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A COMPACTNESS THEOREM FOR SURFACES WITH BOUNDED INTEGRAL CURVATURE

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Abstract We prove a compactness theorem for metrics with bounded integral curvature on a fixed closed surface Σ. As a corollary we obtain a new convergence result for sequences of metrics with conical singularities, where an accumulation of singularities is allowed.

Keywords: differential geometry; metric geometry; conical singularities; Alexandrov surfaces with bounded integral curvature; compactness theorem

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Contents

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Introduction and statement of the Main theorem

In the late 1940s, Alexandrov and the school of Leningrad developed a very rich theory of singular surfaces. These are smooth surfaces, endowed with intrinsic metrics, for which there exists a natural notion of curvature, which is a Radon measure. They are called surfaces (respectively, metrics) with *bounded integral curvature*, denoted by 'B.I.C.' in the sequel. The precise definition is given in § 1.1. For an exposition of the theory, see the book of Alexandrov and Zalgaller [3], the book of Reshetnyak [16], its article [17] or the modern concise survey of Troyanov [19].

The curvature measure is a fundamental object in this singular geometry. It is built from the angular excesses $\overline{a}+\overline{b}+\overline{c}-\pi$ of small triangles ($\overline{a}$, $\overline{b}$ and $\overline{c}$ denote the (upper) angles at A, B and C, see [17]). This theory includes smooth Riemannian metrics: in this case, the curvature measure is $\mathrm{K}_g\, d\mathcal{A}_g$, where K_g stands for the Gauss curvature, as well as metrics with conical singularities, where the curvature measure is $\mathrm{K}_g\, d\mathcal{A}_g +$ a sum of Dirac masses at the cone points (K_g is the Gauss curvature of the smooth part). The next example shows how a sequence of metrics with conical singularities can converge to a surface with B.I.C.:

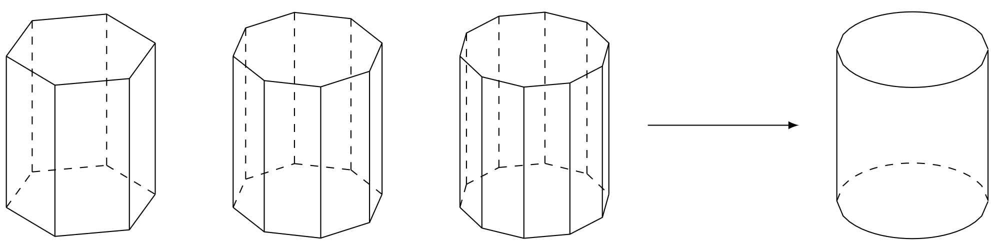

Figure 1. Accumulation of singularities.

The limit space is a cylinder (or a can: the top and the bottom belong to the surface), and the curvature measure of this singular surface is the usual angle measure on the two circles, at the top and at the bottom of the cylinder.

Since these singular surfaces may be defined by approximation by smooth Riemannian surfaces (see Definition 1.1), most of the properties of smooth surfaces extend to this setting: in particular, local conformal coordinates always exist (see Theorem 1.6). This

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property is crucial in our article: the metric is locally induced by a (singular) Riemannian metric $g_{\omega,h} = e^{2V_\omega(z)+2h(z)}|dz|^2$, where V_ω is the logarithmic potential of the curvature measure ω, and h is a harmonic function. Hence, if we know the curvature measure, then we know the local expression of the metric, up to a harmonic function. One of the key steps in this article is to obtain a control on this harmonic term (see Theorem 4.2). When we forget it (that is, we put $h = 0$), we have the following local convergence theorem, due to Reshetnyak (see § 1.2 for the definition of $\overline{d}_{\omega_m,0}$ and $\overline{d}_{\omega,0}$):

Theorem 0.1 (Reshetnyak, see [16, Theorem 7.3.1]). *Let ω_m^+ and ω_m^- be a sequence of non-negative Radon measures with support in $\overline{D}(1/2)$, weakly converging to measures ω^+ and ω^-. Let $\omega_m := \omega_m^+ - \omega_m^-$ and $\omega := \omega^+ - \omega^-$. Then*

$$\overline{d}_{\omega_m,0} \underset{m\to\infty}{\longrightarrow} \overline{d}_{\omega,0},$$

uniformly on any closed set $A \subset \overline{D}(1/2)$ such that $\omega^+(\{z\}) < 2\pi$ for every $z \in A$. That is, if $z_m \to z$ and $z'_m \to z'$, with $\omega^+(\{z\}) < 2\pi$ and $\omega^+(\{z'\}) < 2\pi$, then $\overline{d}_{\omega_m,0}(z_m, z'_m) \to \overline{d}_{\omega,0}(z, z')$.

In this article, we use this local theorem to prove a *global* convergence theorem for surfaces with B.I.C., and as a corollary we obtain a new convergence result for sequences of cone metrics (see Corollary 0.4 below). In the (classical) smooth setting, there are very well-known compactness results. Let Λ, i, V be positive constants, and $\mathcal{M}_n(\Lambda, i, V)$ be the set of compact Riemannian n-manifolds with

(1) |sectional curvature| $\leqslant \Lambda$,

(2) injectivity radius $\geqslant i$,

(3) volume $\leqslant V$.

In the early 1980, Gromov, in [11], stated the precompactness of the set $\mathcal{M}_n(\Lambda, i, V)$, in the Lipschitz topology: for every sequence $(X_m, g_m) \in \mathcal{M}_n(\Lambda, i, V)$, there exists a Riemannian $n-$manifold X, a Riemannian metric g and diffeomorphisms $\varphi_m : X \to X_m$ such that, after passing to a subsequence, $(X, (\varphi_m)^* d_{g_m}) \to (X, d_g)$ in the Lipschitz topology (d_{g_m} and d_g are the length distance associated to the Riemannian metrics g_m and g). This so-called Cheeger–Gromov convergence theorem was already implicit in the thesis of Cheeger in 1970. Since then, many articles were published on the subject, and the initial statement of Gromov was improved in two different ways: one only needs a bound on the Ricci curvature, and the convergence is much stronger than in the Lipschitz topology (see [4, 5, 10, 14, 15]). We need to use harmonic coordinates in order to obtain the optimal regularity in the convergence (see [9, 13]).

For surfaces with B.I.C., the only convergence theorem known to the author deals with a sequence of metrics in a fixed conformal class (see [19, Theorem 6.2]): it is a direct consequence of the local convergence theorem (Theorem 0.1). When we look for a convergence theorem for a sequence of metrics d_m on a surface Σ, at some point one needs to construct the diffeomorphisms $\varphi_m : \Sigma \to \Sigma$. It always involves serious work, for example by embedding the manifolds in some bigger space (see [11, 12] or the present article). Please note that some of the consequences of a uniform convergence $d_m \to d$ (up to diffeomorphisms) are described in [16] and [3].

We want to adapt the three hypothesis of the compactness theorem for smooth Riemannian metrics to our singular setting. The hypothesis (1) deals with the sectional (Gauss) curvature, which does not exist everywhere in the singular setting, hence we ask for a bound on the curvature measure instead. In order to avoid a cusp, that is a point $x \in \Sigma$ where the non-negative part of the curvature measure is $\omega^+(\{x\}) = 2\pi$ (such a point may be at infinite distance to any other point, see Remark 1.4), we ask for the inequality $\omega^+(B(x, \varepsilon)) \leqslant 2\pi - \delta$ for every $x \in \Sigma$ (ε and δ are positive constants). The hypothesis (3), which deals with the volume (or the area in the two-dimensional case), already makes sense for a surface with B.I.C.

So let us look at the hypothesis (2). In the smooth setting, a lower bound on the injectivity radius avoids a pinching of the manifold (as may happen, for example, when one factor of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ shrinks to a point). But for surfaces with B.I.C., the injectivity radius does not make sense, and even for a surface with conical singularities, the injectivity radius of the (open) smooth part is zero (if x is at a distance r of a cone point, then $\mathrm{inj}(x) < r$). Hence we need to define a similar quantity, which makes sense for non-Riemannian metric spaces. We introduce the new notion of *contractibility radius* (see § 2), which is the biggest r such that all the closed balls of radius $s < r$ are homeomorphic to closed discs (hence they are contractible). The important point is that a lower bound on the contractibility radius avoids a pinching of the surface. This notion is very natural: in the classical Cheeger–Gromov convergence theorem, one can replace a lower bound on the injectivity radius by a lower bound on the contractibility radius (see Example 2.8).

From now on, we fix a *closed* surface Σ: that is, a connected compact smooth surface, without boundary. Let $A, c, \varepsilon, \delta$ be some positive constants. Let $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$ be the class of metrics d with B.I.C. on Σ such that:

(1) for every $x \in \Sigma$ we have
$$\omega^+(B(x, \varepsilon)) \leqslant 2\pi - \delta;$$

(2) the contractibility radius of (Σ, d) satisfies
$$\mathrm{cont}(\Sigma, d) \geqslant c;$$

(3) the area of (Σ, d) satisfies
$$\mathcal{A}rea(\Sigma, d) \leqslant A.$$

Remark 0.2. From now on, when considering a set $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, we always assume $\varepsilon < c$ (hence there exists conformal charts on balls of radius ε, see Theorem 1.6 and Proposition 2.6).

The main result of the article is the following.

Main theorem. *The space $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$ is compact, in the uniform metric sense. That is for every sequence $d_m \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, there exists a metric d with B.I.C. such that, after passing to a subsequence, there are diffeomorphisms $\varphi_m : \Sigma \to \Sigma$ with*
$$(\varphi_m)^* d_m \underset{m\to\infty}{\longrightarrow} d \text{ uniformly on } \Sigma.$$

In the classical Cheeger–Gromov theorem, an easy packing argument shows that one can replace an upper bound on the volume by an upper bound on the diameter. In our setting, if we want to do so, we also need to ask for an upper bound on the total measure curvature $|\omega| := \omega^+ + \omega^-$ (see Proposition 4.1):

Corollary 0.3. *The space of metrics with B.I.C. satisfying conditions (1) and (2) above, and with diameter* $\mathrm{diam}(\Sigma, d) \leqslant D$ *and total measure curvature* $|\omega|(\Sigma, d) \leqslant \Omega$ *is compact, in the uniform metric sense. That is, for every sequence of metrics* d_m *satisfying the conditions above, there exists a metric* d *with B.I.C. such that, after passing to a subsequence, there are diffeomorphisms* $\varphi_m : \Sigma \to \Sigma$ *with* $(\varphi_m)^* d_m \to d$ *uniformly on* Σ.

A fortiori, we have compactness in the Gromov–Hausdorff sense of the sequence of metric spaces (Σ, d_m). This property is true under much weaker assumptions: the set of surfaces with B.I.C. with diameter $\leqslant D$ and total measure curvature $\leqslant \Omega$ is precompact in the Gromov–Hausdorff topology, see [18]. Of course, in this case, the limit metric space may not be a surface (a sphere can for example shrink to a point).

For metrics with conical singularities, we obtain the following.

Corollary 0.4. *Consider on* Σ *a sequence* (g_m) *of Riemannian metrics with conical singularities at points* $(p_i^m)_{i \in I_m}$, *with angles* θ_i^m. *Suppose that*

(1) *for every* $x \in \Sigma$, *we have*

$$\int_{B_m(x,\varepsilon)} \mathrm{K}_m^+ \, d\mathcal{A}_m + \sum_i (2\pi - \theta_i^m)^+ \leqslant 2\pi - \delta,$$

where the sum is taken over $i \in I_m$ *such that* $p_i^m \in B_m(x, \varepsilon)$;

(2) *the contractibility radius of* (Σ, d_m) *satisfies*

$$\mathrm{cont}(\Sigma, d_m) \geqslant c;$$

(3) *the area of* (Σ, d_m) *satisfies*

$$\mathcal{A}rea(\Sigma, d_m) \leqslant A.$$

Then, there exists a metric d *with B.I.C. such that, after passing to a subsequence, there are diffeomorphisms* $\varphi_m : \Sigma \to \Sigma$ *with*

$$(\varphi_m)^* d_m \underset{m \to \infty}{\longrightarrow} d \text{ uniformly on } \Sigma.$$

We also obtain some interesting corollaries by considering smooth Riemannian metrics. In the case of non-positive Gauss curvature, the first condition is automatically satisfied, and the injectivity radius is half of the length of the smallest closed geodesic. Hence we obtain

Corollary 0.5. *Consider on* Σ *a sequence* (g_m) *of smooth Riemannian metrics, with non-positive sectional curvature, such that the length of the smallest closed geodesic is bounded below, and the area is bounded above. Then, there exists a metric* d *with B.I.C. such that, after passing to a subsequence, there are diffeomorphisms* $\varphi_m : \Sigma \to \Sigma$ *with* $(\varphi_m)^* d_{g_m} \underset{m \to \infty}{\longrightarrow} d$ *uniformly on* Σ.

The article is organized as follows:

In § 1, we define metrics with B.I.C., as well as metrics with conical singularities. We also state the existence of local conformal charts.

In § 2, we define the new notion of contractibility radius, we give some properties and look at some examples.

In § 3, we prove two properties for surfaces with B.I.C.: one concerns the area of balls (by analogy with the case of smooth Riemannian metrics), and the other one is on the length of a line segment, for a singular Riemannian metric which has no harmonic term.

The heart of the article is § 4: we prove some preliminary properties for the set $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$. Let $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, and let $H : B(x, \varepsilon) \to D(1/2)$ be a conformal chart, with $H(x) = 0$. First, we prove that the harmonic term for the metric is bounded on every compact set of $D(1/2)$ (this is Theorem 4.2). Then, we prove the fundamental Theorem 4.8. Roughly speaking, we have a control on the images by H of balls of 'big' radii $B(x, \varepsilon/2)$ and $B(x, \varepsilon/4)$, and balls of 'small' radii $B(x, \kappa\varepsilon)$ (for some small constant $\kappa > 0$). This control has to be uniform, that is independent of the metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$ we are considering.

In § 5, we prove the Main theorem. We present a detailed sketch of the proof at the beginning of the section: this is an adaptation of the proof of Cheeger–Gromov's compactness theorem presented in [12].

In the Appendix, we recall some standard facts from conformal geometry needed in § 4. This standard material can be found in the book of Ahlfors [1].

Please note that the reader interested in a more detailed study, containing many didactical digressions aimed at explaining the theory of surfaces with B.I.C. as well as the new notion of contractibility radius, may have a look at the author's dissertation [8].

Notations: the usual non-negative and non-positive parts of a real number x are $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$. If f is a function, its non-negative and non-positive parts are $f^+(x) := (f(x))^+$ and $f^-(x) := (f(x))^-$. In this article we deal with signed Radon measures: for such a measure ν, we define the non-negative and non-positive parts by

$$\nu^+(X) := \sup_{A \subset X} \nu(A) \quad \text{and} \quad \nu^-(X) := \sup_{A \subset X} -\nu(A).$$

ν^+ and ν^- are two non-negative measures; we have $\nu = \nu^+ - \nu^-$, and we set $|\nu| := \nu^+ + \nu^-$.

1. Surfaces with bounded integral curvature

We give the definition of a surface with B.I.C. Then we state the fundamental property that they are locally isometric to a (singular) Riemannian metric, conformal to the Euclidean metric $|dz|^2$. Finally we give the definition of a metric with conical singularities. For the notions presented here, see [16, 17, 19].

Let Σ be a closed surface. Recall that a metric d on Σ is *intrinsic* if for every $x, y \in \Sigma$ we have

$$d(x, y) = \inf L(\gamma),$$

where the infimum is taken over all continuous curves $\gamma : [0, 1] \to \Sigma$, with $\gamma(0) = x$ and $\gamma(1) = y$, and where the length of γ is defined by

$$L(\gamma) := \sup_{0=t_0\leqslant\cdots\leqslant t_n=1} \left(\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \right).$$

In our setting, Σ is compact, so if d is an intrinsic metric on Σ compatible with the topology, there always exists a minimizing geodesic between two points. That is for every $x, y \in \Sigma$, there exists a continuous curve $\gamma : [0, 1] \to \Sigma$, with $\gamma(0) = x$ and $\gamma(1) = y$, such that $d(x, y) = L(\gamma)$.

1.1. Definition

Metrics with B.I.C. can be uniformly approximated by Riemannian metrics. Indeed we have the following definition (see [19]):

Definition 1.1. The metric d has bounded integral curvature (abbreviated B.I.C.) on Σ, and we say that (Σ, d) is a surface with B.I.C., if:

(1) d is an intrinsic distance on Σ;

(2) d is compatible with the topology of Σ;

(3) there exists a sequence g_m of Riemannian metrics on Σ, with $(\int_\Sigma |\mathrm{K}_{g_m}| d\mathcal{A}_m)_{m\in\mathbb{N}}$ bounded, such that d is the uniform limit of the metrics d_{g_m} on Σ.

For such a sequence g_m of Riemannian metrics, the sequence of measures $\mathrm{K}_{g_m}\, d\mathcal{A}_m$ converges weakly. The limit is denoted by ω, and is called the curvature measure: it depends on d, but does not depend on the approximating sequence g_m.

Remark 1.2. There is also an (intrinsic) geometric definition of metrics with B.I.C.: for any intrinsic distance on Σ compatible with the topology, one can build the non-negative and non-positive parts of the curvature measure from the angular excesses of small triangles. Then the metric has B.I.C. if this construction gives rise to *finite* measures. See [16, 17] for more details.

If $d = d_g$ is a Riemannian metric, then the curvature measure is $\omega = \mathrm{K}_g\, d\mathcal{A}_g$, and the non-negative and non-positive parts are $\omega^+ = \mathrm{K}_g^+\, d\mathcal{A}_g$ and $\omega^- = \mathrm{K}_g^-\, d\mathcal{A}_g$. We can then define the area measure $d\mathcal{A}$ as the weak limit of $d\mathcal{A}_m$ (it also does not depend of the choice of the sequence g_m). Note that the area measure coincides with the two-dimensional Hausdorff measure of the metric space (Σ, d).

1.2. Conformal charts

In the sequel, for $r > 0$ we set

$$D(r) := \{z \in \mathbb{C}, |z| < r\}$$

and

$$\overline{D}(r) := \{z \in \mathbb{C}, |z| \leqslant r\}.$$

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Let ω be a Radon measure with support in $\overline{D}(1/2)$, and h a harmonic function on $D(1/2)$. Consider the following (singular) Riemannian metric:

$$g_{\omega,h} = e^{2V_\omega(z)+2h(z)}|dz|^2.$$

The function V_ω is the logarithmic potential of the measure ω, and is defined by

$$V_\omega(z) := \iint_{\mathbb{C}} \left(\frac{-1}{2\pi}\right) \ln|z-\xi|\, d\omega(\xi):$$

it is defined for almost every $z \in \mathbb{C}$, and $V_\omega \in L^1_{loc}(\mathbb{C})$. It satisfies $\Delta V_\omega = \omega$ in the weak sense, where the sign convention for the Laplace operator on $\mathbb{C}$ is $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. Since

$$V_\omega(z) = \iint_{D(1/2)} \left(\frac{-1}{2\pi}\right) \ln|z-\xi|\, d\omega(\xi),$$

and $\omega = \omega^+ - \omega^-$, we can write $V_\omega = V_{\omega^+} - V_{\omega^-}$. Moreover, for every $z, \xi \in D(1/2)$ we have $\ln|z-\xi| \leqslant 0$, so for almost every $z \in D(1/2)$ we have $V_{\omega^+}(z) \geqslant 0$ and $V_{\omega^-}(z) \geqslant 0$, hence

$$-V_{\omega^-}(z) \leqslant V_\omega(z) \leqslant V_{\omega^+}(z).$$

Consider $\gamma : [0,1] \to D(1/2)$ a continuous simple curve (that is, γ is injective), parametrized with constant speed s (that is, the Euclidean length of the curve $\gamma_{|[t_1,t_2]}$ is $s \cdot (t_2 - t_1)$ for every $t_1 \leqslant t_2$). We define the length of γ for the singular Riemannian metric $g_{\omega,h}$ by

$$L_{\omega,h}(\gamma) := \int_0^1 e^{V_\omega(\gamma(t))+h(\gamma(t))} s \cdot dt.$$

This integral makes sense, that is, $V_\omega(\gamma(t))$ is well defined for almost every $t \in [0,1]$, because V_ω is the difference of two subharmonic functions (see [16, p. 99]). Then we set

$$d_{\omega,h}(z,z') := \inf L_{\omega,h}(\gamma) \in [0,+\infty],$$

where the infimum is taken over all continuous simple curves $\gamma : [0,1] \to D(1/2)$, parametrized with constant speed, with $\gamma(0) = z$ and $\gamma(1) = z'$. It is clear that $d_{\omega,h}$ satisfies all the properties to be a distance, except that we may have $d_{\omega,h}(z,z') = \infty$. A sufficient condition for $d_{\omega,h}$ to be a distance is the following (see [19, Proposition 5.3]):

Proposition 1.3. *If for every $z \in D(1/2)$ we have $\omega^+(\{z\}) < 2\pi$, then $d_{\omega,h}$ is a distance on $D(1/2)$.*

Remark 1.4. If $\omega^+(\{z_0\}) = 2\pi$ for some $z_0 \in D(1/2)$ (we say that z_0 is a *cusp*), then z_0 may be at infinite distance to any other point $z \in D(1/2)$. For example, if we set $\omega = 2\pi\delta_0$ (δ_0 is the Dirac mass at $0 \in \mathbb{C}$) and $h = 0$, then $g_{\omega,h} = |z|^{-2}|dz|^2$ and we have $d_{\omega,h}(0,z) = \infty$ for every $z \neq 0$.

