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D-SIMPLE RINGS AND PRINCIPAL MAXIMAL IDEALS OF THE WEYL ALGEBRA

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Abstract. We prove that if the order-one differential operator $S = \partial_1 + \sum_{i=2}^{n} \beta_i\partial_i + \gamma$, with $\beta_i, \gamma \in K[x_1, \ldots, x_n]$, generates a maximal left ideal of the Weyl algebra $A_n(K)$, then S does not admit any Darboux differential operator in $K[x_1, \ldots, x_n]\langle\partial_2, \ldots, \partial_n\rangle$; hence in particular, the derivation $\partial_1 + \sum_{i=2}^{n} \beta_i\partial_i$ does not admit any Darboux polynomial in $K[x_1, \ldots, x_n]$. We show that the converse is true when $\beta_i \in K[x_1, x_i]$, for every $i = 2, \ldots, n$. Then, we generalize to $K[x_1, \ldots, x_n]$ the classical result of Shamsuddin that characterizes the simple linear derivations of $K[x_1, x_2]$. Finally, we establish a criterion for the left ideal generated by S in $A_n(K)$ to be maximal in terms of the existence of polynomial solutions of a finite system of differential polynomial equations.

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1. Introduction. Let $A_n(K) = K[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$ be the n-th Weyl algebra over a field K of characteristic zero (here ∂_n denotes the usual derivation $\frac{\partial}{\partial x_n}$). Lately, there has been a lot of research done on the principal maximal left, or right, ideals of $A_n(K)$. (Recall that if τ is the standard involution of $A_n(K)$ and $SA_n(K)$ is a principal maximal right ideal, then $A_n(K)\tau(S)$ is a principal maximal left ideal of $A_n(K)$. Therefore, finding principal maximal right ideals of $A_n(K)$ is the same as finding principal maximal left ideals of $A_n(K)$) .

The first author to address this problem was Stafford who exhibited a family of principal maximal right ideals of $A_n(\mathbb{C})$. In this way he gave the first counterexamples

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to the conjecture that every simple module over $A_n(\mathbb{C})$ should be holonomic. (Since $\frac{A_n(\mathbb{C})}{SA_n(\mathbb{C})}$ is simple but not holonomic if $n \geq 2$). (See [**11**]). Later on, Berstein and Lunts proved that, in a certain sense, the generic operator of $A_n(\mathbb{C})$ generates a maximal left ideal (see [**1**] and [**8**]). Nevertheless, the examples discovered by Stafford were of a different kind of the generic ones of Berstein and Lunts.

Stafford's examples were generalized by Coutinho in [**3**]. Starting with a certain type of simple derivation d of $K[x_1, \ldots, x_n]$, he was able to give a suitable perturbation of d, say $d + \gamma$, $\gamma \in K[x_1, \ldots, x_n]$, such that the left ideal $A_n(K)(d + \gamma)$ is maximal. Here, for the first time, we see that there might be a connection between d-simplicity of $K[x_1, \ldots, x_n]$ and maximality of the left ideal of $A_n(K)$ generated by $d + \gamma$.

One of the objectives of this paper is to address the following question:

QUESTION. Let $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ be a derivation of $K[x_1, \ldots, x_n]$ with $\alpha_i \in K[x_1, \ldots, x_n]$ for every $i = 2, \ldots, n$. Suppose that there exists an element $\gamma \in K[x_1, \ldots, x_n]$ such that the left ideal $A_n(K)(d + \gamma)$ is maximal. Then, is d a simple derivation of $K[x_1, \ldots, x_n]$?

In Section 3, we obtain a positive answer to this question for the class of derivations that satisfy the following extra condition:

$$(*) \qquad \alpha_i \in K[x_1, \ldots, x_i], \text{ for every } i = 2, \ldots, n.$$

This is obtained as a consequence of two results that are interesting in their own right. The first one is that, even when the extra condition $(*)$ is not satisfied, the derivation d does not admit any Darboux polynomial in $K[x_1, \ldots, x_n]$. The second one is a general result on derivations: if $d = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n$ is a derivation of $K[x_1, \ldots, x_n]$ that satisfies condition $(*)$, then d is a simple derivation if (and only if) d does not admit any Darboux polynomial in $K[x_1, \ldots, x_n]$. For the latter result, an example due to Goodearl and Warfield shows that the condition $(*)$ is not superfluous.

Another objective also treated in section 3 is to generalize to $K[x_1, \ldots, x_n]$, for a certain family of derivations (which we call Shamsuddin derivations), the result of Shamsuddin that characterizes the simple linear derivations of $K[x_1, x_2]$ in terms of the existence of a polynomial solution for a certain finite system of differential polynomial equations. We use our criterion to exibit new examples of simple derivations of $K[x_1, \ldots, x_n]$.

In Section 4, for a Shamsuddin derivation $d = \partial_1 + \sum_{i=2}^{n}(a_ix_i + b_i)\partial_i$, with $a_i, b_i \in K[x_1]$ for $i = 2, \ldots, n$ and satisfying the condition $a_i \neq a_j$ for every $i \neq j$, we establish a criterion for the left ideal generated by $d + \gamma$ in $A_n(K)$ to be maximal in terms of the existence of polynomial solutions of a finite system of differential polynomial equations. This generalizes and strengthens a result of Bratti and Takagi for $A_2(d + \gamma)$ (see [**2**]). We give an example to show that the condition $a_i \neq a_j$ for every $i \neq j$ is not superfluous.

In section 2, we prove a general theorem, part of which is needed to obtain the results of section 3. We prove that if the order-one differential operator $S = \partial_1 + \sum_{i=2}^{n}\beta_i\partial_i + \gamma$, with $\beta_i, \gamma \in K[x_1, \ldots, x_n]$, generates a maximal left ideal of the Weyl algebra $A_n(K)$, then S does not admit any Darboux differential operator in $K[x_1, \ldots, x_n]\langle\partial_2, \ldots, \partial_n\rangle$. We show that the converse is true when $\beta_i \in K[x_1, x_i]$ for every $i = 2, \ldots, n$.

Throughout this paper, K will be a field of characteristic zero and $x_1, \ldots, x_n$ some indeterminates over K.

If d is a derivation of a ring B, an ideal I of B is said to be a *d-ideal* if $d(I) \subseteq I$. The ring B is said to be *d-simple* if its only d-ideals are (0) and (1); we shall also say that d is a *simple derivation* of B. A derivation d of $K[x_1, \ldots, x_n]$ is said to be a *Shamsuddin derivation* if $d = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n$ where $\alpha_i = a_ix_i + b_i$, with $a_i, b_i \in K[x_1]$ for every $i = 2, \ldots, n$.

If S is an operator in $A_n(K)$, an element $R \in K[x_1, \ldots, x_n]\langle\partial_2, \ldots, \partial_n\rangle \setminus K$ is called a *Darboux operator of S in* $K[x_1, \ldots, x_n]\langle\partial_2, \ldots, \partial_n\rangle$ if

$$[S, R] \in K[x_1, \ldots, x_n]R.$$

In particular, if $S = d$ is a derivation of $K[x_1, \ldots, x_n]$ and $R = f$ is a polynomial in $K[x_1, \ldots, x_n] \setminus K$, we say that f is a *Darboux polynomial of d* if

$$[d, f] = d(f) \in K[x_1, \ldots, x_n]f.$$

Equivalently, f is a Darboux polynomial of d if (f) is a proper non-zero d-ideal of $K[x_1, \ldots, x_n]$.

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2. Principal maximal left ideals and darboux differential operators. Let K be a field of characteristic zero and let $A_n = A_n(K) = K[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$ be the Weyl algebra in n variables over the field K. Recall that $A_n(K)$ has generators ∂_i, x_j, for $1 \leq i, j \leq n$, satisfying the relations $[\partial_i, x_j] := \partial_ix_j - x_j\partial_i = \delta_{ij}$ and other commutators being zero.

Let A_{n-1} be the K-subalgebra of $A_n(K)$ generated by x_i and ∂_i, for $2 \leq i \leq n$. Then,

$$A_{n-1}[x_1] = K[x_1, \ldots, x_n]\langle\partial_2, \ldots, \partial_n\rangle.$$

DEFINITION 2.1. A *multi-index* α is an element of $\mathbb{N}^n$, say $\alpha = (\alpha_1, \ldots, \alpha_n)$. By ∂^α, we mean the monomial ${\partial_1}^{\alpha_1} \ldots {\partial_n}^{\alpha_n}$. The *order* of this monomial is the length $|\alpha|$ of the multi-index α; namely $|\alpha| = \alpha_1 + \ldots + \alpha_n$. An element $d \in A_n(K)$ may be written uniquely in the form $d = \sum_\alpha q_\alpha\partial^\alpha$, where $q_\alpha \in K[x_1, \ldots, x_n]$. The *order* of d, denoted by *ord*(d), is the largest $|\alpha|$ for which $q_\alpha \neq 0$. We use the convention that the zero element has order $-\infty$. An example will suffice: the order of ${x_1}^3\partial_2 + {x_1}^7x_2{\partial_1}^3{\partial_2}^2$ is equal to 5.

