Combinatorics, Probability and Computing (2017) 26, 423–430.
doi:10.1017/S0963548316000365

Local Conditions for Exponentially Many Subdivisions

HONG LIU,[†] MARYAM SHARIFZADEH[‡] and KATHERINE STADEN[‡]

Mathematics Institute, University of Warwick, Coventry CV4 2AL, UK
(e-mail: {h.liu.9, k.l.staden, m.sharifzadeh}@warwick.ac.uk)

Received 28 July 2016; revised 7 November 2016; first published online 28 November 2016

Given a graph F, let $s_t(F)$ be the number of subdivisions of F, each with a different vertex set, which one can guarantee in a graph G in which every edge lies in at least t copies of F. In 1990, Tuza asked for which graphs F and large t, one has that $s_t(F)$ is exponential in a power of t. We show that, somewhat surprisingly, the only such F are complete graphs, and for every F which is not complete, $s_t(F)$ is polynomial in t. Further, for a natural strengthening of the local condition above, we also characterize those F for which $s_t(F)$ is exponential in a power of t.

2010 *Mathematics subject classification*: Primary 05C10
Secondary 05C35, 05C83

1. Introduction and results

A *subdivision* or *topological minor* of a graph F, denoted by TF, is a graph obtained by replacing the edges of F with internally vertex-disjoint paths between endpoints. A classical result of Mader [6] from 1972 states that there is some function $f(d)$ of d such that any graph with average degree $f(d)$ contains a copy of TK_d. Mader, and independently Erdős and Hajnal [2], conjectured that the correct order of magnitude for $f(d)$ is d^2. This was verified by Bollobás and Thomason [1] and independently by Komlós and Szemerédi [4]. The current best bound is due to Kühn and Osthus [5].

Theorem 1.1 ([5]). *Let G be a graph with average degree at least $(1+o(1))(10/23)d^2$. Then $G \supseteq TK_d$.*

This result is tight up to the constant factor; as Jung [3] observed, the complete bipartite graph $K_{r,r}$ with $r = \lfloor d^2/8 \rfloor$ does not contain a copy of TK_{d+1}.

[†] Supported by EPSRC grant EP/K012045/1, ERC grant 306493 and the Leverhulme Trust Early Career Fellowship ECF-2016-523.
[‡] Supported by ERC grant 306493.

We say that two subgraphs H, H' of a graph G are *distinguishable* if $V(H) \neq V(H')$. In 1981, Komlós (see [7]) conjectured that every graph with minimum degree d contains at least as many distinguishable cycles as K_{d+1}, that is, $2^{d+1} - \binom{d+1}{2} - d - 2$. A weaker conjecture, asking whether minimum degree d forces exponential in d many distinguishable cycles, was verified by Tuza [7] who obtained a lower bound $2^{\lfloor (d+4)/2 \rfloor} - O(d^2)$ for graphs with average degree d. If instead one imposes a different local condition on G, namely that every edge lies in at least t triangles, it is shown in [7] that G contains as least as many distinguishable cycles as K_{t+2}. In other words, G contains at least as many distinguishable subdivisions of the triangle as K_{t+2}.

Our aim in this paper, answering a question of Tuza [7, 8, 9], is to generalize this from triangles to arbitrary fixed graphs F and establish for which F *locally* many copies of F guarantee exponentially many distinguishable subdivisions of F *globally*. To formulate this more precisely, let us say that a graph G is (F,t)-*locally dense* if every edge e of G lies in at least t copies of F. Write $s(F,G)$ for the size of the largest collection of distinguishable subdivisions of F in G. Define

$$s_t(F) = \min\{s(F,G) : G \text{ is } (F,t)\text{-locally dense}\}.$$

Tuza's question [7] was for which graphs F one can find $c, c' > 0$ such that whenever t is a sufficiently large integer compared to $|F|$, we have $s_t(F) \geqslant c^{t^{c'}}$. Our main result answers this question completely, showing that the only such F are complete graphs. For every F that is not complete, we construct graphs that are locally dense in F, with only polynomially many distinguishable subdivisions of F.

Theorem 1.2. *Let $\ell \geqslant 3$ be an integer. Whenever t is sufficiently large, the following hold.*

(i) $s_t(K_\ell) \geqslant 2^{t^{c'}}$, *where* $c' := c'(\ell) \geqslant 1/(2(\ell - 2))$.

