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On finite sections of the multiplicative Hilbert inequalities

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Abstract. We determine the asymptotic behavior of the eigenvalues of finite sections of the multiplicative Hilbert matrices.

1 Introduction

The Hilbert matrix $((m+n+1)^{-1})_{m,n\geqslant 0}$ is an example of Hankel matrix and it can be viewed as the matrix representation of the integral operator

$$Hf(z) := \int_0^1 \frac{f(x)}{1-xz}dx, \qquad |z|<1$$

with respect to the basis $(z^n)_{n\geqslant 0}$ for the Hardy space H^2 on the unit disc $\mathbb{D}$, the space of analytic functions on $\mathbb{D}$ with square-summable Taylor coefficients. The norm of the Hilbert matrix is equal to π. The multiplicative Hilbert matrix $((\sqrt{mn}\log(mn))^{-1})_{m,n\geqslant 2}$ was introduced in the study of Dirichlet series. In [3], Brevig et al. proved that the norm of the multiplicative Hilbert matrix is equal to π. Note that the multiplicative Hilbert matrix is the matrix representation of the integral operator

$$\mathcal{H}f(s) := \int_{1/2}^{+\infty} f(w)(\zeta(w+s)-1)dw, \qquad \Re(s)>1/2$$

with respect to the basis $(n^{-s})_{n\geqslant 2}$ for $\mathcal{H}_0^2$, the Hardy space of Dirichlet series vanishing at $+\infty$ and with square-summable coefficients, where ζ is the Riemann zeta function. For a comprehensive study of multiplicative Hilbert matrices, and more generally Helson matrices (also known as multiplicative Hankel matrices), we refer the reader to [6–8].

The ℓ^p version of the multiplicative Hilbert matrix was also considered in [3], namely, for $p>1$ and $n\geqslant 2$, if $a_2,\dots,a_n,b_2,\dots,b_n$ are positive real numbers which are not all zero, then

$$\sum_{i,j=2}^{n} \frac{a_i\, b_j}{i^{1/q} j^{1/p}\log(ij)} \leqslant \frac{\pi}{\sin(\pi/p)}\left(\sum_{i=2}^{n} a_i^p\right)^{1/p}\left(\sum_{j=2}^{n} b_j^q\right)^{1/q}, \tag{1}$$

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where $q = p/(p-1)$, the best possible constant is $\pi/\sin(\pi/p)$. In this paper, we are interested in the multiplicative Hilbert inequalities restricted to n variables. The ℓ^p version of the finite section of the Hilbert matrix was considered by de Bruijn–Wilf [4, 11] for $p = 2$, and by Bolmarcich [2] for $p \neq 2$. For $p > 1$ and $n \geqslant 3$, we denote by $\lambda_{p,n}$ the best possible constant for the inequality

$$\sum_{i,j=3}^{n} \frac{a_i b_j}{i^{1/q} j^{1/p} \log(ij)} \leqslant \lambda_{p,n} \left(\sum_{i=3}^{n} a_i^p \right)^{1/p} \left(\sum_{j=3}^{n} b_j^q \right)^{1/q}. \tag{2}$$

We get the following result:

Theorem *There exists $\theta = \theta(p)$ such that the best possible constant $\lambda_{p,n}$ for the inequality (2) satisfies*

$$\lambda_{p,n} \leqslant \frac{\pi}{\sin(\pi/p)} - \frac{\pi^5/\sin^3(\pi/p)}{\theta(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty.$$

Furthermore, for $p = 2$, the best possible constant $\lambda_{p,n}$ for the inequality (2) is exactly $\theta = 2$ and

$$\lambda_{2,n} = \pi - \frac{\pi^5}{2(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty.$$

For the general case $p > 1$, we do not know the exact value of θ. But we know that it satisfies $2 \leqslant \theta \leqslant \max(p,q) + \varepsilon$, where ε depending only on p. The proof of our result is based on Bolmarcich's theorem [2] which is a finite section version of the classical inequalities of Hardy et al. [5, Theorem 318]. For the case of $p = 2$, we compare the matrix operator with an integral operator whose spectral asymptotic can be derived from general results of Widom [10], which gives the asymptotic behavior of the eigenvalues of Toeplitz integral operators.

