Bull. Aust. Math. Soc. **103** (2021), 311–317
doi:10.1017/S0004972720001070

UNIQUENESS OF EXTENDABLE TEMPERATURES

NEIL A. WATSON

(Received 7 May 2020; accepted 7 August 2020; first published online 2 October 2020)

Abstract

Let E and D be open subsets of $\mathbb{R}^{n+1}$ such that $\overline{D}$ is a compact subset of E, and let v be a supertemperature on E. A temperature u on D is called *extendable by* v if there is a supertemperature w on E such that $w = u$ on D and $w = v$ on $E \backslash \overline{D}$. From earlier work of N. A. Watson, ['Extendable temperatures', *Bull. Aust. Math. Soc.* **100** (2019), 297–303], either there is a unique temperature extendable by v, or there are infinitely many; a necessary condition for uniqueness is that the generalised solution of the Dirichlet problem on D corresponding to the restriction of v to $\partial_e D$ is equal to the greatest thermic minorant of v on D. In this paper we first give a condition for nonuniqueness and an example to show that this necessary condition is not sufficient. We then give a uniqueness theorem involving the thermal and cothermal fine topologies and deduce a corollary involving only parabolic and coparabolic tusks.

2020 *Mathematics subject classification*: primary 35K05; secondary 31B05, 31B20.

Keywords and phrases: temperature, supertemperature, fine topology, caloric measure.

1. Introduction

Given an open set E in $\mathbb{R}^{n+1}$, a function $u \in C^{2,1}(E)$ that satisfies the standard heat equation on E is called a *temperature*. If

$$W(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

then W is a temperature on $\mathbb{R}^{n+1} \backslash \{0\}$. For any point $(x_0, t_0) \in \mathbb{R}^{n+1}$ and any positive number c, the set

$$\Omega(x_0, t_0; c) = \{(y, s) \in \mathbb{R}^{n+1} : W(x_0 - y, t_0 - s) > (4\pi c)^{-n/2}\}$$

is called the *heat ball* with *centre* (x_0, t_0) and *radius* c. Temperatures can be characterised in terms of mean values over heat balls, since a function $u \in C^{2,1}(E)$ is a temperature if and only if

$$u(x_0, t_0) = (4\pi c)^{-n/2} \iint_{\Omega(x_0,t_0;c)} \frac{|x_0 - x|^2}{4(t_0 - t)^2} u(x, t)\, dx\, dt$$

whenever $\overline{\Omega}(x_0, t_0; c) \subseteq E$.

An extended real-valued function v on E is called a *supertemperature* on E if it satisfies the following four conditions:

(δ_1) $-\infty < v(p) \leq +\infty$ for all $p \in E$;
(δ_2) v is lower semicontinuous on E;
(δ_3) v is finite on a dense subset of E;
(δ_4) given any point $(x_0, t_0) \in E$ and positive number ϵ, there is a positive number $c < \epsilon$ such that the closed heat ball $\overline{\Omega}(x_0, t_0; c) \subseteq E$ and

$$v(x_0, t_0) \geq (4\pi c)^{-n/2} \iint_{\Omega(x_0,t_0;c)} \frac{|x_0 - x|^2}{4(t_0 - t)^2} u(x, t)\, dx\, dt.$$

If v is a supertemperature on E, D is an open subset of E and u is a temperature such that $u \leq v$ on D, then u is called a *thermic minorant* of v on D.

Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let v be a supertemperature on E. If u is a temperature on D such that the function w, defined by

$$w = \begin{cases} u & \text{on } D, \\ v & \text{on } E \backslash \overline{D}, \end{cases}$$

can be extended to a supertemperature on E, we say that u is *extendable by v (to E)*.

The reader is assumed to be familiar with the article [13] where extendable temperatures were introduced. It was noted there that [8, Theorem 6] gave examples of open subsets D, such as heat balls, for which there is only one temperature u on D that is extendable by a given supertemperature v. Otherwise there are infinitely many. The examples in [8] are of very special open sets and it is the purpose of this paper to present a general condition for uniqueness.