We begin with some technical lemmas that will prepare for the proof of Theorem 2.8.

LEMMA 2.2. *Let $S = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n + \gamma$ be an element in A_n, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. If $R \in A_{n-1}[x_1]$, then $[S, R] \in A_{n-1}[x_1]$. In particular, $[\partial_1, R] \in A_{n-1}[x_1]$.*

Proof. This is a straightforward computation. □

LEMMA 2.3. *(Division algorithm). Let $S = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n + \gamma$ be an element in A_n, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. Given $P \in A_n$, we have $P = QS + R$, for some $Q \in A_n$ and $R \in A_{n-1}[x_1]$. Moreover, R and Q are uniquely determined.*

Proof. We will first prove, by induction on n, that $\partial_1^n = QS + R$, for some $Q \in A_n$ and $R \in A_{n-1}[x_1]$.

Note that $\partial_1 = 1S + R$, where $R = -\alpha_2\partial_2 - \cdots - \alpha_n\partial_n - \gamma \in A_{n-1}[x_1]$. Suppose now that the result is true for n. Then

$$\partial_1^{n+1} = \partial_1\partial_1^n = \partial_1(AS + B) = \partial_1 AS + \partial_1 B,\ A \in A_n, B \in A_{n-1}[x_1].$$

By lemma 2.2, we have

$$\partial_1^{n+1} = \partial_1 AS + B\partial_1 + \widetilde{B},\ \widetilde{B} \in A_{n-1}[x_1].$$

Since $\partial_1 = S + R$,

$$\begin{aligned}\partial_1^{n+1} &= \partial_1 AS + B(S + R) + \widetilde{B}\\ &= (\partial_1 A + B)S + BR + \widetilde{B}\\ &= Q'S + R', \quad \text{where } Q' \in A_n \text{ and } R' = BR + \widetilde{B} \in A_{n-1}[x_1].\end{aligned}$$

This completes the induction.

Now, if $P \in A_n$, we can write P in the form $P = E_n\partial_1^n + \cdots + E_1\partial_1 + E_0$, where $E_i \in A_{n-1}[x_1]$.

Thus, $P = E_n(H_nS + B_n) + \cdots + E_1(H_1S + B_1) + E_0$, where $H_1, \ldots, H_n \in A_n$ and $B_1, \ldots B_n \in A_{n-1}[x_1]$. Then,

$$P = (E_nH_n + \cdots + E_1H_1)S + \underbrace{(E_nB_n + \cdots + E_1B_1 + E_0)}_{\in A_{n-1}[x_1]} = QS + R.$$

We claim that R is unique. In fact, let $R, R' \in A_{n-1}[x_1]$ be such that $P = QS + R = Q'S + R'$. So, $R - R' = aS$, for some $a \in A_n$. Writing $a = Q\partial_1 + A$ and $S = \partial_1 + B$, where $Q \in A_n$, $A, B \in A_{n-1}[x_1]$, we have

$$aS = Q{\partial_1}^2 + Q\partial_1 B + A\partial_1 + AB.$$

Since $R, R' \in A_{n-1}[x_1]$, so does aS. Hence, looking at Q as a polynomial in ∂_1 with coefficients in $A_{n-1}[x_1]$ we conclude that $Q = 0$. So, $aS = A\partial_1 + AB$. In the same way, we conclude that $A = 0$. Then $a = 0$, $R = R'$ and $Q = Q'$. □

LEMMA 2.4. *Let A_nS be a principal left ideal of A_n, where $S = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n + \gamma \in A_n$, $\alpha_2, \ldots, \alpha_n$ and $\gamma \in K[x_1, \ldots, x_n]$. Then A_nS is a maximal left ideal of A_n if and only if $A_nS + A_nR = A_n$, for every $R \in A_{n-1}[x_1] \setminus \{0\}$.*

Proof. ($\Rightarrow$) If $R \in A_{n-1}[x_1] \setminus \{0\}$, then $R \notin A_nS$. So, $A_nS + A_nR = A_n$, because A_nS is a maximal left ideal of A_n.

($\Leftarrow$) Of course, A_nS is a maximal left ideal of A_n if and only if $A_nS + A_nP = A_n$, for all $P \notin A_nS$. Now, for $P \notin A_nS$, by lemma 2.3, we have that $P = QS + R$, for some $Q \in A_n$ and $R \in A_{n-1}[x_1]$, $R \neq 0$. Thus, $A_nS + A_nP = A_nS + A_n(QS + R) = A_nS + A_nR = A_n$, by hypothesis. □

LEMMA 2.5. *Let $S = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n + \gamma$ be an element in A_n, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. Then, A_n is a free $A_{n-1}[x_1]$-module with basis $\{1, S, S^2, \ldots\}$.*

Proof. It is known that A_n is a free $A_{n-1}[x_1]$-module with basis $\{1, \partial_1, \partial_1^2, \ldots\}$. Writing $\partial_1 = S - R$ with $R = \alpha_2\partial_2 + \cdots + \alpha_n\partial_n + \gamma \in A_{n-1}[x_1]$ and using lemma 2.2, we see that A_n is generated by $\{1, S, S^2, \ldots\}$ over $A_{n-1}[x_1]$.

Suppose that $r \geq 0$ and $B_0 + B_1 S + \cdots + B_r S^r = 0$ with $B_i \in A_{n-1}[x_1]$ for every $i = 0, \ldots, r$. Substituting S by $\partial_1 + R$ and using lemma 2.2 we have an expression $\widetilde{B}_0 + \widetilde{B}_1 \partial_1 + \cdots + \widetilde{B}_r \partial_1^r = 0$ with $\widetilde{B}_i \in A_{n-1}[x_1]$ for every $i = 1, \ldots, r-1$ and $\widetilde{B}_r = B_r$. Therefore, $B_r = \widetilde{B}_r = 0$. The proof follows by induction on r. □

LEMMA 2.6. *Let $S \in A_n$ with ord$(S) = 1$ and $R \in A_{n-1}[x_1]$ with ord$(R) > 0$. Suppose that $\mu[S, R] = \eta R$ for some $\mu \in K[x_1, \ldots, x_n] \setminus \{0\}$, $\eta \in K[x_1, \ldots, x_n]$. Then, there exists $\widetilde{R} \in A_{n-1}[x_1]$, with ord$(\widetilde{R}) =$ ord(R) and $\widetilde{\eta} \in K[x_1, \ldots, x_n]$ such that $[S, \widetilde{R}] = \widetilde{\eta}\widetilde{R}$.*

Proof. We can write R in the form

$$\sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n} {\partial_2}^{i_2} \ldots {\partial_n}^{i_n}, \text{ where } P_{i_2,\ldots,i_n} \in K[x_1, \ldots, x_n].$$

Let $R = \alpha_0 \widetilde{R}$ where $\alpha_0 \in K[x_1, \ldots, x_n] \setminus \{0\}$ is the greatest common divisor of the elements $P_{i_2,\ldots,i_n}$. By hypothesis we have

$$\mu[S, \alpha_0 \widetilde{R}] = \eta \alpha_0 \widetilde{R}. \tag{1}$$

Since $\mu \in K[x_1, \ldots, x_n]$, by (1), μ divides $\eta\alpha_0$, say $\eta\alpha_0 = \mu\zeta$ for some $\zeta \in K[x_1, \ldots, x_n]$. Since $\mu \neq 0$, $[S, \alpha_0 \widetilde{R}] = \zeta \widetilde{R}$.