(ii) *For all graphs F on ℓ vertices which are not complete,*

$$s_t(F) \leqslant 2^\ell (t+2)^{e(F)+2\ell}.$$

(iii) *Whenever $\ell \geqslant 4$, we have*

$$s_t(K_\ell^-) \geqslant t^{(1-o(1))(e(K_\ell^-)-2\ell+5)},$$

where K_ℓ^- is the graph obtained from K_ℓ by removing an edge.[1]

For part (i), our bound on $c'(\ell)$ is best possible up to the constant factor $1/2$. As noted by Tuza [8], $c'(\ell) \leqslant 1/(\ell - 2)$. This is attained by the graph $G = K_r$, where r satisfies $\binom{r-2}{\ell-2} = t$. Indeed, every edge of G lies in exactly t copies of K_ℓ, and every set $X \subseteq V(G)$ of size at least ℓ is such that there is a copy of TK_ℓ with vertex set X. Therefore,

$$s_t(K_\ell, G) = \sum_{i=\ell}^{r} \binom{r}{i} \leqslant 2^r \leqslant 2^{t^{1/(\ell-2)}}.$$

Nonetheless, we do not believe our bound on $c'(\ell)$ is optimal and so we make no serious attempt to optimize absolute constants.

[1] Here $o(1)$ denotes a function which tends to 0 as $t \to \infty$.

https://doi.org/10.1017/S0963548316000365 Published online by Cambridge University Press

Part (ii) shows that for all non-complete F, the minimum number of distinguishable subdivisions of F among all (F,t)-locally dense graphs is at most a polynomial in t. Our upper bound on $s_t(F)$ in (ii) is close to being optimal, as shown by part (iii).

The rest of the paper is organized as follows. In Section 2 we give the proof of Theorem 1.2. In Section 3 we study how a natural strengthening of the locally dense condition would change the outcome. Some concluding remarks are given in Section 4.

2. Proof of Theorem 1.2

Let us briefly sketch the ideas in the proof. For part (i), first note that a (K_ℓ,t)-locally dense graph G has minimum degree which is polynomial in t. Then Theorem 1.1 furnishes us with a subdivision T of K_d in G such that d is polynomial in t. Every subset of the branch vertices of T which has order at least ℓ gives rise to a subdivision of K_ℓ, and all of these are distinguishable. For part (ii), for each non-complete F we construct a graph G which is (F,t)-locally dense but has a very small dominating set A (so each copy of F has very large overlap). The size of A limits the number of vertices in any subdivision which lie outside A. The proof of (iii) combines ideas from the previous parts together with an averaging argument.

We call those vertices of TF which correspond to those of F the *branch vertices*.[2]

Proposition 2.1. *Let $k,\ell \in \mathbb{N}$ with $\ell \leqslant k$ and suppose that T is a subdivision of K_k. Then T contains a subdivision of K_ℓ which contains all branch vertices of T.*

Proof. Let $x_1,\dots,x_k$ be the branch vertices of K_k in T. Now T is the union of a set

$$\left\{P(i,j) : ij \in \binom{[k]}{2}\right\}$$

of internally vertex-disjoint paths where $P(i,j)$ has endpoints x_i, x_j. Then we can find a copy of TK_ℓ with branch vertices $x_1,\dots,x_\ell$ by taking the paths $P(i,j)$ for all $\{i,j\} \in \binom{[\ell]}{2} \setminus \{1,\ell\}$, and taking the concatenation of the paths $P(\ell,\ell+1), P(\ell+1,\ell+2),\dots,P(k-1,k), P(1,k)$ to obtain a path between x_1 and x_ℓ which contains every other branch vertex in $V(T)$. □

Fix $\ell \in \mathbb{N}$. We will first prove (i). We may assume that $\ell \geqslant 4$ since the statement is known for $\ell = 3$ (see [7]). Let G be (K_ℓ,t)-locally dense, that is, every edge $xy \in E(G)$ lies in at least t copies if K_ℓ. Then

$$t \leqslant \binom{|N(x)\cap N(y)|}{\ell-2} \leqslant \binom{d(x)}{\ell-2} < \left(\frac{d(x)\cdot e}{\ell-2}\right)^{\ell-2} \tag{2.1}$$