We require a classification of the boundary points of E, in which we use the following notation for upper and lower half-balls. Given any point $p_0 = (x_0, t_0)$ in $\mathbb{R}^{n+1}$ and $r > 0$, we put $H(p_0, r) = \{(x, t) : |x - x_0|^2 + (t - t_0)^2 < r^2,\ t < t_0\}$ and $H^*(p_0, r) = \{(x, t) : |x - x_0|^2 + (t - t_0)^2 < r^2,\ t > t_0\}$. Let q be a boundary point of the bounded open set D. In our classification of boundary points, we always suppose that the boundary of D does not contain any polar set whose union with D would give another open set. We call q a *normal* boundary point if every lower half-ball centred at q meets the complement of D. If this condition fails, and also for every $r > 0$ we have $H^*(q, r) \cap D \neq \emptyset$, then q is called a *semi-singular* boundary point. The set of all normal boundary points of D is denoted by $\partial_n D$ and that of all semi-singular points by $\partial_{ss} D$. The *essential boundary* $\partial_e D$ is equal to $\partial_n D \cup \partial_{ss} D$ and the *singular boundary* $\partial_s D$ is $\partial D \backslash \partial_e D$. A similar classification is made relative to the adjoint equation, by interchanging H and H^* throughout. This leads, in particular, to the idea of a point $q \in \partial D$ being a *cothermal* normal boundary point if every upper half-ball centred at q meets the complement of D; the set of all such points is denoted by $\partial_n^* D$.

A function f on $\partial_e D$ is called *resolutive* if it has a PWB (Perron–Wiener–Brelot) solution to the generalised Dirichlet problem, in the sense of [10]. That solution is denoted by S_f^D. Every function $f \in C(\partial_e D)$ is resolutive. For any point $p \in D$ there is a unique nonnegative Borel measure μ_p^D on $\partial_e D$ such that $S_f^D(p) = \int_{\partial_e D} f\, d\mu_p^D$ holds

https://doi.org/10.1017/S0004972720001070 Published online by Cambridge University Press

for every $f \in C(\partial_e D)$. The completion of this measure is called the *caloric measure* relative to D and p; it is also denoted by μ_p^D. A point $q \in \partial_n D$ is called *regular* if $\lim_{p\to q} S_f^D(p) = f(q)$ for all $f \in C(\partial D)$; a point $q \in \partial_n^* D$ is called *coregular* if this holds relative to the adjoint equation.

Further details of all the concepts we use can be found in [10].

2. Nonuniqueness of extendable temperatures

Since each of S_v^D (the generalised solution of the Dirichlet problem on D corresponding to the restriction of v to $\partial_e D$) and GM_v^D (the greatest thermic minorant of v on D) is extendable by v to E, a necessary condition for uniqueness is that these are always equal. By [12, Corollary 14], this happens if and only if every point of ∂D is a coregular point of $\partial_n^* D$ [13, page 301]. We shall give an example to show that this condition is not sufficient, using the following criterion for nonuniqueness.

THEOREM 2.1. *Let D and E be open sets such that $\overline{D}$ is a compact subset of E, and let K be a relatively closed subset of D with no interior points. If K contains a regular point of $\partial_e(D\backslash K)$, then there is a supertemperature v on E for which there are distinct temperatures on $D\backslash K$ that are extendable by v to E.*