But, $[S, \alpha_0 \widetilde{R}] = [S, \alpha_0]\widetilde{R} + \alpha_0 [S, \widetilde{R}]$. Then,

$$\alpha_0 [S, \widetilde{R}] = \underbrace{(\zeta - [S, \alpha_0])}_{\lambda \in K[x_1,\ldots,x_n]} \widetilde{R} = \lambda \widetilde{R}, \text{ where } \lambda \in K[x_1, \ldots, x_n]. \tag{2}$$

It follows from (2), that α_0 divides λ, say $\lambda = \widetilde{\eta}\alpha_0$, for some $\widetilde{\eta} \in K[x_1, \ldots, x_n]$. Since $\alpha_0 \neq 0$, $[S, \widetilde{R}] = \widetilde{\eta}\widetilde{R}$. □

LEMMA 2.7. *Let $S \in A_n$ with ord$(S) = 1$ and $P \in K[x_1, \ldots, x_n] \setminus K[x_1, \ldots, x_{n-1}]$. Suppose that $\mu[S, P] = \eta P$ for some $\mu \in K[x_1, \ldots, x_{n-1}] \setminus \{0\}$ and $\eta \in K[x_1, \ldots, x_n]$. Then, there exists $\widetilde{P} \in K[x_1, \ldots, x_n] \setminus K[x_1, \ldots, x_{n-1}]$ and $\widetilde{\eta} \in K[x_1, \ldots, x_n]$ such that $[S, \widetilde{P}] = \widetilde{\eta}\widetilde{P}$.*

Proof. We can write P in the form

$$\sum_{i=0}^{N} P_i x_n^i, \text{ where } P_i \in K[x_1, \ldots, x_{n-1}] \text{ for every } i.$$

Let $P = \alpha_0 \widetilde{P}$ where $\alpha_0 \in K[x_1, \ldots, x_{n-1}] \setminus \{0\}$ is the greatest common divisor of the elements P_i. By hypothesis we have

$$\mu[S, \alpha_0 \widetilde{P}] = \eta \alpha_0 \widetilde{P}. \tag{3}$$

Since $\mu \in K[x_1, \ldots, x_{n-1}]$, by (3), μ divides $\eta\alpha_0$, say $\eta\alpha_0 = \mu\zeta$ for some $\zeta \in K[x_1, \ldots, x_n]$. Since $\mu \neq 0$, $[S, \alpha_0 \widetilde{P}] = \zeta \widetilde{P}$.

But, $[S, \alpha_0 \widetilde{P}] = [S, \alpha_0]\widetilde{P} + \alpha_0 [S, \widetilde{P}]$, hence $\alpha_0 [S, \widetilde{P}] = \lambda \widetilde{P}$ where $\lambda := \zeta - [S, \alpha_0] \in K[x_1, \ldots, x_n]$. It follows that α_0 divides λ, say $\lambda = \widetilde{\eta}\alpha_0$, for some $\widetilde{\eta} \in K[x_1, \ldots, x_n]$. Since $\alpha_0 \neq 0$, $[S, \widetilde{P}] = \widetilde{\eta}\widetilde{P}$. □

We can now state the main result of this section.

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THEOREM 2.8. *Let* $S = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n + \gamma$ *be in* A_n*, where* $\alpha_2, \ldots, \alpha_n$ *and* $\gamma \in K[x_1, \ldots, x_n]$.

(a) If A_nS *is a maximal left ideal of* A_n*, then* S *has no Darboux operator in* $A_{n-1}[x_1]$.

(b) Reciprocally, if $\alpha_2 \in K[x_1, x_2], \ldots, \alpha_n \in K[x_1, x_n]$ *and* S *has no Darboux operator in* $A_{n-1}[x_1]$*, then* A_nS *is a maximal left ideal of* A_n.

(c) (*Bratti and Takagi,* [**2**, *Theorem 2.2*]) *If* $n = 2$*, then* A_2S *is a maximal left ideal of* A_2 *if and only if* S *has no Darboux operator in* $A_1[x_1]$.

Proof. (a) Let $R \in A_{n-1}[x_1]$. Of course $R \notin A_nS$ and, since A_nS is maximal, there exists $\lambda, \mu \in A_n$ such that $\lambda S + \mu R = 1$. If $ord_{\partial_1}(\lambda) = m$, then $ord_{\partial_1}(\mu) = m + 1$.

By lemma 2.5 we can write λ and μ in the form:

$$\lambda = B_mS^m + \cdots + B_1S + B_0,$$
$$\mu = C_{m+1}S^{m+1} + \cdots + C_1S + C_0,$$

where $B_i, C_j \in A_{n-1}[x_1]$.

So,

$$1 = \lambda S + \mu R = \underbrace{\sum_{k=0}^{m} B_kS^{k+1} + \sum_{k=0}^{m+1} C_kS^kR}_{(*)}$$

Suppose that R is a Darboux operator for S in $A_{n-1}[x_1]$, that is $R \in A_{n-1}[x_1] \setminus K$ and $[S, R] = \eta R$, for some $\eta \in K[x_1, \ldots, x_n]$. Then we have

$$S^{m+1}R = RS^{m+1} + (\xi_mS^m + \xi_{m-1}S^{m-1} + \cdots + \xi_1S + \xi_0)R, \tag{4}$$

with $\xi_j \in K[x_1, \ldots, x_n],\ 0 \le j \le m$. So, the coefficient of S^{m+1} in $(*)$ is

$$B_m + C_{m+1}R.$$

It follows from lemma 2.5 that

$$B_m + C_{m+1}R = 0. \tag{5}$$

Thus,

$$\begin{aligned}
\lambda S + \mu R &= \sum_{k=0}^{m} B_kS^{k+1} + \sum_{k=0}^{m+1} C_kS^kR \\
&= \sum_{k=0}^{m-1} B_kS^{k+1} + \sum_{k=0}^{m} C_kS^kR + B_mS^{m+1} + C_{m+1}S^{m+1}R \\
&\overset{(5)}{=} \sum_{k=0}^{m-1} B_kS^{k+1} + \sum_{k=0}^{m} C_kS^kR - C_{m+1}RS^{m+1} + C_{m+1}S^{m+1}R \\
&= \sum_{k=0}^{m-1} B_kS^{k+1} + \sum_{k=0}^{m} C_kS^kR - C_{m+1}(RS^{m+1} - S^{m+1}R). \qquad (6)
\end{aligned}$$

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Using (4), we can rewrite (6) and obtain:

$$1 = \lambda S + \mu R = \sum_{k=0}^{m-1} B_k S^{k+1} + \sum_{k=0}^{m} \widetilde{C}_k S^k R,$$

for some $\widetilde{C}_k \in A_{n-1}[x_1]$.

This expression has the same form as $(*)$, but it involves only the powers S^i with $i \leq m$.

Repeating the argument m more times, we obtain

$$1 = \lambda S + \mu R = D_0 R,$$

for some $D_0 \in A_{n-1}[x_1]$. Then, R is a unit of the Weyl algebra, hence $R \in K$, a contradiction.

(b) Let R be in $A_{n-1}[x_1]$, such that $\operatorname{ord}(R) = N > 0$. We can write R in the form:

$$R = \sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n} \partial_2{}^{i_2} \ldots \partial_n{}^{i_n}, \text{ where } P_{i_2,\ldots,i_n} \in K[x_1, \ldots, x_n].$$

Since $\operatorname{ord}([S, R]) \leq \operatorname{ord}(S) + \operatorname{ord}(R) - 1$, then $\operatorname{ord}([S, R]) \leq N$. Therefore we can also write $[S, R]$ in the form

$$[S, R] = \sum_{i_2+\ldots+i_n=0}^{N} Q_{i_2,\ldots,i_n} \partial_2{}^{i_2} \ldots \partial_n{}^{i_n}, \text{ where } Q_{i_2,\ldots,i_n} \in K[x_1, \ldots, x_n].$$

As $\operatorname{ord}(R) = N$, there exists $P_{i_{2_0},\ldots,i_{n_0}} \neq 0$, such that

$$i_{2_0} + \cdots + i_{n_0} = N.$$

By hypothesis and by lemma 2.6, we have that

$$\widetilde{R} := P_{i_{2_0},\ldots,i_{n_0}} [S, R] - Q_{i_{2_0},\ldots,i_{n_0}} R \neq 0.$$

Note that, from this equation, we have $0 \leq \operatorname{ord} \widetilde{R}$. Moreover, $\widetilde{R} \in A_n S + A_n R$ and the term of order N involving $\partial_2{}^{i_{2_0}} \ldots \partial_n{}^{i_{n_0}}$ does not appear in $\widetilde{R}$.

CLAIM 2.9. *The multi-indices of maximal length that occur in* $[S, R]$ *already occur in R.*

Let's assume, for a while, that claim 2.9 is true. Then, $\widetilde{R}$ has one term less than R of order N. If $\widetilde{R}$ has another term with order N, we can repeat the process and eliminate it too. Therefore, after a finite number of steps, we have a new $\widetilde{R} \in (A_n S + A_n R) \setminus \{0\}$, with $0 \leq \operatorname{ord}(\widetilde{R}) \leq N - 1$. Proceeding in this way, we obtain

$$(A_n S + A_n R) \cap (K[x_1, \ldots, x_n] \setminus \{0\}) \neq \emptyset.$$

Let $P = \sum_{k=0}^{m} r_k x_n{}^k$, where $r_k \in K[x_1, \ldots, x_{n-1}]$, be a polynomial contained in $(A_n S + A_n R) \cap (K[x_1, \ldots, x_n] \setminus \{0\})$ with the least degree in x_n. If m were strictly greater than 0, then, by the Euclidean Algorithm (applied to $[S, P]$ and P considered as elements in $K(x_1, \ldots, x_{n-1})[x_n]$), there would exist $d \in K[x_1, \ldots, x_{n-1}] \setminus \{0\}$ such that

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$d[S,P] = \eta P + r$, for some $\eta, r \in K[x_1,\ldots,x_n]$ where $deg_{x_n}(r) < deg_{x_n}(P)$ or $r = 0$. This would imply that $r = 0$, by the choice of P, hence that $d[S,P] = \eta P$, which, by lemma 2.7, would lead to a contradiction with the hypothesis. So $m = 0$ and $(A_nS + A_nR) \cap (K[x_1,\ldots,x_{n-1}] \setminus \{0\}) \neq \emptyset$.

