigma}_{s,k}(t)|+|\tilde{C}^{\sigma}_{s,k}(t)|+|\tilde{G}^{\sigma}_{s,k}(t)|)\leqslant\frac{M_1}{\sigma}\Gamma(u,v,s,\sigma), \quad (4.17)$$

where M_1 is a constant independent of s, k and σ, and where Γ is defined in lemma 4.1.

Using the definition (4.8) of T^{σ}_s, recalling that $\nabla u=0$ almost everywhere on $\{(x,t);\ u(x,t)=r\}$ for any $r\in\mathbb{R}$ and since $a(x,t,r,\xi)\xi\geqslant 0$, we obtain that, for any σ and any $k > 0$,

$$\begin{aligned}\frac{1}{k}|\tilde{A}^{\sigma}_{s,k}(t)| &\leqslant \frac{1}{\sigma}\bigg[\int_0^t\int_{\Omega}\chi_{\{s<|u|<s+\sigma\}}a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad + \int_0^t\int_{\Omega}\chi_{\{s<|v|<s+\sigma\}}a(v,\nabla v)\nabla v\,\mathrm{d}x\,\mathrm{d}\tau\bigg] \\ &\leqslant \frac{1}{\sigma}\bigg[\int_{\{s<|u|<s+\sigma\}}a(u,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\{s<|v|<s+\sigma\}}a(v,\nabla v)\nabla v\,\mathrm{d}x\,\mathrm{d}\tau\bigg].\end{aligned} \tag{4.18}$$

Similarly, we have, for any σ and any $k > 0$, that

$$\frac{1}{k}|\tilde{C}^{\sigma}_{s,k}(t)| \leqslant \frac{1}{\sigma}\bigg[\int_{\{s<|u|<s+\sigma\}}|K(u)||\nabla u|\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\{s<|v|<s+\sigma\}}|K(v)||\nabla v|\,\mathrm{d}x\,\mathrm{d}\tau\bigg]. \tag{4.19}$$

As far as $\tilde{G}_{s,k}(t)$ is concerned, for any σ and any $k > 0$ we have that

$$\frac{1}{k}|\tilde{G}_{s,k}(t)| \leqslant \frac{1}{\sigma}\bigg[\int_{\{s<|u|<s+\sigma\}}|g||\nabla u|\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\{s<|v|<s+\sigma\}}|g||\nabla v|\,\mathrm{d}x\,\mathrm{d}\tau\bigg].$$

From (2.2) together with Young's inequality it follows that

$$\begin{aligned}\frac{1}{k}|\tilde{G}_{s,k}(t)| \leqslant \frac{M_1}{\sigma}\bigg(&\int_{\{s<|u|<s+\sigma\}}(a(u,\nabla u)\nabla u + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_{\{s<|v|<s+\sigma\}}(a(v,\nabla v)\nabla v + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}\tau\bigg),\end{aligned} \tag{4.20}$$

where M_1 is a generic constant depending upon p and α_0. Estimates (4.18)–(4.20) allow us to deduce that (4.17) holds.

We now prove that, for almost any $t \in (0,T)$,

$$\limsup_{\sigma\to 0}\frac{1}{k}(|C^{\sigma}_{s,k}(t)| + |G^{\sigma}_{s,k}(t)|) \leqslant \frac{M_1}{k}\Gamma(u,v,s,k) + \omega(k), \tag{4.21}$$

where M_1 is a constant independent of s, k and σ, and where ω is a positive function such that $\lim_{k\to 0}\omega(k) = 0$.

We first write that, for almost any $t \in (0,T)$,

$$\begin{aligned}&\limsup_{\sigma\to 0}\frac{1}{k}|C^{\sigma}_{s,k}(t)| \\ &\quad = \bigg|\frac{1}{k}\int_0^t\int_{\Omega}[\chi_{\{|u|\leqslant s\}}K(u) - \chi_{\{|v|\leqslant s\}}K(v)]\nabla T_k(T_s(u) - T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau\bigg| \\ &\quad \leqslant C^1_{s,k} + C^2_{s,k} + C^3_{s,k},\end{aligned}$$

where

$$C^1_{s,k} = \frac{1}{k}\int_{Q_T}\chi_{\{|u|\leqslant s\}\cap\{|v|>s\}}|K(u)||\nabla T_k(u - s\,\mathrm{sgn}(v))|\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$C^2_{s,k} = \frac{1}{k}\int_{Q_T}\chi_{\{|v|\leqslant s\}\cap\{|u|>s\}}|K(v)||\nabla T_k(v - s\,\mathrm{sgn}(u))|\,\mathrm{d}x\,\mathrm{d}\tau$$

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and

$$C^3_{s,k} = \frac{1}{k}\int_{Q_T} \chi_{\{|v|\leqslant s\}\cap\{|u|\leqslant s\}}|K(u)-K(v)||\nabla T_k(u-v)|\,\mathrm{d}x\,\mathrm{d}\tau.$$

