COMPOSITIO MATHEMATICA

Formality conjecture for minimal surfaces of Kodaira dimension 0

Ruggero Bandiera, Marco Manetti and Francesco Meazzini

Compositio Math. **157** (2021), 215–235.

doi:10.1112/S0010437X20007605

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

Compositio Math. 157 (2021) 215–235
doi:10.1112/S0010437X20007605

Formality conjecture for minimal surfaces of Kodaira dimension 0

Ruggero Bandiera, Marco Manetti and Francesco Meazzini

Abstract

Let $\mathcal{F}$ be a polystable sheaf on a smooth minimal projective surface of Kodaira dimension 0. Then the differential graded (DG) Lie algebra $R\,\mathrm{Hom}(\mathcal{F},\mathcal{F})$ of derived endomorphisms of $\mathcal{F}$ is formal. The proof is based on the study of equivariant L_∞ minimal models of DG Lie algebras equipped with a cyclic structure of degree 2 which is non-degenerate in cohomology, and does not rely (even for K3 surfaces) on previous results on the same subject.

1. Introduction

The main goal of this paper is to provide an elementary proof of the following theorem, which extends an analogous result for K3 surfaces [BZ18].

THEOREM 1.1 (= Theorem 5.3). *Let X be a smooth minimal projective surface of Kodaira dimension* 0*, and consider a polystable sheaf $\mathcal{F}$ on X. Then the differential graded (DG) Lie algebra $R\mathrm{Hom}_X(\mathcal{F},\mathcal{F})$ is formal.*

Moduli spaces of coherent sheaves on K3 surfaces and Abelian surfaces have been intensively studied in recent decades. Among the reasons for the interest in these objects there is certainly the fact due to Mukai that the smooth locus of the moduli space inherits a holomorphic symplectic structure from the symplectic form on the surface [Muk84]. In particular, provided that such a moduli space is smooth and projective, it yields an example of an irreducible holomorphic symplectic manifold. In general the moduli space is singular at a point corresponding to a strictly semistable sheaf; these singularities arise either when the Mukai vector is not primitive or when the polarization on the surface is not general (i.e. it lies on a wall with respect to the walls and chambers decomposition of the ample cone [KLS06, Yos01]). Nevertheless, in some cases there exist symplectic resolutions, which have been investigated for moduli spaces with general polarization and non-primitive Mukai vector. First, O'Grady found two new examples of irreducible holomorphic symplectic manifolds [O'Gr99, O'Gr03] by exhibiting symplectic resolutions of moduli spaces of sheaves on a K3 surface and on an Abelian surface. A few years later Kaledin, Lehn and Sorger showed that, other than the ones in O'Grady's examples, such moduli spaces do not admit symplectic resolutions [KLS06].

More recently, in [AS18] Arbarello and Saccà turned their attention to the case of a K3 surface with a non-general polarization and Mukai vector $(0, c_1, \chi)$. The corresponding moduli

Received 12 January 2020, accepted in final form 24 July 2020, published online 18 February 2021.
2010 Mathematics Subject Classification 14F05, 14D15, 16W50, 18G55 (primary).
Keywords: deformation theory, polystable sheaves, formality, differential graded Lie algebras, L_∞ algebras.

space admits a symplectic resolution, given by moving the polarization (hence changing the notion of stability) into a chamber, and they give a local description of the moduli space around the singularity in terms of a suitable Nakajima quiver variety.

By general deformation theory, an easy description of an analytic neighborhood around a singular point $[\mathcal{F}]$ in the moduli space corresponding to a given (possibly non-general) polarization can be deduced from the formality of the derived endomorphisms of the sheaf $\mathcal{F}$ on the surface X. We now briefly recall the main steps that led to the so-called Kaledin–Lehn *formality conjecture*. It is well known that the base space of the formal semiuniversal deformation of $[\mathcal{F}]$ is the scheme-theoretic fiber of the Kuranishi map

$$k\colon \widehat{\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F})} \to \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F})_0 = \ker(\operatorname{Tr}\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^2(X,\mathcal{O}_X) \cong \mathbb{C})$$

which can be chosen to be G-equivariant (see, for example, [AS18, BMM20, Rim80]) with respect to the action of the automorphisms group modulo the action of the scalars: $G = \operatorname{Aut}(\mathcal{F})/\mathbb{C}^*$. Often it is definitely not trivial to compute the null-fiber of the Kuranishi map; on the other hand. its quadratic part k_2 is nothing more than the Yoneda pairing, so that in general it is much easier to understand $k_2^{-1}(0)$ instead of $k^{-1}(0)$. In [KL07], Kaledin and Lehn essentially conjectured that for a polystable sheaf on a K3 surface the Kuranishi map is *quadratic*, namely $k_2^{-1}(0) \cong k^{-1}(0)$. If this condition is satisfied then the moduli space, locally around $[\mathcal{F}]$, is isomorphic to the geometric invariant theory quotient $k_2^{-1}(0)/\!\!/G$.

In their original paper Kaledin and Lehn gave a first example motivating and inspiring the future work on the subject. The conjecture was then proven in full generality by Yoshioka [Yos17], and partially by Arbarello and Saccà [AS18]. Let us make a few remarks before continuing. First, recall that to any (homotopy class of a) DG Lie algebra there is associated a deformation functor (see, for example, [Man09, Man20]), which in turn provides a Kuranishi map via the Maurer–Cartan equation. Moreover, if the DG Lie algebra L is formal (i.e. it is quasi-isomorphic to its cohomology) then the associated Kuranishi space $k^{-1}(0)$ is the null-fiber of the cup product in cohomology $H^1(L) \to H^2(L)$. Hence, by showing the formality of the DG Lie algebra $R\operatorname{Hom}(\mathcal{F},\mathcal{F})$ one also proves the quadraticity of the Kuranishi map. Notice that since our approach involves techniques of L_∞-algebras we investigate derived endomorphisms as a DG Lie algebra, while the papers [KL07, BZ18] consider $R\operatorname{Hom}(\mathcal{F},\mathcal{F})$ as an associative DG algebra and also Kaledin's refinement of the Massey products works in the associative setting [Kal07]. It is important to point out that the formality in the associative case is a stronger statement, but on the other hand the DG Lie formality is the one needed for applications to moduli spaces.

It is worth mentioning that formality is in general much stronger and harder to prove than the quadraticity property; from the point of view of derived algebraic geometry this can be easily understood since formality implies that the derived moduli space is locally quadratic. Nevertheless, formality of $R\operatorname{Hom}(\mathcal{F},\mathcal{F})$ has been conjectured for polystable sheaves again by Kaledin and Lehn in [KL07], it has been studied in some cases by Zhang [Zha12], and finally completely solved by Budur and Zhang [BZ18] who proved that the conjecture holds true for any polystable sheaf using results about strong uniqueness of DG enhancements.

It is interesting to notice that all of the above-cited formality results actually rely on the famous result due to Kaledin about formality in families [Kal07, Lun10]. Even if the vanishing of Massey products does not guarantee the formality of a DG algebra A (see, for example, [HS79]), Kaledin determined a refinement of them defining the so-called *Kaledin class* in a

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

certain (reduced) Hochschild homology group depending on A. Furthermore, he proved that the variation of such a class in a suitable family of DG algebras $\mathcal{A} \to S$ over an irreducible base S glues to a global section of a certain obstruction bundle $\mathcal{O}b_S$ defined on S. It follows that if $\mathcal{O}b_S$ does not admit non-trivial global sections then all the fibers $\mathcal{A}_s$ are formal [Kal07, Theorem 4.3].

Applying Kaledin's result and twistor spaces, in the paper [KL07] Kaledin and Lehn first obtained the formality of $R\mathrm{Hom}(\mathcal{F},\mathcal{F})$ for sheaves of the form $\mathcal{F} = \mathcal{I}_Z^{\oplus n}$, where $\mathcal{I}_Z$ denotes the ideal sheaf of some zero-dimensional closed subscheme Z. Later, Zhang showed that Kaledin's theorem may be applied to polystable sheaves with some constraints on the ranks of the corresponding stable summands [Zha12, Proposition 1.3], hence enlarging the class of polystable sheaves for which the formality conjecture holds. Eventually in [BZ18] Budur and Zhang established a very interesting result, namely that the formality of derived endomorphisms of any object in $D^b(X)$ is preserved under derived equivalences; hence the formality conjecture follows since by [Yos09] any polystable sheaf can be mapped via a Fourier–Mukai transform to another polystable sheaf satisfying the hypothesis of [Zha12, Proposition 3.1].

In our recent paper [BMM20], we proved that for a sheaf $\mathcal{F}$ whose automorphisms group is reductive (e.g. for any $\mathcal{F}$ polystable), the quadraticity of the Kuranishi map and the formality of the DG Lie algebra $R\,\mathrm{Hom}(\mathcal{F},\mathcal{F})$ are in fact equivalent conditions; our proof has the advantage of relaxing the hypothesis on the surface which no longer needs to be a K3. This provides further evidence of the formality conjecture without involving powerful methods of DG category theory, but instead relying on the work of Yoshioka [Yos17] and of Arbarello and Saccà [AS18]. Actually, both the papers [AS18, Yos17] base their proofs of the quadraticity property on the fundamental work [Zha12], hence again Kaledin's theorem [Kal07] seems to be essential.