[2] We use standard graph-theoretic notation. For natural numbers $k \leqslant \ell$, write $[\ell] := \{1,\dots,\ell\}$ and write $\binom{[\ell]}{k}$ for the set of k-subsets of $[\ell]$.

https://doi.org/10.1017/S0963548316000365 Published online by Cambridge University Press

and so $d(x) > e^{-1}(\ell-2)t^{1/(\ell-2)}$ for all $x \in V(G)$ which are not isolated. Now Theorem 1.1 implies that G contains a copy of TK_d, where

$$d = (1-o(1))\sqrt{\frac{23(\ell-2)}{10e}} \cdot t^{1/(2(\ell-2))}. \tag{2.2}$$

Let $X := \{x_1, \dots, x_d\}$ be the set of branch vertices of this copy of TK_d and let $P(i,j)$, $ij \in \binom{[d]}{2}$, be the internally vertex-disjoint path between x_i and x_j.

Do the following for each $I \subseteq [d]$ with $|I| \geqslant \ell$. Let $X_I := \{x_i : i \in I\}$. Let T_I be the subdivision of $K_{|I|}$ whose set of branch vertices is precisely X_I and which consists of paths $P(i,j)$ with $ij \in \binom{I}{2}$. So $V(T_I) \cap X = X_I$. Proposition 2.1 implies that T_I contains a subgraph S_I which is a subdivision of K_ℓ such that $V(S_I) \cap X = X_I$. Thus obtain a collection $\{S_I : I \subseteq [d], |I| \geqslant \ell\}$ of subdivisions of K_ℓ in G. This collection is distinguishable because $V(S_I) \cap X \neq V(S_{I'}) \cap X$ for distinct $I, I' \subseteq [d]$. So, when t is sufficiently large,

$$s(F,G) \geqslant 2^d - \sum_{i=0}^{\ell-1} \binom{d}{i} \geqslant 2^d - (d+1)^{\ell-1} \geqslant 2^{t^{1/(2(\ell-2))}},$$

where we used the fact that $\ell \geqslant 4$ in (2.2). This completes the proof of part (i) of Theorem 1.2.

We now turn to the proof of (ii), which will follow from the next lemma.

Lemma 2.2. *Let G be a graph with vertex partition $A \cup B$, where B is an independent set. Let F be a graph on $\ell \geqslant |A|$ vertices. Then $s(F,G) \leqslant 2^\ell(|B|+1)^{e(F)+2\ell}$.*

Proof. Write $V(F) = [\ell]$. Let T be a subdivision of F in G, and let $x_1, \dots, x_\ell$ be its branch vertices. So T consists of pairwise internally vertex-disjoint paths $P(i,j)$ for all $ij \in E(F)$, where $P(i,j)$ has endpoints x_i, x_j. Let $\mathcal{P}$ denote the union of the $P(i,j)$. Let $\mathcal{P}(A,A)$ be the set of $P(i,j)$ such that $x_i, x_j \in A$, and define $\mathcal{P}(A,B) = \mathcal{P}(B,A)$ and $\mathcal{P}(B,B)$ analogously. Given $P = P(i,j)$, let a_P be the number of internal vertices of P which lie in A, that is, $a_P := |A \cap (V(P) \setminus \{x_i, x_j\})|$, and define b_P analogously for B. Since B is an independent set, the following relationships are easy to see:

$$\begin{aligned} b_P &\leqslant a_P + 1 && \text{for all } P \in \mathcal{P}(A,A), \\ b_P &\leqslant a_P && \text{for all } P \in \mathcal{P}(A,B), \\ b_P &\leqslant a_P - 1 && \text{for all } P \in \mathcal{P}(B,B). \end{aligned}$$

Then, since $|\mathcal{P}| = e(F)$,

$$\ell \geqslant |A| \geqslant \sum_{P\in\mathcal{P}} a_P \geqslant \sum_{P\in\mathcal{P}} (b_P - 1) = \sum_{P\in\mathcal{P}} b_P - |\mathcal{P}| \;\Rightarrow\; \sum_{P\in\mathcal{P}} b_P \leqslant \ell + |\mathcal{P}| = \ell + e(F).$$