PROOF. Let (x_0, t_0) be a regular point of $\partial_e(D\backslash K)$ that belongs to K. We choose $\delta > 0$ such that the ball $B((x_0, t_0), \delta) \subseteq D$, and denote by v the restriction to E of the characteristic function of $\mathbb{R}^n \times]t_0 - \delta, +\infty[$, which is a supertemperature on E. The temperatures $S_v^{D\backslash K}$ and S_v^D are both extendable by v to E, the former from $D\backslash K$ and the latter from D. Given any $p \in D$ and $r > 0$ such that $B(p, r) \subseteq D$, we have $B(p, r) \cap (D\backslash K) \neq \emptyset$ because K has no interior points, so that $p \in \overline{D\backslash K}$. Thus $D \subseteq \overline{D\backslash K}$, which implies that $\overline{D} = \overline{D\backslash K}$. It follows that the restriction of S_v^D to $D\backslash K$ is also extendable by v to E. We claim that $S_v^D \neq S_v^{D\backslash K}$ on $D\backslash K$. To show this we note that, because (x_0, t_0) is a regular point of $\partial_e(D\backslash K)$, we have $S_v^{D\backslash K}(x, t) \to v(x_0, t_0) = 1$ as $(x, t) \to (x_0, t_0)$ if $(x_0, t_0) \in \partial_n(D\backslash K)$, or as $(x, t) \to (x_0, t_0+)$ if $(x_0, t_0) \in \partial_{ss}(D\backslash K)$; but $S_v^D(x, t) = 0$ whenever $t \leq t_0 - \delta$, so that $S_v^D(x_0, t_0) < 1$ in view of the strong maximum principle and the fact that $S_v^D \leq 1$ on D. □

For our example we also use the following extension of a theorem of Kaufman and Wu [6].

LEMMA 2.2. *Let D be an open set and let $(x^*, t^*) \in \partial D$. Let Ξ be a hyperplane parallel to the t-axis and passing through the point (x^*, t^*). If $\mathbb{R}^{n+1}\backslash D$ contains a set of the form $\Xi \cap (B \times [t', t^*])$ for some neighbourhood B of x^* in $\mathbb{R}^n$ and some $t' < t^*$, then (x^*, t^*) is a regular point of $\partial_n D$.*

PROOF. If $n = 1$ then $\Xi = \{x^*\} \times \mathbb{R}$, and so $\Xi \cap (B \times [t', t^*]) = \{x^*\} \times [t', t^*]$. Here Kaufman and Wu [6, Theorem 5] have proved the result using the function

$$w(x, t) = \sqrt{\frac{\delta}{\pi}} - \int_0^\delta W(x - x^*, t - t^* + s)\, ds$$

https://doi.org/10.1017/S0004972720001070 Published online by Cambridge University Press

on $\mathbb{R}^2$, where $\delta = t^* - t'$. This function is positive except at (x^*, t^*), where it is continuous to zero, and is a temperature outside $\{x^*\} \times [t', t^*]$. Therefore the restriction of w to D is a barrier at (x^*, t^*) and the point is regular by [10, Theorem 8.46].

Now suppose that $n > 1$. We can assume that Ξ is orthogonal to the x_n-axis, and also that $B = \prod_{i=1}^{n}]x_i^* - \epsilon, x_i^* + \epsilon[$ for some $\epsilon > 0$. Then $\Xi = \mathbb{R}^{n-1} \times \{x_n^*\} \times \mathbb{R}$ and $\Xi \cap (B \times [t', t^*]) = \prod_{i=1}^{n-1}]x_i^* - \epsilon, x_i^* + \epsilon[\times \{x_n^*\} \times [t', t^*]$. Taking the above function w, we put $v(x, t) = w(x_n, t)$. Then v is positive except on $\mathbb{R}^{n-1} \times \{x_n^*\} \times \{t^*\}$, where it is continuous to zero, and is a temperature outside $\mathbb{R}^{n-1} \times \{x_n^*\} \times [t', t^*]$. Therefore the restriction of v to $D \cap (\prod_{i=1}^{n-1}]x_i^* - \epsilon, x_i^* + \epsilon[\times \mathbb{R} \times \mathbb{R})$ is a barrier at (x^*, t^*), and the point is regular. □

EXAMPLE 2.3. We let L denote the cylinder $\{(x, t) : |x| < 1, -4 < t < -1\}$, M the strip $\mathbb{R}^{n-1} \times \{0\} \times [-4, -2]$ and N the truncated cone $\{(x, t) : |x| < -t, -1 \leq t < 0\}$. We put $A = N \cup L$. Then $A \backslash M$ is an open set such that every point of ∂A is a regular point of $\partial_n(A \backslash M)$, by the parabolic tusk test [5, 10, Theorem 8.52], and every point of $\partial(A \backslash M) \backslash \partial A$ is a regular point of $\partial_n(A \backslash M)$, by the above lemma. We now put $D = -A$ and $K = (-M) \cap D$, so that $D \backslash K = -(A \backslash M)$. Now every point of $\partial(D \backslash K)$ is a coregular point of $\partial_n^*(D \backslash K)$. Furthermore, if $(y, s) \in K$ and $2 < s < 4$, then (y, s) is a regular point of $\partial_n(D \backslash K)$ by the lemma. If E is any open superset of $\overline{D}$, Theorem 2.1 now shows that there is no uniqueness for D.