The present paper aims to prove the formality conjecture for polystable sheaves on a smooth minimal projective surface of Kodaira dimension 0. Examples of such surfaces include projective K3 surfaces, Enriques surfaces, bielliptic surfaces and Abelian surfaces [BHPV04, Bea94]. One of the main innovations of our proof is that we translate the problem into a purely algebraic statement (see Theorem 3.8) about the formality of DG Lie algebras endowed with some additional structure (see Definition 3.6), which will be proved using only elementary techniques of (strong homotopic) DG Lie algebras. In particular, perhaps surprisingly, in the case of K3 and Abelian surfaces our proof of the formality conjecture only requires a basic knowledge of L_∞ algebras and is self-contained, meaning that it does not involve either Kaledin's result about formality in families or the geometric situations considered by Zhang in [Zha12]. As pointed out by one of the referees, it is similar in spirit to Neisendorfer and Miller's proof of the fact that any six-dimensional simply connected Poincaré duality space is formal [NM78].

The plan of the paper is as follows. In § 2 we fix notation and briefly summarize the results needed in the rest of the paper about formality and L_∞ algebras. In § 3 we introduce the notion of quasi-cyclic DG Lie algebras and discuss examples arising from geometric situations: a DG Lie algebra $(L, d, [-,-])$ with finite-dimensional cohomology equipped with a degree $-n$ symmetric bilinear form $(-,-)\colon L^{\odot 2} \to \mathbb{K}[-n]$ is called *quasi-cyclic of degree* n provided that

$$(dx, y) = (-1)^{|x|+1}(x, dy), \quad ([x,y],z) = (x,[y,z]), \quad \forall\, x,y,z \in L,$$

and the form induced in cohomology $(-,-)\colon H(L)^{\odot 2} \to \mathbb{K}[-n]$ is non-degenerate.

The typical example of a quasi-cyclic DG Lie algebra of degree n is given by the Dolbeault resolution $L = A_X^{0,*}(\mathcal{H}om(\mathcal{E},\mathcal{E}))$ of the sheaf of endomorphisms of a locally free sheaf $\mathcal{E}$ on an n-dimensional manifold X equipped with a nowhere vanishing holomorphic volume form ω_X, with the pairing $(f,g)=\int_X \omega_X\wedge \mathrm{Tr}(fg)$; see Example 3.7. A similar construction can be also performed when $\mathcal{E}$ is replaced by any coherent sheaf; see § 5.

Then § 4 is entirely devoted to the proof of our main algebraic result.

Theorem 1.2 (= Theorem 3.8). *Let $(L,d,[-,-],(-,-))$ be a quasi-cyclic DG Lie algebra of degree $n\leqslant 2$. Assume that there exists a splitting $L=H\oplus d(K)\oplus K$ such that:*

(i) *$H^i=0$ for $i<0$ (and hence also $H^i=0$ for $i>n$);*
(ii) *$H^0\subset L^0$ is closed with respect to the bracket $[-,-]$;*
(iii) *$H^i,K^i\subset L^i$ are H^0-submodules (with respect to the adjoint action) for all $i>0$.*

Then the DG Lie algebra $(L,d,[-,-])$ is formal.

Finally, in § 5 we discuss the applications to moduli spaces of sheaves on minimal projective surfaces of Kodaira dimension 0. We will first prove the formality conjecture for polystable sheaves on K3 and Abelian surfaces as an immediate consequence of Theorem 1.2, where the polystability assumption ensures the existence of the splitting with the required properties.

Then we will extend the formality result to polystable sheaves on surfaces with torsion canonical bundle. Here the idea is to use the cyclic covering trick in order to construct the DG Lie algebra $R\mathrm{Hom}_X(\mathcal{F},\mathcal{F})$ as a subalgebra of a suitable quasi-cyclic DG Lie algebra satisfying the assumptions of Theorem 1.2 and then use the formality transfer theorem due to the second named author [Man15, Theorem 3.4].

2. Review of formality and minimal models of DG Lie algebras

We work over a field $\mathbb{K}$ of characteristic 0 for the algebraic part and over the field $\mathbb{C}$ of complex numbers for the geometric applications. Every complex of vector spaces is intended as a cochain complex.

By definition a DG Lie algebra L is formal if it is quasi-isomorphic to its cohomology DG Lie algebra $H^*(L)$, equipped with the trivial differential and the induced bracket. In order to avoid possible mistakes, it is useful to keep in mind that not every DG Lie algebra is formal and that if L is formal, then in general there does not exist any direct quasi-isomorphism of DG Lie algebras $H^*(L)\to L$. However, since the category of DG Lie algebras admits a model structure where the fibrations (respectively, the weak equivalences) are the surjective maps (respectively, the quasi-isomorphisms) it follows that a DG Lie algebra L is formal if and only if there exists a span of surjective quasi-isomorphisms of DG Lie algebras $L\leftarrow M\to H^*(L)$.

Since two DG Lie algebras are quasi-isomorphic if and only if they are weak equivalent as L_∞ algebras, we also have that a DG Lie algebra L is formal if and only if there exist an L_∞ algebra H and a span of L_∞ weak equivalences

$$L\longleftarrow H\longrightarrow H^*(L). \tag{2.1}$$

We assume that the reader is familiar with the notion and basic properties of L_∞ algebras; see, for example, [Get09, Kon03, LM95, LS93, Man20]. For the reader's convenience and to fix the sign convention, we briefly recall here the definition of L_∞ algebra in the version used for the explicit computations that we shall perform in § 4.

Let V be a graded vector space. Given homogeneous vectors $v_1,\dots,v_n$ of V and a permutation σ of $\{1,\dots,n\}$, we denote by $\chi(\sigma;v_1,\dots,v_n)=\pm 1$ the antisymmetric Koszul sign, defined by

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

the relation

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma; v_1, \ldots, v_n)\, v_1 \wedge \cdots \wedge v_n$$

in the nth exterior power $V^{\wedge n}$. We shall simply write $\chi(\sigma)$ instead of $\chi(\sigma; v_1, \ldots, v_n)$ when the vectors $v_1, \ldots, v_n$ are clear from the context. For instance, if σ is the transposition exchanging 1 and 2 we have $\chi(\sigma) = -(-1)^{|v_1|\,|v_2|}$ where $|v|$ denotes the degree of the homogeneous vector v. Notice that if every v_i has odd degree, then $\chi(\sigma) = 1$ for every σ.

Because of the universal property of wedge powers, we shall constantly interpret every linear map $V^{\wedge p} \to W$ as a graded skew-symmetric p-linear map $V \times \cdots \times V \to W$.

Definition 2.1. An L_∞ algebra is the data of a graded vector space V together with a sequence of (multi)linear maps $\{\cdots\}_n \colon V^{\wedge n} \to V$, $n \geqslant 1$, such that for every n:

(i) $\{\cdots\}_n$ has degree $2 - n$;
(ii) for every $v_1, \ldots, v_n \in V$ homogeneous,

$$\sum_{k=1}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n-k)} \chi(\sigma)\, \{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}_k, v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}\}_{n-k+1} = 0, \tag{2.2}$$

where $S(k, n-k) = \{\sigma \in S_n \mid \sigma(i) < \sigma(i+1),\ \forall\, i \neq k\}$ is the set of $(k, n-k)$-shuffles.

In the above definition we used the sign convention of [Get09, Kon03, Man20], while in [LM95, LS93] the maps $\{\cdots\}_k$ differ by the sign $(-1)^{k(k-1)/2}$. Every DG Lie algebra $(L, d, [-,-])$ is an L_∞ algebra where $\{\cdot\}_1 = d$, $\{\cdot\cdot\}_2 = [-,-]$ and $\{\cdots\}_n = 0$ for every $n > 2$. If $\{\cdot\}_1 = 0$ the L_∞ algebra is called minimal.

There exists a general notion of L_∞ morphism (see, for example, [Man20]), but for simplicity of exposition we only recall here the case of morphisms from an L_∞ algebra to a DG Lie algebra; this particular case will be sufficient for our purposes.

Definition 2.2. Let $(V, \{\cdot\}_1, \{\cdot\cdot\}_2, \{\cdots\}_3, \ldots)$ be an L_∞ algebra and $(L, d, [-,-])$ a DG Lie algebra. An L_∞ morphism $g \colon V \to L$ is a sequence of maps $g_n \colon V^{\wedge n} \to L$, $n \geqslant 1$, with g_n of degree $1 - n$ such that, for every n and every $v_1, \ldots, v_n \in V$ homogeneous, we have

$$\frac{1}{2}\sum_{p=1}^{n-1} \sum_{\sigma \in S(p,n-p)} \chi(\sigma)(-1)^{(1-n+p)(|v_{\sigma(1)}|+\cdots+|v_{\sigma(p)}|-p)} [g_p(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), g_{n-p}(v_{\sigma(p+1)}, \ldots, v_{\sigma(n)})]$$
$$+ d g_n(v_1, \ldots, v_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n-k)} \chi(\sigma) g_{n-k+1}(\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}_k, \ldots, v_{\sigma(n)}).$$

An L_∞ morphism g as in Definition 2.2 is called a weak equivalence or a quasi-isomorphism if $g_1 \colon (V, \{\cdot\}_1) \to (L, d)$ is a quasi-isomorphism of cochain complexes.

By homotopy classification of L_∞ algebras [Kon03], for every DG Lie algebra L there exist a minimal L_∞ algebra H and an L_∞ weak equivalence $\imath \colon H \to L$. The algebra H is called the L_∞ minimal model of L and it is unique up to isomorphism, while the L_∞ morphism i is unique up to homotopy. By homological perturbation theory, every splitting of the complex (L, d) induces canonically a morphism $\imath \colon H \to L$ as above.