The plan of the paper is the following. In the next section, we recall Bolmarcich's Theorem which we use in the case $p \neq 2$, and de Bruijn–Wilf's Theorem which we use in the case $p = 2$. We also give some preliminaries results. In Section 3, we prove the announced results.

2 Preliminaries

2.1 Lemmas for the case where $p \neq 2$

We begin by giving Bolmarcich's Theorem [2].

Theorem 1 (Bolmarchich) *Let $1 < p, q < \infty$ where $1/p + 1/q = 1$ and let $K(x,y)$ be a positive kernel satisfying $K(\alpha x, \alpha y) = \alpha^{-1} K(x,y)$ for $x, y \geqslant 0$ and $\alpha > 0$. Suppose that*

$$M := \int_0^\infty K(1,t) t^{-1/p} \mathrm{d}t, \qquad \delta := \int_0^\infty K(1,t) t^{-1/p} \log t \, \mathrm{d}t,$$
$$\gamma := \int_0^\infty K(1,t) t^{-1/p} (\log t)^2 \mathrm{d}t, \qquad \sigma := \int_0^\infty K(1,t) t^{-1/p} |\log t|^3 \mathrm{d}t$$

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all exist, and

$$\int_0^s K(1,t)t^{-1/p}\mathrm{d}t \sim s^{\delta_1} \qquad \textit{and} \qquad \int_0^s K(t,1)t^{-1/q}\mathrm{d}t \sim s^{\delta_2}, \qquad s \to 0.$$

for some $\delta_1 > 0$ *and* $\delta_2 > 0$. *Then for* $f(x) \geqslant 0$ *and* $n \geqslant 1$, *the best possible constant* M_n *for the inequality*

$$\int_{1/\sqrt{n}}^{\sqrt{n}} \left(\int_{1/\sqrt{n}}^{\sqrt{n}} K(x,y) f(x)\mathrm{d}x \right)^p \mathrm{d}y \leqslant M_n^p \int_{1/\sqrt{n}}^{\sqrt{n}} (f(x))^p \mathrm{d}x \tag{3}$$

satisfies

$$M_n = M - \frac{\pi^2(\gamma - \delta^2/M)}{\theta(\log n)^2} + O((\log n)^{-3}), \qquad n \to \infty,$$

where $2 \leqslant \theta \leqslant \max(p,q) + \varepsilon$, $\varepsilon > 0$. *Furthermore, if* $K(x,y)$ *is decreasing as* $x, y \to \infty$, $a_i \geqslant 0$, *then*

$$\sum_{j=1}^n \left(\sum_{i=1}^n K(i,j) a_i \right)^p \leqslant M_n^p \sum_{i=1}^n a_i^p. \tag{4}$$

The following lemma gives the computation of the integrals M, δ, γ of Theorem 1 if the kernel is $K(x,y) = 1/(x+y)$.

Lemma 2 *Let* $p > 1$. *For* $k = 0,1,2$, *set*

$$J_k = \int_0^\infty \frac{t^{-1/p}(\log t)^k}{1+t}\mathrm{d}t.$$

Then

$$J_0 = \frac{\pi}{\sin(\pi/p)}, \qquad J_1 = \frac{\pi^2\cos(\pi/p)}{\sin^2(\pi/p)}, \qquad \textit{and} \qquad J_2 = \frac{\pi^3(1+\cos^2(\pi/p))}{\sin^3(\pi/p)}.$$