3. A sufficient condition for uniqueness

Our condition requires the concept of the *thermal fine topology*, which is the coarsest topology on $\mathbb{R}^{n+1}$ that makes every supertemperature on $\mathbb{R}^{n+1}$ continuous. It is strictly finer than the Euclidean topology and notions relative to it will be prefixed *thermal fine*. Any supertemperature on any open set is thermal fine continuous. A set S is said to be *thermally thin* at a point if that point is not a thermal fine limit point of S. Further details can be found in [4, 10] and, in much more general contexts, in [2, 3].

Every result about the heat equation has an obvious dual for the adjoint heat equation, obtained by reversing the temporal variable. Such results are not usually stated explicitly, but are referred to as the *cothermal duals* of given results on the heat equation itself. In particular, the thermal fine topology has a dual topology related to the adjoint equation called the *cothermal* fine topology, which we shall also use.

To recap, a necessary condition for uniqueness is that every point of ∂D is a coregular point of $\partial_n^* D$. By the cothermal dual of [10, Theorem 9.40], an equivalent condition is that every point of ∂D is a cothermal fine limit point of $E \backslash D$. This form of the condition is one of the hypotheses of Theorem 3.1 below, which also uses the concept of a caloric measure null set. We recall from [9] that a subset Z of $\partial_e D$ is a *caloric measure null set* for D if $\mu_p^D(Z) = 0$ for all $p \in D$. An equivalent condition is that $S_{\chi_Z}^D = 0$ on D, where χ_Z denotes the characteristic function of Z.

We use m_{n+1} to denote Lebesgue measure on $\mathbb{R}^{n+1}$.

https://doi.org/10.1017/S0004972720001070 Published online by Cambridge University Press

THEOREM 3.1. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let Z be the set of points of $\partial_e D$ where $E\backslash\overline{D}$ is thermally thin. Suppose that:*

(a) *every point of ∂D is a cothermal fine limit point of $E\backslash D$;*
(b) *Z is a caloric measure null set for D; and*
(c) $m_{n+1}(Z) = 0$.

Then for each supertemperature v on E there is a unique temperature u on D that is extendable by v to E.

PROOF. Let v be a supertemperature on E and let $\mathcal{F}$ denote the class of all supertemperatures on E that are equal to v on $E\backslash\overline{D}$. For any function $w \in \mathcal{F}$, we know from [11, Theorem 2.5] that the restriction of w to $\partial_e D$ is resolutive for D, and the function which is equal to S_w^D on D and to $w = v$ on $E\backslash\overline{D}$ can be extended to a supertemperature $w_* \in \mathcal{F}$. Since [10, Lemma 9.3] shows that w is thermal fine continuous on E, we have $w = v$ on $\partial_e D\backslash Z$. Therefore condition (b) implies that $w = v$ on $\partial_e D$ outside a caloric measure null set, so that $S_w^D = S_v^D$ on D in view of [10, Corollary 8.34]. Let μ denote the Riesz measure associated with w and let Y denote the complement in ∂D of the set of coregular points of $\partial_n^* D$. Then [12, Theorem 7] shows that