Let b^* denote the number of branch vertices of T which lie in B. Clearly, $b^* \leqslant \ell$, hence,

$$|V(T) \cap B| = b^* + \sum_{P\in\mathcal{P}} b_P \leqslant 2\ell + e(F).$$

Therefore an upper bound for $s(F,G)$ can be obtained by counting the number of subsets of $V(G)$ containing at most $2\ell + e(F)$ vertices in B. Thus

$$s(F,G) \leqslant 2^{|A|} \cdot \sum_{i=0}^{2\ell+e(F)} \binom{|B|}{i} \leqslant 2^{\ell}(|B|+1)^{e(F)+2\ell},$$

proving the lemma. □

We will construct a graph G for which $s(F,G) \leqslant 2^{\ell}(t+2)^{e(F)+2\ell}$, via Lemma 2.2. Let A, B be disjoint sets of vertices with $|A| = \ell - 2$ and $|B| = t+1$, and let $V(G) = A \cup B$. Add every edge to G with at least one endpoint in A. For any graph F on ℓ vertices which is not complete, we claim that G is (F,t)-locally dense. It suffices to show that G is (K_{ℓ}^{-},t)-locally dense. Let e be the non-adjacent pair in K_{ℓ}^{-} and let $xy \in E(G)$. If $x, y \in A$, then for any of the $\binom{t+1}{2} > t$ pairs $\{w,z\}$ of vertices in B, we have that $G[A \cup \{w,z\}]$ is isomorphic to K_{ℓ}^{-} with wz playing the role of e. Without loss of generality, the only other case is $x \in A$, $y \in B$. Then similarly we can choose any of the t vertices $w \in B \setminus \{y\}$ so that wy plays the role of the missing edge e. Therefore every edge $xy \in E(G)$ lies in at least t copies of K_{ℓ}^{-}, and hence F. Now Lemma 2.2 implies that $s_t(F) \leqslant s(F,G) \leqslant 2^{\ell}(t+2)^{e(F)+2\ell}$, as required.

Finally, we prove (iii). Let $\varepsilon > 0$ and let t be sufficiently large compared to ε. Let G be (K_{ℓ}^{-},t)-locally dense. We will show that $s(K_{\ell}^{-},G) \geqslant t^{(1-2\varepsilon)(e(K_{\ell}^{-})-2\ell+5)}$. For each edge $e \in E(G)$, let $g(e)$ be the number of copies of $K_{\ell-1}$ in G which contain e. Suppose first that $g(e) \geqslant t^{\varepsilon}$ for all $e \in E(G)$. Then G is $(K_{\ell-1},t^{\varepsilon})$-locally dense. So, by (2.2) and Theorem 1.1, G contains a copy of K_d, where $d = (\sqrt{\ell}/3)t^{\varepsilon/(2(\ell-3))}$. The same argument as (i) shows that

$$s(K_{\ell}^{-},G) \geqslant s(K_{\ell},G) \geqslant 2^d - (d+1)^{\ell-1} \geqslant 2^{\frac{1}{2}t^{\varepsilon/(2(\ell-3))}} > t^{\ell^2}$$

whenever t is sufficiently large compared to ε and ℓ (and recall that $\ell \geqslant 4$).

So we may assume that there is some $e \in E(G)$ with $g(e) < t^{\varepsilon}$.

Claim 2.3. *There is a subgraph H of G with vertex partition $A \cup B$ where $|A| = \ell - 1$ and $|B| \geqslant 2t^{1-\varepsilon}/(\ell-1)$, such that $H[A]$ is complete and there is $z \in A$ such that $H[A \setminus \{z\}, B]$ is a complete bipartite graph.*

Proof. Let $\mathcal{J}$ be the set of copies of $K_{\ell-1}$ containing e and let $\mathcal{K}$ be the set of copies of K_{ℓ}^{-} containing e. So $|\mathcal{J}| = g(e)$ and $|\mathcal{K}| \geqslant t$. Now, every copy of K_{ℓ}^{-} containing e contains exactly two distinct $J \in \mathcal{J}$ as subgraphs. So, by averaging, there is some $J \in \mathcal{J}$ such that there is $\mathcal{K}' \subseteq \mathcal{K}$ with the property that $J \subseteq K$ for all $K \in \mathcal{K}'$ and

$$|\mathcal{K}'| \geqslant \frac{2|\mathcal{K}|}{g(e)} \geqslant 2t^{1-\varepsilon}. \tag{2.3}$$