We estimate $C^1_{s,k}$ and $C^2_{s,k}$. By (2.5) we obtain that

$$\begin{aligned} C^1_{s,k} &\leqslant \frac{1}{k}\int_{Q_T} \chi_{\{|u|\leqslant s\}\cap\{|v|>s\}}\chi_{\{|u-s\,\mathrm{sgn}(v)|<k\}}|K(u)||\nabla u|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \frac{1}{k}\int_{\{s-k<|u|\leqslant s\}} |K(u)||\nabla u|\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned} \tag{4.22}$$

and, similarly,

$$C^2_{s,k} \leqslant \frac{1}{k}\int_{\{s-k<|v|\leqslant s\}} |K(v)||\nabla v|\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.23}$$

Finally, since the function K is locally Lipschitz continuous, we have, for some positive L_s element of $L^{p'}(Q_T)$, that

$$\begin{aligned} C^3_{s,k} &= \frac{1}{k}\int_{Q_T} \chi_{\{|v|\leqslant s\}\cap\{|u|\leqslant s\}}|K(u)-K(v)||\nabla T_k(T_s(u)-T_s(v))|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \frac{1}{k}\int_{Q_T} \chi_{\{0<|T_s(v)-T_s(u)|<k\}}L_s(x,\tau)|T_s(u)-T_s(v)| \\ &\qquad\qquad\qquad\qquad \times|\nabla T_k(T_s(u)-T_s(v))|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \int_{Q_T} \chi_{\{0<|T_s(v)-T_s(u)|<k\}}L_s(x,\tau)|\nabla T_k(T_s(u)-T_s(v))|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \int_{Q_T} \chi_{\{0<|T_s(v)-T_s(u)|<k\}}L_s(x,\tau)(|\nabla T_s(u)|+|\nabla T_s(v)|)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

Since L_s belongs to $L^{p'}(Q_T)$ and due to (2.13), the function $L_s(x,\tau)(|\nabla T_s(u)| + |\nabla T_s(v)|)$ belongs to $L^1(Q_T)$. Since $\chi_{\{0<|T_s(v)-T_s(u)|<k\}}$ tends to 0 almost everywhere in Q_T as k goes to 0 and is bounded by 1, the Lebesgue dominated convergence theorem leads to

$$\lim_{k\to 0} C^3_{s,k} = 0 \quad \text{for any } s>0. \tag{4.24}$$

In order to estimate $G_{s,k}(t)$, we obtain, for almost any $t\in(0,T)$, that

$$\begin{aligned} \limsup_{\sigma\to 0}\frac{1}{k}|G^\sigma_{s,k}(t)| &= \left|\frac{1}{k}\int_0^t\int_\Omega [\chi_{\{|u|\leqslant s\}}g - \chi_{\{|v|\leqslant s\}}g]\nabla T_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\leqslant G^1_{s,k}+G^2_{s,k}, \end{aligned}$$

where

$$G^1_{s,k} = \frac{1}{k}\int_{Q_T} \chi_{\{|u|\leqslant s\}\cap\{|v|>s\}}|g||\nabla T_k(u-s\,\mathrm{sgn}(v))|\,\mathrm{d}x\,\mathrm{d}\tau$$

and

$$G^2_{s,k} = \frac{1}{k}\int_{Q_T} \chi_{\{|v|\leqslant s\}\cap\{|u|>s\}}|g||\nabla T_k(v-s\,\mathrm{sgn}(u))|\,\mathrm{d}x\,\mathrm{d}\tau.$$

Since we have that

$$G^1_{s,k} \leqslant \frac{1}{k}\int_{\{s-k<|u|\leqslant s\}} |g||\nabla u| \,\mathrm{d}x\,\mathrm{d}\tau,$$

similar arguments to those used to deal with $\tilde{G}_{s,k}$ yield that

$$G^1_{s,k} \leqslant \frac{M_1}{k}\int_{\{s-k<|u|<s\}} (a(u,\nabla u)\nabla u + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}\tau, \tag{4.25}$$

where M is a constant depending upon p and α_0. With v in place of u in $G^2_{s,k}$ we also have that

$$G^2_{s,k} \leqslant \frac{M_1}{k}\int_{\{s-k<|v|<s\}} (a(v,\nabla v)\nabla v + |g|^{p'})\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.26}$$

Estimates (4.22)–(4.26) imply (4.21).

STEP 3. We are now in a position to prove that $u = v$ almost everywhere in Q_T.