Recall that a splitting of (L, d) is a direct sum decomposition $L = H \oplus d(K) \oplus K$ such that H, K are graded vector subspaces of L and the restrictions of the differential d to H and K are

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

respectively zero and injective; see [Wei94, § 1.4]. In particular, $d(L) = d(K)$, $Z(L) = H \oplus d(K)$ and the natural map $H \to H^*(L)$ is an isomorphism of graded vector spaces. Denoting by $\hookrightarrow$ and $\twoheadrightarrow$ the inclusions and the projections given by the splitting $L = H \oplus d(K) \oplus K$, we define the maps

$$\imath_1 \colon H \hookrightarrow L, \quad \pi \colon L \twoheadrightarrow H, \quad h \colon L \twoheadrightarrow d(K) \xrightarrow{-d^{-1}} K \hookrightarrow L,$$

that satisfy the contraction identities

$$d\imath_1 = 0, \quad \pi d = 0, \quad \pi\imath_1 = \mathrm{id}_H, \quad dh + hd = \imath_1\pi - \mathrm{id}_L, \quad h\imath_1 = 0, \quad \pi h = 0, \quad h^2 = 0.$$

Then a minimal L_∞ algebra $(H, 0, \{\cdot\,\cdot\}_2, \{\cdot\cdot\cdot\}_3, \ldots)$ and an extension of $\imath_1$ to an L_∞ quasi-isomorphism $\imath \colon H \to L$ are defined by the recursive equations

$$\imath_p(\xi_1, \ldots, \xi_p) = \frac{1}{2}\sum_{k=1}^{p-1} \sum_{\sigma \in S(k,p-k)} \chi(\sigma)(-1)^{\alpha(\sigma)} h[\imath_k(\xi_{\sigma(1)}, \ldots), \imath_{p-k}(\ldots, \xi_{\sigma(p)})], \quad p \geqslant 2, \tag{2.3}$$

$$\{\xi_1, \ldots, \xi_p\}_p = \frac{1}{2}\sum_{k=1}^{p-1} \sum_{\sigma \in S(k,p-k)} \chi(\sigma)(-1)^{\alpha(\sigma)} \pi[\imath_k(\xi_{\sigma(1)}, \ldots), \imath_{p-k}(\ldots, \xi_{\sigma(p)})], \quad p \geqslant 2, \tag{2.4}$$

where

$$\alpha(\sigma) = (1 - p + k)\left(k + \sum_{i=1}^{k} |\xi_{\sigma(i)}|\right).$$

Notice that for every $\xi, \eta \in H$ we have

$$\imath_2(\xi, \eta) = h[\imath_1(\xi), \imath_1(\eta)], \quad \{\xi, \eta\}_2 = \pi[\imath_1(\xi), \imath_1(\eta)],$$

the integer $\alpha(\sigma)$ is even for $p = 2$ and $\chi(\sigma)(-1)^{\alpha(\sigma)} = 1$ if $|\xi_i|$ is odd for every i. Formulas (2.3) and (2.4) are well known and essentially date back to Kadeishvili's paper [Kad82]: the choice of signs comes from standard décalage isomorphisms applied to the explicit formulas used in [BM18, Theorem 3.7] and [Man20].

In [Man15] the second named author proved a series of formality criteria for DG Lie algebras. As a consequence of these criteria we have the following formality transfer theorem, where $H^*_{CE}(A, B)$ denotes the Chevalley–Eilenberg cohomology of the graded Lie algebra A with coefficient in the A-module B.

Theorem 2.3 [Man15, Theorem 3.4]. *Let $f \colon M \to L$ be a morphism of DG Lie algebras. Assume that*

(i) *L is formal;*
(ii) *the induced map $f \colon H^2_{CE}(H^*(M), H^*(M)) \to H^2_{CE}(H^*(M), H^*(L))$ is injective.*

Then M is also formal. In particular, if L is formal, f is injective and $f(M)$ is a direct summand of L as M-module, then M is also formal.

It should be noted that for $L = 0$ the above theorem reduces to the classical criterion for intrinsic formality of graded Lie algebras.

3. Cyclic and quasi-cyclic DG Lie algebras

The general notion of cyclic (DG) algebra [GK95] specialized to DG Lie algebras gives the following definition; see also [LS12].

Definition 3.1. Let n be an integer. A *cyclic DG Lie algebra* $(L, d, [-,-], (-,-))$ of degree n is a finite-dimensional DG Lie algebra $(L, d, [-,-])$ equipped with a degree $-n$ non-degenerate graded symmetric bilinear form $(-,-)\colon L^{\odot 2} \to \mathbb{K}[-n]$ such that

$$(dx, y) = (-1)^{|x|+1}(x, dy), \quad ([x,y], z) = (x, [y,z]), \quad \forall\, x, y, z \in L.$$

The condition $(dx, y) = \pm(x, dy)$ implies in particular that $d(L)^{\perp} = \ker(d)$; since L is finite-dimensional we have $d(L) = \ker(d)^{\perp}$, and this implies that also the induced bilinear form in the cohomology $H^*(L)$ is non-degenerate.

Example 3.2 (Symplectic representations). Let (V, ω) be a finite-dimensional symplectic vector space and let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Recall that a left action

$$\mathfrak{g} \times V \to V, \quad (g, v) \mapsto gv,$$

is called symplectic if for every $v, w \in V$ and every $g \in \mathfrak{g}$ we have

$$\omega(gv, w) + \omega(v, gw) = 0.$$

There exists a natural correspondence between (isomorphism classes of) symplectic representations and (isomorphism classes of) cyclic DG Lie algebras of degree 2 with trivial differential and without elements of negative degree: given a symplectic action as above, consider the graded Lie algebra $L = H(L) = L^0 \oplus L^1 \oplus L^2$ whose cyclic Lie structure is defined as follows.

(i) $L^0 = \mathfrak{g}$, $L^1 = V$, and $L^2 = \mathfrak{g}^{\vee} = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathbb{K})$.
(ii) The Lie bracket is defined by
  - $[g, v] = gv$ for every $g \in L^0$, $v \in L^1$,
  - $[v, w]\colon h \mapsto \omega(hv, w)$ for every $v, w \in L^1$, $h \in \mathfrak{g}$,
  - $[g, y]\colon h \mapsto y([h, g]_{\mathfrak{g}})$ for every $g, h \in \mathfrak{g}$, $y \in \mathfrak{g}^{\vee}$.
(iii) The pairing is defined by
  - $(-,-)\colon L^0 \times L^2 \to \mathbb{K}$ is the natural pairing,
  - $(v, w) = \omega(v, w)$ for every $v, w \in L^1$.

The relations below easily follow from the above conditions:

$$(h, [g, y]_L) = ([h, g]_{\mathfrak{g}}, y) \quad \text{for every } h, g \in L^0,\ y \in L^2,$$
$$(g, [v, w]_L) = \omega(gv, w) = \omega([g, v]_L, w) \quad \text{for every } g \in L^0,\ v, w \in L^1.$$

Moreover, the equalities

$$\omega(gv, w) + \omega(v, gw) = (gv, w) - \omega(gw, v) = ([g, v]_L, w) - (g, [v, w]_L), \ \text{ for every } g \in \mathfrak{g},\ v, w \in L^1,$$

show that the *symplectic condition* $\omega(gv, w) + \omega(v, gw) = 0$ is equivalent to the *cyclicity condition* $(g, [v, w]) = ([g, v], w)$. The proof that the data $(L, 0, [-,-], (-,-))$ defined above provides an example of a cyclic DG Lie algebra of degree 2 is now straightforward.

Notice that the Maurer–Cartan functional $\frac{1}{2}[v,v]$ coincides by definition with the moment map $\mu\colon V\to\mathfrak{g}^\vee$ of the symplectic representation.

Example 3.3. Consider the following complex of vector spaces in degrees 0,1,2:

$$L\colon\qquad \operatorname{Span}(a,b)\xrightarrow{d}\operatorname{Span}(x,y,p,db)\xrightarrow{d}\operatorname{Span}(z,dp)$$

equipped with the bilinear form $(-,-)\colon L^{\odot 2}\to\mathbb{K}[-2]$, where the only nontrivial products between basis vectors are

$$(x,y)=-(y,x)=-1,\quad (db,p)=-(p,db)=-1,\quad (a,z)=(z,a)=1,\quad (b,dp)=(dp,b)=1.$$

Next consider the bracket $[-,-]\colon L^{\wedge 2}\to L$, where the only nontrivial brackets between basis vectors are

$$[a,x]=db,\quad [a,p]=y,\quad [x,x]=dp,\quad [p,x]=z,\quad [b,x]=y.$$

The next proposition summarizes the properties or the above example that are relevant for this paper.

Proposition 3.4. *In the above setup:*

(i) *L is a cyclic DG Lie algebra of degree 2;*
(ii) *L is not a formal DG Lie algebra;*
(iii) *there does not exist any splitting $L=H\oplus d(K)\oplus K$ such that $[H^0,H^1]\subset H^1$.*

Proof. The first item is a tedious but straightforward computation. For the second item we observe that the triple Massey power of x is non-trivial since $dp=[x,x]$ and $[p,x]=z$. The third item is clear since for every splitting there exists $\alpha\in\mathbb{K}$ such that $x+\alpha db\in H^1$ and therefore $[a,x+\alpha db]=[a,x]=db\notin H^1$. □

Example 3.5. Consider the following complex of vector spaces in degrees 1,2:

$$L:\qquad \operatorname{Span}(a,b)\xrightarrow{d}\operatorname{Span}(x,db)$$

equipped with the closed bilinear form $(-,-)\colon L^{\odot 2}\to\mathbb{K}[-3]$, where the only non-trivial products between basis vectors are $(a,x)=(b,db)=1$. Next consider the bracket $[-,-]\colon L^{\wedge 2}\to L$, where the only non-trivial brackets between basis vectors are $[a,a]=db$, $[a,b]=x$. The same argument used in the proof of Proposition 3.4 shows that L is a cyclic non-formal DG Lie algebra of degree 3.

It is useful to enlarge the class of cyclic DG Lie algebras by removing the assumption that L is finite-dimensional, which is not satisfied in most geometrical situations. The same weakening of assumption was considered by Kontsevich [Kon94] in the associative case.