Proof Let $q > 1$ such that $1/p + 1/q = 1$. By applying a change of variable $t = \mathrm{e}^u$, we get

$$J_k = \int_{-\infty}^\infty \frac{u^k \mathrm{e}^{u/q}}{1+\mathrm{e}^u}\mathrm{d}u \qquad \text{for } k = 0,1,2.$$

To compute for J_k, we use Cauchy's formula. Let

$$f_k(z) = \frac{z^k \mathrm{e}^{z/q}}{1+\mathrm{e}^z} \qquad \text{for } k = 0,1,2.$$

For $R > 0$, we consider the rectangular contour $\Gamma_R = I_1 \cup I_2 \cup I_3 \cup I_4$ where $I_1 = [-R, R]$, $I_2 = [R, R+2i\pi]$, $I_3 = [R+2i\pi, -R+2i\pi,]$, and $I_4 = [-R+2i\pi, -R]$. By Cauchy's residue theorem,

$$\int_{\Gamma_R} f_k(z) = \sum_{j=1}^4 \int_{I_j} f_k(z)\mathrm{d}z = -2(i\pi)^{k+1}\mathrm{e}^{i\pi/q}. \tag{5}$$

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for $k = 0, 1, 2$. We then obtain

$$\lim_{R\to\infty} \int_{I_1} f_k(z)\mathrm{d}z = J_k, \qquad \lim_{R\to\infty} \int_{I_2} f_k(z)\mathrm{d}z = \lim_{R\to\infty} \int_{I_4} f_k(z)\mathrm{d}z = 0 \tag{6}$$

for all $k = 0, 1, 2$, and

$$\lim_{R\to\infty} \int_{I_3} f_k(z)\mathrm{d}z = \begin{cases} -\mathrm{e}^{2i\pi/q} J_0, & \text{if } k = 0, \\ \mathrm{e}^{2i\pi/q}(-J_1 - 2i\pi J_0), & \text{if } k = 1, \\ \mathrm{e}^{2i\pi/q}(-J_2 - 2i\pi J_1 + 4\pi^2 J_0), & \text{if } k = 2. \end{cases} \tag{7}$$

By (5)–(7) we get the results. Note that the value of J_0 was already known [4, 5, 11]. ■

Remark 2 In Theorem 1, if we apply the substitution

$$u = x\sqrt{n}, \qquad v = y\sqrt{n}, \qquad \text{and} \qquad f(u/\sqrt{n}) = g(u),$$

then (3) takes the form

$$\int_1^n \left(\int_1^n K(u,v) g(u)\mathrm{d}u \right)^p \mathrm{d}v \leqslant M_n^p \int_1^n (g(u))^p \mathrm{d}u. \tag{8}$$

Indeed, we have

$$K(x,y) = K(u/\sqrt{n}, v/\sqrt{n}) = \sqrt{n} K(u,v).$$

Lemma 3 *Let $p > 1$ and $n \geqslant 1$. Then for $g \geqslant 0$, the best possible constant M_n for the inequality*

$$\int_1^n \left(\int_1^n \frac{g(u)}{u+v}\mathrm{d}u \right)^p \mathrm{d}v \leqslant M_n^p \int_1^n (g(u))^p \mathrm{d}u \tag{9}$$

satisfies

$$M_n = \frac{\pi}{\sin(\pi/p)} - \frac{\pi^5 \sin^3(\pi/p)}{\theta(\log n)^2} + O((\log n)^{-3}), \qquad n \to \infty,$$

where $2 \leqslant \theta \leqslant \max(p,q) + \varepsilon$, $\varepsilon > 0$.