$$GM_w^D = S_w^D + G_{\overline{D}}\mu_Y = S_v^D + G_{\overline{D}}\mu_Y,$$

where μ_Y is the restriction to Y of μ. Since condition (a) holds, every point of ∂D is a coregular point of $\partial_n^* D$, by the cothermal dual of [10, Theorem 9.40], so that $Y = \emptyset$ and $GM_w^D = S_v^D = v_*$ on D. Hence, if w is also a temperature on D, then $w = GM_w^D = v_*$ on D, so that $w = v_*$ on $E\backslash\partial D$. From above, $w = v = v_*$ on $\partial_e D\backslash Z$, so that $w = v_*$ on $E\backslash(Z \cup \partial_s D)$. Since [10, Theorem 8.40] shows that $\partial_s D$ is contained in a sequence of hyperplanes of the form $\mathbb{R}^n \times \{t\}$, we have $m_{n+1}(\partial_s D) = 0$, and hence condition (c) implies that $w = v_*$ almost everywhere on E. Therefore $w = v_*$ everywhere on E by [10, Theorem 3.59], which proves the uniqueness. □

REMARK 3.2. In the definition of the set Z in Theorem 3.1, we cannot replace $E\backslash\overline{D}$ by $E\backslash D$. To show this, we take the set D of Example 2.3. The only points of $\partial_e D$ that are not thermal fine limit points of $E\backslash D$ are, by [10, Theorem 9.40], those that are not regular points of $\partial_n D$, which are those in the set $\{(x, t) : |x| < 1, x_n = 0, t = 2\}$. This set is polar [7, page 280], and hence both a caloric measure null set for D and a Lebesgue null subset of $\mathbb{R}^{n+1}$. Hence conditions (a)–(c) of Theorem 3.1 are satisfied if the definition of Z is changed, but there is no uniqueness.

There is no relationship between the two conditions on Z in Theorem 3.1. Clearly $\partial_e D$ is never a caloric measure null set for D, whereas there are many open sets for which $m_{n+1}(\partial_e D) = 0$. We now give an example in which a caloric measure null set Z has Lebesgue measure $m_{n+1}(Z) > 0$. This is similar to [1, Example 6.5.4], which dealt with harmonic measure.

https://doi.org/10.1017/S0004972720001070 Published online by Cambridge University Press

EXAMPLE 3.3. Let $\{t_k\}$ be a dense sequence in $]0,1[$ and let

$$D =]0,1[^n \times \Big(\bigcup_{k=1}^{\infty}]\, t_k, (t_k + 2^{-k-1}) \wedge 1[\Big).$$

The density of $\{t_k\}$ implies that $\overline{D} = [0,1]^{n+1}$. Also,

$$m_{n+1}(\partial_e D) = m_{n+1}(\overline{D}) - m_{n+1}(\partial_s D) - m_{n+1}(D) \geq 1 - \sum_{k=1}^{\infty} 2^{-k-1} = \frac{1}{2}.$$

Let Z be the set of all points in $\partial_e D$ that are not in $\partial_e \Lambda(q;D)$ for any point $q \in D$. It follows from [9, Lemma 4.3] that Z is a caloric measure null set for D. Let $\{q_j\}$ be a sequence of points in D such that $\bigcup_{j=1}^{\infty} \Lambda(q_j;D) = D$. Given any point $q \in D$, we can find j such that $q \in \Lambda(q_j;D)$, so that $\Lambda(q;D) \subseteq \Lambda(q_j;D)$ and hence $\Lambda(q;\Lambda(q_j;D)) = \Lambda(q;D) \subseteq \Lambda(q_j;D)$. It therefore follows from [9, Lemma 2.9] that $\partial_e \Lambda(q;D) \subseteq \partial_e \Lambda(q_j;D)$, which implies that Z is the set of points in $\partial_e D$ that do not belong to $\partial_e \Lambda(q_j;D)$ for any j. Any component of D is an $(n+1)$-dimensional interval, so that the same is true of $\Lambda(q_j;D)$ for every j. Therefore $m_{n+1}(\partial_e \Lambda(q_j;D)) = 0$ for all j, so that

$$m_{n+1}(Z) = m_{n+1}(\partial_e D) - m_{n+1}\Big(\bigcup_{j=1}^{\infty} \partial_e \Lambda(q_j;D) \Big) = m_{n+1}(\partial_e D) \geq \frac{1}{2}.$$