Proceeding in this way, we obtain

$$(A_nS + A_nR) \cap (K[x_1] \setminus \{0\}) \neq \emptyset.$$

Let $P = a_l{x_1}^l + \cdots + a_0$, with $a_i \in K$, be in $(A_nS + A_nR) \cap (K[x_1] \setminus \{0\})$, $a_l \neq 0$. If $l = 0$, then $P \in K \setminus \{0\}$ and therefore $A_nS + A_nR = A_n$. If $l > 0$, we have

$$[S,P] = \partial_1(P) = la_l{x_1}^{l-1} + \cdots + a_1 \in (A_nS + A_nR) \cap (K[x_1] \setminus \{0\}).$$

Repeating this process l times, we have that $l!a_l \in (A_nS + A_nR) \cap (K \setminus \{0\})$. Then $A_nS + A_nR = A_n$. By lemma 2.4, it follows that A_nS is a maximal left ideal of A_n.

To finish the proof, we have to show claim 2.9.

Proof of claim 2.9: Let us suppose that

$$S = d + \gamma \in A_n,$$

where $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ is a derivation of $K[x_1,\ldots,x_n]$ such that $\alpha_i \in K[x_1,x_i]$, $\gamma \in K[x_1,\ldots,x_n]$. Let $R = \sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}$, where $P_{i_2,\ldots,i_n} \in K[x_1,\ldots,x_n]$.

Then,

$$[S,R] = \left[S, \sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}\right] = \sum_{i_2+\ldots+i_n=0}^{N}[S, P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}].$$

Note that: $[\partial_1, P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}] = \partial_1(P_{i_2,\ldots,i_n}){\partial_2}^{i_2}\ldots{\partial_n}^{i_n}$, and

$$\begin{aligned}
[\alpha_2\partial_2, P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}] &= \alpha_2P_{i_2,\ldots,i_n}{\partial_2}^{i_2+1}{\partial_3}^{i_3}\ldots{\partial_n}^{i_n} + \alpha_2\partial_2(P_{i_2,\ldots,i_n}){\partial_2}^{i_2}\ldots{\partial_n}^{i_n}\\
&\quad -P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}\alpha_2\partial_2\\
&= \alpha_2P_{i_2,\ldots,i_n}{\partial_2}^{i_2+1}{\partial_3}^{i_3}\ldots{\partial_n}^{i_n} + \alpha_2\partial_2(P_{i_2,\ldots,i_n}){\partial_2}^{i_2}\ldots{\partial_n}^{i_n} - P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\alpha_2\partial_2{\partial_3}^{i_3}\ldots{\partial_n}^{i_n}\\
&= \alpha_2P_{i_2,\ldots,i_n}{\partial_2}^{i_2+1}{\partial_3}^{i_3}\ldots{\partial_n}^{i_n} + \alpha_2\partial_2(P_{i_2,\ldots,i_n}){\partial_2}^{i_2}\ldots{\partial_n}^{i_n} - P_{i_2,\ldots,i_n}(\alpha_2{\partial_2}^{i_2+1}{\partial_3}^{i_3}\ldots{\partial_n}^{i_n}\\
&\quad +i_2\partial_2(\alpha_2){\partial_2}^{i_2}\ldots{\partial_n}^{i_n} + \text{ terms with lower order})\\
&= (\alpha_2\partial_2(P_{i_2,\ldots,i_n}) - i_2P_{i_2,\ldots,i_n}\partial_2(\alpha_2)){\partial_2}^{i_2}\ldots{\partial_n}^{i_n} + \text{ terms with lower order}
\end{aligned}$$

Hence, the terms with order N in $\sum_{i_2+\ldots+i_n=0}^{N}[\alpha_2\partial_2, P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}]$ are:

$$\sum_{i_2+\ldots i_n=N}(\alpha_2\partial_2(P_{i_2,\ldots,i_n}) - i_2P_{i_2,\ldots,i_n}\partial_2(\alpha_2)){\partial_2}^{i_2}\ldots{\partial_n}^{i_n}.$$

Similarly, the terms with order N in $\sum_{i_2+\ldots+i_n=0}^{N}[\alpha_j\partial_j, P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}]$, where $j = 2,\ldots,n$, are:

$$\sum_{i_2+\ldots i_n=N}(\alpha_j\partial_j(P_{i_2,\ldots,i_n}) - i_jP_{i_2,\ldots,i_n}\partial_j(\alpha_j)){\partial_2}^{i_2}\ldots{\partial_n}^{i_n}.$$

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Note that $\mathrm{ord}([\gamma, R]) \leq N-1$. Then the terms with order N in $[S, R]$ are:

$$\sum_{i_2+\ldots+i_n=N} \Big(d(P_{i_2,\ldots,i_n}) - \sum_{j=2}^{n} i_j P_{i_2,\ldots,i_n} \partial_j(\alpha_j)\Big) {\partial_2}^{i_2} \ldots {\partial_n}^{i_n}. \tag{7}$$

Now, observe that if $P_{i_2,\ldots,i_n}$ is the coefficient of ${\partial_2}^{i_2} \ldots {\partial_n}^{i_n}$ in R, with $i_2 + \ldots + i_n = N$, then the corresponding coefficient in $[S, R]$ is:

$$Q_{i_2,\ldots,i_n} := d(P_{i_2,\ldots,i_n}) - \sum_{j=2}^{n} i_j P_{i_2,\ldots,i_n} \partial_j(\alpha_j).$$

Therefore, if $P_{i_2,\ldots,i_n} = 0$, then $Q_{i_2,\ldots,i_n} = 0$ and the coefficient of ${\partial_2}^{i_2} \ldots {\partial_n}^{i_n}$ in $[S, R]$ is zero. □

3. *d*-simplicity of the ring $K[x_1, \ldots, x_n]$. In this section we study the d-simplicity of the ring $K[x_1, \ldots, x_n]$. Evidently, if $K[x_1, \ldots, x_n]$ is d-simple, there is no non-trivial principal d-ideal (equivalently, there is no Darboux polynomial). If $d(x_1) = 1$, the converse is true when $n = 2$ but is false already when $n = 3$. Indeed, if K is a formally real field (for example if $K = \mathbb{R}$), Goodearl and Warfield observed in ([**6**, p. 61]) that in $K[x_2, x_3]$, the derivation $\delta := (x_2 + x_3)\partial_2 + (x_2^2 + x_3^2)\partial_3$ has no Darboux polynomial even though (x_2, x_3) is a (unique) δ-ideal; then, by a rather straightforward computation, one can see that in $K[x_1, x_2, x_3]$, the derivation $d := \partial_1 + (x_2 + x_3)\partial_2 + (x_2^2 + x_3^2)\partial_3$ has no Darboux polynomial even though $(x_2, x_3)K[x_1, x_2, x_3]$ is a (unique) d-ideal. Our next theorem gives a rather general situation where the converse is true; it points out that the peculiarity of the above example would not have occurred if the coefficient of ∂_2 had been an element of $K[x_1, x_2]$. It generalizes [**9**, Proposition 2.1].

THEOREM 3.1. *Let $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ be a derivation of $K[x_1, \ldots, x_n]$ where $\alpha_i \in K[x_1, \ldots, x_i]$ for every $i = 2, \ldots, n$. Then the following statements are equivalent:*
(i) $K[x_1, \ldots, x_n]$ is d-simple.
(ii) d has no Darboux polynomial.

Proof. It is enough to show that (ii) ⇒ (i). Suppose that $K[x_1, \ldots, x_n]$ is not d-simple. Let I be a proper non-zero d-ideal of $K[x_1, \ldots, x_n]$. Let $P = \sum_{k=0}^{l} r_k {x_n}^k$, where $r_k \in K[x_1, \ldots, x_{n-1}]$, be a non-zero polynomial contained in I with the least degree in x_n.