We say that the metric has no cusp if the condition of Proposition 1.3 is satisfied. The metric $d_{\omega,h}$ is then compatible with the topology of $D(1/2)$ (as a subset of $\mathbb{C}$), and $(D(1/2), d_{\omega,h})$ is a surface with B.I.C. In the sequel, we always assume that the metrics have no cusp. By the hypothesis (1), this is true for every $d \in \mathcal{M}_\Sigma(A,c,\varepsilon,\delta)$.

To state the local convergence theorem (Theorem 0.1), we also need the following definition: if $z, z' \in D(1/2)$, we set

$$\overline{d}_{\omega,0}(z, z') := \inf\{L_{\omega,0}(\gamma)\}, \tag{1}$$

where the infimum is taken over all continuous simple curves $\gamma : [0, 1] \to \overline{D}(1/2)$, parametrized with constant speed, with $\gamma(0) = z$ and $\gamma(1) = z'$. The difference with $d_{\omega,0}(z, z')$ is that the curves we are considering here can meet $\partial D(1/2)$. This technical detail is only needed in the proof of Corollary 4.10. At every other place in the article, we use $d_{\omega,h}$ or $d_{\omega,0}$.

Example 1.5 (Riemannian metric). Let g be a smooth Riemannian metric on Σ. The metric g is locally conformally flat, that is for every $x \in \Sigma$, we can find local coordinates $z \in D(1/2)$ such that the metric reads $g = e^{2u(z)}|dz|^2$. We have the following formulas concerning the Gauss curvature and the area:

$$\mathrm{K}_g = (\Delta u)e^{-2u} \quad \text{and} \quad d\mathcal{A}_g = e^{2u}\, d\lambda(z)$$

(in all the sequel, $d\lambda$ is the Lebesgue measure on $\mathbb{C}$), so the curvature measure is $\omega = \mathrm{K}_g\, d\mathcal{A}_g = \Delta u\, d\lambda(z)$. Let $h := u - V_\omega$: by definition of the logarithmic potential V_ω, the function h is harmonic, so the metric reads

$$g = e^{2V_\omega + 2h}|dz|^2 = g_{\omega,h}.$$

Reschetnyack proved that B.I.C. surfaces are conformally flat (see [17, Theorem 4]):

Theorem 1.6. *Let (Σ, d) be a surface with B.I.C., with no cusp, and let U be an open set, homeomorphic to an open disc, such that $\overline{U}$ is homeomorphic to a closed disc. Then there exists a map H, a measure ω_H, defined in $D(1/2)$, and a harmonic function h on $D(1/2)$ such that*

$$H : (U, d_{|U}) \to (D(1/2), d_{\omega_H,h})$$

is an isometry. Such a map H is called a conformal chart.

We denote by $d_{|U}$ the intrinsic distance induced by d on U (that is, $d_{|U}(x, y)$ is the infimum of the d−length of curves joining x and y in U), and the measure ω_H is defined by $\omega_H = H_\#\omega$ (ω is the curvature measure of (Σ, d)), that is $\omega_H(A) = \omega(H^{-1}(A))$ for every Borel set $A \subset D(1/2)$.

Moreover, the area of any Borel set $A \subset U$ is

$$\mathcal{A}rea(A) = \int_{H(A)} e^{2V_{\omega_H}(z)+2h(z)}\, d\lambda(z).$$

Please note that this theorem implies that the surface Σ has a natural structure of a Riemann surface (see [16] for more details).

1.3. Surfaces with conical singularities

Definition 1.7. A metric with conical singularities is a metric d with B.I.C., with no cusp such that, if the Radon–Nikodym decomposition of the curvature measure ω with respect

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to the area measure $d\mathcal{A}$ reads

$$\omega = \mu + \mathrm{K}\, d\mathcal{A}$$

for some function $\mathrm{K} \in L^1_{loc}(d\mathcal{A})$, then the singular measure μ is a finite sum of Dirac masses.

If $\mu = \sum_{i\in I} k_i \delta_{p_i}$ (where δ_{p_i} is the Dirac mass at p_i, and $k_i < 2\pi$), then in a neighborhood of any p_i there are complex coordinates $z \in D(1/2)$ such that the singular metric reads

$$g = |z|^{2\beta_i} e^{2u_i(z)} |dz|^2,$$

with $\beta_i := -k_i/2\pi > -1$, and $u_i \in L^1_{loc}(D(1/2))$, with $\Delta u_i \in L^1_{loc}(D(1/2))$ in the weak sense.

Please note that the plane, endowed with the metric $g = |z|^{2\beta}|dz|^2$, is isometric to an Euclidean cone of angle $\theta = 2\pi(\beta + 1) = 2\pi - k$. For $\theta \in (0, 2\pi)$, this cone can be obtained by gluing an angular sector of the plane.

2. Contractibility radius

We have already mentioned in the introduction that the contractibility radius is, in some sense, a generalization of the injectivity radius to non-Riemannian metric spaces: the important point is that a lower bound on the contractibility radius prevents a pinching of the surface. We start this section by proving a proposition on the topology of closed balls, needed for the definition.

Let (Σ, d) be a closed surface with B.I.C. If $x \in \Sigma$, we denote by $\overline{B}(x, r)$ the closed ball centered in x and with radius r (that is the set of $y \in \Sigma$ with $d(x, y) \leqslant r$). Since the metric is intrinsic, this is the closure of $B(x, r)$. To define the contractibility radius, we need the following.

Proposition 2.1. *For every $x \in \Sigma$, there exists some $r > 0$ such that for every $s < r$, $\overline{B}(x, s)$ is homeomorphic to a closed disc.*

To prove this proposition, we need a lemma, which is a direct consequence of a result due to Burago and Stratilatova, see [16, Theorem 9.1]. Let $S(x, r)$ be the circle with center x and radius r (that is the set of $y \in \Sigma$ with $d(x, y) = r$). In the general case, the set $S(x, r)$ may be arranged in a rather complicated way.

Theorem 2.2 (Burago and Stratilatova). *Let U be a set homeomorphic to an open disc, with $x \in U$ and $\omega^+(U - \{x\}) < \pi$. If $S(x, r) \subset U$, then $S(x, r)$ is a Jordan curve.*

Lemma 2.3. *Let U be a set homeomorphic to an open disc, with $x \in U$ and $\omega^+(U - \{x\}) < \pi$. If $\overline{B}(x, r) \subset U$, then for every $s \leqslant r$, $\overline{B}(x, s)$ is homeomorphic to a closed disc.*

Proof of Lemma 2.3. Let $h : U \to \mathbb{C}$ be a homeomorphism, and let $s \leqslant r$. Since $S(x, s) \subset \overline{B}(x, r) \subset U$, we can apply Theorem 2.2: $h(S(x, s))$ is a Jordan curve Γ. $\mathbb{C} - \Gamma$ has two connected components: call the bounded component the 'interior' of Γ, and the unbounded component the 'exterior' of Γ. Since $B(x, s)$ is open and closed in $U - S(x, s)$,

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$h(B(x,s))$ is a connected component of $h(U - S(x,s)) = \mathbb{C} - \Gamma$, hence we have either

$$h(B(x,s)) = \text{interior of } \Gamma, \quad \text{or} \quad h(B(x,s)) = \text{exterior of } \Gamma.$$

The second case is impossible, since the closure of $h(B(x,s))$ is $h(\overline{B}(x,s))$, which is compact, and the closure of the exterior of Γ is non-compact.

Hence $h(B(x,s))$ is the interior of Γ, and $h(\overline{B}(x,s))$ is the closure of the interior of Γ. By Jordan–Schoenflies' theorem, we know that these sets are (respectively) homeomorphic to an open (respectively closed) disc on the plane, and this ends the proof of Lemma 2.3. ☐

Proof of Proposition 2.1. By the structure of smooth surface of Σ, we know that we can construct a decreasing sequence of open sets (U_i), such that every U_i is homeomorphic to an open disc, with

$$\{x\} = \bigcap_{i \in \mathbb{N}} U_i.$$

We have

$$0 = \omega^+\left(\bigcap_{i \in \mathbb{N}} (U_i - \{x\})\right) = \lim_{i \to \infty} \omega^+(U_i - \{x\}),$$

hence there exists some $i_0 \in \mathbb{N}$ with $\omega^+(U_{i_0} - \{x\}) < \pi$. Consider some $r > 0$ such that $\overline{B}(x,r) \subset U_{i_0}$: we can apply Lemma 2.3, and for every $s \leqslant r$, $\overline{B}(x,s)$ is homeomorphic to a closed disc. This ends the proof of Proposition 2.1. ☐

We can then define the following *contractibility radius*:

$$\text{cont}(\Sigma, d, x) := \sup\{r > 0 \mid \text{for every } s < r, \overline{B}(x,s) \text{ is homeomorphic to a closed disc}\}$$

(by definition, $\text{cont}(\Sigma, d, x) > 0$) and

$$\text{cont}(\Sigma, d) := \inf_{x \in \Sigma} \text{cont}(\Sigma, d, x).$$

Since $\overline{B}(x, \text{diam}\,\Sigma) = \Sigma$ is not homeomorphic to a closed disc, we have the inequalities

$$\text{cont}(\Sigma, d, x) \leqslant \text{diam}\,\Sigma \quad \text{and} \quad \text{cont}(\Sigma, d) \leqslant \text{diam}\,\Sigma.$$

Remark 2.4. An easy application of Lemma 2.3 is the following: a sufficient condition for having $\text{cont}(\Sigma, d) > 0$ is $\omega^+(\{x\}) < \pi$ for every $x \in \Sigma$. This will not be used in the sequel. See [8] for more details.

A lower bound for $\text{cont}(\Sigma, d)$ avoids a pinching of the surface at the point x. In the picture below, $\text{cont}(\Sigma, d, x)$ can be arbitrarily small:

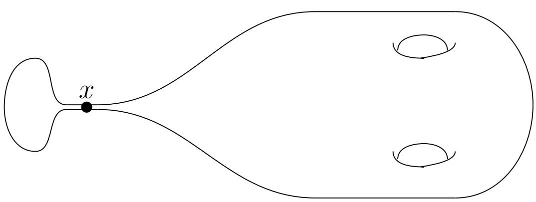

Figure 2. The surface is pinched at the point x.

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Remark 2.5. To avoid a pinching of the surface, we could have defined the natural quantity $\sup\{r > 0 \,|\overline{B}(x,r)$ is homeomorphic to a closed disc$\}$, but it is not relevant, since it is not small in the example given above.

The definition of the contractibility radius deals with *closed* balls. To be able to apply Theorem 1.6 with the *open* balls $B(x,r)$, we need the following.

Proposition 2.6. *For every* $s < \mathrm{cont}(\Sigma, d, x)$*,* $B(x,s)$ *is homeomorphic to an open disc.*

Proof. Let $r \in (s, \mathrm{cont}(\Sigma, d, x))$. Let H a homeomorphism between $\overline{B}(x,r)$ and the closed unit disc $\overline{D}(1)$. Since $H(B(x,s))$ is an open set of the plane, by the Riemann mapping theorem, we only need to show that $H(B(x,s))$ is simply connected.

Let $\gamma : \mathbb{S}^1 \to H(B(x,s))$ be a continuous simple curve. By compactness, the curve $H^{-1}(\gamma)$ in Σ is included in some ball $\overline{B}(x, s-\iota)$ for some $\iota > 0$. Then γ is in $H(\overline{B}(x, s-\iota))$, which is homeomorphic to a closed disc, hence simply connected. γ is homotopic to zero in $H(\overline{B}(x, s-\iota))$, hence is homotopic to zero in $H(B(x,s))$ and this ends the proof. □

Example 2.7 (Euclidean cone). We easily see that an Euclidean cone of cone angle $\theta \in (\pi, 2\pi)$ (that is with curvature $k \in (0, \pi)$) has infinite contractibility radius at every point. On the other hand, if the cone angle is $\theta \in (0, \pi)$ (that is, the curvature is $k \in (\pi, 2\pi)$), then if x is at distance r from the cone point, then $\mathrm{cont}(\Sigma, d)(x) < r$. Hence in that case the contractibility radius of (Σ, d) is zero. See [8] for more details.

Example 2.8 (Riemannian metric). Let g be a smooth Riemannian metric on Σ, and d_g be the associated distance. Then at every point $x \in \Sigma$, the exponential map at x is a homeomorphism between a closed disc in the plane and the closed ball with center x and radius r, hence we have the inequality $\mathrm{cont}(\Sigma, d_g) \geqslant \mathrm{inj}(\Sigma, g)$. Conversely, a well-known result by W. Klingenberg and the Gauss–Bonnet formula imply the following property: for every $\Lambda > 0$ and $c > 0$, there exists an explicit constant $i = i(\Lambda, c)$ such that every smooth Riemannian metric g on Σ with $|\,\mathrm{K}_g\,| \leqslant \Lambda$ and $\mathrm{cont}(\Sigma, d_g) \geqslant c$ satisfies $\mathrm{inj}(\Sigma, g) \geqslant i$. This proves that in the classical Cheeger–Gromov compactness theorem, one can replace a lower bound on the injectivity radius by a lower bound on the contractibility radius. See [8] for more details.

3. Some results on surfaces with B.I.C.

We prove two results for surfaces with B.I.C., needed in the proof of the Main theorem.

3.1. On the area of balls

We need to find an upper bound (respectively, a lower bound) for the area of balls of radius r in surfaces with B.I.C. In Riemannian geometry, this is a well-known fact that a lower bound (respectively, an upper bound) on the sectional curvature is sufficient. To generalize such results for surfaces with B.I.C., we need to have a property in Riemannian geometry which depends only on the curvature measure ω, and not on the pointwise (Gauss) curvature. In [18], Shioya proves the following:

Theorem 3.1. *Let (Σ, g) be a closed Riemannian surface, and let $x \in M$.*

(1) *Let $r > 0$. We have*

$$\mathcal{A}rea(B(x,r)) \leqslant (2\pi + \omega^-(B(x,r)))r^2/2,$$

where $\omega^-(B(x,r)) = \int_{B(x,r)} \mathrm{K}_g^- \, d\mathcal{A}_g$ is the non-positive part of the curvature of $B(x,r)$.

(2) *Let $r > 0$ such that $\mathrm{cont}(\Sigma, d_g, x) \geqslant r$ (that is for every $s < r$, $\overline{B}(x,s)$ is homeomorphic to a closed disc). Then*

$$\mathcal{A}rea(B(x,r)) \geqslant (2\pi - \omega^+(B(x,r)))r^2/2,$$

where $\omega^+(B(x,r)) = \int_{B(x,r)} \mathrm{K}_g^+ \, d\mathcal{A}_g$ is the non-negative part of the curvature of $B(x,r)$.

As a direct consequence, we obtain the

Corollary 3.2. *Let (Σ, d) be a surface with B.I.C., and let $x \in \Sigma$.*

(1) *Let $r > 0$. We have*

$$\mathcal{A}rea(B(x,r)) \leqslant (2\pi + \omega^-(B(x,r)))r^2/2.$$

(2) *Let $r > 0$ such that $\mathrm{cont}(\Sigma, d, x) \geqslant r$. Then*

$$\mathcal{A}rea(B(x,r)) \geqslant (2\pi - \omega^+(B(x,r)))r^2/32.$$

Please note that the second inequality is not optimal, since $r^2/2$ is replaced by $r^2/32$.

Proof of Corollary 3.2. Let g_m be a sequence of Riemannian metrics on Σ, such that $d_m := d_{g_m} \to d$ uniformly on Σ when m goes to infinity. Let $d\mathcal{A}_m$ (respectively $d\mathcal{A}$) be the area measure for (Σ, d_m) (respectively (Σ, d)), and let $\omega_m = \omega_m^+ - \omega_m^-$ (respectively $\omega = \omega^+ - \omega^-$) be the curvature measure of (Σ, d_m) (respectively (Σ, d)), with its non-negative and non-positive parts. We know that $d\mathcal{A}_m \to d\mathcal{A}$ and $\omega_m \to \omega$, weakly on Σ (see [16, Theorems 8.1.9 and 8.4.3]). We do not necessarily have $\omega_m^+ \to \omega^+$ and $\omega_m^- \to \omega^-$ (weakly on Σ), but we can choose a sequence of metrics (g_m) satisfying these properties, see [3, 16].

In the sequel we use the following classical property of converging measures: if U is an open set and $K \subset U$ is a compact set, and if we have a (weakly) convergence of non-negative measures $\mu_m \to \mu$, then for every $\varepsilon > 0$, for m large enough we have $\mu(K) < \mu_m(U) + \varepsilon$ and $\mu_m(K) < \mu(U) + \varepsilon$.

Proof of (1). Remark that $\cup_{\varepsilon>0} \overline{B}(x, r-\varepsilon) = B(x,r)$, so

$$\mathcal{A}rea(B(x,r)) = \lim_{\varepsilon \to 0} \mathcal{A}rea(\overline{B}(x, r-\varepsilon)).$$

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Now, let $\varepsilon > 0$. For m large enough we have

$$\mathcal{A}rea(\overline{B}(x, r-\varepsilon)) < \mathcal{A}rea_m(B(x, r-3\varepsilon/4)) + \varepsilon,$$

and with $B(x, r-3\varepsilon/4) \subset B_m(x, r-\varepsilon/2)$ for m large enough we get

$$\mathcal{A}rea(\overline{B}(x, r-\varepsilon)) < \mathcal{A}rea_m(B_m(x, r-\varepsilon/2)) + \varepsilon.$$

Using Theorem 3.1 with $r-\varepsilon/2$ we obtain

$$\mathcal{A}rea(\overline{B}(x, r-\varepsilon)) \leqslant (2\pi + \omega_m^-(B_m(x, r-\varepsilon/2)))(r-\varepsilon/2)^2/2 + \varepsilon,$$

and with $B_m(x, r-\varepsilon/2) \subset B(x, r-\varepsilon/4)$ for m large enough we get

$$\begin{aligned}\mathcal{A}rea(\overline{B}(x, r-\varepsilon)) &\leqslant (2\pi + \omega_m^-(B(x, r-\varepsilon/4)))(r-\varepsilon/2)^2/2 + \varepsilon \\ &\leqslant (2\pi + \omega_m^-(B(x, r-\varepsilon/4)))r^2/2 + \varepsilon.\end{aligned}$$

For m large enough we have $\omega_m^-(B(x, r-\varepsilon/4)) \leqslant \omega^-(B(x, r)) + \varepsilon$, hence we obtain

$$\mathcal{A}rea(\overline{B}(x, r-\varepsilon)) \leqslant (2\pi + \omega^-(B(x, r)) + \varepsilon)r^2/2 + \varepsilon,$$

and letting $\varepsilon \to 0$ this ends the proof.

Proof of (2). The assertion is trivial if $\omega^+(B(x, r)) \geqslant 2\pi$, so we may assume $\omega^+(B(x, r)) < 2\pi$. We cannot directly apply Theorem 3.1 for the Riemannian metrics d_m, because we may not have $\mathrm{cont}(\Sigma, d_m, x) \geqslant r$ (this is the reason why $r^2/2$ is replaced by $r^2/32$). Let $y \in B(x, r)$ be some point with $d(x, y) = r/2$: we have

$$B(x, r/4) \bigcap B(y, r/4) = \emptyset \quad \text{and} \quad B(x, r/4) \bigcup B(y, r/4) \subset B(x, r).$$

Since $\omega^+(B(x, r)) < 2\pi$, this shows that we have $\omega^+(B(x, r/4)) < \pi$, or $\omega^+(B(y, r/4)) < \pi$. Let z ($z = x$ or y) be a point with

$$\omega^+(B(z, r/4)) < \pi. \tag{2}$$

Let $\varepsilon > 0$. After passing to a subsequence, we may assume

$$\overline{B}_m(z, r/4-\varepsilon) \subset B(z, r/4-\varepsilon/2)$$

and with equation (2) we may also assume

$$\omega_m^+(B(z, r/4-\varepsilon/2)) < \pi.$$

Let $U := B(z, r/4-\varepsilon/2)$. U is homeomorphic to an open disc, and we have $\overline{B}_m(z, r/4-\varepsilon) \subset U$ and $\omega_m^+(U - \{z\}) \leqslant \omega_m^+(U) < \pi$. We can then apply Lemma 2.3 to obtain

$$\mathrm{cont}(\Sigma, d_m, z) \geqslant r/4 - \varepsilon.$$

Now we can apply Theorem 3.1, with the metric d_m, the point z and the radius $r/4-\varepsilon$: for m large enough we have

$$\mathcal{A}rea_m(B_m(z, r/4-\varepsilon)) \geqslant (2\pi - \omega_m^+(B_m(z, r/4-\varepsilon)))(r/4-\varepsilon)^2/2. \tag{3}$$

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For m large enough we also have $B_m(z, r/4-\varepsilon) \subset B(x, r-\varepsilon)$, so for m large enough we get

$$\begin{cases} \mathcal{A}rea(B(x,r)) \geqslant \mathcal{A}rea_m(B(x, r-\varepsilon)) - \varepsilon \\ \omega^+(B(x,r)) \geqslant \omega_m^+(B(x, r-\varepsilon)) - \varepsilon, \end{cases}$$

which gives

$$\begin{cases} \mathcal{A}rea(B(x,r)) \geqslant \mathcal{A}rea_m(B_m(z, r/4-\varepsilon)) - \varepsilon \\ \omega^+(B(x,r)) \geqslant \omega_m^+(B_m(z, r/4-\varepsilon)) - \varepsilon. \end{cases}$$

With equation (3) we get

$$\mathcal{A}rea(B(x,r)) \geqslant (2\pi - \omega^+(B(x,r)) - \varepsilon)(r/4-\varepsilon)^2/2 - \varepsilon,$$

and letting $\varepsilon \to 0$ this ends the proof. □

3.2. An upper bound for the length of a line segment

We want to find an upper bound for the length of a line segment, for a singular Riemannian metric which has 'no harmonic term', that is when $g = e^{2V_\omega(z)}|dz|^2$ for some Radon measure ω.