To this end, we pass to the supremum limit as σ goes to 0 and then to the supremum limit as k goes to 0 in (4.9). Indeed, due to (4.13) we have that

$$\begin{aligned}\int_\Omega |T_s(u)(t) - T_s(v)(t)|\,\mathrm{d}x \leqslant{}& -\liminf_{\sigma\to 0}\limsup_{k\to 0}\frac{1}{k}A^\sigma_{s,k}(t)\\ &+ \limsup_{\sigma\to 0}\limsup_{k\to 0}\frac{1}{k}(|\tilde{A}^\sigma_{s,k}(t)| + |\tilde{C}^\sigma_{s,k}(t)| + |\tilde{G}^\sigma_{s,k}(t)|)\\ &+ \limsup_{\sigma\to 0}\limsup_{k\to 0}\frac{1}{k}(|C^\sigma_{s,k}(t)| + |G^\sigma_{s,k}(t)|)\\ &+ \limsup_{\sigma\to 0}\limsup_{k\to 0}\frac{1}{k}F^\sigma_{s,k}(t)\end{aligned}$$

for any $s > 0$ and for almost any $t \in (0,T)$.

In view of (4.14), (4.15), (4.17) and (4.21) we deduce that

$$\begin{aligned}\int_\Omega |T_s(u)(t) - T_s(v)(t)|\,\mathrm{d}x \leqslant{}& M_1 \limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k)\\ &+ M_1\limsup_{\sigma\to 0}\frac{1}{\sigma}\Gamma(u,v,s,\sigma) + \omega(s)\end{aligned} \tag{4.27}$$

for any $s > 0$, for almost any $t \in (0,T)$ and where $\omega(s) \to 0$ as $s \to \infty$.

Recalling that u (respectively, v) is finite almost everywhere in Q_T, $T_s(u)(t)$ (respectively, $T_s(v)(t)$) converges almost everywhere to $u(t)$ (respectively, $v(t)$) as s goes to infinity for almost any $t \in (0,T)$. By Fatou's lemma we can pass to the infimum limit as s goes to $+\infty$ in (4.27) and we obtain, for almost any $t \in (0,T)$, that

$$\int_\Omega |u(t) - v(t)|\,\mathrm{d}x \leqslant 2M_1 \liminf_{s\to+\infty}\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k). \tag{4.28}$$

Lemma 4.1 allows us to conclude that

$$\int_\Omega |u(t) - v(t)|\,\mathrm{d}x = 0$$

for almost any $t \in (0,T)$, so $u = v$ almost everywhere in Q_T. □

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In the case of the complete operator we need the following lemma, which concerns Boccardo–Gallouët-type estimates in Lorentz spaces.

Lemma 4.2. *Assume that* $Q_T = \Omega \times (0,T)$, *with* Ω *an open subset of* $\mathbb{R}^N$ *of finite measure and* $p > 1$. *Let* u *be a measurable function satisfying* $T_k(u) \in L^\infty((0,T);L^2(\Omega)) \cap L^p((0,T);W_0^{1,p}(\Omega))$ *for every* $k > 0$, *and such that, for* $\alpha > 2(N+1)/(N+2)$,

$$\sup_{t\in(0,T)} \int_\Omega |T_k(u(t))|^2 \leqslant kM \quad \text{and} \quad \int_0^T \int_\Omega |\nabla T_k(u)|^\alpha \leqslant C_0 k^{\alpha/2} M^{\alpha/2}, \tag{4.29}$$

where M *and* C_0 *are positive constants. Then,*

$$\|u\|_{L^{\alpha(N+2)/2N,\infty}(Q_T)} \leqslant CM \tag{4.30}$$

and

$$\||\nabla u|\|_{L^{\alpha(N+2)/2(N+1),\infty}(Q_T)} \leqslant CM, \tag{4.31}$$

where C *is a constant depending only on* N *and* C_0.

Such a result being standard, we omit the proof of lemma 4.2 (see, for example, the proof of [13, lemma A.1] with a few modifications).

Proof of theorem 3.2. The proof is divided into four steps. As in the previous theorem we consider two renormalized solutions u and v of (2.1) for the same data f, g and u_0. In step 1, we plug the test function $T_k(T_s^\sigma(u) - T_s^\sigma(v))$ into the difference of equations (2.15) for u and v with $S = T_s^\sigma$ (defined in (4.8)) and we obtain (4.32). Step 2 is devoted to estimating the terms of (4.32). In step 3 we pass to the limit as $\sigma \to 0$ and $s \to +\infty$, k being fixed. Finally, in step 4, using lemma 4.2, we give an estimate of $\nabla u - \nabla v$ in some suitable Lorentz spaces, which allows us to conclude that $u = v$.