Definition 3.6. A *quasi-cyclic DG Lie algebra* $(L,d,[-,-],(-,-))$ of degree n is a DG Lie algebra $(L,d,[-,-])$ with finite-dimensional cohomology, together with a degree $-n$ symmetric bilinear form $(-,-)\colon L^{\odot 2}\to\mathbb{K}[-n]$ which satisfies

$$(dx,y)=(-1)^{|x|+1}(x,dy),\quad ([x,y],z)=(x,[y,z]),\quad \forall\, x,y,z\in L,$$

and such that the induced form $(-,-)\colon H(L)^{\odot 2}\to\mathbb{K}[-n]$ is non-degenerate.

If $(L, d, [-,-], (-,-))$ is a quasi-cyclic DG Lie algebra then its cohomology $H^*(L)$ is naturally endowed with a structure of cyclic graded Lie algebra of the same degree.

Example 3.7 (Vector bundles on manifolds with trivial canonical bundle). Let $\mathcal{E}$ be a locally free sheaf of a smooth complex projective manifold X of dimension n with trivial canonical bundle, and denote by ω_X be a holomorphic volume form. Then the Dolbeault complex

$$L = A_X^{0,*}(\mathcal{H}om(\mathcal{E}, \mathcal{E}))$$

of the sheaf of endomorphisms of $\mathcal{E}$ is a quasi-cyclic DG Lie algebra of degree n, where

$$(f, g) = \int_X \omega_X \wedge \operatorname{Tr}(fg).$$

We have $H^i(L) = \operatorname{Ext}^i_X(\mathcal{E}, \mathcal{E})$ and by Serre duality the induced pairing

$$(-,-)\colon\ \operatorname{Ext}^i_X(\mathcal{E}, \mathcal{E}) \times \operatorname{Ext}^{n-i}_X(\mathcal{E}, \mathcal{E}) \to \mathbb{C}$$

is non-degenerate. In § 5 we extend this construction to coherent sheaves.

We are now ready to state one of the main results of this paper, namely a sufficient condition for formality of quasi-cyclic DG Lie algebras of degree at most 2.

Theorem 3.8. *Let $(L, d, [-,-], (-,-))$ be a quasi-cyclic DG Lie algebra of degree $n \leqslant 2$. Assume that there exists a splitting $L = H \oplus d(K) \oplus K$ such that:*

(i) *$H^i = 0$ for $i < 0$ (and hence also $H^i = 0$ for $i > n$);*
(ii) *$H^0 \subset L^0$ is closed with respect to the bracket $[-,-]$;*
(iii) *$H^i, K^i \subset L^i$ are H^0-submodules (with respect to the adjoint action) for all $i > 0$.*

Then the DG Lie algebra $(L, d, [-,-])$ is formal.

For $n \leqslant 0$ the above theorem is trivial since $H^i = 0$ for every $i > 0$ and then the embedding $H^0 \to L$ is a quasi-isomorphism of DG Lie algebras. The next section will be entirely devoted to the (long) proof in the case $n = 2$, whose first step also provides a complete proof for $n = 1$. Examples 3.5 and 3.3 show that formality fails if either $n > 2$ or without the assumption (iii), even for cyclic DG Lie algebras.

4. Proof of Theorem 3.8

Let $(L, d, [-,-], (-,-))$ be as in Theorem 3.8. If A, B are two subsets of L we shall write $A \perp B$ if $(x, y) = 0$ for every $x \in A$, $y \in B$. For instance, it follows immediately from the relation $(dx, y) = (-1)^{|x|+1}(x, dy)$ that $d(K) \perp d(K)$ and $H \perp d(K)$.

Lemma 4.1. *Up to a possible restriction to a quasi-isomorphic DG Lie subalgebra of L we may assume that the splitting $L = H \oplus d(K) \oplus K$ satisfies the following conditions:*

(i) *$H^i = 0$ for $i < 0$;*
(ii) *$H^0 \subset L^0$ is closed with respect to the bracket $[-,-]$;*
(iii) *$H^i, K^i \subset L^i$ are H^0-submodules (with respect to the adjoint action) for all $i \in \mathbb{Z}$;*
(iv) *$H \perp K$.*

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

Proof. Since $H^i = 0$ for every $i < 0$ the DG Lie subalgebra

$$H^0 \oplus (H^1 \oplus K^1) \oplus (H^2 \oplus d(K^1) \oplus K^2) \oplus \cdots$$

is quasi-cyclic and quasi-isomorphic to L. This proves that, up to a possible restriction to a quasi-isomorphic DG Lie subalgebra, it is not restrictive to assume the validity of condition (iii). Next, for every integer i consider the vector subspace

$$C^i = \{x \in H^i \oplus K^i \mid (x, y) = 0 \quad \forall\, y \in H^{n-i}\}.$$

Since $(-,-)\colon H^i \times H^{n-i} \to \mathbb{K}$ is a perfect pairing, the map

$$\begin{gathered} H^i \oplus C^i \to H^i \oplus K^i \\ (h_1, h_2, k) \mapsto (h_1 - h_2, k) \end{gathered}$$

is an isomorphism. If $x \in C^i$ and $a \in H^0$, then $[a, x] \in H^i \oplus K^i$; for every $y \in H^{n-i}$ we have $(y, [a, x]) = ([y, a], x) = 0$ and therefore $[a, x] \in C^i$. Finally, replacing K^i with C^i, we may assume $H \perp K$.

For later use it should be pointed out that the non-degeneracy of $(-,-)\colon H^{\odot 2} \to \mathbb{K}$ immediately implies

$$x \in K \oplus d(K) \iff (x, y) = 0 \quad \text{for every } y \in H. \tag{4.1}$$

□

From now on we assume that $(L, d, [-,-], (-,-))$ is a quasi-cyclic DG Lie algebra of degree $n \leqslant 2$ equipped with a splitting $L = H \oplus d(K) \oplus K$ satisfying the conditions of Lemma 4.1. The fist step is to use such a splitting in order to produce a minimal L_∞ model of L. Following the recipe described in § 2, we introduce the maps

$$\imath_1\colon H \hookrightarrow L, \quad \pi\colon L \twoheadrightarrow H, \quad h\colon L \twoheadrightarrow d(K) \xrightarrow{-d^{-1}} K \hookrightarrow L$$

that satisfy the relations

$$(\imath_1(x), \imath_1(y)) = (x, y), \quad (h(l), \imath_1(x)) = 0, \quad (\pi(l), x) = (l, \imath_1(x)), \quad \forall\, x, y \in H,\ l \in L.$$

The first one is obvious and the second one follows from the orthogonality condition $H \perp K$. Since $\operatorname{Im}(\imath_1) = H$, $\operatorname{Im}(h) = K$ and $H \perp d(K) \oplus K$, the first two imply the third:

$$(\pi(l), x) = (\imath_1\pi(l), \imath_1(x)) = ((\mathrm{id}_L + dh + hd)(l), \imath_1(x)) = (l, \imath_1(x)).$$

The maps $\imath_1, \pi, h$ induce via homotopy transfer a minimal L_∞-algebra structure on H, together with an L_∞ quasi-isomorphism $\imath\colon H(L) \to L$ of L_∞-algebras with linear part $\imath_1$. The quadratic components are given by

$$\imath_2(\xi_1, \xi_2) = h[\imath(\xi_1), \imath(\xi_2)], \qquad \{\xi_1, \xi_2\}_2 = \pi[\imath(\xi_1), \imath(\xi_2)]\,,$$

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

while the higher brackets $\{\cdots\}_p\colon H^{\wedge p}\to H[2-p]$ and the higher Taylor coefficients $\imath_p\colon H^{\wedge p}\to L[1-p]$, $p\geqslant 2$, are explicitly (and recursively) defined by

$$\imath_p(\xi_1,\ldots,\xi_p)=\frac{1}{2}\sum_{k=1}^{p-1}\sum_{\sigma\in S(k,p-k)}\pm h[\imath_k(\xi_\sigma(1),\ldots),\imath_{p-k}(\ldots,\xi_{\sigma(p)})],\tag{4.2}$$

$$\{\xi_1,\ldots,\xi_p\}_p=\frac{1}{2}\sum_{k=1}^{p-1}\sum_{\sigma\in S(k,p-k)}\pm \pi[\imath_k(\xi_\sigma(1),\ldots),\imath_{p-k}(\ldots,\xi_{\sigma(p)})],\tag{4.3}$$

where $\pm$ is the appropriate Koszul sign described explicitly in (2.3) and (2.4). These signs will simplify in our specific case, for instance $\pm 1=+1$ whenever $\xi_i\in H^1$ for every i, and we do not need to make them explicit.

Notice that $\{a,b\}_2=[a,b]$ for $a,b\in H^0$ and under the natural isomorphism $H\cong H^*(L)$, the quadratic bracket $\{x,y\}_2=\pi[\imath_1(x),\imath_1(y)]$ on H is just the bracket induced by $[-,-]$ in cohomology.