First, consider a Riemannian metric on $D(1/2)$ with a conical singularity at 0: $g = |z|^{2\beta}|dz|^2$ (for some $\beta > -1$), and let γ be the line segment joining 0 and a point $z \in D(1/2)$. The length of γ is

$$L(\gamma) = \int_0^1 |tz|^\beta |z|\, dt = \frac{1}{1+\beta}|z|^{1+\beta}.$$

Moreover, the curvature measure is $\omega = -2\pi\beta \cdot \delta_0$ (where δ_0 is the Dirac mass at 0), so the non-negative part of the curvature measure is $\omega^+(D(1/2)) = \max(0, -2\pi\beta) = 2\pi\beta^-$ and we get $1+\beta \geqslant 1-\beta^- = 1-\omega^+(D(1/2))/2\pi$, hence

$$L(\gamma) \leqslant \frac{1}{1-\omega^+(D(1/2))/2\pi}|z|^{1-\omega^+(D(1/2))/2\pi}.$$

The next proposition shows how we can extend this result to arbitrary curvature measures.

Proposition 3.3. *Let ω be a Radon measure defined in $D(1/2)$, with $\omega^+(D(1/2)) < 2\pi$. Let $z, z' \in D(1/2)$, and let $\gamma(t) := (1-t)z + tz'$ be the line segment $[zz']$. Let $L(\gamma)$ be the length of this line segment for the singular metric $g = e^{2V_\omega(z)}|dz|^2$. Then we have*

$$L(\gamma)\left(= \int_0^1 e^{V_\omega(\gamma(t))}|z-z'|\, dt\right) \leqslant \frac{2}{1-\omega^+(D(1/2))/2\pi}|z-z'|^{1-\omega^+(D(1/2))/2\pi}. \tag{4}$$

Proof. *First step.* We first show that this is sufficient to prove the proposition in the case where ω is a sum of Dirac masses. If so, let ω be a Radon measure with $\omega^+(D(1/2)) < 2\pi$, and write ω as $\omega = \omega^+ - \omega^-$, where ω^+ and ω^- are non-negative Radon measures. Let ω_m^+ and ω_m^- be a sequence of sums of Dirac masses such that $\omega_m^+ \to \omega^+$ and $\omega_m^- \to \omega^-$ weakly, and let $\omega_m := \omega_m^+ - \omega_m^-$.

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Let $L_m(\gamma)$ be the length of the line segment γ for the singular metric $g_m = e^{2V_{\omega_m}(z)}|dz|^2$. For almost every $t \in [0, 1]$ we have

$$V_{\omega_m}(\gamma(t)) = \iint_{D(1/2)} \left(\frac{-1}{2\pi}\right) \ln|\gamma(t) - \xi|\, d\omega_m(\xi) \underset{m\to\infty}{\longrightarrow} \iint_{D(1/2)} \left(\frac{-1}{2\pi}\right) \ln|\gamma(t) - \xi|\, d\omega(\xi) = V_\omega(\gamma(t)),$$

hence by Fatou's lemma we get

$$L(\gamma) = \int_0^1 e^{V_\omega(\gamma(t))}|z - z'|\, dt = \int_0^1 \liminf_{m\to\infty}(e^{V_{\omega_m}(\gamma(t))}|z - z'|)\, dt \leqslant \liminf_{m\to\infty}\left(\int_0^1 e^{V_{\omega_m}(\gamma(t))}|z - z'|\, dt\right) = \liminf_{m\to\infty} L_m(\gamma).$$

If we apply inequality (4) with the measures ω_m (which are sums of Dirac masses, with $\omega_m^+(D(1/2)) < 2\pi$ for m large enough), we get

$$L_m(\gamma) \leqslant \frac{2}{1 - \omega_m^+(D(1/2))/2\pi}|z - z'|^{1-\omega_m^+(D(1/2))/2\pi},$$

hence

$$L(\gamma) \leqslant \liminf_{m\to\infty}\left(\frac{2}{1 - \omega_m^+(D(1/2))/2\pi}|z - z'|^{1-\omega_m^+(D(1/2))/2\pi}\right) = \frac{2}{1 - \omega^+(D(1/2))/2\pi}|z - z'|^{1-\omega^+(D(1/2))/2\pi}.$$

Second step. We may now assume that ω is a sum of Dirac masses: there exists $p_1, \ldots, p_n \in D(1/2)$ and $k_1, \ldots, k_n \in \mathbb{R}$ such that

$$\omega = \sum_{s=1}^n k_s \delta_{p_s} \quad \text{and} \quad \omega^+(D(1/2)) = \sum_{s=1}^n k_s^+ < 2\pi$$

(δ_{p_s} denotes the Dirac mass at p_s). For almost every $z \in D(1/2)$ we have

$$V_\omega(z) = \iint_D \left(\frac{-1}{2\pi}\right) \ln|z - \xi|\, d\omega(\xi) = \sum_{s=1}^n \left(\frac{-k_s}{2\pi}\right) \ln|z - p_s| \leqslant \sum_{s=1}^n \left(\frac{-k_s^+}{2\pi}\right) \ln|z - p_s|,$$

so if we set $\beta_s := -k_s/2\pi$ we have $\beta_s^- = \max(0, -\beta_s) = \max(0, k_s/2\pi) = k_s^+/2\pi$, hence

$$e^{V_\omega(z)} \leqslant \prod_{s=1}^n |z - p_s|^{-\beta_s^-},$$

and this gives

$$L(\gamma) = \int_0^1 e^{V_\omega(\gamma(t))}|z - z'|\, dt \leqslant \int_0^1 \left(\prod_{s=1}^n |\gamma(t) - p_s|^{-\beta_s^-}\right)|z - z'|\, dt.$$

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Let $S := \{s \text{ such that } \beta_s^- > 0\}$. If $S = \emptyset$, then $\beta_s^- = 0$ for every s: we have $\omega^+(D(1/2)) = 0$, and the last inequality shows that $L(\gamma) \leqslant |z - z'|$, so inequality (4) is true.

Hence we may assume $S \neq \emptyset$. Let $M := -\sum_{s\in S} \beta_s^-$: by hypothesis we have $-1 < M < 0$. For $s \in S$, let $q_s := -M/\beta_s^-$: we have

$$q_s \geqslant 1 \quad \text{and} \quad \sum_{s\in S} \frac{1}{q_s} = 1.$$

Since $|\gamma(t) - p_s|^{-\beta_s^-} \leqslant 1$ if $\beta_s^- \leqslant 0$, we can apply Hölder's inequality as follows:

$$\begin{aligned}\int_0^1 \left(\prod_{s=1}^n |\gamma(t) - p_s|^{-\beta_s^-}\right) dt &\leqslant \int_0^1 \left(\prod_{s\in S} |\gamma(t) - p_s|^{-\beta_s^-}\right) dt \qquad (5)\\ &\leqslant \prod_{s\in S}\left(\int_0^1 |\gamma(t) - p_s|^{-q_s\beta_s^-}\, dt\right)^{1/q_s}\\ &= \prod_{s\in S}\left(\int_0^1 |\gamma(t) - p_s|^{M}\, dt\right)^{1/q_s}. \qquad (6)\end{aligned}$$

Now, fix some $s \in S$ and consider $\int_0^1 |\gamma(t) - p_s|^M dt$. Let p_s' be the projection of the point p_s on the line (zz'):

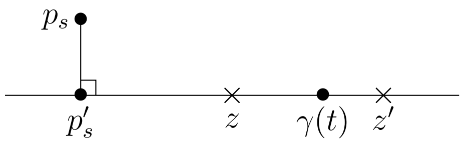

Figure 3.

We have $|\gamma(t) - p_s'| \leqslant |\gamma(t) - p_s|$, so with $M < 0$ we get

$$\int_0^1 |\gamma(t) - p_s|^M\, dt \leqslant \int_0^1 |\gamma(t) - p_s'|^M\, dt.$$

Since p_s' belongs to the line (zz'), we can write $p_s' = (1 - \lambda_s)z + \lambda_s z'$ for some $\lambda_s \in \mathbb{R}$, hence

$$|\gamma(t) - p_s'| = |(1-t)z + tz' - (1-\lambda_s)z - \lambda_s z'| = |t - \lambda_s||z - z'|.$$

Moreover, we easily see that the function $\lambda \mapsto \int_0^1 |t - \lambda|^M\, dt$ admits its maximum for $\lambda = 1/2$ (recall that $M < 0$), hence

$$\int_0^1 |t - \lambda_s|^M\, dt \leqslant \int_0^1 |t - \frac{1}{2}|^M\, dt = \frac{1}{(M+1)2^{M+1}},$$

so we obtain

$$\int_0^1 |\gamma(t) - p_s|^M\, dt \leqslant \frac{|z - z'|^M}{(M+1)2^M}.$$

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If we put this in inequality (6), with $\sum_{s\in S} 1/q_s = 1$ we get

$$\int_0^1 \left(\prod_{s\in S} |\gamma(t) - p_s|^{-\beta_s^-}\right) dt \leqslant \frac{|z-z'|^M}{(M+1)2^M}.$$

Since $M > -1$ we have $2^M \geqslant 1/2$, and with the equality $M = -\omega^+(D(1/2))/2\pi$ we obtain

$$L(\gamma) \leqslant \frac{2}{1-\omega^+(D(1/2))/2\pi} |z-z'|^{1-\omega^+(D(1/2))/2\pi}. \qquad \square$$

4. Preliminary properties of $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$

Before starting the proof of the Main theorem, we prove some important preliminary properties for the set $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$.

4.1. Another definition of $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$

By analogy with Cheeger–Gromov's convergence theorem, where we can replace a bound on the volume by a bound on the diameter, the following proposition shows that in the Main theorem, we can replace a bound on the area by a bound on the diameter and on the total curvature (this is Corollary 0.3). Recall that $|\omega|(\Sigma, d) = \omega^+(\Sigma, d) + \omega^-(\Sigma, d)$.

Proposition 4.1. *Let Σ be a closed surface and let $c, \varepsilon, \delta > 0$. Let d be a metric with B.I.C. on Σ, satisfying properties (1) and (2) in the definition of $\mathcal{M}_\Sigma$, that is*

(1) *for every $x \in \Sigma, \omega^+(B(x, \varepsilon)) \leqslant 2\pi - \delta$*
(2) *$cont(\Sigma, d) \geqslant c$.*

Then:

- *for every $A > 0$, there exists some positive constants D and Ω such that*

$$\mathcal{A}rea(\Sigma, d) \leqslant A(\Longleftrightarrow d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)) \implies \begin{cases} \operatorname{diam}(\Sigma, d) \leqslant D \\ |\omega|(\Sigma, d) \leqslant \Omega. \end{cases}$$

- *Conversely, for every $\Omega, D > 0$, there exists a positive constant A such that*

$$\left.\begin{matrix} \operatorname{diam}(\Sigma, d) \leqslant D \\ |\omega|(\Sigma, d) \leqslant \Omega \end{matrix}\right\} \implies \mathcal{A}rea(\Sigma, d) \leqslant A(\Longleftrightarrow d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)).$$

Proof. To prove the first property, consider some $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$. Let $B(x_i, \varepsilon/2)$, for $i \in \{1, \dots, N\}$, be a maximal number of disjoint balls of radius $\varepsilon/2$ in Σ. By Corollary 3.2, all these balls have an area bounded below; since the area of (Σ, d) is bounded above, the integer N is also bounded. If the diameter was arbitrarily large, then we could find an arbitrarily large number of disjoint balls: this shows that there exists some $D > 0$ such that $\operatorname{diam}(\Sigma, d) \leqslant D$. And by an elementary covering argument, the N balls $B(x_i, \varepsilon)$ cover Σ; but the non-negative curvatures of these balls are bounded above, so the

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non-negative curvature $\omega^+(\Sigma, d)$ of Σ is also bounded above. The Gauss–Bonnet formula gives us $\omega^+(\Sigma, d) - \omega^-(\Sigma, d) = 2\pi\chi(\Sigma)$ (where $\chi(\Sigma)$ is the Euler characteristic of Σ), so $\omega^-(\Sigma, d)$ is also bounded above, and this shows that there exists some $\Omega > 0$ such that $|\omega|(\Sigma, d) \leqslant \Omega$.

Conversely, if $\mathrm{diam}(\Sigma, d) \leqslant D$ and $|\omega|(\Sigma, d) \leqslant \Omega$, then Σ is equal to some ball $B(x, D+1)$. Since $\omega^-(\Sigma)$ is bounded, by Corollary 3.2, we know that the area of such a ball is bounded above, and this shows that there exists a constant $A > 0$ such that $\mathcal{A}rea(\Sigma, d) \leqslant A$. □

From now on, we fix a closed surface Σ and $A, c, \varepsilon, \delta > 0$; we then have some positive constants D and Ω satisfying the first part in Proposition 4.1.

4.2. A bound for the harmonic term

This section is devoted to the proof of the following theorem, which gives a bound for the harmonic term h when we express (locally) any metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$ as a singular Riemannian metric $g = e^{2V_\omega + 2h}|dz|^2$. This bound has to be *uniform*, that is independent of the metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$. This theorem is very important in the sequel and relies on the conformal geometry of an annulus (see the Appendix).

Theorem 4.2. *Let $K \subset D(1/2)$ be a compact set. There exists a constant $M(K) = M(K, \Sigma, A, c, \varepsilon, \delta)$ satisfying the following property.*

Let d be a metric in $\mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, and let $H : B(x, \varepsilon) \to D(1/2)$ be a conformal chart, with $H(x) = 0$. As usual, we denote by h the harmonic term for the metric in this chart (see Theorem 1.6). We then have

$$|h(z)| \leqslant M(K) \quad \textit{for every } z \in K.$$

We first give an explicit *upper* bound for h, which will be used in the next section.

Proposition 4.3. *Under the hypothesis of Theorem 4.2 we have, for every $z \in D(1/2)$,*

$$e^{h(z)} \leqslant \frac{\varepsilon}{(1/2 - |z|)^{1+\Omega/2\pi}} \cdot C(\Omega), \tag{7}$$

where $C(\Omega) := \sqrt{1 + \Omega/2\pi} \cdot e^{\Omega/4\pi}$.

Proof. Let $a \in D(1/2)$, and set $s = 1/2 - |a| > 0$: we have $D(a, s) \subset D(1/2)$. Let $u := V_{\omega_H} + h$, so that the singular Riemannian metric reads $g = e^{2u}|dz|^2$. By Jensen's inequality we get

$$\exp\left(\iint_{D(a,s)} 2u(z) \frac{d\lambda(z)}{\pi s^2}\right) \leqslant \iint_{D(a,s)} e^{2u(z)} \frac{d\lambda(z)}{\pi s^2} \leqslant \frac{\mathcal{A}rea(B(x, \varepsilon))}{\pi s^2},$$

and by Corollary 3.2 we have $\mathcal{A}rea(B(x, \varepsilon)) \leqslant (\pi + \Omega/2) \cdot \varepsilon^2$, hence

$$\frac{1}{\pi s^2} \iint_{D(a,s)} 2u(z)\, d\lambda(z) \leqslant \ln\left((1 + \Omega/2\pi)\frac{\varepsilon^2}{s^2}\right).$$

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The function h is harmonic, hence it satisfies the mean-value property:

$$h(a) = \frac{1}{\pi s^2} \iint_{D(a,s)} h(z)\, d\lambda(z) = \frac{1}{\pi s^2} \iint_{D(a,s)} u(z)\, d\lambda(z) - \frac{1}{\pi s^2} \iint_{D(a,s)} V_{\omega_H}(z)\, d\lambda(z) \quad (8)$$

$$\leqslant \frac{1}{2} \ln\left((1+\Omega/2\pi)\frac{\varepsilon^2}{s^2}\right) - \frac{1}{\pi s^2} \iint_{D(a,s)} V_{\omega_H}(z)\, d\lambda(z), \quad (9)$$

and to conclude we need to find a lower bound for $\iint_{D(a,s)} V_{\omega_H}(z)\, d\lambda(z)$.

We know that $V_{\omega_H}(z) = V_{\omega_H^+}(z) - V_{\omega_H^-}(z) \geqslant -V_{\omega_H^-}(z)$ for almost every $z \in D(1/2)$, hence

$$\iint_{D(a,s)} V_{\omega_H}(z)\, d\lambda(z) \geqslant -\iint_{D(a,s)} \left(\iint_{D(1/2)} \left(\frac{-1}{2\pi}\right) \ln|z-\xi|\, d\omega_H^-(\xi)\right) d\lambda(z) \quad (10)$$

$$= -\iint_{D(1/2)} \left(\iint_{D(a,s)} \left(\frac{-1}{2\pi}\right) \ln|z-\xi|\, d\lambda(z)\right) d\omega_H^-(\xi). \quad (11)$$

Moreover, for $\xi \in D(1/2)$ we have

$$\iint_{D(a,s)} \left(\frac{-1}{2\pi}\right) \ln|z-\xi|\, d\lambda(z) = \iint_{D(a-\xi,s)} \left(\frac{-1}{2\pi}\right) \ln|z|\, d\lambda(z),$$

and we easily see that this function of ξ is maximum for $\xi = a$ (that is when the disc is centered in 0). So for every $\xi \in D(1/2)$,

$$\iint_{D(a,s)} \left(\frac{-1}{2\pi}\right) \ln|z-\xi|\, d\lambda(z) \leqslant \iint_{D(0,s)} \left(\frac{-1}{2\pi}\right) \ln|z|\, d\lambda(z)$$

$$= -\int_0^s r \ln r\, dr = \frac{s^2}{4} - \frac{s^2 \ln s}{2}.$$

Using inequality (11) we get

$$\iint_{D(a,s)} V_{\omega_H}(z)\, d\lambda(z) \geqslant \left(-\frac{s^2}{4} + \frac{s^2 \ln s}{2}\right) \cdot \omega_H^-(D) \geqslant \left(-\frac{s^2}{4} + \frac{s^2 \ln s}{2}\right) \cdot \Omega$$

(recall that Ω satisfies $\omega_H^+(D) + \omega_H^-(D) \leqslant \Omega$). With the inequality (9) we obtain

$$h(a) \leqslant \frac{1}{2} \ln\left((1+\Omega/2\pi)\frac{\varepsilon^2}{s^2}\right) + \frac{\Omega}{4\pi} - \frac{\Omega \ln s}{2\pi}$$

and this ends the proof. □

We now prove that the image by the conformal chart H of the ball $B(x, \varepsilon/2)$ cannot go close to the boundary of the disc $D(1/2)$. We use the results stated in the Appendix, by looking at the annulus $D(1/2) - H(\overline{B}(x, \varepsilon/2))$. On the one hand, by definition, this annulus has a modulus bounded below; and on the other hand, by Grötzsch's theorem (see Theorem A.2 in the Appendix), if $H(\overline{B}(x, \varepsilon/2)$ was arbitrarily close to $\partial D(1/2)$, then the modulus of $D(1/2) - H(\overline{B}(x, \varepsilon/2))$ would be arbitrarily close to zero. We need this result to prove Theorem 4.2, but we also need it later (see Theorem 4.8).

Proposition 4.4. *Let $r < 1/2$ be such that the modulus of the Grötzsch annulus $G(2r)$ satisfies $\mathrm{mod}(G(2r)) < \varepsilon^2/4A$ (see the Appendix). Let $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, and let $H : B(x, \varepsilon) \to D(1/2)$ be a conformal chart, with $H(x) = 0$. Then*

$$H(B(x, \varepsilon/2)) \subset D(r).$$

Proof. The following subset of $\mathbb{C}$:

$$U := D(1/2) - H(\overline{B}(x, \varepsilon/2))$$

is an annulus (in the sense given in the Appendix, see Definition A.1). We know that H is an isometry between $(B(x, \varepsilon), d_{|B(x,\varepsilon)})$ and $(D(1/2), d_{\omega_H,h})$. Recall that the modulus of U is

$$\mathrm{mod}(U) = \sup_\rho \frac{\inf_{\gamma\in\Gamma} L_\rho(\gamma)^2}{A_\rho(U)},$$

(see the Appendix), where Γ is the set of continuous simple curves γ in U, parametrized by arc length, joining $\partial D(1/2)$ and a point in $H(\overline{B}(x, \varepsilon/2))$. We take $\rho := e^{V_{\omega_H}+h}$.