Step 1. Let u and v be two renormalized solutions to (2.1) for the same data f, g and u_0. For every real number $s > 0$ and $\sigma > 0$ we take $S = T_s^\sigma$ in (2.15) for u and v. By plugging in the test function $T_k(T_s^\sigma(u) - T_s^\sigma(v))$ into the difference of these two equations, we obtain, upon integration on $(0,t)$, that

$$\begin{aligned}\int_0^t \left\langle \frac{\partial}{\partial t}[T_s^\sigma(u) - T_s^\sigma(v)], T_k(T_s^\sigma(u) - T_s^\sigma(v)) \right\rangle \mathrm{d}\tau + A_{s,k}^\sigma(t) + \tilde{A}_{s,k}^\sigma(t) \\ = B_{s,k}^\sigma(t) + C_{s,k}^\sigma(t) + \tilde{C}_{s,k}^\sigma(t) + F_{s,k}^\sigma(t) + G_{s,k}^\sigma(t) + \tilde{G}_{s,k}^\sigma(t)\end{aligned} \tag{4.32}$$

for every $k > 0$, $s > 0$, $\sigma > 0$ and for almost any $t \in (0,T)$, where

$$B_{s,k}^\sigma(t) = -\int_0^t \int_\Omega [(T_s^\sigma)'(u)H(\nabla u) - (T_s^\sigma)'(v)H(\nabla u)]T_k(T_s^\sigma(u) - T_s^\sigma(v))\,\mathrm{d}x\,\mathrm{d}\tau$$

and the remainder terms are defined in the proof of theorem 3.1. We now pass to the limit in (4.32) as σ goes to zero and then as s goes to $+\infty$.

STEP 2. We recall that, for almost any $t \in (0,T)$,

$$\int_0^t \left\langle \frac{\partial}{\partial t}[T_s^\sigma(u) - T_s^\sigma(v)], T_k(T_s^\sigma(u) - T_s^\sigma(v)) \right\rangle \mathrm{d}\tau = \int_\Omega \Psi_k(T_s^\sigma(u)(t) - T_s^\sigma(v)(t)) \, \mathrm{d}x.$$

Due to the definition of T_s^σ we obtain that

$$\lim_{\sigma\to 0} \int_0^t \left\langle \frac{\partial}{\partial t}[T_s^\sigma(u) - T_s^\sigma(v)], T_k(T_s^\sigma(u) - T_s^\sigma(v)) \right\rangle \mathrm{d}\tau = \int_\Omega \Psi_k(T_s(u)(t) - T_s(v)(t)) \, \mathrm{d}x$$

and, since u and v are finite almost everywhere in Q_T, from Fatou's lemma it follows that

$$\liminf_{s\to+\infty} \lim_{\sigma\to 0} \int_0^t \left\langle \frac{\partial}{\partial t}[T_s^\sigma(u) - T_s^\sigma(v)], T_k(T_s^\sigma(u) - T_s^\sigma(v)) \right\rangle \mathrm{d}\tau \geqslant \int_\Omega \Psi_k(u(t) - v(t)) \, \mathrm{d}x. \quad (4.33)$$

Since $H(\nabla u)$ and $H(\nabla v)$ belong to $L^1(Q_T)$ and since u and v are finite almost everywhere in Q_T, the Lebesgue theorem yields that

$$\lim_{s\to+\infty} \lim_{\sigma\to 0} B_{s,k}^\sigma(t) = -\int_0^t \int_\Omega [H(\nabla u) - H(\nabla v)] T_k(u - v) \, \mathrm{d}x \, \mathrm{d}\tau.$$

Using the Lipschitz condition (3.4) on H and (3.6) we obtain that

$$\int_0^t \int_\Omega |[H(\nabla u) - H(\nabla v)] T_k(u - v)| \, \mathrm{d}x \, \mathrm{d}\tau \leqslant k \|b\|_{L^{\lambda,1}(Q_t)} \|1 + |\nabla u| + |\nabla v|\|_{L^{q,\infty}(Q_t)}^\sigma \||\nabla u - \nabla v|\|_{L^{\theta,\infty}(Q_t)},$$

with

$$\frac{1}{\lambda} + \frac{\sigma}{q} + \frac{1}{\theta} = 1, \quad 1 \leqslant q \leqslant \frac{N(p-1)+p}{N+1}, \quad \theta = \frac{N+2}{N+1} \quad \text{and} \quad \lambda \geqslant N+2.$$

It follows that, for almost any $t \in (0,T)$,

$$\lim_{s\to+\infty} \lim_{\sigma\to 0} |B_{s,k}^\sigma(t)| \leqslant k \|b\|_{L^{\lambda,1}(Q_t)} \|1 + |\nabla u| + |\nabla v|\|_{L^{q,\infty}(Q_t)}^\sigma \||\nabla u - \nabla v|\|_{L^{\theta,\infty}(Q_t)}. \quad (4.34)$$

Since f belongs to $L^1(Q_T)$ while u and v are finite almost everywhere in Q_T we have that

$$\lim_{s\to+\infty} \lim_{\sigma\to 0} F_{s,k}^\sigma(t) = \lim_{s\to+\infty} \int_0^t \int_\Omega f[\chi_{\{|u|\leqslant s\}} - \chi_{\{|v|\leqslant s\}}] T_k(T_s(u) - T_s(v)) \, \mathrm{d}x \, \mathrm{d}\tau = 0. \quad (4.35)$$