Lemma 4.2. *In the above setup, for every $p\geqslant 2$ and every $g\in H^0$ we have*

$$\imath_p(g,\ldots)=0,\quad \{g,\ldots\}_{p+1}=0.$$

Proof. If $g\in H^0$, then $[\imath_1(g),\imath_1(\xi)]\in H\subset \mathrm{Ker}(h)$ for all $\xi\in H$, since H is an H^0-submodule of L, thus $\imath_2(g,\xi)=0$ for all $g\in H^0$ and $\xi\in H$. In general, by formulas (4.2), (4.3) and induction on p, for all $p\geqslant 2$, $g\in H^0$ and $\xi_1,\ldots,\xi_p\in H$, we have

$$\{\xi_1,\ldots,\xi_p,g\}_{p+1}=\pm\pi[\imath_p(\xi_1,\ldots,\xi_p),\imath_1(g)],\quad \imath_{p+1}(\xi_1,\ldots,\xi_p,g)=\pm h[\imath_p(\xi_1,\ldots,\xi_p),\imath_1(g)].$$

Finally, notice that for $p\geqslant 2$ we have $\mathrm{Im}(\imath_p)\subset K\subset \mathrm{Ker}(h)\bigcap \mathrm{Ker}(\pi)$, and that K is by hypothesis an H^0-submodule of L. This implies that $[\imath_p(\xi_1,\ldots,\xi_p),\imath_1(g)]\in K$ and therefore

$$\{\xi_1,\ldots,\xi_p,g\}_{p+1}=\imath_{p+1}(\xi_1,\ldots,\xi_p,g)=0.$$ □

Lemma 4.2 provides a complete proof of formality for $n=1$, since $H^i=0$ for every $i\neq 0,1$ and therefore for degree reasons $\{\cdots\}_{p+1}=0$ for every $p\geqslant 2$. From now on we assume that *the degree of the quasi-cyclic DG Lie algebra L of Theorem 3.8 is equal to $n=2$.*

Lemma 4.3. *In the above situation, for every $\xi_1,\ldots,\xi_p\in H^1$, $p\geqslant 3$, we have*

$$\pi[\imath_p(\xi_1,\ldots,\xi_{p-1}),\imath_1(\xi_p)]=0,\tag{4.4}$$

$$\{\xi_1,\ldots,\xi_p\}_p=\frac{1}{2}\sum_{k=2}^{p-2}\sum_{\sigma\in S(k,p-k)}\pi[\imath_k(\xi_\sigma(1),\ldots),\imath_{p-k}(\ldots,\xi_{\sigma(p)})],\tag{4.5}$$

and therefore $\{\xi_1,\xi_2,\xi_3\}_3=0$ for every $\xi_1,\xi_2,\xi_3\in H^1$.

Proof. It is sufficient to prove (4.4). We first note that the image of $\imath_1$ is contained in H and the image of $\imath_j$ is contained in K for every $j>1$. Moreover, we can rewrite (4.1) in the form $\pi(x)=0$ if and only if $(x,y)=0$ for every $y\in H$: now it is sufficient to observe that for any $g\in H^0$, $x\in H^1$ and $y\in K^1$ we have $(g,[x,y])=([g,x],y)=0$ since H^1 is an H^0-module and K^1 is orthogonal to H^1. □

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

For degree reasons, Lemma 4.2 implies that for $p \geqslant 2$ we have $\{\xi_1, \dots, \xi_{p+1}\}_{p+1} = 0$ unless $\xi_1, \dots, \xi_{p+1} \in H^1$, and then by Lemma 4.3 we have $\{\cdots\}_3 \equiv 0$. However, it should be noted that in general the higher brackets $\{\cdots\}_p$ will not vanish for $p \geqslant 4$ and therefore the proof of Theorem 3.8 is still very far from concluded.

Notation. From now on we shall denote by $\mathfrak{g}$ the Lie algebra $(H^0, \{-,-\})$.

Now we notice that item (iii) in the hypotheses of Lemma 4.1 implies that the maps $\imath\colon H \to L$, $\pi\colon L \to H$ and $h : L \to L[-1]$ are equivariant with respect to the induced $\mathfrak{g}$-module structures.

Since $\imath = (\imath_1, \imath_2, \dots)$ is a morphism of L_∞ algebras we have

$$\sum_{k=1}^{p} \sum_{\sigma \in S(p+2-k,k-1)} \pm \imath_k(\{\xi_{\sigma(1)}, \dots\}_{p+2-k}, \dots, \xi_{\sigma(p+1)})$$
$$= \pm d\imath_{p+1}(\xi_1, \dots, \xi_{p+1}) + \frac{1}{2} \sum_{j=1}^{p} \sum_{\sigma \in S(j,p+1-j)} \pm [\imath_j(\xi_{\sigma(1)}, \dots), \imath_{p+1-j}(\dots, \xi_{\sigma(p+1)})],$$

and taking $p \geqslant 2$, $\xi_1, \dots, \xi_p \in H^1$, $\xi_{p+1} = g \in \mathfrak{g}$, by Lemma 4.2 the above expression reduces to

$$[\imath_p(\xi_1, \dots, \xi_p), \imath_1(g)] = \imath_p(\{\xi_1, g\}_2, \dots, \xi_p) + \cdots + \imath_p(\xi_1, \dots, \{\xi_p, g\}_2). \tag{4.6}$$

Notice that formula (4.6) is also trivially satisfied for $p = 1$. For later use it is useful to introduce, for every $0 < j < p$, the function

$$I_j^p\colon (H^1)^{\odot j} \otimes (H^1)^{\odot p-j} \to \mathbb{K}, \quad I_j^p(\xi_1, \dots, \xi_p) = (\imath_j(\xi_1, \dots, \xi_j), \imath_{p-j}(\xi_{j+1}, \dots, \xi_p)).$$

Then for every $0 < j < p$, $\xi_1, \dots, \xi_p \in H^1$ and $g \in \mathfrak{g}$, we have

$$\sum_{i=1}^{p} I_j^p(\xi_1, \dots, \{\xi_i, g\}_2, \dots, \xi_p) = 0. \tag{4.7}$$

The proof of (4.7) is an immediate consequence of (4.6) together with the identity $([l_1, \imath_1(g)], l_2) + (l_1, [l_2, \imath_1(g)]) = 0$ for all $g \in \mathfrak{g}$, $l_1, l_2 \in L^1$. Moreover, the orthogonality condition $H \perp K$ implies that for every $p \geqslant 2$ we have $I_1^{p+1} = I_p^{p+1} = 0$.

Notation. We denote by $\{-,-\}\colon H^*(L)^{\wedge 2} \to H^*(L)$ the Lie bracket induced by the bracket $[-,-]\colon L^{\wedge 2} \to L$ on L. We have already observed that via the natural identification $H = H^*(L)$ we have $\{-,-\} = \{\cdot\cdot\}_2$ and it is straightforward to check that it continues to satisfy the condition $(\{x,y\}, z) = (x, \{y,z\})$ for all $x, y, z \in H^*(L)$.

By homotopy classification of DG Lie and L_∞ algebras, in order to prove the formality of L it is enough to exhibit an L_∞ isomorphism

$$f\colon (H, 0, \{-,-\}, 0, \{\cdots\}_4, \{\cdots\}_5, \dots) \to (H^*(L), 0, \{-,-\}, 0, 0, \dots)$$

between H with the transferred L_∞ algebra structure and $H^*(L)$ with the induced graded Lie algebra structure. Denoting by $f_p\colon H^{\wedge p} \to H[1-p]$ the Taylor coefficients of f, the necessary

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

relations these have to satisfy in order for f to be an L_∞ morphism read

$$\begin{aligned}\sum_{k=1}^{p}\sum_{\sigma\in S(p+2-k,k-1)} &\pm f_k(\{\xi_{\sigma(1)},\dots\},\dots,\xi_{\sigma(p+1)})\\ &=\frac{1}{2}\sum_{j=1}^{p}\sum_{\sigma\in S(j,p+1-j)}\pm\{f_j(\xi_{\sigma(1)},\dots),f_{p+1-j}(\dots,\xi_{\sigma(p+1)})\}\end{aligned}\tag{4.8}$$

for all $p\geqslant 2$ and $\xi_1,\dots,\xi_{p+1}\in H$. If these are satisfied, for f to be an isomorphism of L_∞ algebras it is necessary and sufficient that its linear part $f_1\colon H\to H$ is an isomorphism of graded spaces. We look for an L_∞ isomorphism f as above such that moreover $f_1=\mathrm{id}_H$ and $f_p(\xi_1,\dots,\xi_p)=0$ for $p\geqslant 2$ unless $p\geqslant 3$ and $\xi_1,\dots,\xi_p\in H^1$. With these hypotheses, many of the previous relations (4.8) become trivial, and the only non-trivial ones we are left to verify are

$$\big\{f_p(\xi_1,\dots,\xi_p),g\big\}=f_p(\{\xi_1,g\},\dots,\xi_p)+\dots+f_p(\xi_1,\dots,\{\xi_p,g\}),\tag{4.9}$$

$$\big\{\xi_1,\dots,\xi_{p+1}\big\}_{p+1}=\frac{1}{2}\sum_{j=1}^{p}\sum_{\sigma\in S(j,p+1-j)}\big\{f_j(\xi_{\sigma(1)},\dots),f_{p+1-j}(\dots,\xi_{\sigma(p+1)})\big\},\tag{4.10}$$

for all $p\geqslant 2$, $\xi_1,\dots,\xi_{p+1}\in H^1$ and $g\in\mathfrak{g}$ (as in the case of transfer formulas, Koszul signs have disappeared since $|\xi_1|=\dots=|\xi_{p+1}|=1$). Since $f_2=0$ by definition and we already know that $\{\cdots\}_3=0$, relations (4.9) and (4.10) are trivially satisfied for $p=2$.

For every $p\geqslant 3$ and every $1<j<p$, we define recursively the linear maps

$$f_p\colon (H^1)^{\odot p}\to H^1,\quad F_j^{p+1}\colon (H^1)^{\odot j}\otimes (H^1)^{\odot p-j+1}\to\mathbb{K},$$

by the formulas

$$F_j^{p+1}(\xi_1,\dots,\xi_{p+1})=(f_j(\xi_1,\dots,\xi_j),f_{p-j+1}(\xi_{j+1},\dots,\xi_{p+1})),\tag{4.11}$$

$$\big(f_p(\xi_1,\dots,\xi_p),\xi_{p+1}\big)=\frac{1}{2}\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j)}(I_j^{p+1}-F_j^{p+1})(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)},\xi_{p+1}),\tag{4.12}$$

for all $\xi_1,\dots,\xi_{p+1}\in H^1$. The validity of (4.9) is proved in the following lemma.