For every $\gamma \in \Gamma$, we cannot be sure that the ρ-length of γ is equal to its d-length. But we know that, by definition of the length distance $d_{\omega_H,h}$ (let us recall here that $\gamma(0) \notin D(1/2)$) :

$$\begin{aligned} L_\rho(\gamma) \geqslant \limsup_{t\to 0} d_{\omega_H,h}(\gamma(t), \gamma(1)) &= \limsup_{t\to 0} d_{B(x,\varepsilon)}(H^{-1}(\gamma(t)), H^{-1}(\gamma(1))) \\ &\geqslant \limsup_{t\to 0} d(H^{-1}(\gamma(t)), H^{-1}(\gamma(1))) \end{aligned}$$

(the last inequality is a direct consequence of the definition of the induced metric). Consider a sequence $t_k \to 0$ such that $H^{-1}(\gamma(t_k))$ converges to $y \in \Sigma$: since $\gamma(0) \notin D(1/2)$, we have $y \notin B(x, \varepsilon)$, and since $H^{-1}(\gamma(1)) \in \overline{B}(x, \varepsilon/2)$ this gives

$$L_\rho(\gamma) \geqslant d(y, H^{-1}(\gamma(1))) \geqslant \varepsilon/2.$$

Since the $\rho-$area of U is less than or equal to the area of (Σ, d) we get

$$\mathrm{mod}(U) \geqslant \frac{(\varepsilon/2)^2}{A} = \varepsilon^2/4A.$$

We prove the lemma by contradiction: assume we have $H(\overline{B}(x, \varepsilon/2)) \not\subset D(r)$. Then, after a rotation (such that the complex number of maximum modulus of $H(\overline{B}(x, \varepsilon/2))$, which is greater than or equal to r, belongs to the real axis), and after an homothety with scale factor 2, we see that U is conformally equivalent to an annulus $U' \subset D(1)$, not containing 0 (since $H(x) = 0$) and $2r$. By Grötzsch's theorem (see Theorem A.2 in the Appendix), we have $\mathrm{mod}(U) = \mathrm{mod}(U') \leqslant \mathrm{mod}(G(2r))$, which is impossible with the choice of r we made. We then have $H(\overline{B}(x, \varepsilon/2)) \subset D(r)$ and this ends the proof of the proposition. □

Let $p^* := 4\pi/\delta > 1$. To obtain a lower bound for the harmonic term h, we need the following lemma, which gives a lower bound for a certain integral involving h:

Lemma 4.5. *Under the hypothesis of Theorem 4.2 we have*

$$\left(\frac{\delta^2\varepsilon^2}{512\pi^2}\right)^{p^*} \leqslant \iint_{D(r)} e^{2p^*h(z)}\, d\lambda(z).$$

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Proof. We use the following proposition, see [20]:

Proposition 4.6 (Troyanov). *Let ν be a non-negative Radon measure defined in $D(1/2)$. Suppose there exists some $p>1$ such that $\nu(D(1/2))<2\pi/p$. Then*

$$\iint_{D(1/2)} e^{2pV_\nu(z)}d\lambda(z) \leqslant \frac{\pi}{1-\frac{p}{2\pi}\nu(D(1/2))}.$$

Let $p>1$ be such that $1/p+1/p^*=1$. By property (1) in the definition of $\mathcal{M}_\Sigma(A,c,\varepsilon,\delta)$ we have

$$\omega_H^+(D(1/2)) = \omega^+(B(x,\varepsilon)) \leqslant 2\pi-\delta = 2\pi(1-2/p^*) = 2\pi(1/p-1/p^*) < 2\pi/p.$$

Proposition 4.6 shows that

$$\iint_{D(1/2)} e^{2pV_{\omega_H^+}(z)}d\lambda(z) \leqslant \frac{\pi}{1-\frac{p}{2\pi}\omega_H^+(D(1/2))},$$

and with $1-\frac{p}{2\pi}\omega_H^+(D(1/2)) \geqslant \frac{p}{p^*} \geqslant \frac{1}{p^*} = \delta/4\pi$ we obtain

$$\iint_{D(1/2)} e^{2pV_{\omega_H^+}(z)}d\lambda(z) \leqslant \frac{4\pi^2}{\delta}. \tag{12}$$

By Proposition 4.4 we have $H(B(x,\varepsilon/2))\subset D(r)$, hence $\iint_{D(r)} e^{2u(z)}d\lambda(z) \geqslant \mathcal{A}rea\,(B(x,\varepsilon/2))$, and Corollary 3.2 shows that

$$\mathcal{A}rea(B(x,\varepsilon/2)) \geqslant (2\pi-\omega^+(B(x,\varepsilon/2)))\cdot(\varepsilon/2)^2/32 \geqslant \delta\varepsilon^2/128.$$

Since $u=V_{\omega_H}+h$, by Hölder's inequality we obtain

$$\begin{aligned}\delta\varepsilon^2/128 \leqslant \mathcal{A}rea(B(x,\varepsilon/2)) &\leqslant \iint_{D(r)} e^{2u(z)}\,d\lambda(z)\\ &\leqslant \left(\iint_{D(r)} e^{2pV_{\omega_H}(z)}\,d\lambda(z)\right)^{1/p}\left(\iint_{D(r)} e^{2p^*h(z)}\,d\lambda(z)\right)^{1/p^*}\\ &\leqslant \left(\iint_{D(1/2)} e^{2pV_{\omega_H}(z)}\,d\lambda(z)\right)^{1/p}\left(\iint_{D(r)} e^{2p^*h(z)}\,d\lambda(z)\right)^{1/p^*}.\end{aligned}$$

With the inequality $V_{\omega_H}(z)=V_{\omega_H^+}(z)-V_{\omega_H^-}(z)\leqslant V_{\omega_H^+}(z)$ valid for almost every $z\in D(1/2)$, and with equation (12) we get

$$\delta\varepsilon^2/128 \leqslant \left(\frac{4\pi^2}{\delta}\right)^{1/p}\cdot\left(\iint_{D(r)} e^{2p^*h(z)}\,d\lambda(z)\right)^{1/p^*} \leqslant \frac{4\pi^2}{\delta}\cdot\left(\iint_{D(r)} e^{2p^*h(z)}\,d\lambda(z)\right)^{1/p^*}$$

(recall that $\frac{4\pi^2}{\delta}>1$) and this ends the proof of the lemma. □

To prove Theorem 4.2, we finally need the following Harnack's lemma for non-negative harmonic functions (see [6]):

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Theorem 4.7 (Harnack's lemma). *Let f_m be a sequence of non-negative harmonic functions on a connected open set $U \subset \mathbb{C}$. We then have the following alternative: either (1) $f_m \to +\infty$ locally uniformly on U, or (2) there exists a subsequence m_j of m such that $f_{m_j} \to f$ locally uniformly on U, where f is a harmonic function on U.*

We can now finish the proof of Theorem 4.2.

Proof of Theorem 4.2. Proposition 4.3 ensures that for every compact set $K \subset D(1/2)$, there exists a constant $M'(K)$ such that under the hypothesis of Theorem 4.2 we have $h(z) \leqslant M'(K)$ for every $z \in K$ (recall that $M'(K)$ is independent of the metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$).

We prove Theorem 4.2 by contradiction. Suppose there exists a compact set $K \subset D(1/2)$ (we may assume $D(r) \subset K$), a sequence $d_m \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, a sequence of points $x_m \in \Sigma$, and a sequence of conformal charts $H_m : B_m(x_m, \varepsilon) \to D(1/2)$, with $H_m(x_m) = 0$, such that the harmonic term h_m for the metric in this chart satisfies

$$\min_{z \in K} h_m(z) \underset{m \to \infty}{\to} -\infty.$$

Choose some $z_m \in K$ such that $\min_{z \in K} h_m(z) = h_m(z_m)$. After passing to a subsequence, we may assume $z_m \to z \in K$. Let U be a connected open set, with $K \subset U \subset \overline{U} \subset D(1/2)$. Since $h_m(z) \leqslant M'(\overline{U})$ for every $z \in U$, we can consider the following sequence of non-negative harmonic functions on U:

$$f_m := M'(\overline{U}) - h_m.$$

Since $f_m(z_m) \to +\infty$, alternative (2) in Theorem 4.7 cannot occur. So we have $f_m \to +\infty$ locally uniformly on U, hence $h_m \to -\infty$ locally uniformly on U. But Lemma 4.5 tells us that

$$\left(\frac{\delta^2 \varepsilon^2}{512\pi^2}\right)^{p^*} \leqslant \iint_{D(r)} e^{2p^* h_m(z)}\, d\lambda(z).$$

This is a contradiction, since the right-hand side term goes to zero as $m \to \infty$, by Lebesgue's dominated convergence theorem: we have $e^{2p^* h_m(z)} \to 0$ for every $z \in D(r) \subset U$, and since $D(r) \subset \overline{U}$, we can dominate $e^{2p^* h_m(z)}$ by the constant $e^{2p^* M'(\overline{U})}$. □

4.3. Conformal images of balls

This section is devoted to the proof of the next theorem. It is a key step in our article, and also relies on the conformal geometry of an annulus. Roughly, it says the following. Let $H : B(x, \varepsilon) \to D(1/2)$ be a conformal chart for some metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, with $H(x) = 0$. Then we have a control on the images of balls of 'large' radii $B(x, \varepsilon/4)$ (that is, we have $D(2\alpha) \subset H(B(x, \varepsilon/4))$), and balls of 'small' radii $B(x, \kappa\varepsilon)$ (that is, we have $H(B(x, \kappa\varepsilon)) \subset D(\alpha)$), for some positive constants α and κ (the picture in the theorem explains the situation). Of course, for any metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$ such constants α and κ do exist, but the hard part of the work is to show that they can be chosen *uniformly*: they do not depend on the metric $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$.

Theorem 4.8. *There exists constants* $\alpha = \alpha(\Sigma, A, c, \varepsilon, \delta) > 0$ *and* $\kappa = \kappa(\Sigma, A, c, \varepsilon, \delta) > 0$ *satisfying the following property. Let* $d \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$*, and let* $H : B(x, \varepsilon) \to D(1/2)$ *be a conformal chart, with* $H(x) = 0$*. We are in the following situation:*

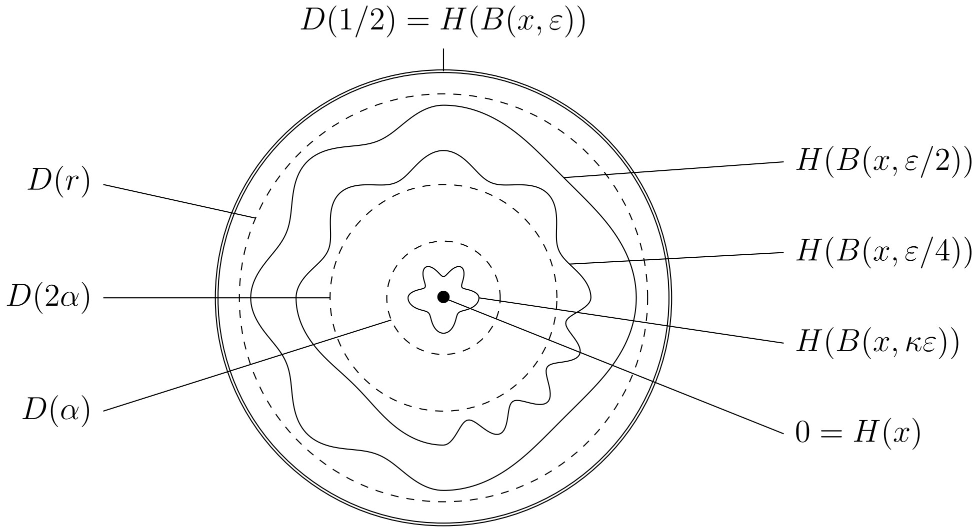

Figure 4.

(The inclusion $H(B(x, \varepsilon/2)) \subset D(r)$ *has already been proved in Proposition 4.4.)*
Thus, this is sufficient to prove the following two properties:

(1) *If* γ *is the line segment joining 0 and a point in* $D(2\alpha)$*, then the length of the curve* $H^{-1}(\gamma)$ *in* Σ *is smaller than* $\varepsilon/4$*. This proves that* $D(2\alpha) \subset H(B(x, \varepsilon/4))$*;*

(2) *we have the inclusion* $H(B(x, \kappa\varepsilon)) \subset D(\alpha)$*.*

We first choose α small enough so that property (1) is true: by Theorem 4.3, we have an (explicit) upper bound for the harmonic function h, and Proposition 3.3 gives an upper bound for the length of a line segment, when there is no harmonic term in the expression of the singular metric.

We then prove a convergence theorem for distances (a corollary of the local convergence theorem due to Reshetnyak, Theorem 0.1). We need it to prove part (2) of Theorem 4.8, but we also need it later (see § 5.5.1).

With this convergence theorem, we are able to choose κ small enough so that the annulus $D(1/2) - H(\overline{B}(x, \kappa\varepsilon))$ has a modulus big enough. Hence by Grötzsch's theorem, $H(\overline{B}(x, \kappa\varepsilon))$ will be 'far away' from the boundary $\partial D(1/2)$, that is we have $H(B(x, \kappa\varepsilon)) \subset D(\alpha)$.

4.3.1. Choice of α. Recall that $C(\Omega) = \sqrt{1 + \Omega/2\pi} \cdot e^{\Omega/4\pi}$ is the constant which appears in Proposition 4.3. Choose $0 < \alpha < 1/8$ such that

$$\frac{(2\alpha)^{\delta/2\pi} \cdot 4\pi}{\delta} \cdot \frac{C(\Omega)}{(1/4)^{1+\Omega/2\pi}} \leqslant 1/4. \tag{13}$$

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Proof of the first part of Theorem 4.8. Let $|z| < 2\alpha$, and $\gamma(t) := tz$ the line segment $[0, z]$. We want to find an upper bound for the length of $H^{-1}(\gamma)$:

$$L(H^{-1}(\gamma)) = \int_0^1 e^{V_{\omega_H}(\gamma(t))+h(\gamma(t))}|z|\,dt.$$

We have the following.

Fact 4.9. For every $z' \in D(2\alpha)$ we have

$$\frac{(2\alpha)^{\delta/2\pi}\cdot 4\pi}{\delta}\cdot e^{h(z')} \leqslant \varepsilon/4.$$

Proof. Proposition 4.3 shows that, for every $z' \in D(1/2)$,

$$e^{h(z')} \leqslant \frac{\varepsilon}{(1/2-|z'|)^{1+\Omega/2\pi}}\cdot C(\Omega),$$

so by multiplying with inequality (13) we get

$$\frac{(2\alpha)^{\delta/2\pi}\cdot 4\pi}{\delta}\cdot\frac{C(\Omega)}{(1/4)^{1+\Omega/2\pi}}\cdot e^{h(z')} \leqslant \varepsilon/4\cdot\frac{C(\Omega)}{(1/2-|z'|)^{1+\Omega/2\pi}}.$$

For every $z' \in D(2\alpha)$ we have $|z'| \leqslant 1/4$, hence $(1/2-|z'|)^{1+\Omega/2\pi} \geqslant (1/4)^{1+\Omega/2\pi}$. After simplification we obtain the inequality announced in Fact 4.9. □

The line segment $[0, z]$ is included in $D(2\alpha)$ so by Fact 4.9 we have $e^{h(\gamma(t))} \leqslant \frac{\delta}{(2\alpha)^{\delta/2\pi}\cdot 4\pi}\varepsilon/4$, hence

$$L(H^{-1}(\gamma)) \leqslant \frac{\delta}{(2\alpha)^{\delta/2\pi}\cdot 4\pi}\cdot\varepsilon/4\cdot\int_0^1 e^{V_{\omega_H}(\gamma(t))}|z|\,dt. \tag{14}$$

Moreover, Proposition 3.3 shows that

$$\begin{aligned}\int_0^1 e^{V_{\omega_H}(\gamma(t))}|z|\,dt &\leqslant \frac{2}{1-\omega_H^+(D(1/2))/2\pi}\cdot|z|^{1-\omega_H^+(D(1/2))/2\pi}\\ &= \frac{2}{1-\omega^+(B(x,\varepsilon))/2\pi}\cdot|z|^{1-\omega^+(B(x,\varepsilon))/2\pi}.\end{aligned}$$

With the inequality $\omega^+(B(x,\varepsilon)) \leqslant 2\pi-\delta$ we obtain $\int_0^1 e^{V_{\omega_H}(\gamma(t))}|z|\,dt < \frac{4\pi}{\delta}\cdot(2\alpha)^{\delta/2\pi}$, and with (14) we finally obtain $L(H^{-1}(\gamma)) < \varepsilon/4$. This ends the proof of the first part of Theorem 4.8. □

4.3.2. Convergence of metrics: a corollary of Theorem 0.1.

We now prove a corollary of Theorem 0.1, which is needed to finish the proof of Theorem 4.2. This result will also be a key step at the end of this article (see § 5.5.1).

Corollary 4.10. *Let $d_m \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$ be a sequence of metrics, and for every $x_m \in \Sigma$, consider some conformal chart $H_m : B_m(x_m, \varepsilon) \to D(1/2)$, with $H_m(x_m) = 0$. Let ω_m be*

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the curvature measure of (Σ, d_m)*, and let* $\omega_{H_m} := (H_m)_\#\omega_m$ *be the measure and* h_m *be the harmonic function such that* H_m *is an isometry between* $(B_m(x_m,\varepsilon), d_{m|B_m(x_m,\varepsilon)})$ *and* $(D(1/2), d_{\omega_{H_m},h_m})$*. Then after passing to a subsequence, the following is true.*

There is a constant $C > 0$ *and a measure* $\widetilde{\omega}$*, with support in* $\overline{D}(1/2)$*, such that*

$$d_{\omega_{H_m},h_m} \text{ converges to } C \cdot \overline{d}_{\widetilde{\omega},0}, \text{ locally uniformly on } D(2\alpha)$$

(that is, if $z_m \to z \in D(2\alpha)$ *and* $z'_m \to z' \in D(2\alpha)$*, then* $d_{\omega_{H_m},h_m}(z_m, z'_m) \to C \cdot \overline{d}_{\widetilde{\omega},0}(z,z')$*).*

For the proof, we need to apply Theorem 0.1, which is a convergence theorem for distances, when there is no harmonic term in the metric. Hence we need to get rid of h_m: to do so, we express h_m as the logarithmic potential of some measure, with support on a circle.

Proof. We know that we can express an harmonic function in terms of its normal derivatives along a circle: for $z \in D(r)$ we have

$$h_m(z) = h_m(0) - \frac{1}{\pi}\int_{\partial D(r)} \ln|z-\xi| \cdot \frac{\partial h_m}{\partial \nu}(\xi)|d\xi|,$$

where $\frac{\partial h_m}{\partial \nu}$ is the radial derivative of h_m. Hence for $z \in D(r)$, we can write h_m as

$$h_m(z) = h_m(0) + V_{\mu_m}(z), \tag{15}$$

where μ_m is the following measure with support in $\partial D(r) : \mu_m := \frac{1}{2}\frac{\partial h_m}{\partial \nu}|d\xi|$. Let

$$\widetilde{\omega}_m := \omega_{H_m} + \mu_m.$$

We have the following fact, which needs some justifications, since representation (15) is only valid for $z \in D(r)$:

Fact 4.11. For $u, u' \in D(2\alpha)$ we have

$$d_{\omega_{H_m},h_m}(u,u') = e^{h_m(0)} \cdot \overline{d}_{\widetilde{\omega}_m,0}(u,u').$$

Proof. By definition,

$$d_{\omega_{H_m},h_m}(u,u') = \inf_{\gamma}\int_0^l e^{V_{\omega_{H_m}}(\gamma(t))+h_m(\gamma(t))}\,dt, \tag{16}$$

where the infimum is taken over all simple continuous curves $\gamma : [0,l] \to D(1/2)$, parametrized by arc length, with $\gamma(0) = u$ and $\gamma(l) = u'$. Let $y_m = (H_m)^{-1}(u)$ and $y'_m = (H_m)^{-1}(u')$. Since property (1) of Theorem 4.8 has already been proved, we have $(H_m)^{-1}(D(2\alpha)) \subset B_m(x_m, \varepsilon/4)$, hence $d_m(x_m, y_m) < \varepsilon/4$ and $d_m(x_m, y'_m) < \varepsilon/4$.

Now, assume that $\gamma : [0,l] \to D(1/2)$ is a continuous simple curve between u and u', parametrized by arc length, which is not included in $D(r)$. Let $\widetilde{\gamma} := (H_m)^{-1}(\gamma)$: this is a curve between y_m and y'_m, and since $H_m(B_m(x_m,\varepsilon/2)) \subset D(r)$ (this is Proposition 4.4), $\widetilde{\gamma}$ is not included in $B_m(x_m,\varepsilon/2)$. But $\widetilde{\gamma}$ is a curve joining two points in $B_m(x_m,\varepsilon/4)$, and has to leave $B_m(x_m,\varepsilon/2)$, so its length is greater than or equal to $2 \cdot \varepsilon/4 = \varepsilon/2$:

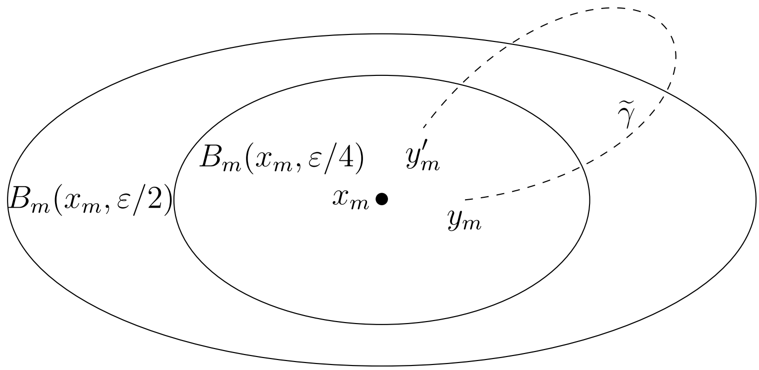

Figure 5.