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We now deal with $A^{\sigma}_{s,k}$, $C^{\sigma}_{s,k}$ and $G^{\sigma}_{s,k}$. From the definition of T^{σ}_{s} and (3.5) we get that

$$\begin{aligned}\lim_{\sigma\to 0} A^{\sigma}_{s,k}(t)&\\ &= \int_0^t\int_{\Omega}[\chi_{\{|u|\leqslant s\}}a(\nabla u)-\chi_{\{|v|\leqslant s\}}a(\nabla v)]\nabla T_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau\\ &= \int_0^t\int_{\Omega}\chi_{\{T_s(u)-T_s(v)|<k\}}[a(\nabla T_s(u))-a(\nabla T_s(v))](\nabla T_s(u)-\nabla T_s(v))\,\mathrm{d}x\,\mathrm{d}\tau\\ &\geqslant \beta\int_0^t\int_{\Omega}\chi_{\{T_s(u)-T_s(v)|<k\}}(1+|\nabla T_s(u)|+|\nabla T_s(v)|)^{p-2}\\ &\qquad\qquad\times|\nabla T_s(u)-\nabla T_s(v)|^2\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{4.36}$$

Since u and v are finite almost everywhere, Fatou's lemma then implies that

$$\begin{aligned}\liminf_{s\to+\infty}\lim_{\sigma\to 0} A^{\sigma}_{s,k}&\\ &\geqslant \beta\int_0^t\int_{\Omega}\chi_{\{|u-v|<k\}}(1+|\nabla u|+|\nabla(v)|)^{p-2}|\nabla u-\nabla v|^2\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{4.37}$$

Using assumption (3.3) we have that

$$\begin{aligned}\lim_{\sigma\to 0}|C^{\sigma}_{s,k}(t)| &\leqslant \int_0^t\int_{\Omega}|\chi_{\{|u|\leqslant s\}}K(u)-\chi_{\{|v|\leqslant s\}}K(v)|\\ &\qquad\times|\nabla T_k(T_s(u)-T_s(v))|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant \int_0^t\int_{\Omega}\chi_{\{|u|\leqslant s\}\cap\{|v|\leqslant s\}}c(x,\tau)(1+|u|+|v|)^{\tau}\\ &\qquad\times|u-v||\nabla T_k(u-v)|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad+\int_0^t\int_{\Omega}\chi_{\{s-k<|v|\leqslant s\}}|K(v)||\nabla v|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad+\int_0^t\int_{\Omega}\chi_{\{s-k<|u|\leqslant s\}}|K(u)||\nabla u|\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{4.38}$$

From Hölder's inequality and (3.6) we obtain that

$$\begin{aligned}\int_0^t\int_{\Omega}c(x,\tau)(1+|u|+|v|)^{\tau}|\nabla u-\nabla v|\,\mathrm{d}x\,\mathrm{d}\tau&\\ \leqslant \|c\|_{L^{r,1}(Q_t)}\|1+|u|+|v|\|^{\tau}_{L^{\bar q,\infty}(Q_t)}\||\nabla u-\nabla v|\|_{L^{\theta,\infty}(Q_t)},&\end{aligned}\tag{4.39}$$

with

$$\frac{1}{r}+\frac{\tau}{\bar q}+\frac{1}{\theta}=1,\quad 1\leqslant\bar q\leqslant\frac{N(p-1)+p}{N},\quad \theta=\frac{N+2}{N+1}\quad\text{and}\quad r>\frac{N+p}{p-1}.$$

From the regularities of c, u, v, ∇u and ∇v it follows that $c(x,\tau)(1+|u|+|v|)^{\tau}|\nabla u-\nabla v|$ belongs to $L^1(Q_t)$ for any $t\in(0,T)$. Recalling the definition (4.2) of Θ in lemma 4.1 leads to

$$\lim_{\sigma\to 0}|C^\sigma_{s,k}(t)| \leqslant k\|c\|_{L^{r,1}(Q_t)}\|1+|u|+|v|\|^{\tau}_{L^{\bar q,\infty}(Q_t)} \times \||\nabla u-\nabla v|\|_{L^{\theta,\infty}(Q_t)} + \Theta(u,v,s,k) \tag{4.40}$$

for any $k>0$.