Suppose that $l > 0$. By the usual Euclidean Algorithm (applied to $d(P)$ and P considered as elements in $K(x_1, \ldots, x_{n-1})[x_n]$), there exists $g \in K[x_1, \ldots, x_{n-1}] \setminus \{0\}$ such that $gd(P) = hP + r$, for some $h, r \in K[x_1, \ldots, x_n]$, where $deg_{x_n}(r) < deg_{x_n}(P)$ or $r = 0$. This implies that $r = 0$, by the choice of P. Thus, $gd(P) = hP$. Since $d(P) = [d, P]$, then by lemma 2.7, there exist $\widetilde{h} \in K[x_1, \ldots, x_n]$ and $\widetilde{P} \in K[x_1, \ldots, x_n] \setminus K[x_1, \ldots, x_{n-1}]$ such that $[d, \widetilde{P}] = \widetilde{h}\widetilde{P}$. As $d(\widetilde{P}) = [d, \widetilde{P}]$, $\widetilde{P}$ is a Darboux polynomial of d, a contradiction to the hypothesis.

Thus $l = 0$ and $I \cap K[x_1, \ldots, x_{n-1}] \neq (0)$. Note that $d\,|_{K[x_1,\ldots,x_{n-1}]} (K[x_1, \ldots, x_{n-1}]) \subseteq K[x_1, \ldots, x_{n-1}]$, since $\alpha_i \in K[x_1, \ldots, x_i]$. Then we can repeat the argument.

Going on this way, we obtain that $I \cap K[x_1] \neq (0)$. But this is impossible since d restricted to $K[x_1]$ is ∂_1 and ∂_1 is a simple derivation of $K[x_1]$. □

COROLLARY 3.2. *Let $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ be a derivation of $K[x_1, \ldots, x_n]$ where $\alpha_i \in K[x_1, \ldots, x_i]$ for every $i = 2, \ldots, n$. Then, $K[x_1, \ldots, x_n]$ is d-simple if and only if no prime ideal of height one is a d-ideal.*

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Proof. By Theorem 3.1, $K[x_1, \ldots, x_n]$ is d-simple if and only if no non-zero proper principal ideal is a d-ideal. But if an ideal I is a d-ideal, then every minimal prime of I is also a d-ideal. By Krull's Principal Ideal Theorem, every minimal prime ideal of a principal ideal has height one. □

COROLLARY 3.3. *Let $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ be a derivation of $K[x_1, \ldots, x_n]$, with $\alpha_i \in K[x_1, \ldots, x_n]$ for every $i = 2, \ldots, n$. Suppose that there exists $\gamma \in K[x_1, \ldots, x_n]$ such that $A_n(d + \gamma)$ is a maximal left ideal of A_n. Then,*

(a) d has no Darboux polynomial.
(b) d is a simple derivation if $\alpha_i \in K[x_1, \ldots, x_i]$, for every $i = 2, \ldots, n$.

Proof. (a): Suppose that $A_n(d + \gamma)$ is a maximal left ideal of A_n. By Theorem 2.8(a), we have

$$[d + \gamma, R] \notin K[x_1, \ldots, x_n]R, \ \forall\, R \in A_{n-1}[x_1] \setminus K.$$

So, in particular

$$[d + \gamma, P] \notin K[x_1, \ldots, x_n]P, \ \forall\, P \in K[x_1, \ldots, x_n] \setminus K.$$

Since $[d + \gamma, P] = [d, P] = d(P)$, d has no Darboux polynomial.

(b): It follows from item (a) and Theorem 3.1 . □

Examples of simple derivations of the polynomial ring $K[x_1, \ldots, x_n]$ are not easy to find. A family of linear simple derivations was discovered by Coutinho in [**3**] (generalizing an example of Stafford) and is based on a result of Shamsuddin (see [**10**]). Families of simple quadratic derivations of $K[x_1, x_2]$ were found by Maciejewski, Moulin-Ollagnier and Nowicki in [**9**].

EXAMPLE 3.4. Let $K \supsetneq \mathbb{Q}$ be a field and S be the element in $A_2(K)$, given by

$$S = \partial_1 + (x_1x_2 + \lambda {x_2}^2 + 1)\partial_2 + \lambda\mu x_2, \ \lambda \in K \setminus \mathbb{Q} \text{ and } \mu \notin \mathbb{Z}.$$

Stafford proved in [**11**, Proposition 2.2] that A_2S is a maximal left ideal of $A_2(K)$ (actually, our operator is obtained from Stafford's after a transposition and a change of indices).

Consider the derivation $d = \partial_1 + (x_1x_2 + \lambda {x_2}^2 + 1)\partial_2$ of $K[x_1, x_2]$ extracted from S. By Corollary 3.3, we have that $K[x_1, x_2]$ is d-simple. Note that in this case, we get an example where $K[x_1, x_2]$ is d-simple and d is **not** a Shamsuddin derivation.

The following lemma will be used in the proof of the next theorem.

If $\mathcal{A}$ denotes a commutative domain, let $qf(\mathcal{A})$ denote its field of quotients and let $\mathcal{A}^*$ denote its group of units.

LEMMA 3.5. *Let $\mathcal{A}$ be a K-algebra which is a factorial domain and d a K-derivation of $\mathcal{A}$. Suppose that $\mathcal{A}$ has no non-zero proper principal d-ideals. Given $f, g \in \mathcal{A}$ consider the following differential equation:*

$$d(u) + fu = g. \tag{8}$$

(a) If $u \in qf(\mathcal{A})$ is a solution of (8), then $u \in \mathcal{A}$
(b) If $g = 0$ and $u \in \mathcal{A}$ is a nontrivial solution of (8), then $u \in \mathcal{A}^*$. In particular, if $\mathcal{A}^* = K^*$ and $f \neq 0$, then equation (8) has only the trivial solution.

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Proof. (a) Suppose that $u = \frac{p}{q} \in qf(\mathcal{A})$, with gcd $(p, q) = 1$, is such that

$$d\left(\frac{p}{q}\right) + f\frac{p}{q} = g.$$

Then

$$q(d(p) + fp - gq) = pd(q).$$

As gcd $(p, q) = 1$, there exists $r \in \mathcal{A}$ such that

$$\begin{cases} d(p) + fp - gq = rp \\ d(q) = rq \end{cases}$$

Therefore, $(q) \subset \mathcal{A}$ is a d-ideal of $\mathcal{A}$. Then $(q) = (0)$ or $(q) = (1)$. As $q \neq 0$, it follows that $q \in \mathcal{A}^*$. Hence $u = \frac{p}{q} \in \mathcal{A}$.

(b) Let $p \in \mathcal{A}$ be a solution of $d(u) + fu = 0$. Then (p) is a d-ideal and $p = 0$ or $p \in \mathcal{A}^*$. If $\mathcal{A}^* = K^*$, as d is a K-derivation, we have that $fp = 0$. Then $p = 0$. □

A characterization of the d-simplicity of the ring $K[x_1, x_2]$, where d is a Shamsuddin derivation, is given in [**10**] in terms of the existence of a polynomial solution of a certain ODE. The following theorem generalizes this result for an arbitrary number of variables.

THEOREM 3.6. *Let $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ be a Shamsuddin derivation of $K[x_1, \ldots, x_n]$, where $\alpha_i(x_1, x_i) = a_i(x_1)x_i + b_i(x_1) \in K[x_1, x_i]$, $2 \leq i \leq n$. Suppose that $a_i \neq a_j$, for $2 \leq i < j \leq n$. Then the following statements are equivalent:*

(i) $K[x_1, \ldots, x_n]$ is d-simple.
(ii) $\partial_1(v) \neq a_i \cdot v + b_i$, for every $v \in K(x_1)$, for all $i = 2, \ldots, n$.
(iii) $\partial_1(v) \neq a_i \cdot v + b_i$, for every $v \in K[x_1]$, for all $i = 2, \ldots, n$.
(iv) $K[x_1, x_i]$ is $d|_{K[x_1,x_i]}$-simple, for all $i = 2, \ldots, n$.

Proof. (ii) ⇔ (iii) is given by Lemma 3.9 applied with $\mathcal{A} = K[x_1]$ and $d = \partial_1$.

(iii) ⇔ (iv) is given by Shamsuddin's Theorem ([**3**, Proposition 3.2]).

(i) ⇒ (iv). If I is a non-zero proper $d|_{K[x_1,x_i]}$-ideal of $K[x_1, x_i]$, then $IK[x_1, \ldots, x_n]$ is a non-zero proper d-ideal of $K[x_1, \ldots, x_n]$.