Since the distance between y_m and y'_m is less than $\varepsilon/2$, this shows that in the formula (16) we can only consider curves γ included in $D(r)$. For such curves γ we can use the representation (15), so we get

$$d_{\omega_{H_m},h_m}(u,u') = e^{h_m(0)} \cdot \inf_{\gamma} \int_0^l e^{V_{\widetilde{\omega}_m}(\gamma(t))}\, dt,$$

where the infimum is taken over all continuous simple curves $\gamma : [0,l] \to D(r)$, parametrized by arc length, with $\gamma(0) = u$ and $\gamma(l) = u'$. For the same reason as before, this is equal to the infimum of the same quantity, over all the curves $\gamma : [0,l] \to \overline{D}(1/2)$, and this is exactly $e^{h_m(0)} \cdot \overline{d}_{\widetilde{\omega}_m,0}(u,u')$ (see the equation (1) in § 1.2 for the definition of $\overline{d}_{\widetilde{\omega}_m,0}$). This ends the proof of Fact 4.11. □

By Theorem 4.2, the sequence $(h_m(0))_{m\in\mathbb{N}}$ is bounded, so after passing to a subsequence, we may assume that $e^{h_m(0)} \to C > 0$. Moreover, for (bounded) harmonic functions, Cauchy's formula gives a bound for the derivatives (at some point x) of the function, in terms of a bound for the modulus of the function (on some ball centered in x). Since the harmonic functions h_m are bounded on every compact subset of $D(1/2)$, this shows that $\frac{\partial h_m}{\partial \nu}$ is bounded on $\partial D(r)$ by a quantity which does not depend on m, hence $\mu_m^+(D(1/2))$ and $\mu_m^-(D(1/2))$ are bounded, so after passing to a subsequence we may assume that $\mu_m^+ \to \mu^+$ and $\mu_m^- \to \mu^-$ weakly. Since the supports of μ_m^+ and μ_m^- are included in $\partial D(r)$, the supports of μ^+ and μ^- are also included in $\partial D(r)$. Let $\mu := \mu^+ - \mu^-$.

We may also assume that $\omega_{H_m}^+ \to \omega^+$, and $\omega_{H_m}^- \to \omega^-$ weakly. We set $\omega := \omega^+ - \omega^-$. Since $\widetilde{\omega}_m = \omega_{H_m} + \mu_m$, we have $\widetilde{\omega}_m \to \widetilde{\omega} := \omega + \mu$ weakly.

Since $\omega_{H_m}^+(D(1/2)) = \omega_m^+(B_m(x_m,\varepsilon)) \leqslant 2\pi - \delta$, we have $\omega^+(\{z\}) < 2\pi$ for every $z \in D(2\alpha)$, and μ^+ has its support in $\partial D(r)$, hence $\mu^+(\{z\}) = 0$ for every $z \in D(2\alpha)$. We have obtained $\widetilde{\omega}^+(\{z\}) < 2\pi$ for every $z \in D(2\alpha)$, we can then apply Theorem 0.1: if $z_m \to z \in D(2\alpha)$ and $z'_m \to z' \in D(2\alpha)$, then $\overline{d}_{\widetilde{\omega}_m,0}(z_m,z'_m) \to \overline{d}_{\widetilde{\omega},0}(z,z')$, and Fact 4.11 gives

$$d_{\omega_{H_m},h_m}(z_m,z'_m) = e^{h_m(0)} \cdot \overline{d}_{\widetilde{\omega}_m,0}(z_m,z'_m) \to C \cdot \overline{d}_{\widetilde{\omega},0}(z,z'). \qquad \square$$

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4.3.3. Choice of κ. We first choose κ small enough so that the annulus $D(1/2) - H(\overline{B}(x, \kappa\varepsilon))$ has a modulus big enough (this is Lemma 4.12); we can then use Grötzsch theorem to finish the proof of Theorem 4.8. Recall that $\mathrm{mod}(G(2\alpha))$ is the modulus of the Grötzsch annulus $G(2\alpha)$.

Lemma 4.12. *There exists a constant $\kappa = \kappa(\Sigma, A, c, \varepsilon, \delta) > 0$ satisfying the following property. Under the hypothesis of Theorem 4.8, the annulus (in the sense given in the Appendix) $D(1/2) - H(\overline{B}(x, \kappa\varepsilon))$ has a modulus greater than $\mathrm{mod}(G(2\alpha))$.*

Remark 4.13. This property is obvious in the Euclidean plane: an annulus $A(R_1, R_2)$ of boundary two concentric circles of radii $R_1 < R_2$ has modulus $\frac{1}{2\pi}\ln(R_2/R_1)$, so this quantity goes to infinity when R_1 goes to zero.

Proof. We prove the lemma by contradiction. Suppose there exists a sequence $d_m \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, a sequence of points $x_m \in \Sigma$, a sequence of harmonic charts $H_m : B_m(x_m, \varepsilon) \to D(1/2)$, with $H_m(x_m) = 0$, such that

$$\mathrm{mod}(D(1/2) - H_m(\overline{B}_m(x_m, \varepsilon/m))) \leqslant \mathrm{mod}(G(2\alpha)).$$

Let $\iota > 0$ be such that $\frac{1}{2\pi}\ln(1/2\iota) > \mathrm{mod}(G(2\alpha))$; we may assume $\iota < 2\alpha$. We have the following.

Fact 4.14. We have $H_m(B_m(x_m, \varepsilon/m)) \not\subset D(\iota)$.

Proof. If $H_m(B_m(x_m, \varepsilon/m)) \subset D(\iota)$, then $D(1/2) - \overline{D}(\iota) \subset D(1/2) - H_m(\overline{B}_m(x, \varepsilon/m))$, hence

$$\mathrm{mod}(D(1/2) - H_m(\overline{B}_m(x, \varepsilon/m))) \geqslant \mathrm{mod}(D(1/2) - \overline{D}(\iota)) = \frac{1}{2\pi}\ln(1/2\iota) > \mathrm{mod}(G(2\alpha))$$

and this is a contradiction. □

Consider the singular Riemannian metric $g_m = e^{2V_{\omega_m} + 2h_m}|dz|^2$, such that H_m is an isometry between $(B_m(x_m, \varepsilon), d_{m|B_m(x_m,\varepsilon)})$ and $(D(1/2), d_{\omega_m,h_m})$. By Fact 4.14, there exists complex numbers z_m, with $|z_m| \geqslant \iota$, and $d_{\omega_m,h_m}(0, z_m) \leqslant \varepsilon/m$. By considering the intersection point between a geodesic from 0 to z_m and $\partial D(\iota)$, we may even assume that $|z_m| = \iota$ and $d_{\omega_m,h_m}(0, z_m) \leqslant \varepsilon/m$. By compactness, after passing to a subsequence we may assume $z_m \to z \neq 0$. Since $|z| = \iota < 2\alpha$, by Corollary 4.10, we know that there exists a constant $C > 0$ and a Radon measure $\widetilde{\omega}$, with support in $\overline{D}(1/2)$, such that, after passing to a subsequence, we have

$$d_{\omega_m,h_m}(0, z_m) \underset{m\to\infty}{\longrightarrow} C \cdot \overline{d}_{\widetilde{\omega},0}(0, z) \neq 0,$$

and this is absurd since $d_{\omega_m,h_m}(0, z_m) \leqslant \varepsilon/m \to 0$. □

Proof of the second part of Theorem 4.8. We prove the inclusion

$$H(\overline{B}(x, \kappa\varepsilon)) \subset D(\alpha)$$

by contradiction: suppose we have $H(\overline{B}(x, \kappa\varepsilon)) \not\subset D(\alpha)$. After a rotation, we may assume that $H(\overline{B}(x, \kappa\varepsilon))$ does not contain the point $\alpha \in \mathbb{R} \subset \mathbb{C}$. Then, after a homothety of scale factor 2, we see that the annulus $D(1/2) - H(\overline{B}(x, \kappa\varepsilon))$ is conformal to an annulus $U \subset D(1)$, not containing 0 and 2α. By Grötzsch's theorem (see Theorem A.2 in the Appendix), we know that the modulus of this annulus is

$$\mathrm{mod}(U) = \mathrm{mod}(D(1/2) - H(\overline{B}(x, \kappa\varepsilon))) \leqslant \mathrm{mod}(G(2\alpha)),$$

and this is a contradiction by Lemma 4.12. This ends the proof of Theorem 4.8. □

5. Proof of the Main theorem

We can now start the proof of the Main theorem. From now on, we consider a sequence of metrics $d_m \in \mathcal{M}_\Sigma(A, c, \varepsilon, \delta)$, that is

(1) for every $x \in \Sigma$ we have $\omega_m^+(B_m(x, \varepsilon)) \leqslant 2\pi - \delta$;

(2) $\mathrm{cont}(\Sigma, d_m) \geqslant c$;

(3) $\mathcal{A}rea(\Sigma, d_m) \leqslant A$.

Recall that we always assume $\varepsilon < c$. By Proposition 4.1, we also have some constants $D > 0$ and $\Omega > 0$ such that

$$\mathrm{diam}(\Sigma, d_m) \leqslant D \quad \text{and} \quad |\omega|(\Sigma, d_m) \leqslant \Omega,$$

and we have some constants $\alpha > 0$ and $\kappa > 0$ such that Theorem 4.8 is true.

We often consider subsequences of the original sequence (d_m): we never change the name of the sequence, and we assume that the sequence has the desired properties from the beginning.

Sketch of the proof

The proof is an adaptation of the proof of Cheeger–Gromov's compactness theorem, presented in [12]. We give an outline of the proof here: to understand it in its globality, we have simplified many of the arguments. See the proof below for precise statements.

(1) We cover Σ by open sets $B_m(x_i^m, \varepsilon)$, for $i \in \{1, \ldots, N\}$ (by volume arguments, the number N is independent of m). We can then define conformal charts $H_i^m : B_m(x_i^m, \varepsilon) \to D$ with $H_i^m(x_i^m) = 0$.

(2) We extend the charts H_i^m to the whole surface Σ by defining maps $\overline{H_i^m} : \Sigma \to D$, which are equal to H_i^m near x_i^m. We then embed Σ into an Euclidean space $\mathbb{R}^q$:

$$\Psi^m(x) :\simeq (\overline{H_1^m}(x), \ldots, \overline{H_N^m}(x)) \in \mathbb{R}^q.$$

The embedded surface Σ^m is locally a graph over some subset $D' \subset D$: for example with $i = 1$, we have, for $z \in D'$,

$$\Psi^m((H_1^m)^{-1}(z)) \simeq (z, \Theta_1^m(z)), \tag{17}$$

where $\Theta_1^m(z) = (\overline{H_2^m} \circ (H_1^m)^{-1}(z), \ldots, \overline{H_N^m} \circ (H_1^m)^{-1}(z))$.

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(3) The maps $\overline{H_j^m} \circ (H_i^m)^{-1}$ are either zero, or looks like $H_j^m \circ (H_i^m)^{-1}$ when this last expression makes sense. Moreover, since the charts H_i^m are conformal, the transition maps $H_j^m \circ (H_i^m)^{-1}$ are conformal maps between open sets of $\mathbb{C}$: since they are bounded, by Montel's theorem they will converge uniformly (up to a subsequence). We can pass to the limit in the representation of Σ^m as a union of graphs (see the equation (17)), and define a subset $\Sigma^\infty \subset \mathbb{R}^q$. We prove that Σ^∞ is an embedded surface.

(4) For m large enough, the embedded surfaces Σ^m are in a tubular neighborhood of Σ^∞. We can then project Σ^m along the normals onto Σ^∞ and define a map $\Pi^m : \Sigma^m \to \Sigma^\infty$. Since Σ^m converge to Σ^∞ (in the sense given above), we prove that Π^m is actually a diffeomorphism.

(5) We have diffeomorphisms $\Sigma \xrightarrow{\sim} \Sigma^m \subset \mathbb{R}^q \xrightarrow{\sim} \Sigma^\infty \subset \mathbb{R}^q$. We transport the initial metric d_m to a metric $\widetilde{d_m}$ on Σ^∞, so that (Σ, d_m) is isometric to $(\Sigma^\infty, \widetilde{d_m})$. We finally show that the metric $\widetilde{d_m}$ converge, by using the local convergence theorem due to Reshetnyak (Theorem 0.1).

5.1. Covering of Σ and notations

Let $m \in \mathbb{N}$. Consider a maximal number $N(m)$ of disjoint balls $B_m(x_i^m, \kappa\varepsilon/4)$ in Σ. By Corollary 3.2, we know that the area of these balls are bounded below. Since the area of (Σ, d_m) is bounded above, this shows that the integer $N(m)$ is bounded: after passing to a subsequence, we can assume that $N(m) = N$ is constant. Moreover, by an elementary doubling property we have

$$\Sigma = \bigcup_{i=1}^{N} B_m(x_i^m, \kappa\varepsilon/2). \tag{18}$$

By Proposition 2.6 and Theorem 1.6 we can consider conformal charts

$$H_i^m : B_m(x_i^m, \varepsilon) \to D(1/2),$$

with $H_i^m(x_i^m) = 0$. In the sequel, we set $\omega_i^m := (H_i^m)_\# \omega$, and h_i^m is the harmonic function on $D(1/2)$ such that the singular Riemannian metric writes

$$g_i^m := e^{2V_{\omega_i^m}(z) + 2h_i^m(z)} |dz|^2.$$

Property (2) in Theorem 4.8 shows that we have $H_i^m(B_m(x_i^m, \kappa\varepsilon)) \subset D(\alpha)$, and with relation (18) we obtain the following.

Fact 5.1. We have the covering

$$\Sigma = \bigcup_{i=1}^{N} (H_i^m)^{-1}(D(\alpha)).$$

5.2. Embedding in an Euclidean space

We use a cut-off function $\varphi : \mathbb{C} \to [0, 1]$ to extend the charts H_i^m to the whole surface Σ. Then, by an analog to Whitney's embedding theorem, we embed Σ into an Euclidean space, and we show that this set is locally a graph.

Let $\varphi : \mathbb{C} \to [0, 1]$ be a smooth function, with value 1 on $D(5\alpha/3)$ and 0 outside $D(2\alpha)$ (see the picture in Theorem 4.8). For $1 \leqslant i \leqslant N$, we define on Σ the following smooth maps:

- $\varphi_i^m := \varphi \circ H_i^m : \Sigma \to [0, 1]$. By Theorem 4.8, this function has value 1 on $B_m(x_i^m, \kappa\varepsilon)$, and 0 outside $B_m(x_i^m, \varepsilon/4)$.
- $\overline{H_i^m} := \varphi_i^m H_i^m : \Sigma \to D(1/2)$. This map extends H_i^m. It is equal to H_i^m on $B_m(x_i^m, \kappa\varepsilon)$, and is 0 outside $B_m(x_i^m, \varepsilon/4)$.

Let us now describe Withney's embedding. Let $q := 2N + N$. We define

$$\begin{aligned} \Psi^m : \Sigma &\longrightarrow \mathbb{R}^q \\ x &\longmapsto \Psi^m(x) = (\overline{H_1^m}(x), \dots, \overline{H_N^m}(x), \varphi_1^m(x), \dots, \varphi_N^m(x)). \end{aligned}$$

This is an easy verification to show that Ψ^m is a smooth embedding, from Σ into $\mathbb{R}^q$: the $2N$ first coordinates ensure the immersion property, and with the N last coordinates we obtain injectivity. We denote by

$$\Sigma^m := \Psi^m(\Sigma)$$

the submanifold of $\mathbb{R}^q$ we have obtained.

Since $\varphi = 1$ on $D(5\alpha/3)$, we remark that these submanifolds are locally graphs, parametrized by $D(5\alpha/3)$. Indeed, for $1 \leqslant i \leqslant N$, the open sets $(H_i^m)^{-1}(D(5\alpha/3))$ cover Σ (this is Fact 5.1), and for $z \in D(5\alpha/3)$ we have

$$\varphi_i^m((H_i^m)^{-1}(z)) = \varphi(z) = 1 \quad \text{and} \quad \overline{H_i^m}((H_i^m)^{-1}(z)) = z,$$

hence

$$\begin{aligned} \Phi_i^m(z) &:= \Psi^m((H_i^m)^{-1}(z)) \\ &= \big(\overline{H_1^m}((H_i^m)^{-1}(z)), \dots, z, \dots, \overline{H_N^m}((H_i^m)^{-1}(z)), \varphi_1^m((H_i^m)^{-1}(z)), \dots, 1, \dots, \varphi_N^m((H_i^m)^{-1}(z))\big) \end{aligned} \tag{19}$$

(this is a graph since the ith coordinate is z). Σ^m is then the union of N pieces of graphs:

$$\Sigma^m = \bigcup_{i=1}^N \Phi_i^m(D(5\alpha/3)).$$

If $x \in \Phi_i^m(D(5\alpha/3))$, we say that x is in the graph number i.

5.3. Convergence of the embedded surfaces Σ^m to an embedded surface Σ^∞

We want to show the convergence of the maps defining Σ^m as graphs, that is the convergence of the maps $\overline{H_j^m} \circ (H_i^m)^{-1}$ and $\varphi_j^m \circ (H_i^m)^{-1}$. We first show that, on some good open sets V, these maps are either zero on V, for every $m \in \mathbb{N}$, or $H_j^m \circ (H_i^m)^{-1}$ is well defined on V (that is $(H_i^m)^{-1}(V) \subset B_m(x_j^m, \varepsilon)$), for every $m \in \mathbb{N}$. In the first case, the sequence of maps $\overline{H_j^m} \circ (H_i^m)^{-1}$ converges trivially; and in the second case, Montel's theorem allows us to conclude that this sequence of bounded conformal maps converges locally uniformly on V. By passing to the limit, we can define a subset $\Sigma^\infty \subset \mathbb{R}^q$. We prove that this set is an embedded surface.

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5.3.1. A preliminary study of the maps $\overline{H_j^m} \circ (H_i^m)^{-1}$ and $\varphi_j^m \circ (H_i^m)^{-1}$. Let $z \in D(1/2)$. If the expression $H_j^m \circ (H_i^m)^{-1}(z)$ makes sense (that is, if $(H_i^m)^{-1}(z) \in B_m(x_j^m, \varepsilon)$), then we have

$$\begin{cases} \overline{H_j^m} \circ (H_i^m)^{-1}(z) = \varphi(H_j^m \circ (H_i^m)^{-1}(z)) \cdot H_j^m \circ (H_i^m)^{-1}(z) \\ \varphi_j^m \circ (H_i^m)^{-1}(z) = \varphi(H_j^m \circ (H_i^m)^{-1}(z)), \end{cases}$$

otherwise (that is, if $(H_i^m)^{-1}(z) \notin B_m(x_j^m, \varepsilon)$) we have

$$\overline{H_j^m} \circ (H_i^m)^{-1}(z) = 0 \quad \text{and} \quad \varphi_j^m \circ (H_i^m)^{-1}(z) = 0.$$

In some sense, we want to show that this dichotomy is valid *uniformly in* $m \in \mathbb{N}$. After passing to a subsequence, the following proposition is true:

Proposition 5.2. *There exists a finite covering with open sets*

$$D(2\alpha) = \bigcup_{t \in T} V_t$$

such that the following property is true (after passing to a subsequence). We fix $i, j \in \{1, \ldots, N\}$ *and* $t \in T$*. Then at least one of the following two properties is true:*

- (A) *for every* $m \in \mathbb{N}$*,* $V_t \subset H_i^m(B_m(x_i^m, \varepsilon) \cap B_m(x_j^m, \varepsilon))$*.*
  Then $H_j^m \circ (H_i^m)^{-1}$ *is well defined on* V_t*.*
- (B) *for every* $m \in \mathbb{N}$*,* $\varphi_j^m \circ (H_i^m)^{-1} = 0$ *on* V_t*.*
  Then for every $m \in \mathbb{N}$*,* $\overline{H_j^m} \circ (H_i^m)^{-1} = 0$ *on* V_t*.*

Remark 5.3. An open set V_t may satisfy the two properties. Moreover, property (B) is satisfied as soon as there are no transition maps between $B_m(x_i^m, \varepsilon)$ and $B_m(x_j^m, \varepsilon)$, that is when $B_m(x_i^m, \varepsilon) \cap B_m(x_j^m, \varepsilon) = \emptyset$.

This proposition is a direct consequence of the following lemma. We set $\eta > 0$ the constant satisfying the following equation:

$$e^{M(\overline{D}(2\alpha))} \cdot \frac{4\pi}{\delta} \cdot \eta^{\delta/2\pi} = 3\varepsilon/4 \tag{20}$$

(recall that $M(\overline{D}(2\alpha))$ is the constant which appears in Theorem 4.2 for the compact set $K = \overline{D}(2\alpha)$).