Lemma 4.4. *In the above situation, for every $p\geqslant 2$, every $1<j<p$, every $\xi_1,\dots,\xi_{p+1}\in H^1$ and every $g\in\mathfrak{g}$, we have*

$$\sum_{i=1}^{p}f_p(\xi_1,\dots,\{\xi_i,g\},\dots,\xi_p)=\{f_p(\xi_1,\dots,\xi_p),g\},\tag{4.13}$$

$$\sum_{i=1}^{p+1}F_j^{p+1}(\xi_1,\dots,\{\xi_i,g\},\dots,\xi_{p+1})=0.\tag{4.14}$$

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

Proof. The above formula are trivially satisfied for $p=2$, since $f_2=0$ and (4.14) is empty. Assuming (4.13) valid for all integers smaller than p, we have

$$\sum_{i=1}^{p+1} F_j^{p+1}(\xi_1,\ldots,\{\xi_i,g\},\ldots,\xi_{p+1})$$
$$=(\{f_j(\xi_1,\ldots,\xi_j),g\},f_{p-j-1}(\xi_1,\ldots,\xi_j))+(f_j(\xi_1,\ldots,\xi_j),\{f_{p-j-1}(\xi_1,\ldots,\xi_j),g\})=0,$$

where the second equality follows from the cyclic condition $(\{x,g\},y)+(x,\{y,g\})=0$ for all $g\in\mathfrak{g}$ and $x,y\in H^1$. For the same reason we have

$$\begin{aligned}\big(\{f_p(\xi_1,\ldots,\xi_p),g\},\xi_{p+1}\big) &= -\big(f_p(\xi_1,\ldots,\xi_p),\{\xi_{p+1},g\}\big)\\ &=\Big(-\frac{1}{2}\Big)\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j)}(I_j^{p+1}-F_j^{p+1})\big(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)},\{\xi_{p+1},g\}\big)\\ &=\frac{1}{2}\sum_{i=1}^{p}\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j)}(I_j^{p+1}-F_j^{p+1})\big(\xi_{\sigma(1)},\ldots,\{\xi_{\sigma(i)},g\},\ldots,\xi_{p+1}\big)\\ &=\sum_{h=1}^{p}\big(f_p(\xi_1,\ldots,\{\xi_h,g\},\ldots,\xi_p),\xi_{p+1}\big),\end{aligned}$$

where in the second and fourth equalities we have used the defining formula (4.12), while the third equality is a consequence of (4.7) and (4.14). Since $(-,-)$ is non-degenerate in H^1, the formula

$$\big(\{f_p(\xi_1,\ldots,\xi_p),g\},\xi_{p+1}\big)=\sum_{h=1}^{p}\big(f_p(\xi_1,\ldots,\{\xi_h,g\},\ldots,\xi_p),\xi_{p+1}\big)$$

is completely equivalent to (4.13). ☐

Finally, we prove (4.10), or equivalently that

$$2\big(\{\xi_1,\ldots,\xi_{p+1}\}_{p+1},g\big)=\sum_{j=1}^{p}\sum_{\sigma\in S(j,p+1-j)}\big(\{f_j(\xi_{\sigma(1)},\ldots),f_{p+1-j}(\ldots,\xi_{\sigma(p+1)})\},g\big),$$

for every $\xi_1,\ldots,\xi_{p+1}\in H^1$ and $g\in\mathfrak{g}$. By (4.5) we have

$$\begin{aligned}2\big(\{\xi_1,\ldots,\xi_{p+1}\}_{p+1},g\big)&=\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p+1-j)}\big(\pi\big[\imath_j\big(\xi_{\sigma(1)},\ldots\big),\imath_{p+1-j}\big(\ldots,\xi_{\sigma(p+1)}\big)\big],g\big)\\ &=\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p+1-j)}\big(\big[\imath_j\big(\xi_{\sigma(1)},\ldots\big),\imath_{p+1-j}\big(\ldots,\xi_{\sigma(p+1)}\big)\big],\imath_1(g)\big).\end{aligned}$$

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

By using the cyclic relation $([l_1,l_2],l_3)=(l_1,[l_2,l_3])$, $\forall\, l_1,l_2,l_3\in L$, and (4.6) we get

$$\begin{aligned}
2\big(\{\xi_1,\dots,\xi_{p+1}\}_{p+1},g\big) &= \sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p+1-j)}\big(\imath_j\big(\xi_{\sigma(1)},\dots\big),[\imath_{p+1-j}\big(\dots,\xi_{\sigma(p+1)}\big),\imath(g)]\big)\\
&= \sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j,1)}\big(\imath_j\big(\xi_{\sigma(1)},\dots\big),\imath_{p+1-j}\big(\dots,\xi_{\sigma(p)},\{\xi_{\sigma(p+1)},g\}\big)\big)\\
&= \sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j,1)} I_j^{p+1}\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)},\{\xi_{\sigma(p+1)},g\}\big),
\end{aligned}$$

where $S(j,p-j,1)$ is the set of permutations σ of $1,\dots,p+1$ such that

$$\sigma(1)<\cdots<\sigma(j),\quad \sigma(j+1)<\cdots<\sigma(p).$$

On the other hand,

$$\begin{aligned}
&\sum_{j=1}^{p}\sum_{\sigma\in S(j,p+1-j)}\big(\{f_j(\xi_{\sigma(1)},\dots),f_{p+1-j}(\dots,\xi_{\sigma(p+1)})\},g\big)\\
&\quad=\sum_{j=1}^{p}\sum_{\sigma\in S(j,p+1-j)}\big(f_j(\xi_{\sigma(1)},\dots),\{f_{p+1-j}(\dots,\xi_{\sigma(p+1)}),g\}\big)\\
&\quad=\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j,1)}\big(f_j(\xi_{\sigma(1)},\dots),\{f_{p+1-j}(\dots,\xi_{\sigma(p+1)}),g\}\big)\\
&\qquad+\sum_{\sigma\in S(p,1)}\big(f_p\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}\big),\{\xi_{\sigma(p+1)},g\}\big).
\end{aligned}$$

Reasoning as before and using the already proved (4.9), we have

$$\begin{aligned}
&\sum_{j=1}^{p}\sum_{\sigma\in S(j,p+1-j)}\big(\{f_j(\xi_{\sigma(1)},\dots),f_{p+1-j}(\dots,\xi_{\sigma(p+1)})\},g\big)\\
&\quad=\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j,1)}\big(f_j\big(\xi_{\sigma(1)},\dots\big),f_{p+1-j}\big(\dots,\xi_{\sigma(p)},\{\xi_{\sigma(p+1)},g\}\big)\big)\\
&\qquad+\sum_{\sigma\in S(p,1)}\big(f_p\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}\big),\{\xi_{\sigma(p+1)},g\}\big)\\
&\quad=\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j,1)}F_j^{p+1}\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)},\{\xi_{\sigma(p+1)},g\}\big)\\
&\qquad+\sum_{\sigma\in S(p,1)}\big(f_p\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}\big),\{\xi_{\sigma(p+1)},g\}\big)\\
&\quad=\sum_{j=2}^{p-1}\sum_{\sigma\in S(j,p-j,1)}I_j^{p+1}\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)},\{\xi_{\sigma(p+1)},g\}\big),
\end{aligned}$$

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

where in the last equality we used the recursive definition (4.12) of f_p. The proof of Theorem 3.8 is now complete.

Corollary 4.5. *Let $(L, d, [-,-], (-,-))$ be a quasi-cyclic DG Lie algebra of degree 2 with $H^i(L) = 0$ for every $i \neq 0, 1, 2$. Assume that there exists a Lie subalgebra $H^0 \subset Z^0(L)$ such that:*

(i) *the projection $Z^0(L) \to H^0(L)$ induces an isomorphism $H^0 \simeq H^0(L)$;*
(ii) *L^i is a completely reducible H^0-module (with respect to the adjoint action) for all i.*

Then the DG Lie algebra $(L, d, [-,-])$ is formal.

Proof. Construct a splitting by choosing for every $i \neq 0$ a direct sum decomposition $Z^i(L) = B^i(L) \oplus H^i$ of H^0-modules and then, for every i, a direct sum decomposition $L^i = Z^i(L) \oplus K^i$ of H^0-modules. This splitting satisfies the conditions of Theorem 3.8. □

5. Derived endomorphisms and their formality

For every coherent sheaf $\mathcal{F}$ on a smooth complex projective manifold X there is a well-defined homotopy class of DG Lie algebras denoted by $R\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$ and called, with a little abuse of language, the *DG Lie algebra of derived endomorphisms* of $\mathcal{F}$. There exist several possible (quasi-isomorphic) representatives for $R\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$, and we refer to [Mea18] for an explicit and concrete description of many of them. The importance of the DG Lie algebra of derived endomorphisms relies on the fact that it controls the deformation theory of $\mathcal{F}$ in the usual way via Maurer–Cartan equation modulus gauge action; cf. [AS18, BZ18, IM19, Mea18]. Moreover, $H^i(R\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})) = \mathrm{Ext}^i(\mathcal{F}, \mathcal{F})$ for every i.