(ii) ⇒ (i). Let I be a non-zero d-ideal of $K[x_1, \ldots, x_n]$ and let $P \in I$, $P \neq 0$. We can suppose that P is not a constant. We write P in the form:

$$P = \sum_{i_2+\cdots+i_n=0}^{N} P_{i_2,\ldots,i_n} {x_2}^{i_2} \cdots {x_n}^{i_n}, \text{ where } P_{i_2,\ldots,i_n} \in K[x_1].$$

Then, a simple calculation gives the following expression for $d(P)$:

$$\begin{aligned} d(P) = \sum_{i_2+\cdots+i_n=0}^{N} &\big\{\big(\partial_1(P_{i_2,\ldots,i_n}) + i_2P_{i_2,\ldots,i_n}a_2 + \cdots + i_nP_{i_2,\ldots,i_n}a_n\big){x_2}^{i_2} \cdots {x_n}^{i_n} + \\ &+ \big(i_2P_{i_2,\ldots,i_n}b_2\big){x_2}^{i_2-1} \cdots {x_n}^{i_n} + \cdots + \big(i_nP_{i_2,\ldots,i_n}b_n\big){x_2}^{i_2} \cdots {x_n}^{i_n-1}\big\} \end{aligned} \tag{9}$$

Let us choose $P \in I$ such that N is minimum. If $N = 0$ then $P \in K[x_1] \setminus K$ and we are done; indeed, if degree $P = r$, then $d^{(r)}(P)$ is a unit that belongs to I.

Suppose that $N > 0$. So, there exists $P_{j_2,\dots,j_n} \neq 0$ for some $j_2 + \cdots + j_n = N$. Without loss of generality we may suppose that $j_2 > 0$. Note that

$$\rho := \partial_1(P_{j_2,\dots,j_n}) + j_2 P_{j_2,\dots,j_n} a_2 + \cdots + j_n P_{j_2,\dots,j_n} a_n$$

is the coefficient of the monomial ${x_2}^{j_2} \cdots {x_n}^{j_n}$ in $d(P)$. We consider

$$P_1 := P_{j_2,\dots,j_n} d(P) - \rho P \in I.$$

Evidently, P_1 has no term in $x_2^{j_2} \cdots x_n^{j_n}$, while the coefficient of the term $x_2^{j_2-1} \cdots x_n^{j_n}$, say $\zeta_{j_2-1,\dots,j_n} \in K[x_1]$, is the following:

$$\zeta_{j_2-1,\dots,j_n} = P^2_{j_2,\dots,j_n}\left(\partial_1\left(\frac{P_{j_2-1,\dots,j_n}}{P_{j_2,\dots,j_n}}\right) - a_2\frac{P_{j_2-1,\dots,j_n}}{P_{j_2,\dots,j_n}} + j_2 b_2 \right.$$
$$\left. + \ (j_3+1)b_3\frac{P_{j_2-1,j_3+1,j_4,\dots,j_n}}{P_{j_2,\dots,j_n}} + \cdots + (j_n+1)b_n\frac{P_{j_2-1,j_3,\dots,j_{n-1},j_n+1}}{P_{j_2,\dots,j_n}}\right). \quad (10)$$

We will analyze two cases.

First case: If $P_{j_2-1,j_3+1,j_4,\dots,j_n} = \cdots = P_{j_2-1,j_3,\dots,j_{n-1},j_n+1} = 0$.

We claim that $P_1 \neq 0$. Indeed, in this case, equation (10) simplifies and the coefficient of the term $x_2^{j_2-1} \cdots x_n^{j_n}$ is

$$\zeta_{j_2-1,\dots,j_n} = P^2_{j_2,\dots,j_n} j_2\left(\partial_1\left(\frac{P_{j_2-1,\dots,j_n}}{j_2 P_{j_2,\dots,j_n}}\right) - a_2\frac{P_{j_2-1,\dots,j_n}}{j_2 P_{j_2,\dots,j_n}} + b_2\right)$$

which is non-zero by hypothesis (ii).

Therefore, the ideal I contains a non-zero element P_1 without the term ${x_2}^{j_2} \cdots {x_n}^{j_n}$.

Second case: If $P_{j_2-1,\dots,j_k+1,\dots,j_n} \neq 0$, for some k, $3 \leq k \leq n$.

Note that

$$\begin{aligned}\psi_{j_2-1,\dots,j_k+1,\dots,j_n} :&= \partial_1(P_{j_2-1+,\dots,j_k+1,\dots,j_n}) + (j_2-1)P_{j_2-1,\dots,j_k+1,\dots,j_n} a_2 \\ &+ j_3 P_{j_2-1,\dots,j_k+1,\dots,j_n} a_3 + \cdots + (j_k+1)P_{j_2-1,\dots,j_k+1,\dots,j_n} a_k + \cdots \\ &+ j_n P_{j_2-1,\dots,j_k+1,\dots,j_n} a_n\end{aligned}$$

is the coefficient of the monomial ${x_2}^{j_2-1} \cdots {x_k}^{j_k+1} \cdots {x_n}^{j_n}$ in $d(P)$.

Consider

$$P_2 := P_{j_2-1,\dots,j_k+1,\dots,j_n} d(P) - \psi_{j_2-1,\dots,j_k+1,\dots,j_n} P \in I.$$

Evidently, P_2 has no term in $x_2^{j_2-1} \cdots {x_k}^{j_k+1} \cdots x_n^{j_n}$, while the coefficient of the term $x_2^{j_2} \cdots x_n^{j_n}$, say $\vartheta_{j_2,\dots,j_n} \in K[x_1]$, is the following:

$$\vartheta_{j_2,\dots,j_n} = P^2_{j_2-1,\dots,j_k+1,\dots,j_n}\left(\partial_1\left(\frac{P_{j_2,\dots,j_n}}{P_{j_2-1,\dots,j_k+1,\dots,j_n}}\right) + (a_2 - a_k)\frac{P_{j_2,\dots,j_n}}{P_{j_2-1,\dots,j_k+1,\dots,j_n}}\right).$$

Hence, from Lemma 3.5 and from the fact that $a_2 \neq a_k$, we obtain that $\vartheta_{j_2,\dots,j_n} \neq 0$. Then, the coefficient of $x_2^{j_2} \cdots x_n^{j_n}$ in P_2 is nonzero, while its coefficient in $x_2^{j_2-1} \cdots {x_k}^{j_k+1} \cdots x_n^{j_n}$ is zero. Repeating this argument for every $k = 3, \dots, n$ such that $P_{j_2-1,\dots,j_k+1,\dots,j_n} \neq 0$, we obtain a nonzero element $\tilde{P} \in I$ such that its coefficient of

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$x_2^{j_2}\cdots x_n^{j_n}$ is non-zero while all the coefficients of $x_2^{j_2-1}\cdots x_k{}^{j_k+1}\cdots x_n^{j_n}$, for $3\leq k\leq n$, are zero. We are back to the first case.

In any case, we get a nonzero element in I that does not involve the monomial $x_2^{j_2}\cdots x_n^{j_n}$. Iterating this argument, we have that I contains a nonzero element Q of the form $Q=\sum_{i_2+\cdots+i_n=0}^{N-1}Q_{i_2,\ldots,i_n}x_2{}^{i_2}\cdots x_n{}^{i_n}$. This is a contradiction with the minimality of N. □

The next example shows that the hypothesis $a_i\neq a_j$, for $i\neq j$, in Theorem 3.6 cannot be dropped in general.

EXAMPLE 3.7. Let $d=\partial_1+(x_1x_2+1)\partial_2+(x_1x_3+1)\partial_3$ be a derivation of $K[x_1,x_2,x_3]$. Let $I=(x_2-x_3)K[x_1,x_2,x_3]$. Then $d(x_2-x_3)=x_1(x_2-x_3)$ and I is a non-zero, proper d-ideal. Therefore, d is not a simple derivation of $K[x_1,x_2,x_3]$, even though $K[x_1,x_i]$ is $d|_{K[x_1,x_i]}$-simple for $i=2,3$.

We will now use our theorem 3.6 to recover [**3**, Theorem 3.3]. Coutinho considers, for $2\leq i\leq n$, non-zero polynomials $a_i,b_i\in K[x_1]$ such that:

(1) $\frac{a_i}{a_j}\notin\mathbb{Q}$ whenever $2\leq i<j\leq n$ and

(2) $deg(a_i)>deg(b_i)$ for $i=2,\ldots,n$.

He shows that $d=\partial_1+\sum_{i=2}^{n}(x_ia_i+b_i)\partial_i$ is a simple derivation of the ring $K[x_1,\ldots,x_n]$. One advantage of our approach is that we can weaken the conditions on the polynomials $a_2,\ldots,a_n$.

EXAMPLE 3.8. Consider, for $2\leq i\leq n$, non-zero polynomials $a_i,b_i\in K[x_1]$ such that $deg(a_i)>deg(b_i)$ and $a_i\neq a_j$ for $2\leq i<j\leq n$. Then,

$$d=\partial_1+\sum_{i=2}^{n}(x_ia_i+b_i)\partial_i$$

is a simple derivation of the ring $K[x_1,\ldots,x_n]$.

In fact, we must check if

$$\partial_1(v)\neq a_i\cdot v+b_i,$$

for every $v\in K[x_1]$ and for every $i=2,\ldots,n$.