Every $K \in \mathcal{K}'$ has exactly one non-adjacent pair, and exactly one member of this pair lies in $V(J)$. Averaging once again we see that there is some $z \in V(J)$ and $\mathcal{K}'' \subseteq \mathcal{K}'$ such that for all $K \in \mathcal{K}''$ every $y \in V(K)$ is incident with every $x \in V(J) \setminus \{z\}$, and

$$|\mathcal{K}''| \geqslant \frac{|\mathcal{K}'|}{\ell - 1} \overset{(2.3)}{\geqslant} \frac{2t^{1-\varepsilon}}{\ell - 1}.$$

Now, let $A := V(J)$ and let $B := \bigcup_{K \in \mathcal{K}''} V(K) \setminus V(J)$. Every $K \in \mathcal{K}''$ contains precisely one vertex x_K outside $V(J)$ and the only non-adjacent pair in K is $x_K z$. Therefore the vertices x_K are distinct for distinct K. So $|B| = |\mathcal{K}''|$, as required. The remaining properties are clear. □

It will suffice to give a lower bound for $s(K_\ell^-, H)$. We will count the number of subdivisions T of K_ℓ^- in H with specific properties. Label the vertices of K_ℓ^- by $1, \ldots, \ell$, where $E(K_\ell^-) = \binom{[\ell]}{2} \setminus \{\ell - 1, \ell\}$. Write $A = \{x_1, \ldots, x_{\ell-1} := z\}$. Let x_ℓ and x_{ij} for all $ij \in \binom{[\ell-2]}{2}$ be arbitrary distinct vertices in B and let X be the set consisting of these vertices. Using these we can find a subdivision $T(X)$ of K_ℓ^- in H with vertex set $A \cup X$, branch vertices $x_1, \ldots, x_\ell$ and paths $P(i,j)$ for all $ij \in E(K_\ell^-)$, as follows.

- For all $i \in [\ell - 1]$, the vertex $x_i \in A$ corresponds to $i \in V(K_\ell^-)$, and the vertex $x_\ell \in B$ corresponds to $\ell \in V(K_\ell^-)$.
- For all $ij \in \binom{[\ell-2]}{2}$, each $P(i,j)$ has precisely one internal vertex x_{ij}, which lies in B. For all $i \in [\ell - 2]$ and $j \in \{\ell - 1, \ell\}$, $P(i,j)$ is the edge $x_i x_j$.

Note that $T(X)$ exists because $H[A]$ is complete and $H[A \setminus \{x_{\ell-1}\}, B]$ is complete bipartite. Different choices of X give rise to distinguishable $T(X)$ since $V(T(X)) = A \cup X$. Each X has order $\binom{\ell-2}{2} + 1$, so

$$\begin{aligned} s(K_\ell^-, G) \geqslant s(K_\ell^-, H) \geqslant \binom{|B|}{\binom{\ell-2}{2} + 1} &\geqslant \left(\frac{2t^{1-\varepsilon}}{(\ell - 1)\left(\binom{\ell-2}{2} + 1\right)} \right)^{\binom{\ell-2}{2}+1} \\ &\geqslant t^{(1-2\varepsilon)\left(\binom{\ell-2}{2}+1\right)} = t^{(1-2\varepsilon)(e(K_\ell^-) - 2\ell + 5)}, \end{aligned}$$

completing the proof of (iii). This concludes the proof of Theorem 1.2. □

3. A spectrum of local conditions

We now investigate a spectrum of progressively stronger conditions for host graphs G, and characterize those graphs F such that every graph G satisfying the condition contains exponentially many copies of TF. Given a graph F on ℓ vertices, for each $3 \leqslant k \leqslant \ell$, we make the following definition. We say that a graph G is (F, k, t)-*locally dense* if every edge of G lies in at least t copies of F' for all subgraphs $F' \subseteq F$ with $|F'| \geqslant k$. Let $s_t(F, k)$ be the minimum of $s(F, G)$ over all (F, k, t)-locally dense graphs G.