Proof Put $K(u,v) = 1/(u+v)$. The kernel K is positive and satisfies $K(\alpha u, \alpha v) = \alpha^{-1} K(u,v)$. By Lemma 2, the integrals M, δ, γ, σ converge, and the values of M, δ, γ are given by J_0, J_1, J_2. Moreover,

$$\int_0^s K(1,t) t^{-1/p}\mathrm{d}t = \int_0^s \frac{t^{-1/p}}{1+t}\mathrm{d}t \sim s^q,$$

and

$$\int_0^s K(t,1) t^{-1/q}\mathrm{d}t = \int_0^s \frac{t^{-1/q}}{t+1}\mathrm{d}t \sim s^p, \qquad s \to 0.$$

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Then by Theorem 1 and Remark 2, we obtain

$$\begin{aligned} M_n &= M - \frac{\pi^2(\gamma - \delta^2/M)}{\theta(\log n)^2} + O((\log n)^{-3}) \\ &= \frac{\pi}{\sin(\pi/p)} - \frac{\pi^5 \sin^3(\pi/p)}{\theta(\log n)^2} + O((\log n)^{-3}) \qquad (n \to \infty). \end{aligned}$$ ■

Lemma 4 *Let $p > 1$ and $\alpha \geqslant 2$. Then for $f \geqslant 0$, the best possible constant $\widetilde{M}_n$ for the inequality*

$$\int_\alpha^n \left(\int_\alpha^n \frac{f(x)}{x^{1/q} y^{1/p} \log(xy)} \mathrm{d}x \right)^p \mathrm{d}y \leqslant \widetilde{M}_n^p \int_\alpha^n (f(x))^p \, \mathrm{d}x \tag{10}$$

satisfies

$$\widetilde{M}_n = \frac{\pi}{\sin(\pi/p)} - \frac{\pi^5 \sin^3(\pi/p)}{\theta(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty,$$

where $2 \leqslant \theta \leqslant \max(p,q) + \varepsilon$, $\varepsilon > 0$.

Proof By setting

$$f(x) = x^{-1/p} g(\log x/\log \alpha), \quad u = \log x/\log \alpha, \quad \text{and} \quad v = \log y/\log \alpha,$$

the inequality (10) takes the form

$$\int_1^{\log n/\log \alpha} \left(\int_1^{\log n/\log \alpha} \frac{g(u)}{u+v} \mathrm{d}u \right)^p \mathrm{d}v \leqslant \widetilde{M}_n^p \int_1^{\log n/\log \alpha} (g(v))^p \mathrm{d}v,$$

and by Lemma 3, the best constant possible $\widetilde{M}_n$ is

$$\begin{aligned} \widetilde{M}_n &= M_{\log n/\log \alpha} \\ &= \frac{\pi}{\sin(\pi/p)} - \frac{\pi^5 \sin^3(\pi/p)}{\theta(\log(\log n/\log \alpha))^2} + O((\log(\log n/\log \alpha))^{-3}) \\ &= \frac{\pi}{\sin(\pi/p)} - \frac{\pi^5 \sin^3(\pi/p)}{\theta(\log\log n)^2} + O((\log\log n)^{-3}) \qquad (n \to \infty). \end{aligned}$$ ■

2.2 Lemmas for the case where $p = 2$

We begin by providing the statement of following version of Widom's Theorem [10] given by de Bruijn–Wilf [4] (see also [11, Theorem 2.9] and Section 2.5 p. 32)

Theorem 5 *Let $n \geqslant 1$, $g \in L([1,n])$, $g \geqslant 0$. The largest eigenvalue μ_n of the integral equation*

$$\int_1^n \frac{g(u)}{u+v} \mathrm{d}u = \mu g(v), \qquad 1 \leqslant v \leqslant n,$$

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satisfies

$$\mu_n = \pi - \frac{\pi^5}{2(\log n)^2} + O((\log n)^{-3}), \qquad n \to \infty.$$

Lemma 6 *Let $n \geqslant \alpha$ $(\alpha \geqslant 2)$, $f \in L([\alpha, n])$, $f \geqslant 0$. The largest eigenvalue $\widetilde{\mu}_n$ of the integral equation*

$$\int_\alpha^n \frac{f(x)}{\sqrt{xy}\log(xy)}\mathrm{d}x = \widetilde{\mu} f(y), \qquad \alpha \leqslant y \leqslant n, \tag{11}$$

satisfies

$$\widetilde{\mu}_n = \pi - \frac{\pi^5}{2(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty.$$