We now study $G^\sigma_{s,k}(t)$. We first have that

$$\lim_{\sigma\to 0} G^\sigma_{s,k}(t) = \int_0^t\int_\Omega g[\chi_{\{|u|<s\}} - \chi_{\{|v|<s\}}]\nabla T_k(T_s(u)-T_s(v))\,\mathrm{d}x\,\mathrm{d}t.$$

It follows that

$$\lim_{\sigma\to 0}|G^\sigma_{s,k}(t)| \leqslant \int_0^t\int_\Omega \chi_{\{s-k<|u|<s\}}|g||\nabla u|\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega \chi_{\{s-k<|v|<s\}}|g||\nabla v|\,\mathrm{d}x\,\mathrm{d}\tau.$$

With Young's inequality and integrating on Q_T in place of $\Omega\times(0,t)$ we obtain that

$$\lim_{\sigma\to 0}|G^\sigma_{s,k}(t)| \leqslant \frac{1}{p'}\int_{Q_T}(\chi_{\{s-k<|u|<s\}} + \chi_{\{s-k<|v|<s\}})|g|^{p'}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{p}\int_{\{s-k<|u|<s\}}|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{p}\int_{\{s-k<|v|<s\}}|\nabla v|^p\,\mathrm{d}x\,\mathrm{d}\tau.$$

Since u and v are finite almost everywhere in Q_T the function $(\chi_{\{s-k<|u|<s\}} + \chi_{\{s-k<|v|<s\}})|g|^{p'}$ converges to zero as s goes to $+\infty$ in $L^1(Q_T)$. Since the operator a is elliptic (see assumption (2.2)) and recalling the definition of Θ in lemma 4.1, we then obtain that

$$\lim_{\sigma\to 0}|G^\sigma_{s,k}(t)| \leqslant \frac{1}{\alpha_0}\Theta(u,v,s,k) + \omega(s), \tag{4.41}$$

where $\omega(s)$ is a generic function that converges to 0 as s goes to infinity.

We recall (see (4.17) in the proof of theorem 3.1) that, for almost any $t\in(0,T)$,

$$|\tilde A^\sigma_{s,k}(t)| + |\tilde C^\sigma_{s,k}(t)| + |\tilde G^\sigma_{s,k}(t)| \leqslant \frac{M_1 k}{\sigma}\Gamma(u,v,s,\sigma). \tag{4.42}$$

From estimates (4.40)–(4.42) it follows that

$$\begin{aligned}\limsup_{\sigma\to 0}(|C^\sigma_{s,k}(t)| + |\tilde A^\sigma_{s,k}(t)| + |G^\sigma_{s,k}(t)| + |\tilde C^\sigma_{s,k}(t)| + |\tilde G^\sigma_{s,k}(t)|)\\ \leqslant k\|c\|_{L^{r,1}(Q_t)}\|1+|u|+|v|\|^{\tau}_{L^{\bar q,\infty}(Q_t)}\||\nabla u-\nabla v|\|_{L^{\theta,\infty}(Q_t)}\\ + \Theta(u,v,s,k) + \frac{1}{\alpha_0}\Theta(u,v,s,k) + \omega(s)\\ + M_1 k\limsup_{\sigma\to 0}\frac{1}{\sigma}\Gamma(u,v,s,\sigma).\end{aligned}$$

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By the above inequality and lemma 4.1 we can conclude that, for almost any $t \in (0, T)$,

$$\liminf_{s\to+\infty} \limsup_{\sigma\to 0}(|C^{\sigma}_{s,k}(t)| + |\tilde{A}^{\sigma}_{s,k}(t)| + |G^{\sigma}_{s,k}(t)| + |\tilde{C}^{\sigma}_{s,k}(t)| + |\tilde{G}^{\sigma}_{s,k}(t)|) \leqslant k\|c\|_{L^{r,1}(Q_t)}\|1 + |u| + |v|\|^{\tau}_{L^{\bar{q},\infty}(Q_t)}|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_t)}. \quad (4.43)$$

Step 3. We are now able to pass to the limit in (4.32). Indeed, gathering (4.33)–(4.35), (4.37) and (4.43), we get that

$$\int_{\Omega} \Psi_k(u(t) - v(t))\,\mathrm{d}x + \frac{\beta}{2}\int_0^t \int_{\Omega} \chi_{\{|u-v|<k\}}(1 + |\nabla u| + |\nabla(v)|)^{p-2}|\nabla u - \nabla v|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant k\|c\|_{L^{r,1}(Q_t)}\|1 + |u| + |v|\|^{\tau}_{L^{\bar{q},\infty}(Q_t)}|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_t)} + k\|b\|_{L^{\lambda,1}(Q_t)}\|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^{q,\infty}(Q_t)}|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_t)}$$

for almost any $t \in (0, T)$. It is worth noting that the above inequality implies that

$$\chi_{\{|u-v|<k\}}(1 + |\nabla u| + |\nabla(v)|)^{p-2}|\nabla u - \nabla v|^2 \in L^1(Q_T).$$

Since $(1 + |\xi| + |\xi'|)^{p-2}|\xi - \xi'|^2 \geqslant |\xi - \xi'|^2$ for any ξ, ξ' in $\mathbb{R}^N$, we obtain that $T_k(u - v)$ belongs to $L^2((0, T); H^1_0(\Omega))$.