The next theorem generalizes Theorem 1.2 by describing the set of all graphs F on ℓ vertices for which one can find $c, c' > 0$ such that whenever t is a sufficiently large integer compared to ℓ, we have $s_t(F, k) \geqslant c^{t^{c'}}$. Note that (F, ℓ, t)-locally dense is the same as (F, t)-locally dense. So this notion seems to be the most natural strengthening of (F, t)-locally dense. As expected, the family of those F of order ℓ giving rise to exponentially many subdivisions when the host graph is (F, k, t)-locally dense gets strictly larger as k decreases. In fact F lies in this family if and only if $F \supseteq K_k$.

https://doi.org/10.1017/S0963548316000365 Published online by Cambridge University Press

Theorem 3.1. *Let $\ell, k \in \mathbb{N}$ and $3 \leqslant k \leqslant \ell$. Then, whenever t is sufficiently large, the following hold.*

(i) *For all graphs F containing a copy of K_k, we have that $s_t(F,k) \geqslant 2^{t^{c'(k)}}$, where $c'(k) \geqslant 1/(2(k-2))$.*
(ii) *For all K_k-free graphs F, we have that $s_t(F,k) \leqslant 2^{\ell}(t+2)^{e(F)+2\ell}$.*

Proof. For (i), let F be a graph on ℓ vertices containing a copy of K_k. Let G be a (F,k,t)-locally dense graph. Then G is certainly (K_k,t)-locally dense. The conclusion follows immediately from Theorem 1.2(i).

For (ii), note that F is K_k-free if and only if every subgraph $F' \subseteq F$ with $|F'| \geqslant k$ is such that F' contains a pair of non-adjacent vertices. Therefore the graph G we constructed in the proof of Theorem 1.2(ii) is in fact (F,k,t)-dense. (To see this, one can use the same argument, as there is always a non-adjacent pair in F' which we can embed into B in at least t different ways.) The conclusion follows from Lemma 2.2. □

4. Concluding remarks

We have shown that among all graphs G which are (F,t)-locally dense, for large t the minimum number of distinguishable subdivisions of F in G is exponential in a power of t if and only if F is a complete graph. Whenever F contains a non-adjacent pair of vertices, there is a simple construction of an (F,t)-locally dense graph G of order $O(t)$ with a very small dominating set A of order less than $|F|$. This property means that any subdivision of F in G cannot contain many vertices outside A, which in turn implies $s(F,G)$ is polynomial in t.

When one strengthens the local condition on G (in order to increase the family of graphs F which give rise to exponentially many subdivisions) in the most natural way, it is easy to characterize this family using our earlier results. Therefore it would be of interest to restate the question of Tuza with an entirely different local condition. To be interesting, such a condition on the host graph G cannot imply for all F that G has minimum degree at least some polynomial in t; otherwise Theorem 1.1 and the argument in Theorem 1.2(i) imply the desired conclusion.

We remark that our proof of Theorem 1.2 also works when 'edge' is replaced by 'vertex' in our local condition: that is, we consider host graphs G in which every vertex lies in at least t copies of F.

References

[1] Bollobás, B. and Thomason, A. (1996) Highly linked graphs. *Combinatorica* **16** 313–320.
[2] Erdős, P. and Hajnal, A. (1969) On topological complete subgraphs of certain graphs. *Ann. Univ. Sci. Budapest* 193–199.
[3] Jung, H. A. (1970) Eine Verallgemeinerung des *n*-fachen Zusammenhangs für Graphen. *Math. Ann.* **187** 95–103.
[4] Komlós, J. and Szemerédi, E. (1996) Topological cliques in graphs II. *Combin. Probab. Comput.* **5** 79–90.

https://doi.org/10.1017/S0963548316000365 Published online by Cambridge University Press

[5] Kühn, D. and Osthus, D. (2006) Extremal connectivity for topological cliques in bipartite graphs. *J. Combin. Theory Ser. B* **96** 73–99.
[6] Mader, W. (1972) Hinreichende Bedingungen für die Existenz von Teilgraphen, die zu einem vollständingen Graphen homomorph sind. *Math. Nachr.* **53** 145–150.
[7] Tuza, Z. (1990) Exponentially many distinguishable cycles in graphs. *J. Combin. Inform. Syst. Sci.* **15** 281–285.
[8] Tuza, Z. (2001) *Unsolved Combinatorial Problems Part I* BRICS Lecture Series LS-01-1.
[9] Tuza, Z. (2013) Problems on cycles and colorings. *Discrete Math.* **313** 2007–2013.