Observe that if $v\in K[x_1]$ is a solution of $\partial_1(v)=a_i\cdot v+b_i$, then,

$$\underbrace{\partial_1(v)}_{deg(v)-1}-\underbrace{a_i\cdot v}_{deg(a_i)+deg(v)}=\underbrace{b_i}_{deg(b_i)},\ i=2,\ldots,n.$$

Since $deg(a_i)>deg(b_i)$, $i=2,\ldots,n$, none of these equations has a solution in $K[x_1]$.

By theorem 3.6, it follows that $K[x_1,\ldots,x_n]$ is d-simple.

Now we give another new family of simple derivations of the ring $K[x_1,\ldots,x_n]$. They are Shamsuddin derivations.

EXAMPLE 3.9. For $2\leq i\leq n$, let f_i,g_i be monic polynomials in $K[x_1]$ such that $deg(f_i)=deg(g_i)$, and $f_i\neq f_j$, $2\leq i<j\leq n$. Then the following derivation

$$d=\partial_1+(x_1{}^2g_2+x_1f_2x_2)\partial_2+\cdots+(x_1{}^2g_n+x_1f_nx_n)\partial_n$$

is a simple derivation of the ring $K[x_1,\ldots,x_n]$.

In fact, we must check if

$$\partial_1(v) \neq x_1 f_i v + x_1{}^2 g_i,$$

for every $v \in K[x_1]$ and for every $i = 2, \ldots, n$. Let $k_i := deg(f_i) = deg(g_i)$. If $v \in K[x_1]$ is such that $\partial_1(v) = x_1 f_i v + x_1{}^2 g_i$, then

$$\underbrace{\partial_1(v)}_{deg(v)-1} - \underbrace{x_1 f_i v}_{deg(v)+k_i+1} = \underbrace{x_1{}^2 g_i}_{k_i+2}.$$

Hence $deg(v) = 1$.

We can write f_i, g_i and v in the form:

$$f_i = x_1{}^{k_i} + {}_if_{k_i-1} x_1{}^{k_i-1} + \cdots + {}_if_0$$
$$g_i = x_1{}^{k_i} + {}_ig_{k_i-1} x_1{}^{k_i-1} + \cdots + {}_ig_0$$
$$v = cx_1 + e$$

with ${}_if_j,\ {}_ig_j \in K$ for every i, j and $c, e \in K$. It follows that $(-c+1)x_1{}^{k_i+2} + \cdots + c = 0$, which is a contradiction with the fact that $c \neq 0$. Therefore, none of the these equations has a solution in $K[x_1]$. By theorem 3.6, we have that $K[x_1, \ldots, x_n]$ is d-simple.

4. A differential criterion for maximality. In this section we establish a criterion for the ideal $A_n(d+\gamma)$ to be maximal in terms of polynomial solutions of a finite system of partial differential equations over the polynomial ring $K[x_1, \cdots, x_n]$. Our result generalizes and strengthens a theorem of Bratti and Takagi ([**2**]).

THEOREM 4.1. *Let $d = \partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n$ be a Shamsuddin derivation of $K[x_1, \ldots, x_n]$, where $\alpha_i(x_1, x_i) = a_i(x_1)x_i + b_i(x_1) \in K[x_1, x_i]$, $i = 2, \ldots, n$. Let $\gamma \in K[x_1, \ldots, x_n]$.*

- *(a) If $A_n(d+\gamma)$ is a maximal left ideal of A_n, then the following conditions are satisfied:*
  - *(i) $\partial_1(v) - a_i \cdot v \neq b_i$, for every $v \in K(x_1)$, $2 \leq i \leq n$.*
  - *(i') $\partial_1(v) - a_i \cdot v \neq b_i$, for every $v \in K[x_1]$, $2 \leq i \leq n$.*
  - *(ii) $d(u) + a_i \cdot u \neq \partial_i(\gamma)$, for every $u \in K(x_1, \ldots, x_n)$, $2 \leq i \leq n$.*
  - *(ii') $d(u) + a_i \cdot u \neq \partial_i(\gamma)$, for every $u \in K[x_1, \ldots, x_n]$, $2 \leq i \leq n$.*
- *(b) Reciprocally, suppose that conditions (i) and (ii) are satisfied and moreover that $a_i \neq a_j$, for every $i \neq j$. Then, $A_n(d+\gamma)$ is a maximal left ideal of A_n.*

Proof. (a): (i'): Since $A_n(d+\gamma)$ is a maximal left ideal, it follows from corollary 3.3 that $K[x_1, \ldots, x_n]$ is d-simple. Then, by theorem 3.6, $K[x_1, x_i]$ is $d|_{K[x_1,x_i]}$-simple, for every $i = 2, \ldots, n$. By Shamsuddin's theorem ([**3**, Proposition 3.2]) we have that $\partial_1(v) - a_i \cdot v \neq b_i$, for every $v \in K[x_1]$, $2 \leq i \leq n$

(i): It follows from (i') and Lemma 3.5.

(ii'): Suppose that $p \in K[x_1, \ldots, x_n]$ satisfies $d(p) + a_i \cdot p = \partial_i(\gamma)$, for some $i \in \{2, \ldots, n\}$. Let $R = \partial_i + p$. Then,

$$[d+\gamma, R] = -a_i\partial_i + (d(p) - \partial_i(\gamma)).$$

Hence,

$$[d+\gamma, R] + a_i R = d(p) + a_i \cdot p - \partial_i(\gamma) = 0.$$

Therefore, R is a Darboux operator of $d+\gamma$ in $A_{n-1}[x_1]$. This is contrary to theorem 2.8.

(ii): We have noted already (proof of item (i')) that d is simple derivation of $K[x_1, \dots, x_n]$. Then, (ii) follows from (ii') and lemma 3.5.

(b): Let $R = \sum_{i_2+\cdots+i_n=0}^{N} P_{i_2,\dots,i_n} \partial_2{}^{i_2} \cdots \partial_n{}^{i_n} \in A_{n-1}[x_1]$, where $P_{i_2,\dots,i_n} \in K[x_1, \dots, x_n]$, be an operator of order N. Then, a simple calculation gives the following expression for $[d+\gamma, R]$:

$$\begin{aligned}[d+\gamma, R] = \sum_{i_2+\cdots+i_n=0}^{N} \{ & [d(P_{i_2,\dots,i_n}) - i_2 P_{i_2,\dots,i_n} a_2 - \cdots - i_n P_{i_2,\dots,i_n} a_n] \partial_2{}^{i_2} \cdots \partial_n{}^{i_n} \\ & + [-i_2 P_{i_2,\dots,i_n} \partial_2(\gamma)] \partial_2{}^{i_2-1} \cdots \partial_n{}^{i_n} \\ & + \cdots \\ & + [-i_n P_{i_2,\dots,i_n} \partial_n(\gamma)] \partial_2{}^{i_2} \cdots \partial_n{}^{i_n-1} \\ & + \text{ terms with order lower than } (i_2 + \cdots + i_n) - 1 \}. \end{aligned} \tag{11}$$

Suppose that $N > 0$. So, there exists $P_{j_2,\dots,j_n} \neq 0$, for some $j_2 + \cdots + j_n = N$. Without loss of generality we may suppose that $j_2 > 0$. Note that $\lambda_{j_2,\dots,j_n} := d(P_{j_2,\dots,j_n}) - j_2 P_{j_2,\dots,j_n} a_2 - \cdots - j_n P_{j_2,\dots,j_n} a_n$ is the coefficient of the monomial $\partial_2{}^{j_2} \cdots \partial_n{}^{j_n}$ in $[d+\gamma, R]$.

We consider

$$R_1 := P_{j_2,\dots,j_n}[d+\gamma, R] - \lambda_{j_2,\dots,j_n} R.$$

Evidently, R_1 has no term in $\partial_2{}^{j_2} \cdots \partial_n{}^{j_n}$, while the coefficient of the term $\partial_2{}^{j_2-1} \cdots \partial_n{}^{j_n}$ is the following:

$$\begin{aligned} q_{j_2-1,\dots,j_n} = P^2_{j_2,\dots,j_n} \Bigg\{ & d\left(\frac{P_{j_2-1,j_3,\dots,j_n}}{P_{j_2,\dots,j_n}}\right) + a_2 \frac{P_{j_2-1,\dots,j_n}}{P_{j_2,\dots,j_n}} - j_2 \partial_2(\gamma) \\ & - (j_3+1) \frac{P_{j_2-1,j_3+1,\dots,j_n}}{P_{j_2,\dots,j_n}} \partial_3(\gamma) - \cdots - (j_n+1) \frac{P_{j_2-1,\dots,j_n+1}}{P_{j_2,\dots,j_n}} \partial_n(\gamma) \Bigg\}. \end{aligned} \tag{12}$$

We will analyze two cases.