Due to the definition of Ψ_k, taking the supremum for $t \in (0, t_1)$, where $t_1 \in (0, T)$ will be chosen later, leads to

$$\frac{1}{2}\sup_{t\in(0,t_1)} \int_{\Omega} |T_k(u - v)|^2\,\mathrm{d}x + \frac{\beta}{2}\int_0^{t_1} \int_{\Omega} |\nabla T_k(u - v)|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant kM, \quad (4.44)$$

where

$$M = \|b\|_{L^{\lambda,1}(Q_{t_1})}\|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^{q,\infty}(Q_{t_1})}|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})} + \|c\|_{L^{r,1}(Q_{t_1})}\|1 + |u| + |v|\|^{\tau}_{L^{\bar{q},\infty}(Q_{t_1})}|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})}. \quad (4.45)$$

By (4.44) and lemma 4.2 we get that

$$\|\nabla u - \nabla v\|_{L^{\theta,\infty}(Q_{t_1})} \leqslant CM \quad (4.46)$$

for some constant $C > 0$ independent of u and v and $\theta = (N + 2)/(N + 1)$.

Step 4. Using (4.45) and (4.46) we obtain that

$$|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})} \leqslant C[\|b\|_{L^{\lambda,1}(Q_{t_1})}\|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^{q,\infty}(Q_{t_1})} + \|c\|_{L^{r,1}(Q_{t_1})}\|1 + |u| + |v|\|^{\tau}_{L^{\bar{q},\infty}(Q_{t_1})}]|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})}. \quad (4.47)$$

Since c belongs to $L^{r,1}(Q_T)$ and since b belongs to $L^{\lambda,1}(Q_T)$, if we choose t_1 small enough such that

$$1-C(\|b\|_{L^{\lambda,1}(Q_{t_1})}\|1+|\nabla u|+|\nabla v|\|^{\sigma}_{L^{q,\infty}(Q_{t_1})}+\|c\|_{L^{r,1}(Q_{t_1})}\|1+|u|+|v|\|^{\tau}_{L^{\bar{q},\infty}(Q_{t_1})}) > 0, \quad (4.48)$$

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then (4.47) gives that

$$|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})} \leqslant 0, \tag{4.49}$$

with $\theta = (N+2)/(N+1)$.

Now, we use the same technique as in [23] (see also [13]). We consider a partition of the entire interval $[0,T]$ into a finite number of intervals $[0,t_1]$, $[t_1,t_2]$, ..., $[t_{n-1},T]$ such that for each interval $[t_{i-1},t_i]$ a similar condition to (4.48) holds. In this way, in each cylinder $Q_{t_i} = \Omega \times [t_{i-1},t_i]$ we obtain estimates of type (4.49). We can then deduce that

$$|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_T)} \leqslant 0 \quad \text{for some } \theta \geqslant 1,$$

which implies that $u = v$ almost everywhere in Q_T. □

Proof of theorem 3.3. The strategy of the proof is the same as in theorem 3.2 and relies on passing to the limit in (4.32). The main differences are in dealing with the terms $A^{\sigma}_{s,k}(t)$, $B^{\sigma}_{s,k}$ and $C^{\sigma}_{s,k}(t)$ and the estimate on $\nabla T_k(u-v)$. We recall (4.32):

$$\int_0^t \left\langle \frac{\partial}{\partial t}[T_s^{\sigma}(u) - T_s^{\sigma}(v)], T_k(T_s^{\sigma}(u) - T_s^{\sigma}(v)) \right\rangle \mathrm{d}\tau + A^{\sigma}_{s,k}(t) + \tilde{A}^{\sigma}_{s,k}(t) \\ = B^{\sigma}_{s,k}(t) + C^{\sigma}_{s,k}(t) + \tilde{C}^{\sigma}_{s,k}(t) + F^{\sigma}_{s,k}(t) + G^{\sigma}_{s,k}(t) + \tilde{G}^{\sigma}_{s,k}(t)$$

for any $s > 0$, any $k > 0$ and any $\sigma > 0$ and for almost any $t \in (0,T)$. Reasoning as in theorem 3.2, by assumption (3.7) we obtain that

$$\liminf_{s\to+\infty} \lim_{\sigma\to 0} A^{\sigma}_{s,k}(t) \geqslant \beta \int_0^t \int_{\Omega} \chi_{\{|u-v|<k\}} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.50}$$