Since the notion of quasi-cyclic DG Lie algebra is not stable under general quasi-isomorphisms, in view of a possible application of Theorem 3.8 it is useful to consider the Dolbeault representatives for $R\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$. Consider a *finite locally free* resolution $\mathcal{E}^* = \{\cdots \mathcal{E}^{-1} \to \mathcal{E}^0\} \to \mathcal{F}$ and denote by

$$\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) = \bigoplus_d \mathcal{H}om^d_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) = \bigoplus_d \bigoplus_p \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{E}^{d+p})$$

the (DG) sheaf of endomorphisms of $\mathcal{E}^*$. Then $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)$ is a sheaf of DG Lie algebras over X. It is important to notice that the bracket $[f, g] = fg - (-1)^{|f||g|} gf$ is $\mathcal{O}_X$-bilinear and therefore it can be extended naturally to Dolbeault's resolution

$$L = A^{0,*}_X(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)) = \bigoplus_{p,q,r} A^{0,p}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^q, \mathcal{E}^r)),$$

where $A^{0,p}_X(\mathcal{G})$ denotes the space of global differential forms of type $(0, q)$ with values in the locally free sheaf $\mathcal{G}$. Similarly, the usual trace map (see, for example, [IM19] and references therein)

$$\mathrm{Tr}\colon \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*) \to \mathcal{O}_X, \quad \mathrm{Tr}(f) = \sum_i (-1)^i \, \mathrm{Tr}(f^i_i), \quad \text{where } f = \sum_{i,j} f^j_i, \quad f^j_i \colon \mathcal{E}^i \to \mathcal{E}^j,$$

is a morphism of sheaves of DG Lie algebras, and extends to a morphism of DG Lie algebras

$$\mathrm{Tr}\colon L = A^{0,*}_X(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*)) \to A^{0,*}_X.$$

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

Since the bracket on $A_X^{0,*}$ is trivial we have

$$\operatorname{Tr}([f,g])=0,\quad \operatorname{Tr}(df)=\bar{\partial}\operatorname{Tr}(f),$$

for every $f,g\in L$, and this immediately implies that $\operatorname{Tr}([f,g]h)=\operatorname{Tr}(f[h,g])$ for every $f,g,h\in L$. If ω is a non-trivial section of the canonical bundle of X, the graded symmetric bilinear form

$$(-,-)\colon L^{\odot 2}\to\mathbb{C}[-\dim X],\quad (f,g)=\int_X \omega\wedge\operatorname{Tr}(fg), \tag{5.1}$$

is a cyclic bilinear form, where this means that it satisfies the conditions $(df,g)+(-1)^{|f|}(f,dg)=0$ and $([f,g],h)=(f,[g,h])$. Finally, if ω is a holomorphic volume form, by Serre duality the above bilinear form is non-degenerate in cohomology and therefore $(L,(-,-))$ is a quasi-cyclic DG Lie algebra of degree $\dim X$.

From now on we consider only coherent sheaves on projective surfaces with torsion canonical bundle. According to the Enriques–Kodaira classification of surfaces (see, for example, [BHPV04, Bea94]), a smooth projective surface has torsion canonical bundle K if and only if it is minimal of Kodaira dimension 0. According to the values of irregularity q and geometric genus p_g, these surfaces are classified into four (non-empty) distinguished classes:

- projective K3 surfaces, with $q=0$, $p_g=1$ and $K=0$;
- Enriques surfaces, with $q=0$, $p_g=0$ and $2K=0$;
- bielliptic surfaces, with $q=1$, $p_g=0$ and $nK=0$ for some $n=2,3,4,6$;
- Abelian surfaces, with $q=2$, $p_g=1$ and $K=0$.

We are now ready to prove the Kaledin–Lehn formality conjecture for the above surfaces, namely that $R\operatorname{Hom}_X(\mathcal{F},\mathcal{F})$ is formal whenever $\mathcal{F}$ is polystable with respect to any (possibly non-generic) polarization; see, for example, [HL10, Chapter 1]. It is useful and instructive to give first a separate proof for the cases of K3 and Abelian surfaces.

Theorem 5.1. *Let X be a complex projective surface with trivial canonical bundle and let $\mathcal{F}$ be a coherent sheaf on X. If the group of automorphisms of $\mathcal{F}$ is linearly reductive (e.g. if $\mathcal{F}$ is polystable), then the DG Lie algebra $R\operatorname{Hom}_X(\mathcal{F},\mathcal{F})$ is formal.*

Proof. Let us denote by G the linearly reductive group of automorphisms of $\mathcal{F}$. Since X is smooth projective it is not difficult to see that there exists a G-equivariant finite locally free resolution $\mathcal{E}^*=\{0\to\mathcal{E}^{-2}\to\mathcal{E}^{-1}\to\mathcal{E}^0\}\to\mathcal{F}$; a detailed proof is given, for instance, in [BMM20]. We claim that the DG Lie algebra $L=A_X^{0,*}(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*))$ satisfies the condition of Theorem 3.8, when equipped with the cyclic non-degenerate structure (5.1).

Assume for the moment that the induced action of G on L^i is rational for every i; since the action of G commutes with the differential of the resolution $\mathcal{E}^*$. we have a natural inclusion $G\subset Z^0(L)$ and we can take $H^0=T_{\mathrm{Id}}G\subset Z^0(L)\subset L^0$ as the Lie algebra of G. Then, since G is assumed to be linearly reductive we may extend H^0 to a G-equivariant splitting of L that clearly satisfies the hypotheses of Theorem 3.8.

It remains to be shown that L^i is a rational representation of G for every i. This follows immediately from the results of [BMM20], and we give here only a sketch of the proof. The key point is that if G acts on a coherent sheaf $\mathcal{G}$ then, for every open affine subset U, the space $\mathcal{G}(U)$ is a rational *finitely supported* representation of G [BMM20, Lemma 3.5]. Recall that a representation is finitely supported if it is isomorphic to a finite direct sum $\bigoplus_{i=1}^n H_i\otimes W_i$, for some irreducible rational (hence finite-dimensional) representations H_i and some trivial representations W_i; every subrepresentation and every quotient of a rational finitely supported representation remains finitely supported [BMM20, Lemma 2.7 and Remark 2.8].

Let $X=\bigcup_j U_j$ be a finite open affine cover such that $\mathcal{E}^*$ is free over U_j for every j. Then $\Gamma\big(U_j,\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)\big)$ is rational and finitely supported for every j, therefore

$$L\subset\bigoplus_j A^{0,*}_{U_j}\big(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)\big)\subset\bigoplus_j A^{0,*}_{U_j}\otimes_{\mathbb{C}}\Gamma\big(U_j,\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)\big)$$

is also a rational finitely supported representation of G. □

Let us now return to our initial situation, namely with $\mathcal{F}$ a polystable sheaf on a smooth projective surface X with torsion canonical bundle $K_X=\Omega^2_X$. We denote by n be the smallest positive integer such that $K_X^{\otimes n}\simeq\mathcal{O}_X$ (we already know that $n=1,2,3,4,6$).

Every choice of an isomorphism $K_X^{\otimes n}\xrightarrow{\simeq}\mathcal{O}_X$ induces naturally a structure of commutative $\mathcal{O}_X$-algebra on the locally free sheaf of rank n,

$$\mathcal{C}:=\mathcal{O}_X\oplus K_X\oplus K_X^{\otimes 2}\oplus\cdots\oplus K_X^{\otimes n-1}.$$

Since K_X is a torsion line bundle we have that $\mathcal{F}\otimes\mathcal{C}$ is also polystable.

Let $\mathcal{E}^*=\{\cdots\mathcal{E}^{-1}\to\mathcal{E}^0\}\to\mathcal{F}$ be any finite locally free resolution. Then $\mathcal{E}^*\otimes\mathcal{C}$ is a finite locally free resolution of $\mathcal{F}\otimes\mathcal{C}$. Moreover,

$$\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*\otimes\mathcal{C},\mathcal{E}^*\otimes\mathcal{C})=\bigoplus_{i,j=0}^{n-1}\mathcal{H}om^*_{\mathcal{O}_X}\big(\mathcal{E}^*\otimes K_X^{\otimes i},\mathcal{E}^*\otimes K_X^{\otimes j}\big),$$

and every direct summand is a $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*,\mathcal{E}^*)$-module via the adjoint action. The trace map extends naturally to a morphism of sheaves

$$\widetilde{\mathrm{Tr}}\colon\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*\otimes\mathcal{C},\mathcal{E}^*\otimes\mathcal{C})\to\mathcal{C},$$

with components

$$\widetilde{\mathrm{Tr}}\colon\mathcal{H}om^*_{\mathcal{O}_X}\big(\mathcal{E}^*\otimes K_X^{\otimes i},\mathcal{E}^*\otimes K_X^{\otimes j}\big)=\mathcal{H}om^*_{\mathcal{O}_X}\big(\mathcal{E}^*,\mathcal{E}^*\big)\otimes K_X^{\otimes j-i}\xrightarrow{\mathrm{Tr}\otimes\mathrm{Id}}K_X^{\otimes j-i}.$$

The DG Lie algebra $L=A_X^{0,*}(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*\otimes\mathcal{C},\mathcal{E}^*\otimes\mathcal{C}))$ is quasi-cyclic of degree 2, when equipped with the pairing

$$(f,g)=\int_X p_K\widetilde{\mathrm{Tr}}(fg),$$

where $p_K\colon A_X^{0,*}(\mathcal{C})\to A_X^{0,2}(K_X)=A_X^{2,2}$ is the projection. In fact, for every $0\leqslant i,j<n$ the above pairing induces the Serre duality isomorphism

$$\mathrm{Ext}^h_X\left(\mathcal{F}\otimes K_X^{\otimes i},\mathcal{F}\otimes K_X^{\otimes j}\right)\simeq\mathrm{Ext}^{2-h}_X\left(\mathcal{F}\otimes K_X^{\otimes j},\mathcal{F}\otimes K_X^{\otimes i+1}\right)^\vee$$

so that it is non-degenerate in cohomology.