Lemma 5.4. *Let* $m \in \mathbb{N}$*,* $i, j \in \{1, \ldots, N\}$ *and* $z_0, z \in D(2\alpha)$*. Suppose* $\varphi_j^m \circ (H_i^m)^{-1}(z_0) \neq 0$*. Then*

$$|z - z_0| \leqslant \eta \implies (H_i^m)^{-1}(z) \in B_m(x_j^m, \varepsilon) \ \textit{(hence } H_j^m \circ (H_i^m)^{-1}(z) \textit{ exists).}$$

Proof. Since $\varphi_j^m \circ (H_i^m)^{-1}(z_0) = \varphi(H_j^m \circ (H_i^m)^{-1}(z_0)) \neq 0$, we have $H_j^m \circ (H_i^m)^{-1}(z_0) \in D(2\alpha)$, so by Theorem 4.8 we have $H_j^m \circ (H_i^m)^{-1}(z_0) \in H_j^m(B_m(x_j^m, \varepsilon/4))$, hence

$$d_m(x_j^m, (H_i^m)^{-1}(z_0)) < \varepsilon/4. \tag{21}$$

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Now we use the same arguments as in the proof of the first part of Theorem 4.8. Let $\gamma(t) := (1-t)z_0 + tz$ be the line segment between z_0 and z. We have

$$L_m((H_i^m)^{-1}(\gamma)) = \int_0^1 e^{V_{\omega_i^m}(\gamma(t)) + h_i^m(\gamma(t))} |z - z_0| \, dt.$$

Since the line segment γ is included in $D(2\alpha)$, we have $e^{h_i^m(\gamma(t))} \leqslant e^{M(\overline{D}(2\alpha))}$. Using Proposition 3.3 we have

$$\begin{aligned} \int_0^1 e^{V_{\omega_i^m}(\gamma(t))} |z - z_0| \, dt &\leqslant \frac{2}{1 - (\omega_i^m)^+(D(1/2))/2\pi} \cdot |z - z_0|^{1 - (\omega_i^m)^+(D(1/2))/2\pi} \\ &\leqslant \frac{4\pi}{\delta} \cdot |z - z_0|^{\delta/2\pi}, \end{aligned}$$

thus we obtain

$$L_m((H_i^m)^{-1}(\gamma)) \leqslant e^{M(\overline{D}(2\alpha))} \cdot \frac{4\pi}{\delta} \cdot |z - z_0|^{\delta/2\pi} \leqslant e^{M(\overline{D}(2\alpha))} \cdot \frac{4\pi}{\delta} \cdot \eta^{\delta/2\pi} = 3\varepsilon/4$$

(we have chosen η so that the last equality is true). Since $(H_i^m)^{-1}(\gamma)$ is a continuous curve in Σ joining $(H_i^m)^{-1}(z_0)$ and $(H_i^m)^{-1}(z)$, we have $d_m((H_i^m)^{-1}(z_0), (H_i^m)^{-1}(z)) \leqslant 3\varepsilon/4$, and with inequality (21) we obtain $d_m(x_j^m, (H_i^m)^{-1}(z)) < 3\varepsilon/4 + \varepsilon/4 = \varepsilon$, and this ends the proof. □

Proof of Proposition 5.2. The following fact is a direct consequence of Lemma 5.4:

Fact 5.5. Let $m \in \mathbb{N}$, $i, j \in \{1, \ldots, N\}$, and $V \subset D(2\alpha)$ be an open set with diameter less than η. Then at least one of the following two properties is true:

- (A′) we have $V \subset H_i^m(B_m(x_i^m, \varepsilon) \cap B_m(x_j^m, \varepsilon))$,
- (B′) we have $\varphi_j^m \circ (H_i^m)^{-1} = 0$ on V.

Proof. Assume (B') is not true. Then there exists some $z_0 \in V$ with $\varphi_j^m \circ (H_i^m)^{-1}(z_0) \neq 0$. And every $z \in V$ satisfies $|z - z_0| \leqslant \eta$, so $(H_i^m)^{-1}(z) \in B_m(x_j^m, \varepsilon)$ by Lemma 5.4: this shows that property (A') is true. □

Now, cover $D(2\alpha)$ by a finite number of open sets V_t, for $t \in T$, with diameter less than η. For every $i, j \in \{1, \ldots, N\}$ and every $t \in T$, by the preceding fact, there exists an infinite number of integer m satisfying the same proposition, (A') or (B'). Hence there exists a subsequence m' of m such that this property ((A') or (B')) is satisfied for every m'. Taking a finite number of successive extractions, when $i, j \in \{1, \ldots, N\}$ and $t \in T$, we obtain Proposition 5.2. □

5.3.2. Convergence of the transition maps. Let $i, j \in \{1, \ldots, N\}$. We want to show the convergences of the sequences of maps $\overline{H_j^m} \circ (H_i^m)^{-1}$ and $\varphi_j^m \circ (H_i^m)^{-1}$ on $D(2\alpha)$.

On an open set V_t satisfying property (B) in Proposition 5.2, we have, for every $m \in \mathbb{N}$, $\overline{H_j^m} \circ (H_i^m)^{-1} = 0$ and $\varphi_j^m \circ (H_i^m)^{-1} = 0$ on V_t, hence the sequences converge trivially.

Consider now some open set V_t such that property (A) in Proposition 5.2 is satisfied. Then $H_j^m \circ (H_i^m)^{-1}$ is well defined on V_t. The maps H_i^m and H_j^m are conformal charts, so

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$H_j^m \circ (H_i^m)^{-1}$ is a conformal map between open subsets of $\mathbb{C}$: this classical property for surfaces with smooth Riemannian metrics extends to the class of surfaces with B.I.C. (this is in [16, Theorem 7.3.1]). $(H_j^m \circ (H_i^m)^{-1})_{m\in\mathbb{N}}$ is then a sequence of uniformly bounded holomorphics (or anti-holomorphics) maps on V_t: by Montel's theorem, we know that after passing to a subsequence, $H_j^m \circ (H_i^m)^{-1}$ converge locally uniformly (as well as the derivatives) to some holomorphic (or anti-holomorphic) map on V_t.

Let A_{ji} := the union of the open sets V_t such that property (A) in Proposition 5.2 is satisfied, and B_{ji} := the union of the open sets V_t such that property (B) is satisfied. We have $D(2\alpha) = A_{ji} \cup B_{ji}$. After considering successive subsequences, we can define a smooth map H_{ji} on A_{ji} by $H_{ji}(z) := \lim_{m\to\infty} H_j^m \circ (H_i^m)^{-1}(z)$. For every $i, j \in \{1, \dots, N\}$, after passing to a subsequence we have the following properties:

- There exists a smooth function φ_{ji} on $D(2\alpha)$ such that

$$\varphi_j^m \circ (H_i^m)^{-1} \underset{m\to\infty}{\longrightarrow} \varphi_{ji} \text{ locally uniformly (as well as the derivatives) on } D(2\alpha):$$

  φ_{ji} is defined by $\varphi_{ji} = \varphi \circ H_{ji}$ on A_{ji}, and $\varphi_{ji} = 0$ on B_{ji}.
- There exists a smooth map $\overline{H_{ji}}$ on $D(2\alpha)$ such that

$$\overline{H_j^m} \circ (H_i^m)^{-1} \underset{m\to\infty}{\longrightarrow} \overline{H_{ji}} \text{ locally uniformly (as well as the derivatives) on } D(2\alpha):$$

  $\overline{H_{ji}}$ is defined by $\overline{H_{ji}} = \varphi_{ji} H_{ji}$ on A_{ji}, and $\overline{H_{ji}} = 0$ on B_{ji}.

5.3.3. Construction of the limit embedded surface Σ^∞. Let m tend to infinity in relation (19): for $z \in D(5\alpha/3)$ we set

$$\Phi_i^\infty(z) := \big(\overline{H_{1i}}(z), \dots, z, \dots, \overline{H_{Ni}}(z), \varphi_{1i}(z), \dots, 1, \dots, \varphi_{Ni}(z)\big). \tag{22}$$

We also define the following subset of $\mathbb{R}^q$:

$$\Sigma^\infty := \bigcup_{i=1}^N \Phi_i^\infty(D(5\alpha/3)).$$

If $x \in \Sigma^\infty$ is in the open set $\Phi_i^\infty(D(5\alpha/3))$, we say that x is in the graph number i. Since Σ^m is covered by the sets $(H_i^m)^{-1}(D(\alpha))$, for $1 \leqslant i \leqslant N$, the following proposition is a straightforward verification:

Proposition 5.6. *We have*

$$\Sigma^\infty = \bigcup_{i=1}^N \Phi_i^\infty(\overline{D}(\alpha)),$$

hence

$$\Sigma^\infty = \bigcup_{i=1}^N \Phi_i^\infty(D(4\alpha/3)).$$

Proof. Let $x \in \Sigma^\infty$: by definition, there exists some points $x_m \in \Sigma^m$ with $x_m \to x$. Every x_m belongs to some open set $\Phi_{i(m)}^m(D(\alpha))$, for some $i(m) \in \{1, \dots, N\}$: after passing to a

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subsequence, we may assume this $i(m)$ is constant. For simplicity, assume $i(m) = 1$. Thus there exists a sequence of complex numbers $z_m \in D(\alpha)$ with

$$x_m = \Phi_1^m(z_m) = \left(z_m, \ldots, \overline{H_N^m}((H_1^m)^{-1}(z_m)), 1, \ldots, \varphi_N^m((H_1^m)^{-1}(z_m))\right).$$

By compactness, after passing to a subsequence, we may assume $z_m \to z \in \overline{D}(\alpha)$, and by uniform convergence of all the maps which appear in the last equality, we get $x = \Phi_1^\infty(z)$, and this ends the proof. □

We easily see that such a 'limit' of embedded submanifolds may not be a submanifold:

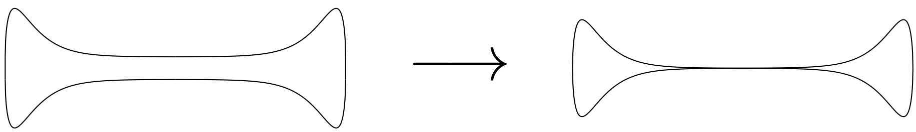

Figure 6.

But in this case, we have the following.

Proposition 5.7. Σ^∞ *is a (possibly disconnected) smooth embedded compact surface in* $\mathbb{R}^q$.

This is a straightforward consequence of the following technical lemma, which will also be used later: points of Σ^m (or points of Σ^∞) which are close to points in the graph number i, are also in the graph number i.

Lemma 5.8. *The following properties hold for every* $i \in \{1, \ldots, N\}$:

(1) *Let* $m \in \mathbb{N}$, *and* $x_0 \in \Phi_i^m(D(4\alpha/3)) \subset \Sigma^m$. *For every* $x \in \Sigma^m$,

$$\|x - x_0\| < \alpha/6 \Longrightarrow x \in \Phi_i^m(D(5\alpha/3)).$$

(2) *Let* $x_0 \in \Phi_i^\infty(D(4\alpha/3)) \subset \Sigma^\infty$. *For every* $x \in \Sigma^\infty$,

$$\|x - x_0\| < \alpha/6 \Longrightarrow x \in \Phi_i^\infty(D(5\alpha/3)).$$

(2') *Let* $x_0 \in \Phi_i^\infty(D(\alpha)) \subset \Sigma^\infty$. *For every* $x \in \Sigma^\infty$,

$$\|x - x_0\| < \alpha/6 \Longrightarrow x \in \Phi_i^\infty(D(7\alpha/6)).$$

($\|.\|$ is the Euclidean norm of $\mathbb{R}^q$.*)*

Proof of Proposition 5.7. Compactness follows from Proposition 5.6. Now, let $x_0 \in \Sigma^\infty$. By Proposition 5.6, there exists $i \in \{1, \ldots, N\}$ such that $x_0 \in \Phi_i^\infty(D(4\alpha/3))$. Second part of Lemma 5.8 shows that

$$B_{\text{euc}}(x_0, \alpha/6) \cap \Sigma^\infty = B_{\text{euc}}(x_0, \alpha/6) \cap \Phi_i^\infty(D(5\alpha/3))$$

($B_{\text{euc}}(x_0, \alpha/6)$ is the Euclidean ball with center x_0 and radius $\alpha/6$). Since $\Phi_i^\infty(z)$ is a graph of a map (see equality (22)), this shows that Σ^∞ is a submanifold of $\mathbb{R}^q$. □

Proof of Lemma 5.8. We do the computations in the case $i = 1$.

Proof of (1). There exists some $z_0 \in D(4\alpha/3))$ such that

$$x_0 = \Phi_1^m(z_0) = \big(z_0, \dots, \overline{H_N^m}((H_1^m)^{-1}(z_0)), 1, \dots, \varphi_N^m((H_1^m)^{-1}(z_0))\big),$$

and we consider some $x \in \Sigma^m$ with $\|x - x_0\| < \alpha/6$. There exists an integer $i \in \{1, \dots, N\}$ and $z \in D(4\alpha/3)$ such that

$$x = \big(\overline{H_1^m}((H_i^m)^{-1}(z)), \dots, z, \dots, \overline{H_N^m}((H_i^m)^{-1}(z)), \\ \varphi_1^m((H_i^m)^{-1}(z)), \dots, 1, \dots, \varphi_N^m((H_i^m)^{-1}(z))\big).$$

Set $z' = \overline{H_1^m}((H_i^m)^{-1}(z))$: we want to show $|z'| < 5\alpha/3$ and $x = \Phi_1^m(z')$.

Since $|z' - z_0| \leqslant \|x - x_0\| < \alpha/6$, we have $|z'| < 4\alpha/3 + \alpha/6 = 3\alpha/2$. For the same reason, $|\varphi_1^m((H_i^m)^{-1}(z)) - 1| \leqslant \|x - x_0\| < \alpha/6 < 1/10$, so $\varphi_1^m((H_i^m)^{-1}(z)) > 9/10$.

Since $\varphi_1^m((H_i^m)^{-1}(z)) \neq 0$, we know that $H_1^m((H_i^m)^{-1}(z))$ exists, so we have $z' = \overline{H_1^m}((H_i^m)^{-1}(z)) = \varphi_1^m((H_i^m)^{-1}(z)) \cdot H_1^m((H_i^m)^{-1}(z))$. Since $\varphi_1^m((H_i^m)^{-1}(z)) \geqslant 9/10$ we have

$$|z'| \geqslant \tfrac{9}{10} \cdot |H_1^m((H_i^m)^{-1}(z))|,$$

hence

$$|H_1^m((H_i^m)^{-1}(z))| \leqslant \frac{10}{9} \cdot |z'| < \frac{10}{9} \cdot \frac{3\alpha}{2} = \frac{5\alpha}{3}.$$

Since $\varphi = 1$ on $D(5\alpha/3)$, we get

$$\varphi_1^m((H_i^m)^{-1}(z)) = \varphi(H_1^m((H_i^m)^{-1}(z))) = 1,$$

and we finally obtain $z' = H_1^m((H_i^m)^{-1}(z))$. We already have $|z'| < 5\alpha/3$. To show the equality $x = \Phi_1^m(z')$, we need to show

$$x = \big(z', \dots, \overline{H_N^m}((H_1^m)^{-1}(z')), 1, \dots, \varphi_N^m((H_1^m)^{-1}(z'))\big),$$

so we need to prove the following equalities, for $j \geqslant 2$:

$$\overline{H_j^m}((H_1^m)^{-1}(z')) = \overline{H_j^m}((H_i^m)^{-1}(z))$$

and

$$\varphi_j^m((H_1^m)^{-1}(z') = \varphi_j^m((H_i^m)^{-1}(z)),$$

and these are directs consequences of the equality $(H_1^m)^{-1}(z') = (H_i^m)^{-1}(z)$.

Proof of (2). (The proof of (2)' is perfectly analogous.) The proof is similar to the proof of (1) (but the end of the proof is different).

There exists some $z_0 \in D(4\alpha/3))$ such that

$$x_0 = \Phi_1^\infty(z_0) = \big(z_0, \overline{H_{21}}(z_0), \dots, \overline{H_{N1}}(z_0), 1, \varphi_{21}(z_0), \dots, \varphi_{N1}(z_0)\big),$$

and some $i \in \{1, \dots, N\}$ and $z \in D(4\alpha/3)$ with

$$x = \big(\overline{H_{1i}}(z), \dots, z, \dots, \overline{H_{Ni}}(z), \varphi_{1i}(z), \dots, 1, \dots, \varphi_{Ni}(z)\big).$$

We set $z' = \overline{H_{1i}}(z)$, and we want to show that $|z'| < 5\alpha/3$, and $x = \Phi_1^\infty(z')$.

For the same reasons than in the proof of (1), we have $|z'| < 3\alpha/2$, and $\varphi_{1i}(z) > 9/10$: $H_{1i}(z)$ exists (that is z is in some open set V_t satisfying property (A) in Proposition 5.2), and we have

$$|H_{1i}(z)| < 5\alpha/3.$$

We get $\varphi_{1i}(z) = 1$, and finally $z' = \varphi_{1i}(z) \cdot H_{1i}(z) = H_{1i}(z)$. We have to show the equality:

$$x = \Phi_1^\infty(z') = \big(z', \overline{H_{21}}(z'), \dots, \overline{H_{N1}}(z'), 1, \varphi_{21}(z'), \dots, \varphi_{N1}(z')\big).$$

So we need to show the following equalities, for $j \geqslant 2$:

$$\overline{H_{j1}}(z') = \overline{H_{ji}}(z) \quad \text{and} \quad \varphi_{j1}(z') = \varphi_{ji}(z).$$

The first equality writes

$$\lim_{m\to\infty} \overline{H_j^m} \circ (H_1^m)^{-1}\Big(\lim_{m'\to\infty} H_1^{m'} \circ (H_i^{m'})^{-1}(z)\Big) = \lim_{m\to\infty} \overline{H_j^m} \circ (H_i^m)^{-1}(z),$$

and this equality is true because all the convergences are uniform. The second equality writes

$$\lim_{m\to\infty} \varphi_j^m \circ (H_1^m)^{-1}\Big(\lim_{m'\to\infty} H_1^{m'} \circ (H_i^{m'})^{-1}(z)\Big) = \lim_{m\to\infty} \varphi_j^m \circ (H_i^m)^{-1}(z),$$

and is true for the same reason. □

5.4. Construction of a diffeomorphism $\Pi^m : \Sigma^m \to \Sigma^\infty$

For m large enough, Σ^m is in a tubular neighborhood of Σ^∞: hence we can define a projection $\Sigma^m \to \Sigma^\infty$. We show that, since Σ^m converges to Σ^∞ (in the sense given above), this projection is actually a diffeomorphism.

5.4.1. Construction of a projection $\Pi^m : \Sigma^m \to \Sigma^\infty$. Σ^∞ is a smooth compact embedded surface in $\mathbb{R}^q$, possibly disconnected, with only a finite number of connected components. We can thus consider the normal projection onto Σ^∞: there exists $\tau > 0$ (we may assume $\tau < \alpha/12$), a tubular neighborhood

$$\mathcal{V} = \{x \in \mathbb{R}^q \mid d_{\text{euc}}(x, \Sigma^\infty) < \tau\}$$

and a smooth projection $\Pi : \mathcal{V} \to \Sigma^\infty$ satisfying the following property (see [7]): if $x \in \mathcal{V}$, then $\Pi(x)$ is the closest point of Σ^∞. For every $x \in \mathcal{V}$ we then have

$$x - \Pi(x) \in (T_{\pi(x)}\Sigma^\infty)^\perp$$

(see the picture below), where we denote by $T_{\pi(x)}\Sigma^\infty \subset \mathbb{R}^q$ the tangent space of Σ^∞ at the point $\pi(x)$, and $(T_{\pi(x)}\Sigma^\infty)^\perp$ its orthogonal in $\mathbb{R}^q$.

Thanks to § 5.3.2, we know that after passing to a subsequence the following is true:

Fact 5.9. For every $m \in \mathbb{N}, i \in \{1, \dots, N\}$ and $z \in D(5\alpha/3)$ we have

$$\|\Phi_i^m(z) - \Phi_i^\infty(z)\| < \tau.$$

Since every $x \in \Sigma^m$ can be written $x = \Phi_i^m(z)$ for some $i \in \{1, .., N\}$ and some $z \in D(\alpha)$, we have $d_{\text{euc}}(x, \Sigma^\infty) < \tau$, so $\Sigma^m \subset \mathcal{V}$. We can thus consider the following restriction:

$$\Pi^m := \Pi_{|\Sigma^m} : \Sigma^m \to \Sigma^\infty.$$

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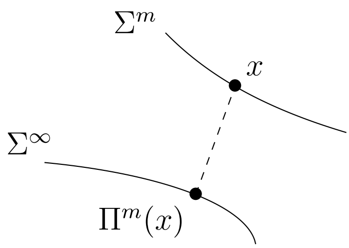

Figure 7.

5.4.2. $\Pi^m : \Sigma^m \to \Sigma^\infty$ is a diffeomorphism. We want to show the following.