As far as $B^{\sigma}_{s,k}(t)$ is concerned, a few computations, estimates (2.18) and (2.19), condition (3.8) and Hölder's inequality lead to

$$\int_0^t \int_{\Omega} |[H(\nabla u) - H(\nabla v)]T_k(u-v)| \,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant k\|b\|_{L^{\lambda,1}(Q_t)} \|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^{q,\infty}(Q_t)} |||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_t)},$$

with

$$\frac{1}{\lambda} + \frac{\sigma}{q} + \frac{1}{\theta} = 1, \quad 1 \leqslant q \leqslant \frac{N(p-1)+p}{N+1},$$
$$\theta = \frac{\alpha(N+2)}{2(N+1)}, \quad \lambda \geqslant N+2 \quad \text{and} \quad \alpha < \frac{2p(N+1) - 2N}{N+2}.$$

Similarly, we obtain

$$\liminf_{s\to+\infty} \lim_{\sigma\to 0} |C^{\sigma}_{s,k}(t)| \leqslant k\|c\|_{L^{r,1}(Q_t)} \|1 + |u| + |v|\|^{\tau}_{L^{\bar{q},\infty}(Q_t)} |||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_t)}, \tag{4.51}$$

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with

$$\frac{1}{r}+\frac{\tau}{\bar q}+\frac{1}{\theta}=1,\quad 1\leqslant \bar q\leqslant \frac{N(p-1)+p}{N},$$

$$\theta=\frac{\alpha(N+2)}{2(N+1)},\quad r>\frac{N+p}{p-1}\quad\text{and}\quad \alpha<\frac{2p(N+1)-2N}{N+2}.$$

Then, the analogous expression to (4.44) is

$$\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u-v)|^2\,\mathrm{d}x+\beta\int_0^{t_1}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,\mathrm{d}x\,\mathrm{d}t\leqslant kM, \tag{4.52}$$

where t_1 will be chosen later and where M is defined in the proof of theorem 3.2 (see (4.45)). We then obtain that

$$\beta\int_0^{t_1}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,\mathrm{d}x\,\mathrm{d}t\leqslant Mk, \tag{4.53}$$

$$\frac{1}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u-v)|^2\leqslant Mk. \tag{4.54}$$

If $1\leqslant\alpha<p$, by Hölder's inequality and (4.53) we have that

$$\begin{aligned}
&\int_0^{t_1}\int_\Omega |\nabla T_k(u-v)|^\alpha\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^{t_1}\int_\Omega |\nabla T_k(u-v)|^\alpha\frac{(|\nabla u|+|\nabla v|)^{(2-p)\alpha/2}}{(|\nabla u|+|\nabla v|)^{(2-p)\alpha/2}}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant\biggl(\int_0^{t_1}\int_\Omega \frac{|\nabla T_k(u-v)|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,\mathrm{d}x\,\mathrm{d}t\biggr)^{\alpha/2}\\
&\qquad\times\biggl(\int_0^{t_1}\int_\Omega (|\nabla u|+|\nabla v|)^{(2-p)\alpha/(2-\alpha)}\,\mathrm{d}x\,\mathrm{d}t\biggr)^{(2-\alpha)/2}\\
&\quad\leqslant (Mk)^{\alpha/2}\biggl(\int_0^{t_1}\int_\Omega (|\nabla u|+|\nabla v|)^{(2-p)\alpha/(2-\alpha)}\,\mathrm{d}x\,\mathrm{d}t\biggr)^{(2-\alpha)/2}.
\end{aligned}\tag{4.55}$$

By (2.18), the last integral in (4.55) is finite if

$$\alpha<\frac{2p(N+1)-2N}{N+2}. \tag{4.56}$$

We observe that (4.56) and the condition on α in lemma 4.2 are compatible only if $p>2-1/(N+1)$. Then, by (4.55), (4.56) and by Hölder's inequality we have that

$$\int_0^{t_1}\int_\Omega |\nabla T_k(u-v)|^\alpha\,\mathrm{d}x\,\mathrm{d}t\leqslant C(Mk)^{\alpha/2}, \tag{4.57}$$

where C is a constant independent of t_1.

By (4.54), (4.57) and the definition of M, for $\theta = \alpha(N+2)/2(N+1)$, lemma 4.2 yields

$$\begin{aligned}|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})} \leqslant C[\|b\|_{L^{\lambda,1}(Q_{t_1})} \|1 + |\nabla u| + |\nabla v|\|^{\sigma}_{L^{q,\infty}(Q_{t_1})} \\ + \|c\|_{L^{r,1}(Q_{t_1})} \|1 + |u| + |v|\|^{\tau}_{L^{\bar{q},\infty}(Q_{t_1})}] \\ \times |||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})}. \end{aligned} \tag{4.58}$$

Under hypotheses (3.8) we can choose t_1 small enough such that (4.48) holds. Then, by (4.58) and (4.48) it follows that, for $\theta = \alpha(N+2)/2(N+1)$,

$$|||\nabla u - \nabla v|||_{L^{\theta,\infty}(Q_{t_1})} \leqslant 0.$$

Arguing as in theorem 3.2, we conclude that $u = v$ almost everywhere in Q_T. □

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