Lemma 5.2. *In the above situation there exists a finite locally free resolution $\mathcal{E}^*\to\mathcal{F}$ such that every endomorphism of $\mathcal{F}\otimes\mathcal{C}$ lifts canonically to an endomorphism of the complex $\mathcal{E}^*\otimes\mathcal{C}$.*

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

Proof. By assumption $\mathcal{F}$ is a pure coherent sheaf that is a direct sum of stable sheaves with the same reduced Hilbert polynomial:

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n.$$

In particular, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{F}_j) = 0$ for every $i \neq j$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{F}_i) = \mathbb{C}$ for every i. Consider the following equivalence relation on the set of direct summands

$$\mathcal{F}_i \sim \mathcal{F}_j \iff \mathcal{F}_i \otimes \mathcal{C} \cong \mathcal{F}_j \otimes \mathcal{C}.$$

Equivalently, $\mathcal{F}_i \sim \mathcal{F}_j$ if and only if $\mathcal{F}_i$ is isomorphic to $\mathcal{F}_j \otimes K^{\otimes h}$ for some h. Up to permutation of indices we may assume that $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are a set or representatives for this equivalence relation. We may write

$$\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{F}_i \otimes \mathcal{W}_i,$$

where every $\mathcal{W}_i$ is a direct sum of line bundles of type $K_X^{\otimes h}$. We have $\mathcal{F} \otimes \mathcal{C} = \bigoplus_{i=1}^{r} \mathcal{F}_i \otimes \mathcal{C}^{\oplus w_i}$ where w_i is the rank of $\mathcal{W}_i$. Every non-trivial endomorphism of $\mathcal{F}_i$ is a scalar multiple of the identity and then the group of automorphisms of $\mathcal{F} \otimes \mathcal{C}$ is the product of n copies of $\prod_{i=1}^{r} \mathrm{GL}_{w_i}(\mathbb{C})$.

Choose r a finite locally free resolution $\mathcal{E}_i^* \to \mathcal{F}_i$: every endomorphism of $\mathcal{F}_i$ is a scalar multiple of the identity and then lifts canonically to $\mathcal{E}_i^*$. It is now easy to verify that

$$\mathcal{E}^* = \bigoplus_{i=1}^{r} \mathcal{E}_j^* \otimes \mathcal{W}_i$$

is resolution of $\mathcal{F}$ with the required properties. □

Theorem 5.3. *In the above situation both the DG Lie algebras $R\mathrm{Hom}_X(\mathcal{F} \otimes \mathcal{C}, \mathcal{F} \otimes \mathcal{C})$ and $R\mathrm{Hom}_X(\mathcal{F}, \mathcal{F})$ are formal.*

Proof. Let $\mathcal{E} \to \mathcal{F}$ be a resolution as in Lemma 5.2 and consider the quasi-cyclic DG Lie algebra

$$L = A_X^{0,*}(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^* \otimes \mathcal{C}, \mathcal{E}^* \otimes \mathcal{C}))$$

as a representative in the homotopy class of $R\,\mathrm{Hom}(\mathcal{F} \otimes \mathcal{C}, \mathcal{F} \otimes \mathcal{C})$. The same arguments used in the proof of Theorem 5.1 imply that L is a rational and finitely supported representation of the linearly reductive group of automorphisms of $\mathcal{F} \otimes \mathcal{C}$.

By assumption there exists a natural inclusion of Lie algebras

$$\mathrm{Hom}_X(\mathcal{F} \otimes \mathcal{C}, \mathcal{F} \otimes \mathcal{C}) \simeq H^0 \subset \mathrm{Hom}_X(\mathcal{E}^* \otimes \mathcal{C}, \mathcal{E}^* \otimes \mathcal{C}) = Z^0(L)$$

that induces an isomorphism $H^0 \simeq H^0(L)$. The adjoint action of $\mathrm{Hom}_X(\mathcal{F} \otimes \mathcal{C}, \mathcal{F} \otimes \mathcal{C})$ on L is induced by a rational action of a linearly reductive algebraic group, hence the action of H^0 on L is completely reducible and the formality of L follows from Corollary 4.5.

Taking

$$M = A_X^{0,*}(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^*, \mathcal{E}^*))$$

as a representative in the homotopy class of $R\,\mathrm{Hom}(\mathcal{F}, \mathcal{F})$, we have already observed that there exists a natural inclusion of DG Lie algebra $M \subset L$ together a decomposition of L as a direct

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

sum of M-modules:

$$L = \bigoplus_{i,j=0}^{n-1} A_X^{0,*}(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}^* \otimes K_X^{\otimes i}, \mathcal{E}^* \otimes K_X^{\otimes j})).$$

Now the formality of M is a direct consequence of the formality of L and of the formality transfer theorem (Theorem 2.3). □

Acknowledgements

This work has been carried out in the framework of the PRIN project 'Moduli and Lie theory' 2017YRA3LK.

References

AS18 E. Arbarello and G. Saccà, *Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties*, Adv. Math. **329** (2018), 649–703.

BM18 R. Bandiera and M. Manetti, *Algebraic models of local period maps and Yukawa algebras*, Lett. Math. Phys. **108** (2018), 2055–2097.

BMM20 R. Bandiera, M. Manetti and F. Meazzini, *Deformations of polystable sheaves on surfaces: quadraticity implies formality*, Mosc. Math. J. (2020), to appear.

BHPV04 W. Barth, K. Hulek, C. Peters and A. van de Ven, Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, second edition (Springer, 2004).

Bea94 A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Lecture Note Series, vol. 68 (Cambridge University Press, 1994).

BZ18 N. Budur and Z. Zhang, *Formality conjecture for K3 surfaces*, Compos. Math. **155** (2018), 902–911.

Get09 E. Getzler, *Lie theory for nilpotent L_∞ algebras*, Ann. of Math. (2) **170** (2009), 271–301.

GK95 E. Getzler and M. M. Kapranov, *Cyclic operads and cyclic homology*, in *Geometry, topology, and physics* (International Press, Cambridge, MA, 1995), 167–201.

HS79 S. Halperin and J. Stasheff, *Obstructions to homotopy equivalences*, Adv. Math. **32** (1979), 233–279.

HL10 D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves* (Cambridge University Press, 2010).

IM19 D. Iacono and M. Manetti, *On deformations of pairs (manifold, coherent sheaf)*, Canad. J. Math. **71** (2019), 1209–1241.

Kad82 T. V. Kadeishvili, *The algebraic structure in the cohomology of $A(\infty)$-algebras*, Soobshch. Akad. Nauk Gruzin. SSR **108** (1982), 249–252 (Russian).

Kal07 D. Kaledin, *Some remarks on formality in families*, Mosc. Math. J. **7** (2007), 643–652.

KL07 D. Kaledin and M. Lehn, *Local structure of hyperkähler singularities in O'Grady's examples*, Mosc. Math. J. **7** (2007), 653–672.

KLS06 D. Kaledin, M. Lehn and Ch. Sorger, *Singular symplectic moduli spaces*, Invent. Math. **164** (2006), 591–614.

Kon94 M. Kontsevich, *Feynman diagram and low dimensional topology*, Proc. First Eur. Congr. Math. (1994), 97–122.

Kon03 M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, Lett. Math. Phys. **66** (2003), 157–216.

LM95 T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. Algebra **23** (1995), 2147–2161.

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press

LS93 T. Lada and J. D. Stasheff, *Introduction to sh Lie algebras for physicists*, Int. J. Theor. Phys. **32** (1993), 1087–1104.

LS12 A. Lazarev and T. Schedler, *Curved infinity-algebras and their characteristic classes*, J. Topol. **5** (2012), 503–528.

Lun10 V. A. Lunts, *Formality of DG algebras (after Kaledin)*, J. Algebra **323** (2010), 878–898.

Man09 M. Manetti, *Differential graded Lie algebras and formal deformation theory*, Algebraic Geometry: Seattle 2005. Proc. Sympos. Pure Math. **80** (2009), 785–810.

Man15 M. Manetti, *On some formality criteria for DG-Lie algebras*, J. Algebra **438** (2015), 90–118.

Man20 M. Manetti, *Lie methods in deformation theory.* Draft version (2020).

Mea18 F. Meazzini, *A DG-enhancement of* $D(\mathrm{QCoh}(X))$ *with applications in deformation theory*, Preprint (2018), arXiv:1808.05119.

Muk84 S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.

NM78 J. Neisendorfer and T. Miller, *Formal and coformal spaces*, Illinois J. Math. **22** (1978), 565–580.

O'Gr99 K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49–117.

O'Gr03 K. G. O'Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), 435–505.

Rim80 D. S. Rim, *Equivariant* G*-structure on versal deformations*, Trans. Amer. Math. Soc. **257** (1980), 217–226.

Wei94 C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38 (Cambridge University Press, 1994).

Yos01 K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), 817–884.

Yos09 K. Yoshioka, *Stability and the Fourier-Mukai transform. II*, Compos. Math. **145** (2009), 112–142.

Yos17 K. Yoshioka, *Fourier-Mukai duality for K3 surfaces via Bridgeland stability condition*, J. Geom. Phys. **122** (2017), 103–118.

Zha12 Z. Zhang, *A note on formality and singularities of moduli spaces*, Mosc. Math. J. **12** (2012), 863–879.

Ruggero Bandiera bandiera@mat.uniroma1.it
Dipartimento di Matematica Guido Castelnuovo, Università degli studi di Roma La Sapienza, P.le Aldo Moro 5, I-00185 Roma, Italy

Marco Manetti manetti@mat.uniroma1.it
Dipartimento di Matematica Guido Castelnuovo, Università degli studi di Roma La Sapienza, P.le Aldo Moro 5, I-00185 Roma, Italy

Francesco Meazzini meazzini@mat.uniroma1.it
Dipartimento di Matematica Guido Castelnuovo, Università degli studi di Roma La Sapienza, P.le Aldo Moro 5, I-00185 Roma, Italy

https://doi.org/10.1112/S0010437X20007605 Published online by Cambridge University Press