Proposition 5.10. *After passing to a subsequence, for every $m \in \mathbb{N}$, $\Pi^m : \Sigma^m \to \Sigma^\infty$ is a $\mathcal{C}^\infty$ diffeomorphism.*

For technical reasons, we first show that for every $i \in \{1, \dots, N\}$, points in Σ^m in the graph number i are sent to points in Σ^∞ in the graph number i, and conversely:

Proposition 5.11. *The following inclusions hold for every $m \in \mathbb{N}$ and every $i \in \{1, \dots, N\}$:*

$$(1)\ (\Pi^m)^{-1}(\Phi_i^\infty(D(4\alpha/3))) \subset \Phi_i^m(D(5\alpha/3)),$$
$$(2)\ \Pi^m(\Phi_i^m(D(4\alpha/3))) \subset \Phi_i^\infty(D(5\alpha/3)),$$

and

$$(2')\ \Pi^m(\Phi_i^m(D(\alpha))) \subset \Phi_i^\infty(D(7\alpha/6)).$$

Proof. This is a straightforward consequence of Lemma 5.8 and Fact 5.9.

Proof of (1). Let $x \in (\Pi^m)^{-1}(\Phi_i^\infty(D(4\alpha/3)))$. There exists $z \in D(4\alpha/3)$ such that $\Pi^m(x) = \Phi_i^\infty(z)$. By Proposition 5.9 we have

$$\|\Pi^m(x) - \Phi_i^m(z)\| = \|\Phi_i^\infty(z) - \Phi_i^m(z)\| < \tau < \alpha/12,$$

and we also have $\|\Pi^m(x) - x\| < \tau < \alpha/12$ (by definition of the normal projection Π). Hence $\|x - \Phi_i^m(z)\| < \alpha/6$, and the identity (1) in Lemma 5.8 shows that $x \in \Phi_i^m(D(5\alpha/3))$.

Proof of (2). (The proof of (2') is perfectly analogous, using (2') in Lemma 5.8 instead of (2).) Let $z \in D(4\alpha/3)$ and $x = \Phi_i^m(z)$. By Proposition 5.9 we have

$$\|x - \Phi_i^\infty(z)\| = \|\Phi_i^m(z) - \Phi_i^\infty(z)\| < \tau < \alpha/12,$$

and we also have $\|x - \Pi^m(x)\| < \tau < \alpha/12$. Hence $\|\Pi^m(x) - \Phi_i^\infty(z)\| < \alpha/6$, and we can use the identity (2) in Lemma 5.8 to show that $\Pi^m(x) \in \Phi_i^\infty(D(5\alpha/3))$. □

We can now prove the following.

Proposition 5.12. Σ^∞ *is path-connected.*

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Proof. Let $x = \Phi_i^\infty(z)$ and $y = \Phi_j^\infty(z')$ be two points in Σ^∞, with $z, z' \in D(4\alpha/3)$. For $m = 1$, let γ be a continuous path in Σ^m joining $\Phi_i^m(z)$ and $\Phi_j^m(z')$ (recall that Σ^m is connected). Then, $\Pi^m \circ \gamma$ is a continuous path in Σ^∞ joining $\Pi^m(\Phi_i^m(z))$ and $\Pi^m(\Phi_j^m(z'))$. We have

$$\Pi^m(\Phi_i^m(z)) \in \Pi^m(\Phi_i^m(D(4\alpha/3))) \subset \Phi_i^\infty(D(5\alpha/3)),$$

and since $x \in \Phi_i^\infty(D(5\alpha/3))$ and $\Phi_i^\infty(D(5\alpha/3))$ is path-connected, we can join $\Pi^m(\Phi_i^m(z))$ and x by a continuous path. For the same reason we can also join $\Pi^m(\Phi_j^m(z'))$ and y by a continuous path, thus we can join x and y by a continuous path. □

To prove Proposition 5.10, we only need to prove the following.

Lemma 5.13. *After passing to a subsequence, for every $m \in \mathbb{N}$, Π^m is an injective immersion.*

Proof of Proposition 5.10. Π^m is a diffeomorphism onto its image, which is an open and closed subset of Σ^∞, thus is Σ^∞ itself by connectedness.

Proof of Lemma 5.13. There are two distinct steps. We prove both steps by contradiction: roughly speaking, since Σ^m converges to Σ^∞ (in the sense given above), the tangent spaces have to converge as well, and this will give a contradiction.

First step. We show that after passing to a subsequence, $\Pi^m : \Sigma^m \to \Sigma^\infty$ is an immersion.

Suppose this is not true. Then Π^m is an immersion only for a finite number of $m \in \mathbb{N}$: there exists $M_0 \in \mathbb{N}$ such that Π^m is not an immersion for $m \geqslant M_0$. From now on we assume $m \geqslant M_0$: then there exists a sequence $x_m \in \Sigma^m$ satisfying $\ker(D\Pi^m(x_m)) \neq \{0\}$. By compactness, we may assume $x_m \to x \in \Sigma^\infty$; we also have $\Pi^m(x_m) = \Pi(x_m) \to \Pi(x) = x$ (recall that Π is the normal projection onto Σ^∞, and Π^m its restriction to Σ^m).

Moreover, $x \in \Phi_i^\infty(D(4\alpha/3)$ for some $i \in \{1, \ldots, N\}$: for simplicity, we may assume $i = 1$. Let $z \in D(4\alpha/3)$ such that $x = \Phi_1^\infty(z)$. For m large enough, $\Pi^m(x_m) \in \Phi_1^\infty(D(4\alpha/3))$, so by Proposition 5.11 we get

$$x_m \in (\Pi^m)^{-1}(\Phi_1^\infty(D(4\alpha/3))) \subset \Phi_1^m(D(5\alpha/3)).$$

We then have sequences z_m and z'_m in $D(5\alpha/3)$ such that $x_m = \Phi_1^m(z_m)$ and $\Pi^m(x_m) = \Phi_1^\infty(z'_m)$. For simplicity, we write x_m and $\Pi^m(x_m)$ under the following form:

$$x_m = (z_m, \Theta^m(z_m)) \quad \text{and} \quad \Pi^m(x_m) = (z'_m, \Theta^\infty(z'_m)),$$

where Θ^m and Θ^∞ are smooth maps, and $\Theta^m \to \Theta^\infty$ uniformly (and all the derivatives) on every compact set of $D(2\alpha)$ (see § 5.3.2). We have

$$\ker(D\Pi^m(x_m)) = T_{x_m}\Sigma^m \cap (T_{\Pi^m(x_m)}\Sigma^\infty)^\perp \neq \{0\},$$

so we can consider a unit vector u_m in this vector space. We know a basis of $T_{x_m}\Sigma^m$, so there exists real numbers a_m and b_m such that

$$u_m = a_m(1, 0, \partial_x\Theta^m(z_m)) + b_m(0, 1, \partial_y\Theta^m(z_m)).$$

Since u_m is a unit vector, we have $|a_m| \leqslant 1$ and $|b_m| \leqslant 1$. Now consider the following vector in $T_{\Pi^m(x_m)}\Sigma^\infty$:

$$v_m = a_m(1, 0, \partial_x\Theta^\infty(z'_m)) + b_m(0, 1, \partial_y\Theta^\infty(z'_m)).$$

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Since u_m and v_m are orthogonal, we have $1 = \|u_m\|^2 \leqslant \|u_m - v_m\|^2$, so $1 \leqslant \|u_m - v_m\|$ and

$$\begin{aligned} 1 &\leqslant |a_m| \cdot \|\partial_x \Theta^m(z_m) - \partial_x \Theta^\infty(z'_m)\| + |b_m| \cdot \|\partial_y \Theta^m(z_m) - \partial_y \Theta^\infty(z'_m)\| \\ &\leqslant \|\partial_x \Theta^m(z_m) - \partial_x \Theta^\infty(z'_m)\| + \|\partial_y \Theta^m(z_m) - \partial_y \Theta^\infty(z'_m)\|. \end{aligned}$$

This is a contradiction: when m goes to infinity, x_m and $\Pi^m(x_m)$ converge to x, so z_m and z'_m converge to z, and we have uniform convergence of the derivatives of Θ^m to the derivatives of Θ^∞, which shows that the right-hand side term of the inequality goes to zero.

Second step. We show that after passing to a subsequence, $\Pi^m : \Sigma^m \to \Sigma^\infty$ is injective.

Suppose this is not true. Then Π^m is injective only for a finite number of $m \in \mathbb{N}$: there exists $M_0 \in \mathbb{N}$ such that Π^m is not injective for every $m \geqslant M_0$. From now on we assume that $m \geqslant M_0$: there exists sequences $x_m, x'_m \in \Sigma^m$, with $x_m \neq x'_m$, such that $\Pi^m(x_m) = \Pi^m(x'_m)$. Hence we have $x_m - x'_m \in (T_{\Pi^m(x_m)}\Sigma^\infty)^\perp$:

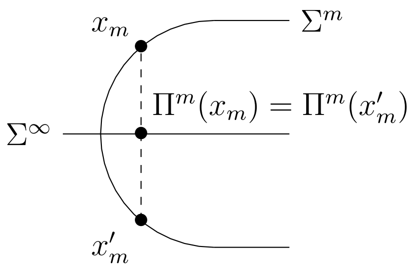

Figure 8.

We can also suppose that x_m and x'_m converge, and these sequences have the same limit $x \in \Sigma^\infty$, since $\lim x_m = \lim \Pi^m(x_m)$ and $\lim x'_m = \lim \Pi^m(x'_m)$. We know that there exists some $i \in \{1, \ldots, N\}$ such that $x \in \Phi_i^\infty(D(4\alpha/3))$; for simplicity, we may assume $i = 1$. There exists $z \in D(4\alpha/3)$ such that $x = \Phi_1^\infty(z)$. If m is large enough, x_m and x'_m also belong to $\Phi_1^\infty(D(4\alpha/3))$, so there exists z_m and z'_m such that

$$x_m = (z_m, \Theta^m(z_m)) \quad \text{and} \quad x'_m = (z'_m, \Theta^m(z'_m))$$

(with the notations as above); we then have

$$x_m - x'_m = (z_m - z'_m, \Theta^m(z_m) - \Theta^m(z'_m)).$$

If m is large enough, $\Pi^m(x_m) = \Pi^m(x'_m)$ also belongs to $\Phi_1^\infty(D(4\alpha/3))$, so there exists $u_m \in D(4\alpha/3)$ such that

$$\Pi^m(x_m) = \Pi^m(x'_m) = (u_m, \Theta^\infty(u_m)).$$

Write $z_m = a_m + ib_m$ and $z'_m = a'_m + ib'_m$ for $a_m, b_m \in \mathbb{R}$ and consider the following vector in $T_{\Pi^m(x_m)}\Sigma^\infty$:

$$\frac{a_m - a'_m}{|z_m - z'_m|^2} \cdot (1, 0, \partial_x \Theta^\infty(u_m)) + \frac{b_m - b'_m}{|z_m - z'_m|^2} \cdot (0, 1, \partial_y \Theta^\infty(u_m))$$

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($x_m \neq x'_m$ implies $z_m \neq z'_m$). By taking the scalar product with $x_m - x'_m \in (T_{\Pi^m(x_m)}\Sigma^\infty)^\perp$ we obtain

$$0 = 1 + \left\langle \frac{\Theta^m(z_m) - \Theta^m(z'_m)}{|z_m - z'_m|}, D\Theta^\infty(u_m)\left(\frac{z_m - z'_m}{|z_m - z'_m|}\right)\right\rangle$$

(we denote by $\langle , \rangle$ the Euclidean scalar product in $\mathbb{R}^{q-2}$). We want to use the mean-value theorem, hence we need to consider real-valued functions. We can write the components of Θ^m and Θ^∞ as

$$\Theta^m = (\Theta^{m,1}, \ldots, \Theta^{m,q-2}) \quad \text{and} \quad \Theta^\infty = (\Theta^{\infty,1}, \ldots, \Theta^{\infty,q-2})$$

with functions $\Theta^{m,j}, \Theta^{\infty,j} : D(5\alpha/3) \to \mathbb{R}$. We can then write

$$0 = 1 + \sum_{j=1}^{q-2} \left(\frac{\Theta^{m,j}(z_m) - \Theta^{m,j}(z'_m)}{|z_m - z'_m|}\right) \cdot D\Theta^{\infty,j}(u_m)\left(\frac{z_m - z'_m}{|z_m - z'_m|}\right).$$

Since the $\Theta^{m,j}$ are functions with values in $\mathbb{R}$, by the mean-value theorem, we know that for every $j \in \{1, \ldots, q-2\}$, there exists some $\zeta_m^j \in [z_m, z'_m]$ such that $\Theta^{m,j}(z_m) - \Theta^{m,j}(z'_m) = D\Theta^{m,j}(\zeta_m^j) \cdot (z_m - z'_m)$: we then have

$$0 = 1 + \sum_{j=1}^{q-2} D\Theta^{m,j}(\zeta_m^j) \cdot \left(\frac{z_m - z'_m}{|z_m - z'_m|}\right) \cdot D\Theta^{\infty,j}(u_m)\left(\frac{z_m - z'_m}{|z_m - z'_m|}\right).$$

By compactness we can suppose that $\frac{z_m - z'_m}{|z_m - z'_m|} \to v \in \mathbb{S}^1$, and since Θ^m (and its derivatives) converge to Θ^∞, we get $0 = 1 + \|D\Theta^\infty(z) \cdot (v)\|^2$ and this is a contradiction. □

5.5. End of the proof of the Main theorem

We have constructed the following diffeomorphisms:

$$\Sigma \xrightarrow[\Psi^m]{\sim} \Sigma^m \subset \mathbb{R}^q \xrightarrow[\Pi^m]{\sim} \Sigma^\infty \subset \mathbb{R}^q.$$

Recall that Ψ^m is obtained by Whitney's embedding, and Π^m is the restriction to Σ^m of the normal projection onto Σ^∞. We can consider the following metric on Σ^∞:

$$\widetilde{d_m} := ((\Pi^m \circ \Psi^m)^{-1})^* d_m,$$

that is

$$\widetilde{d_m}(x, y) = d_m((\Pi^m \circ \Psi^m)^{-1}(x), (\Pi^m \circ \Psi^m)^{-1}(y)),$$

so that (Σ, d_m) is isometric to $(\Sigma^\infty, \widetilde{d_m})$. To finish the proof of the Main theorem, we need to show that $\widetilde{d_m}$ converges uniformly to some metric with B.I.C. $\widetilde{d}$ on Σ^∞.

In § 5.5.1, we prove that $(\widetilde{d_m}(x, y))$ converges when x and y are in the same graph $\Phi_i^\infty(D(4\alpha/3))$. Then, in § 5.5.2, we prove that $(\widetilde{d_m}(x, y))$ converges for every x and y in Σ: we can define $\widetilde{d}(x, y) := \lim_{m\to\infty} \widetilde{d_m}(x, y)$. To finish the proof of the Main theorem, we show that $\widetilde{d_m}$ converges *uniformly* to $\widetilde{d}$, and that $\widetilde{d}$ has B.I.C.

5.5.1. Local properties. By Proposition 5.11, for every $m \in \mathbb{N}$ and every $i \in \{1, \ldots, N\}$ we have

$$(\Pi^m)^{-1}(\Phi_i^\infty(D(4\alpha/3))) \subset \Phi_i^m(D(5\alpha/3)),$$

so we can consider the map $f_i^m : D(4\alpha/3) \to D(5\alpha/3)$ such that the following diagram commutes:

$$\begin{array}{ccc} D(5\alpha/3) & \xleftarrow{\sim} & \Phi_i^m(D(5\alpha/3)) \\ f_i^m \uparrow & & (\Pi^m)^{-1} \uparrow \\ D(4\alpha/3) & \xrightarrow{\sim} & \Phi_i^\infty(D(4\alpha/3)) \end{array}$$

that is $f_i^m = (\Phi_i^m)^{-1} \circ (\Pi^m)^{-1} \circ \Phi_i^\infty$.

Proposition 5.14. *For every $i \in \{1, \ldots, N\}$, $f_i^m : D(4\alpha/3) \to D(5\alpha/3)$ converges uniformly to the inclusion $D(4\alpha/3) \hookrightarrow D(5\alpha/3)$.*

Proof. For every $z \in D(4\alpha/3)$, let $z' = f_i^m(z) \in D(5\alpha/3)$. Since z (respectively z') is the ith component of $\Phi_i^\infty(z)$ (respectively $\Phi_i^m(z')$), we have

$$|z - z'| \leqslant \|\Phi_i^\infty(z) - \Phi_i^m(z')\| = \|\Pi^m(\Phi_i^m(z')) - \Phi_i^m(z')\| \leqslant \|\Phi_i^\infty(z') - \Phi_i^m(z')\| :$$

the last inequality comes from the fact that $\|\Pi^m(\Phi_i^m(z')) - \Phi_i^m(z')\|$ is the distance between $\Phi_i^m(z')$ and the embedded surface Σ^∞, and we have $\Phi_i^\infty(z') \in \Sigma^\infty$. We then have

$$|z - f_i^m(z)| \leqslant \sup_{u \in D(5\alpha/3)} \|\Phi_i^\infty(u) - \Phi_i^m(u)\|,$$

and we know that the right-hand side goes to zero as m goes to infinity. □

We know that for every $m \in \mathbb{N}$ and every $i \in \{1, \ldots, N\}$ we have an isometry

$$(B_m(x_i^m, \varepsilon), d_{m|B_m(x_i^m,\varepsilon)}) \underset{H_i^m}{\xrightarrow{\sim}} (D(1/2), d_{\omega_i^m, h_i^m}). \tag{23}$$

Moreover, for every $i \in \{1, \ldots, N\}$, by the important Corollary 4.10, there exists a measure $\widetilde{\omega}_i$, with support in $\overline{D}(1/2)$, and a constant $C_i > 0$ such that, after passing to a subsequence, $d_{\omega_i^m, h_i^m}$ converge locally uniformly on $D(2\alpha)$ to the metric $d^i := C_i \cdot \overline{d}_{\widetilde{\omega}_i, 0}$ (when m goes to infinity).

Now, if we consider the diagram at the beginning of § 5.5 we can consider the following metric on $\Phi_i^\infty(D(4\alpha/3))$:

$$\widetilde{d^i} := ((\Phi_i^\infty)^{-1})^* d^i.$$

Proposition 5.15. *Let $i \in \{1, \ldots, N\}$ and $(x_m), (y_m)$ be two sequences of points in Σ^∞ such that $x_m \to x \in \Phi_i^\infty(D(4\alpha/3))$ and $y_m \to y \in \Phi_i^\infty(D(4\alpha/3))$. Then*

$$\widetilde{d_m}(x_m, y_m) \underset{m \to \infty}{\longrightarrow} \widetilde{d^i}(x, y).$$

Proof. Suppose m is large enough so that $x_m, y_m \in \Phi_i^\infty(D(4\alpha/3))$. By Corollary 4.10, $d_{\omega_i^m, h_i^m}$ converge locally uniformlyto d^i on $D(2\alpha)$, and $f_i^m : D(4\alpha/3) \to D(5\alpha/3)$ converges uniformly to the inclusion $D(4\alpha/3) \hookrightarrow D(5\alpha/3)$ (this is Proposition 5.14), so

$$\begin{aligned}\widetilde{d^i}(x, y) &= d^i((\Phi_i^\infty)^{-1}(x), (\Phi_i^\infty)^{-1}(y) \\ &= \lim_{m\to\infty} d_{\omega_i^m, h_i^m}(f_i^m \circ (\Phi_i^\infty)^{-1}(x_m), f_i^m \circ (\Phi_i^\infty)^{-1}(y_m)).\end{aligned}$$

And by isometry (23), we have, for every $z, z' \in D(5\alpha/3)$,

$$d_{\omega_i^m, h_i^m}(z, z') = d_{m|B_m(x_i^m,\varepsilon)}((H_i^m)^{-1}(z), (H_i^m)^{-1}(z')) = d_m((H_i^m)^{-1}(z), (H_i^m)^{-1}(z')):$$

indeed, by Theorem 4.8, we have $(H_i^m)^{-1}(z), (H_i^m)^{-1}(z') \in B_m(x_i^m, \varepsilon/4)$, so a curve which (almost) minimizes the distance $d_m((H_i^m)^{-1}(z), (H_i^m)^{-1}(z'))$ has to stay inside $B_m(x_i^m, \varepsilon)$ (we have $d_m((H_i^m)^{-1}(z), (H_i^m)^{-1}(z')) \leqslant \varepsilon/2$, and a curve which joins $(H_i^m)^{-1}(z)$ and $(H_i^m)^{-1}(z')$, and which is not contained in $B_m(x_i^m, \varepsilon)$ has a length $\geqslant 2\cdot(\varepsilon - \varepsilon/4) > \varepsilon/2$). Hence we get

$$\widetilde{d^i}(x, y) = \lim_{m\to\infty} d_m((H_i^m)^{-1} \circ f_i^m \circ (\Phi_i^\infty)^{-1}(x_m), (H_i^m)^{-1} \circ f_i^m \circ (\Phi_i^\infty)^{-1}(y_m)).$$

By definition of Φ_i^m, we have $(H_i^m)^{-1} = (\Psi^m)^{-1} \circ \Phi_i^m$, and with the equality $f_i^m \circ (\Phi_i^\infty)^{-1} = (\Phi_i^m)^{-1} \circ (\Pi^m)^{-1}$ (see the commutative diagram at the beginning of this section) we obtain

$$\widetilde{d^i}(x, y) = \lim_{m\to\infty} d_m((\Psi^m)^{-1} \circ (\Pi^m)^{-1}(x_m), (\Psi^m)^{-1} \circ (\Pi^m)^{-1}(y_m)) = \lim_{m\to\infty} \widetilde{d_m}(x_m, y_m),$$

and this ends the proof. □

5.5.2. Construction of the limit metric $\widetilde{d}$ and conclusion. We already know that $\lim_{m\to\infty} \widetilde{d_m}(x, y)$ exists if x and y are in the same graph (that is, if there exists some $i \in \{1, \dots, N\}$ with $x, y \in \Phi_i^\infty(D(4\alpha/3))$), see Proposition 5.15. To prove that this limit exists for every $x, y \in \Sigma$, we consider $\widetilde{d_m}$−geodesics between x and y, $\gamma_m : [0, 1] \to \Sigma$, and we cut the segment $[0, 1]$ into subintervals for which we can apply Proposition 5.15.