Proof By setting

$$f(x) = g(\log x/\log\alpha)/\sqrt{x}, \quad u = \log x/\log\alpha, \quad \text{and} \quad v = \log y/\log\alpha,$$

the equation (11) becomes

$$\int_1^{\log n/\log\alpha} \frac{g(u)}{u+v} du = \widetilde{\mu} g(v).$$

Using Theorem 5, we obtain

$$\begin{aligned}\widetilde{\mu}_n &= \mu_{\log n/\log\alpha}\\ &= \pi - \frac{\pi^5}{2(\log(\log n/\log\alpha))^2} + O((\log(\log n/\log\alpha))^{-3})\\ &= \pi - \frac{\pi^5}{2(\log\log n)^2} + O((\log\log n)^{-3}) \qquad (n \to \infty).\end{aligned}$$ ■

Lemma 7 *Let $n \geqslant 2$, $f \in L([3, n+1])$, $f \geqslant 0$. The largest eigenvalue $\widetilde{\mu}_n$ of the integral equation*

$$\int_3^{n+1} \frac{f(x)}{\sqrt{(x-1)(y-1)}\log((x-1)(y-1))}\mathrm{d}x = \widetilde{\mu} f(y), \qquad 3 \leqslant y \leqslant n+1, \tag{12}$$

satisfies

$$\widetilde{\mu}_n = \pi - \frac{\pi^5}{2(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty.$$

Proof By setting

$$s = x-1, \quad t = y-1, \quad \text{and} \quad h(s) = f(x+1),$$

the equation (12) becomes

$$\int_2^n \frac{h(s)}{\sqrt{st}\log(st)}\mathrm{d}s = \widetilde{\mu} h(t).$$

By Lemma 6, we get the result. ■

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3 Proof of the main theorem

3.1 Proof of the Theorem for the case $p \neq 2$

Let $\lambda_{p,n}$ denote the best possible constant for the inequality

$$\sum_{j=3}^{n}\left(\sum_{i=3}^{n}\frac{a_i}{i^{1/q}j^{1/p}\log(ij)}\right)^p \leqslant \lambda_{p,n}^p\sum_{i=3}^{n}a_i^p.$$

We have

$$\lambda_{p,n}^p = \max_{\|(a_i)\|_{\ell^p}=1}\sum_{j=3}^{n}\left(\sum_{i=3}^{n}\frac{a_i}{i^{1/q}j^{1/p}\log(ij)}\right)^p.$$

To determine $\lambda_{p,n}$, the method consists in relating $\lambda_{p,n}$ to $\widetilde{M}_n$, where $\widetilde{M}_n$ is the best possible constant for the inequality

$$\int_2^n\left(\int_2^n\frac{f(x)}{x^{1/q}y^{1/p}\log(xy)}dx\right)^p\mathrm{d}y \leqslant \widetilde{M}_n^p\int_2^n(f(x))^p\mathrm{d}x.$$

Let $(a_i)_{3\leqslant i\leqslant n}$ be a positive sequence of ℓ^p such that $\|(a_i)\|_{\ell^p}=1$. Put $f(x)=a_{i+1}$ for $i\leqslant x\leqslant i+1$, we have $\|f\|_{L^p}=1$. Then

$$\widetilde{M}_n^p \geqslant \int_2^n\left(\int_2^n\frac{f(x)}{x^{1/q}y^{1/p}\log(xy)}\mathrm{d}x\right)^p\mathrm{d}y \geqslant \sum_{j=3}^{n}\left(\sum_{i=3}^{n}\frac{a_i}{i^{1/q}j^{1/p}\log(ij)}\right)^p.$$