It is desirable to have a uniqueness theorem which does not explicitly involve the thermal and cothermal fine topologies, and so can be applied without knowledge of those topologies. One can be given using parabolic and coparabolic tusks, which are defined as follows. Let $q = (y,s)$, let $r < s$ and let B be a closed n-dimensional ball. Given any point $x \in B$, we denote by γ_x the parabolic curve with vertex at q that passes through the point (x,r). The set

$$\Gamma = \bigcup_{x \in B} \gamma_x$$

is called a *parabolic tusk with vertex at q*. Dually, if $r > s$, the set

$$\Delta = \bigcup_{x \in B} \gamma_x$$

is called a *coparabolic tusk with vertex at q*.

Usually $m_{n+1}(\partial D) = 0$, in which case condition (c) in Theorem 3.1 is superfluous. We incorporate this into our final result.

COROLLARY 3.4. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E and $m_{n+1}(\partial_e D) = 0$. Suppose that each point of ∂D is the vertex of some coparabolic tusk $\Delta \subseteq E \backslash D$, and that each point of $\partial_e D$ outside a set Z of caloric measure zero is the vertex of some parabolic tusk $\Gamma \subseteq E \backslash \overline{D}$. Then for each supertemperature v on E there is a unique temperature u on D that is extendable by v to E.*

PROOF. Let $q \in \partial D$, so that q is the vertex of some coparabolic tusk $\Delta_q \subseteq E\backslash D$. By the cothermal dual of [10, Example 9.42], q is a cothermal fine limit point of Δ_q, so that q is also a cothermal fine limit point of $E\backslash D$. Furthermore, each point $q \in \partial_e D\backslash Z$ is the vertex of some parabolic tusk $\Gamma_q \subseteq E\backslash \overline{D}$. By [10, Example 9.42], q is a thermal fine limit point of Γ_q, so that it is also a thermal fine limit point of $E\backslash \overline{D}$. Since $m_{n+1}(Z) \leq m_{n+1}(\partial_e D) = 0$, the result now follows from Theorem 3.1. □

References

[1] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory* (Springer-Verlag, London, 2001).

[2] H. Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture Notes in Mathematics, 22 (Springer-Verlag, Berlin, 1966).

[3] J. Bliedtner and W. Hansen, *Potential Theory: An Analytic and Probabilistic Approach to Balayage* (Springer-Verlag, Berlin, 1986).

[4] J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Grundlehren der mathematischen Wissenschaften, 262 (Springer-Verlag, New York, 1984).

[5] E. G. Effros and J. L. Kazdan, 'On the Dirichlet problem for the heat equation', *Indiana Univ. Math. J.* **20** (1970–1), 683–693.

[6] R. Kaufman and J. Wu, 'Parabolic potential theory', *J. Differential Equations* **43** (1982), 204–234.

[7] N. A. Watson, 'Green functions, potentials, and the Dirichlet problem for the heat equation', *Proc. London Math. Soc.* **33** (1976), 251–298.

[8] N. A. Watson, 'Mean values of subtemperatures over level surfaces of Green functions', *Ark. Mat.* **30** (1992), 165–185.

[9] N. A. Watson, 'Caloric measure for arbitrary open sets', *J. Aust. Math. Soc.* **92** (2012), 391–407.

[10] N. A. Watson, *Introduction to Heat Potential Theory*, Mathematical Surveys and Monographs, 182 (American Mathematical Society, Providence, RI, 2012).

[11] N. A. Watson, 'Thermic minorants and reductions of supertemperatures', *J. Aust. Math. Soc.* **99** (2015), 128–144.

[12] N. A. Watson, 'Extensions of Green functions and the representation of greatest thermic minorants', *New Zealand J. Math.* **47** (2017), 97–110.

[13] N. A. Watson, 'Extendable temperatures', *Bull. Aust. Math. Soc.* **100** (2019), 297–303.

NEIL A. WATSON, School of Mathematics and Statistics,
University of Canterbury, Private Bag, Christchurch, New Zealand
e-mail: neil.watson@canterbury.ac.nz

https://doi.org/10.1017/S0004972720001070 Published online by Cambridge University Press