Proposition 5.16. *Let $x, y \in \Sigma^\infty$: the limit $\lim_{m\to\infty} \widetilde{d_m}(x, y)$ exists, and we set*

$$\widetilde{d}(x, y) := \lim_{m\to\infty} \widetilde{d_m}(x, y) \in [0, +\infty).$$

Proof. Let $x, y \in \Sigma^\infty$. Consider some subsequence (m_j) of (m) such that

$$\widetilde{d_{m_j}}(x, y) \underset{j\to\infty}{\longrightarrow} \liminf_{m\to\infty} \widetilde{d_m}(x, y).$$

Let $\gamma_{m_j} : [0, 1] \to \Sigma^\infty$ be minimizing geodesics for the metric $\widetilde{d_{m_j}}$, between x and y (with $\gamma_{m_j}(0) = x$ and $\gamma_{m_j}(1) = y$), and parametrized with constant speed: for every $t, t' \in [0, 1]$ we have

$$\widetilde{d_{m_j}}(\gamma_{m_j}(t), \gamma_{m_j}(t')) = \widetilde{d_{m_j}}(x, y) \cdot |t - t'|.$$

After passing to a subsequence of m_j (we do not change the name of the sequence), the following fact is true:

Fact 5.17. Let $n \in \mathbb{N}$ be an integer such that $D/n \leqslant \kappa\varepsilon/2$. For every $k \in \{0, \dots, n-1\}$, there exists $i(k) \in \{1, \dots, N\}$ such that for every $j \in \mathbb{N}$,

$$\gamma_{m_j}(k/n) \quad \text{and} \quad \gamma_{m_j}((k+1)/n) \text{ belong to } \Phi_{i(k)}^\infty(D(7\alpha/6)).$$

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Proof. By the covering (18), we know that for every $k \in \{0, \dots, n-1\}$ and for every $j \in \mathbb{N}$, there exists an integer $i(k, j) \in \{1, \dots, N\}$ such that

$$(\Pi^{m_j} \circ \Psi^{m_j})^{-1}(\gamma_{m_j}(k/n)) \in B_{m_j}(x^{m_j}_{i(k,j)}, \kappa\varepsilon/2). \tag{24}$$

Since $i(k, j)$ belongs to a finite set, after passing to a subsequence of (m_j), we may assume that for every $k \in \{0, \dots, n-1\}$, $i(k, j)$ does not depend on j: we can write $i(k, j) = i(k)$. To finish the proof, we show that

$$\gamma_{m_j}([k/n, (k+1)/n]) \subset \Phi^{\infty}_{i(k)}(D(7\alpha/6)).$$

Let $t \in [k/n, (k+1)/n]$: we have

$$\widetilde{d_{m_j}}(\gamma_{m_j}(t), \gamma_{m_j}(k/n)) = d_{m_j}(x, y) \cdot |t - k/n|,$$

so

$$d_{m_j}((\Pi^{m_j} \circ \Psi^{m_j})^{-1}(\gamma_{m_j}(t)), (\Pi^{m_j} \circ \Psi^{m_j})^{-1}(\gamma_{m_j}(k/n))) \leqslant D \cdot 1/n \leqslant \kappa\varepsilon/2,$$

and with (24) this shows that

$$(\Pi^{m_j} \circ \Psi^{m_j})^{-1}(\gamma_{m_j}(t)) \in B_{m_j}(x^{m_j}_{i(k)}, \kappa\varepsilon).$$

So

$$(\Pi^{m_j})^{-1}(\gamma_{m_j}(t)) \in \Psi^{m_j}(B_{m_j}(x^{m_j}_{i(k)}, \kappa\varepsilon)) = \Phi^{m_j}_{i(k)} \circ H^{m_j}_{i(k)}(B_{m_j}(x^{m_j}_{i(k)}, \kappa\varepsilon)) \subset \Phi^{m_j}_{i(k)}(D(\alpha))$$

(the last inclusion comes from Theorem 4.8). With the identity (1) in Proposition 5.11, we obtain

$$\gamma_{m_j}(t) \in \Pi^{m_j}(\Phi^{m_j}_{i(k)}(D(\alpha))) \subset \Phi^{\infty}_{i(k)}(D(7\alpha/6)),$$

and this ends the proof of Fact 5.17. □

After passing to a subsequence of (m_j), we may also assume that for every $k \in \{0, \dots, n\}$,

$$\gamma_{m_j}(k/n) \underset{j\to\infty}{\longrightarrow} \alpha_{k/n} \in \overline{\Phi^{\infty}_{i(k)}(D(7\alpha/6))} \subset \Phi^{\infty}_{i(k)}(D(4\alpha/3)),$$

where we have $\alpha_0 = x$ and $\alpha_1 = y$. For every $k \in \{0, \dots, n-1\}$ we have $\alpha_{k/n}, \alpha_{(k+1)/n} \in \Phi^{\infty}_{i(k)}(D(4\alpha/3))$, and Proposition 5.15 gives

$$\widetilde{d_{m_j}}(\gamma_{m_j}(k/n), \gamma_{m_j}((k+1)/n)) \underset{j\to\infty}{\longrightarrow} \widetilde{d^{i(k)}}(\alpha_{k/n}, \alpha_{(k+1)/n}).$$

Since the curves γ_{m_j} are minimizing geodesics, we have

$$\widetilde{d_{m_j}}(x, y) = \sum_{k=0}^{n-1} \widetilde{d_{m_j}}(\gamma_{m_j}(k/n), \gamma_{m_j}((k+1)/n)),$$

hence when j goes to infinity we obtain

$$\begin{aligned}\liminf_{m\to\infty} \widetilde{d_m}(x, y) &= \sum_{k=0}^{n-1} \widetilde{d^{i(k)}}(\alpha_{k/n}, \alpha_{(k+1)/n}) = \sum_{k=0}^{n-1} \limsup_{m\to\infty} \widetilde{d_m}(\alpha_{k/n}, \alpha_{(k+1)/n}) \\ &\geqslant \limsup_{m\to\infty} \sum_{k=0}^{n-1} \widetilde{d_m}(\alpha_{k/n}, \alpha_{(k+1)/n}) \geqslant \limsup_{m\to\infty} \widetilde{d_m}(x, y).\end{aligned}$$

Hence $\lim_{m\to\infty} \widetilde{d_m}(x, y)$ exists in $[0, +\infty]$, and this limit is finite since $\widetilde{d_m}(x, y) \leqslant D$. This ends the proof of Proposition 5.16. □

We know prove the *uniform* convergence of $(\widetilde{d_m})$ to $\widetilde{d}$:

Corollary 5.18. *Let (x_m) and (y_m) be two sequences in Σ^∞, such that $x_m \to x \in \Sigma^\infty$ and $y_m \to y \in \Sigma^\infty$. Then*

$$\widetilde{d_m}(x_m, y_m) \underset{m\to\infty}{\longrightarrow} \widetilde{d}(x, y).$$

Proof. We have

$$\begin{aligned} |\widetilde{d_m}(x_m, y_m) - \widetilde{d}(x, y)| &\leqslant |\widetilde{d_m}(x_m, y_m) - \widetilde{d_m}(x, y)| + |\widetilde{d_m}(x, y) - \widetilde{d}(x, y)| \\ &\leqslant \widetilde{d_m}(x_m, x) + \widetilde{d_m}(y_m, y) + |\widetilde{d_m}(x, y) - \widetilde{d}(x, y)|. \end{aligned}$$

By definition, $|\widetilde{d_m}(x, y) - \widetilde{d}(x, y)|$ goes to zero. And if $i \in \{1, \dots, N\}$ is such that $x \in \Phi_i^\infty(D(4\alpha/3))$, then Proposition 5.15 shows that $\widetilde{d_m}(x_m, x) \to \widetilde{d^i}(x, x) = 0$. For the same reason $\widetilde{d_m}(y_m, y) \to 0$, and this ends the proof. □

To finish the proof of the Main theorem, we need to show that $\widetilde{d}$ has B.I.C. on Σ^∞.

Proposition 5.19. *$\widetilde{d}$ is a distance on Σ^∞.*

Proof. By definition of $\widetilde{d}(x, y) = \lim_{m\to\infty} \widetilde{d_m}(x, y)$, symmetry and triangular inequality are clear. Now, consider some $x, y \in \Sigma^\infty$ with $\widetilde{d}(x, y) = 0$. Then $\widetilde{d_m}(x, y) \to 0$. For every $m \in \mathbb{N}$, there exists an integer $i(m) \in \{1, \dots, N\}$ such that

$$(\Pi^m \circ \Psi^m)^{-1}(x) \in B_m(x^m_{i(m)}, \kappa\varepsilon/2);$$

since $i(m)$ belong to a finite set, there exists a subsequence m_j of (m) and an integer $i \in \{1, \dots, N\}$ such that $i(m_j) = i$ for all $j \in \mathbb{N}$:

$$(\Pi^{m_j} \circ \Psi^{m_j})^{-1}(x) \in B_{m_j}(x_i^{m_j}, \kappa\varepsilon/2).$$

If j is large enough so that $\widetilde{d_{m_j}}(x, y) \leqslant \kappa\varepsilon/2$, we have

$$d_{m_j}((\Pi^{m_j} \circ \Psi^{m_j})^{-1}(x), (\Pi^{m_j} \circ \Psi^{m_j})^{-1}(y)) \leqslant \kappa\varepsilon/2,$$

hence

$$(\Pi^{m_j} \circ \Psi^{m_j})^{-1}(x) \quad \text{and} \quad (\Pi^{m_j} \circ \Psi^{m_j})^{-1}(y) \text{ belong to } B_{m_j}(x_i^{m_j}, \kappa\varepsilon).$$

Then as in the end of the proof of Fact 5.17 we have

$$(\Pi^{m_j})^{-1}(x) \quad \text{and} \quad (\Pi^{m_j})^{-1}(y) \text{ belong to } \Psi^{m_j}(B_{m_j}(x_i^{m_j}, \kappa\varepsilon)),$$

and we have

$$\Psi^{m_j}(B_{m_j}(x_i^{m_j}, \kappa\varepsilon)) = \Phi_i^{m_j}(H_i^{m_j}(B_{m_j}(x_i^{m_j}, \kappa\varepsilon))) \subset \Phi_i^{m_j}(D(\alpha))$$

(the last inclusion comes from Theorem 4.8). Hence we obtain

$$x \text{ and } y \text{ belong to } \Pi^{m_j}(\Phi_i^{m_j}(D(\alpha))) \subset \Phi_i^\infty(D(7\alpha/6))$$

(the last inclusion comes from the identity (2) in Proposition 5.11). Since $x, y \in \Phi_i^\infty(D(4\alpha/3))$, we can apply Proposition 5.15 to obtain $\widetilde{d_{m_j}}(x, y) \to \widetilde{d^i}(x, y)$, so $\widetilde{d^i}(x, y) = 0$ and $x = y$. □

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The fact that $\widetilde{d}$ is an intrinsic distance comes from the following lemma, which has its own interest:

Proposition 5.20. *Let $x, y \in \Sigma^\infty$ and $\gamma_m : [0,1] \to \Sigma^\infty$ be minimizing geodesics for the metric $\widetilde{d_m}$, between x and y (with $\gamma_m(0) = x$ and $\gamma_m(1) = y$), and parametrized with constant speed. Then there exists a subsequence (m_j) of (m) such that γ_{m_j} converges uniformly to a continuous curve $\gamma : [0,1] \to \Sigma^\infty$, and for the metric $\widetilde{d}$, γ is a minimizing geodesic between x and y.*

Proof. We adapt the proof of Arzela–Ascoli's lemma to our setting. We have

$$\widetilde{d_m}(\gamma_m(t), \gamma_m(t')) = \widetilde{d_m}(x, y) \cdot |t - t'| \leqslant D \cdot |t - t'|.$$

Now let $\{t_k, k \in \mathbb{N}\}$ be a dense subset of $[0,1]$: by a diagonal argument, we can construct a subsequence (m_j) of (m) such that for every $k \in \mathbb{N}$, $\gamma_{m_j}(t_k) \to_{j\to\infty} \alpha_k \in \Sigma^\infty$. Let $\gamma : \{t_k, k \in \mathbb{N}\} \to \Sigma^\infty$ be the map defined by $\gamma(t_k) := \alpha_k$. We have

$$\widetilde{d_{m_j}}(\gamma_{m_j}(t_{k_1}), \gamma_{m_j}(t_{k_2})) \leqslant D \cdot |t_{k_1} - t_{k_2}|,$$

and when j goes to infinity, with Corollary 5.18 we get

$$\widetilde{d}(\gamma(t_{k_1}), \gamma(t_{k_2})) \leqslant D \cdot |t_{k_1} - t_{k_2}|.$$

γ is then Lipschitz on $\{t_k, k \in \mathbb{N}\}$, which is dense in $[0,1]$, so there exists a (unique) Lipschitz extension $\gamma : [0,1] \to \Sigma^\infty$. Then, when j goes to infinity,

$$\gamma_{m_j} \text{ converges uniformly on } [0,1] \text{ to } \gamma.$$

Indeed, let (u_{m_j}) be a sequence in $[0,1]$ such that $u_{m_j} \to u \in [0,1]$: we want to show that $\gamma_{m_j}(u_{m_j}) \to \gamma(u)$ when j goes to infinity. Let $\varepsilon > 0$, and suppose that j is large enough so that $\widetilde{d} \leqslant \widetilde{d_{m_j}} + \varepsilon$ on $\Sigma^\infty \times \Sigma^\infty$. Choose $k \in \mathbb{N}$ such that $|t_k - u| \leqslant \varepsilon/D$. Then we have

$$\begin{aligned}
\widetilde{d}(\gamma_{m_j}(u_{m_j}), \gamma(u)) &\leqslant \widetilde{d}(\gamma_{m_j}(u_{m_j}), \gamma_{m_j}(t_k)) + \widetilde{d}(\gamma_{m_j}(t_k), \gamma(t_k)) + \widetilde{d}(\gamma(t_k), \gamma(u)) \\
&\leqslant \widetilde{d_{m_j}}(\gamma_{m_j}(u_{m_j}), \gamma_{m_j}(t_k)) + \varepsilon + \widetilde{d}(\gamma_{m_j}(t_k), \gamma(t_k)) + \widetilde{d}(\gamma(t_k), \gamma(u)) \\
&\leqslant D \cdot |u_{m_j} - t_k| + \varepsilon + \widetilde{d}(\gamma_{m_j}(t_k), \gamma(t_k)) + D \cdot |t_k - u| \\
&\leqslant D \cdot |u_{m_j} - t_k| + \varepsilon + \widetilde{d}(\gamma_{m_j}(t_k), \gamma(t_k)) + \varepsilon,
\end{aligned}$$

and the right-hand side is $\leqslant 3\varepsilon$ if j is large enough, so $\gamma_{m_j}(u_{m_j})$ converges to $\gamma(u)$ as j goes to infinity.

Finally, for the metric $\widetilde{d}$, γ is a minimizing geodesic between x and y: for every subdivision $0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_p = 1$ of $[0,1]$, since γ_{m_j} is a minimizing geodesic we have

$$\widetilde{d_{m_j}}(x, y) = \sum_{k=0}^{p-1} \widetilde{d_{m_j}}(\gamma_{m_j}(\lambda_k), \gamma_{m_j}(\lambda_{k+1})).$$

When j goes to infinity we get

$$\widetilde{d}(x, y) = \sum_{k=0}^{p-1} \widetilde{d}(\gamma(\lambda_k), \gamma(\lambda_{k+1})),$$

and this proves the claim, taking the supremum over all subdivisions $0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_p = 1$ of $[0,1]$: $\widetilde{d}(x, y)$ is equal to the $\widetilde{d}-$length of the curve γ. □

The next proposition finishes the proof of the Main theorem.

Proposition 5.21. *The metric $\widetilde{d}$ has B.I.C. on Σ^∞.*

Proof. The metric $\widetilde{d}$ is intrinsic, and is compatible with the topology of Σ^∞: for every $\varepsilon > 0$, if m is large enough, for any $x \in \Sigma$ we have $B_{\widetilde{d}}(x, \varepsilon/2) \subset B_{\widetilde{d_m}}(x, \varepsilon) \subset B_{\widetilde{d}}(x, 2\varepsilon)$ (with obvious notations), and the metric $\widetilde{d_m}$ is compatible with the topology of Σ^∞, hence $\widetilde{d}$ is also compatible with the topology of Σ^∞.

Moreover, for every $\varepsilon > 0$, consider some $m \in \mathbb{N}$ such that $\|\widetilde{d_m} - \widetilde{d}\|_\infty \leqslant \varepsilon$. Since $\widetilde{d_m}$ has B.I.C., there exists some smooth Riemannian metric g on Σ^∞ with $\|\widetilde{d_m} - d_g\|_\infty \leqslant \varepsilon$, and with $\int_{\Sigma^\infty} |\,\mathrm{K}_g\,| d\mathcal{A}_g \leqslant \Omega + 1$. We then have $\|\widetilde{d} - d_g\|_\infty \leqslant 2\varepsilon$: $\widetilde{d}$ can be uniformly approximate by Riemannian metrics, with $\int_\Sigma |\,\mathrm{K}_g\,| d\mathcal{A}_g$ bounded, hence $\widetilde{d}$ has B.I.C. (see Definition 1.1). □

Appendix. Conformal geometry of an annulus

The material presented here is standard, we recall it to fix the notations used in this article. A classical reference is the book of Ahlfors [1].

Definition A.1. An *annulus* U is a subset of the plane which is bounded, open, and such that $\mathbb{C} - U$ has only one bounded component, and this component is not reduced to a point.

Every annulus U is conformally equivalent to a standard annulus

$$A(R_1, R_2) = \{z \in \mathbb{C} |\ R_1 < |z| < R_2\},$$

for some $0 < R_1 < R_2 < \infty$, and the ratio R_2/R_1 is uniquely determined by U (see [2]).

Modulus of an annulus U

Let U be an annulus. Let Γ be the set of continuous simple curves $\gamma : [0, l] \to U$, parametrized by arc length (that is for every $t_1 \leqslant t_2$, the Euclidean length of $\gamma_{|[t_1,t_2]}$ is $t_2 - t_1$), joining the bounded and the unbounded components of $\mathbb{C} - U$: that is, $\gamma(0)$ (respectively, $\gamma(1)$) belongs to the bounded (respectively, unbounded) component of $\mathbb{C} - U$, and $\gamma(t) \in U$ for $t \in (0, 1)$. If $\rho : U \to [0, +\infty]$ is a measurable function, we define the $\rho-$length of γ by

$$L_\rho(\gamma) := \int_\gamma \rho |dz| = \int_0^l \rho(\gamma(t))\, dt,$$

and the $\rho-$area of U by

$$A_\rho(U) := \iint_U \rho^2\, d\lambda.$$

These are the length of γ and the area of U for the (singular) Riemannian metric $g = \rho^2 |dz|^2$. We define the modulus of U as follows:

$$\mathrm{mod}(U) := \sup_\rho \frac{\inf_{\gamma \in \Gamma} L_\rho(\gamma)^2}{A_\rho(U)},$$

where the supremum is taken over all measurable functions ρ with $0 < A_\rho(U) < +\infty$.

The modulus of an annulus is a conformal invariant, and it measures the 'thickness' of the annulus: if U and U' are two annuli with $U \subset U'$, then $\mathrm{mod}(U) \leqslant \mathrm{mod}(U')$. For example for a regular annulus $U = A(R_1, R_2)$ we have $\mathrm{mod}(U) = \frac{1}{2\pi} \ln(R_2/R_1)$.

The Grötzsch annulus

Let $0 < r < 1$. The *Grötzsch annulus* is the annulus

$$G(r) := D(1) - [0, r].$$

The function $r \in (0, 1) \mapsto \mathrm{mod}(G(r))$ is decreasing, and we have $\lim_{r \to 0^+} \mathrm{mod}(G(r)) = +\infty$ and $\lim_{r \to 1^-} \mathrm{mod}(G(r)) = 0$. We need the following theorem (see [1]):

Theorem A.2 (Grötzsch). *Let $U \subset D(1)$ be an annulus, not containing 0 and r. Then $\mathrm{mod}(U) \leqslant \mathrm{mod}(G(r))$.*

Please note that when the annulus U does not intersect the whole line segment $[0, r]$, we have $U \subset G(r)$ and the conclusion of the theorem is obvious.

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