FIRST CASE: If $P_{j_2-1,j_3+1,\dots,j_n} = P_{j_2-1,j_3,j_4+1,\dots,j_n} = \cdots = P_{j_2-1,j_3,\dots,j_n+1} = 0$.

We claim that $R_1 \neq 0$. Indeed, in this case, (12) simplifies and the coefficient of the term $\partial_2{}^{j_2-1} \cdots \partial_n{}^{j_n}$ is

$$q_{j_2-1,\dots,j_n} = P^2_{j_2,\dots,j_n} j_2 \left(d\left(\frac{P_{j_2-1,j_3,\dots,j_n}}{j_2 P_{j_2,\dots,j_n}}\right) + a_2 \frac{P_{j_2-1,\dots,j_n}}{j_2 P_{j_2,\dots,j_n}} - \partial_2(\gamma) \right),$$

which is non-zero by hypothesis.

Therefore, the ideal $A_n(d+\gamma) + A_n R$ contains a nonzero element R_1 without the term $\partial_2{}^{j_2} \cdots \partial_n{}^{j_n}$, and clearly R_1 does not have any monomial of order N that was not already a monomial of R.

SECOND CASE: If $P_{j_2-1,\dots,j_k+1,\dots,j_n} \neq 0$, for some k, $3 \leq k \leq n$.

Note that

$$\begin{aligned} \mu_{j_2-1,\dots,j_k+1,\dots,j_n} := {} & d(P_{j_2-1,\dots,j_k+1,\dots,j_n}) - (j_2-1) P_{j_2-1,\dots,j_k+1,\dots,j_n} a_2 - \cdots \\ & - (j_k+1) P_{j_2-1,\dots,j_k+1,\dots,j_n} a_k - \cdots - j_n P_{j_2-1,\dots,j_k+1,\dots,j_n} a_n \end{aligned}$$

is the coefficient of the term $\partial_2^{j_2-1} \cdots \partial_k^{j_k+1} \cdots \partial_n^{j_n}$ in $[d+\gamma, R]$.

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Consider

$$R_2 := P_{j_2-1,\dots,j_k+1,\dots,j_n}[d+\gamma, R] - \mu_{j_2-1,\dots,j_k+1,\dots,j_n} R.$$

Evidently, R_2 has no term in $\partial_2^{j_2-1}\cdots\partial_k^{j_k+1}\cdots\partial_n^{j_n}$, while the coefficient of the term $\partial_2^{j_2}\cdots\partial_k^{j_k}\cdots\partial_n^{j_n}$ is the following:

$$\xi_{j_2,\dots,j_n} = P^2_{j_2-1,\dots,j_k+1,\dots,j_n}\left(d\left(\frac{P_{j_2,\dots,j_n}}{P_{j_2-1,\dots,j_k+1,\dots,j_n}}\right) + (a_k - a_2)\frac{P_{j_2,\dots,j_n}}{P_{j_2-1,\dots,j_k+1,\dots,j_n}}\right).$$

Now, by hypothesis and theorem 3.6, d is a simple derivation of $K[x_1,\dots,x_n]$. Applying lemma 3.5 and noticing that $a_2 \neq a_k$, we obtain that $\xi_{j_2,\dots,j_n}$ is non-zero. Then, the coefficient of the term $\partial_2^{j_2}\cdots\partial_n^{j_n}$ in R_2 is non-zero, while its coefficient of the term $\partial_2^{j_2-1}\cdots\partial_k^{j_k+1}\cdots\partial_n^{j_n}$ is zero. Repeating this argument, for every $k = 3,\dots,n$ such that $P_{j_2-1,\dots,j_k+1,\dots,j_n} \neq 0$, we obtain a non-zero element $\tilde{R} \in A_n(d+\gamma) + A_n R$ such that its coefficient of $\partial_2^{j_2}\cdots\partial_n^{j_n}$ is non-zero while all the coefficients of $\partial_2^{j_2-1}\cdots\partial_k{}^{j_k+1}\cdots\partial_n^{j_n}$, for $3 \leq k \leq n$, are zero. We are back to the first case.

In any case, the ideal $A_n(d+\gamma) + A_n R$ contains a nonzero element $\tilde{Q}$ with no monomial $\partial_2{}^{j_2}\cdots\partial_n{}^{j_n}$ and whose monomials of order N are among the monomials of R. Note that, by (11), any element of the form $f(x_1,\dots,x_n)[d+\gamma,\tilde{Q}] + g(x_1,\dots,x_n)\tilde{Q}$, where $f(x_1,\dots,x_n), g(x_1,\dots,x_n) \in K[x_1,\dots,x_n]$, does not have the term $\partial_2{}^{j_2}\cdots\partial_n{}^{j_n}$ either.

Proceeding in this way, we can eliminate all the monomials of positive order and we get a non-zero element with order zero, that is

$$(A_n(d+\gamma) + A_n R) \cap (K[x_1,\dots,x_n] \setminus \{0\}) \neq \emptyset.$$

Now, let

$$P = \sum_{i_2+\cdots+i_n=0}^{N} P_{i_2,\dots,i_n} x_2{}^{i_2}\cdots x_n{}^{i_n} \in (A_n(d+\gamma) + A_n R) \cap (K[x_1,\dots,x_n] \setminus \{0\}),$$

where $P_{i_2,\dots,i_n} \in K[x_1]$. Notice that, since P is a polynomial, $[d+\gamma, P] = d(P)$. Then, repeating the argument of the proof of theorem 3.6, (ii) $\Rightarrow$ (i), we see that

$$(A_n(d+\gamma) + A_n R) \cap (K[x_1] \setminus \{0\}) \neq \emptyset.$$

Therefore, $A_n(d+\gamma) + A_n R = A_n$. By lemma 2.4, $A_n(d+\gamma)$ is a maximal left ideal of $A_n(K)$. □

EXAMPLE 4.2. Let d be a Shamsuddin derivation of $K[x_1,\dots,x_n]$ and $\gamma \in K[x_1,\dots,x_n]$. If $deg_{x_i}(\gamma) = 0$, for some $i \in \{2,\cdots,n\}$, then $A_n(d+\gamma)$ is not a maximal left ideal of A_n. Indeed,in this case, the equation $d(u) + a_i \cdot u = \partial_i(\gamma)$ has $u = 0$ as a solution.

We show next that the conditions $a_i \neq a_j$, for $i \neq j$, in part **(b)** of theorem 4.1 cannot be dropped in general.

EXAMPLE 4.3. Let $d = \partial_1 + (ax_2 + b_2)\partial_2 + (ax_3 + b_3)\partial_3$ be a simple Shamsuddin derivation of $K[x_1, x_2, x_3]$ with deg $a \geq 1$. Let $\gamma := x_2 + x_3$. Then,

(1) Conditions (i) and (ii) of Theorem 4.1(a) are satisfied.

(2) $A_3(d+\gamma)$ is not a maximal left ideal of A_3.

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Proof. (1): By theorem 3.6, to say that d is simple is equivalent to say that the equations $\partial_1(v) - a_i \cdot v = b_i$, $i = 2, \cdots, n$, have no solution in $K(x_1)$. Then, condition (a)(i) of theorem 4.1 is satisfied. Now we consider condition (a)(ii). By lemma 3.5 and the fact that d is simple, this is equivalent to condition (a)(ii′).

Suppose that there exists $u \in K[x_1, x_2, x_3]$ such that

$$d(u) + a \cdot u = 1. \tag{13}$$

Let $i \in \{2, 3\}$. Applying ∂_i to (13) we have,

$$\partial_i(d(u)) = -a\partial_i(u).$$

Hence,

$$d(\partial_i(u)) + a\partial_i(u) = -a\partial_i(u),$$
$$d(\partial_i(u)) = -2a\partial_i(u).$$

Therefore $\partial_i(u) \in K$ since d is a simple derivation of $K[x_1, x_2, x_3]$. Then $d(\partial_i(u)) = 0$ and $\partial_i(u) = 0$, since $a \neq 0$.

Since this is valid for $i = 2, 3$, we obtain that $u \in K[x_1]$. Then, (13) becomes

$$u' = -au + 1.$$

This is absurd since $\deg a \geq 1$.

(2): Let $R := \partial_2 - \partial_3$. We have $[d + \gamma, R] = -aR \in K[x_1, x_2, x_3]R$. Thus, by theorem 2.8, $A_3(d + \gamma)$ is not a maximal left ideal of A_3. □

REMARK 4.4. Simple Shamsuddin derivations of $K[x_1, x_2, x_3]$ with $a_2 = a_3$ exist. For example, $d = \partial_1 + (x_1^2 x_2 + x_1^3)\partial_2 + (x_1^2 x_3 + x_1 + 1)\partial_3$ is one of them. (See [7]).

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