This is true for all $(a_i)_i$ positive such that $\|(a_i)\|_{\ell^p}=1$. Hence, by Lemma 4,

$$\lambda_{p,n}\leqslant\widetilde{M}_n=\frac{\pi}{\sin(\pi/p)}-\frac{\pi^5\sin^3(\pi/p)}{\theta(\log\log n)^2}+O((\log\log n)^{-3}).$$

3.2 Proof of the theorem for the case $p = 2$

Let $\lambda_{2,n}$ denote the best possible constant for the inequality

$$\sum_{i,j=3}^{n}\frac{a_ia_j}{\sqrt{ij}\log(ij)}\leqslant\lambda_{2,n}\sum_{i=3}^{n}a_i^2.$$

$\lambda_{2,n}$ is just the largest eigenvalue of the matrix equation

$$\sum_{i=3}^{n}\frac{a_i}{\sqrt{ij}\log(ij)}=\lambda a_j. \tag{13}$$

To determine $\lambda_{2,n}$, the method consists in relating $\lambda_{2,n}$ to $\widetilde{\mu}_n$, where $\widetilde{\mu}_n$ is the largest eigenvalue of the integral equation

$$\int_3^n\frac{f(x)}{\sqrt{xy}\log(xy)}\mathrm{d}x=\widetilde{\mu}f(y).$$

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Put $f(x) = a_i$ for $i \leqslant x \leqslant i+1$, we have

$$\int_3^{n+1} \frac{f(x)}{\sqrt{[x][y]}\log([x][y])}\mathrm{d}x = \sum_{i=3}^{n} \frac{a_i}{\sqrt{ij}\log(ij)}$$

and the equation (13) takes the form

$$\int_3^{n+1} \frac{f(x)}{\sqrt{[x][y]}\log([x][y])}\mathrm{d}x = \lambda f(y).$$

Now, for $3 \leqslant x, y \leqslant n+1$, we have

$$\frac{1}{\sqrt{[x][y]}\log([x][y])} \geqslant \frac{1}{\sqrt{xy}\log(xy)}.$$

Since all the kernels are positive, then by Perron–Frobenius theory, the largest eigenvalue of the kernel on the left is not less than the largest eigenvalue of the kernel on the right. Hence, by Lemma 6,

$$\lambda_{2,n} \geqslant \widetilde{\mu}_{n+1} := \pi - \frac{\pi^5}{2(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty.$$

On the other hand, for $3 \leqslant x, y \leqslant n+1$, we have

$$\frac{1}{\sqrt{[x][y]}\log([x][y])} \leqslant \frac{1}{\sqrt{(x-1)(y-1)}\log((x-1)(y-1))}.$$

Then, by Perron–Frobenius theory and by Lemma 7,

$$\lambda_{2,n} \leqslant \widetilde{\mu}_n := \pi - \frac{\pi^5}{2(\log\log n)^2} + O((\log\log n)^{-3}), \qquad n \to \infty.$$

4 Remark

It would be interesting to study the ℓ^p version of the finite section of the Beurling inequalities. In [1, pp. 367–368] (see also [9]) Beurling showed the following inequality: Let $a_1, a_2, \dots$ be non-negative real numbers such that $\sum_{n=1}^{\infty} na_n^2 < \infty$, then

$$\sum_{i,j=1}^{\infty} \frac{a_i a_j}{\log(i+j)} \leqslant K \sum_{n=1}^{\infty} na_n^2, \qquad \text{where } K = \frac{4e}{e-1}.$$

Since $\log(i+j) \geqslant \log(\sqrt{ij})$ for all $i, j \in \mathbb{N}$, if we set $a_n = \widetilde{a}_n/\sqrt{n}$, $(\widetilde{a}_n) \in \ell^2$, then by the multiplicative Hilbert inequality (1), we get a better constant $K = 2\pi$. However, it is still unknown if the best possible constant is 2π.

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