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A GENERALIZATION OF THE BERNSTEIN POLYNOMIALS

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Dedicated to Philip J. Davis

This paper is concerned with a generalization of the classical Bernstein polynomials where the function is evaluated at intervals which are in geometric progression. It is shown that, when the function is convex, the generalized Bernstein polynomials B_n are monotonic in n, as in the classical case.

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1. Introduction

Recently the second author proposed (see [7]) the following generalization of the Bernstein polynomials, based on the q-integers. For each positive integer n, we define

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \tag{1.1}$$

where an empty product denotes 1 and $f_r = f([r]/[n])$. The notation requires some explanation. The function f is evaluated at ratios of the q-integers $[r]$ and $[n]$, where q is a positive real number and

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1. \end{cases} \tag{1.2}$$

Then, in a natural way, we define the q-factorial $[r]!$ by

$$[r]! = \begin{cases} [r].[r-1]\dots[1], & r = 1, 2, \dots, \\ 1, & r = 0 \end{cases} \tag{1.3}$$

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and the q-binomial coefficient $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!} \tag{1.4}$$

for integers $n \geq r \geq 0$. The Pascal identities

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix} \tag{1.5}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} \tag{1.6}$$

are readily verified from (1.4). The q-binomial coefficients are also called Gaussian polynomials (see Andrews **[1]**) and an induction argument using either (1.5) or (1.6) readily shows that $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ is a polynomial of degree $r(n-r)$ in q with positive integral coefficients.

When $q = 1$, the q-binomial coefficient reduces to the ordinary binomial coefficient and (1.1) gives the classical Bernstein polynomial. (See, for example, Cheney **[2]**, Davis **[3]**, Rivlin **[10]**.) It is clear from (1.1) that, as in the case when $q = 1$, the generalized Bernstein polynomial interpolates the function f at $x = 0$ and 1 and that, for $0 < q \leq 1$, B_n is a monotone linear operator.

The generalized Bernstein polynomial defined by (1.1) can be expressed in terms of q-differences. For any function f we define

$$\Delta^0 f_i = f_i$$

for $i = 0, 1, \ldots n$ and, recursively,

$$\Delta^{k+1} f_i = \Delta^k f_{i+1} - q^k \Delta^k f_i \tag{1.7}$$

for $k = 0, 1, \ldots, n-i-1$, where f_i denotes $f([i]/[n])$. See Schoenberg **[11]**, Lee and Phillips **[6]**. When $q = 1$, these q-differences reduce to ordinary forward differences and it is easily established by induction that

$$\Delta^k f_i = \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r}. \tag{1.8}$$

Then we may write, as shown in Phillips **[7]**,

$$B_n(f; x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r \tag{1.9}$$

which generalizes the well-known result (see, for example, Davis [3]) for the classical Bernstein polynomial. We may deduce from (1.9), as in Phillips [7], that if $f \in P_k$, the linear space of polynomials of degree at most k, then $B_n(f; x) \in P_k$. In particular, if $f \in P_1$ its second and higher q-differences are zero and we may deduce from (1.9) that, for any real numbers a and b,

$$B_n(ax+b; x) = ax + b. \tag{1.10}$$

For what follows, we also require the Euler identity

$$(1+x)(1+qx)\dots(1+q^{k-1}x) = \sum_{r=0}^{k} q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} x^r. \tag{1.11}$$

We observe that this identity, which may be verified by induction, generalizes the binomial expansion.

In [7] there is a discussion on convergence and a Voronovskaya type theorem on the rate of convergence. Results concerning the convergence of derivatives of the generalized Bernstein polynomials are given in [8]. The following de Casteljau type algorithm (see [9]) may be used for evaluating generalized Bernstein polynomials iteratively.

ALGORITHM

for $r = 0$ to n
  $f_r^{[0]} := f([r]/[n])$
next r
for $m = 1$ **to** n
  for $r = 0$ **to** $n - m$
    $f_r^{[m]} := (q^r - q^{m-1}x) f_r^{[m-1]} + x f_{r+1}^{[m-1]}$
  next r
next m

It is shown in [9] that $f_0^{[n]} = B_n(f; x)$. This generalizes the well known de Casteljau algorithm (see [5]) for evaluating the classical Bernstein polynomials.

2. Non-negative differences

In Davis [3] it is shown that, for any convex function f, the classical Bernstein polynomial (that is, (1.1) with $q = 1$) is also convex and the sequence of Bernstein polynomials is monotonic decreasing. It is also shown in [3] that if the kth ordinary differences of f are non-negative then the kth derivative of the classical Bernstein polynomial $B_n(f; x)$ is non-negative on [0,1]. We will discuss extensions of these results to the generalized Bernstein polynomials in this and the following section. We begin by recalling the following definition.

Definition 2.1. A function f is said to be *convex* on $[0, 1]$ if, for any t_0, t_1 such that $0 \le t_0 < t_1 \le 1$ and any $\lambda, 0 < \lambda < 1$,

$$f(\lambda t_0 + (1-\lambda)t_1) \le \lambda f(t_0) + (1-\lambda)f(t_1). \tag{2.1}$$

Geometrically, this definition states that no chord of f lies below the graph of f. With $\lambda = q/(1+q)$, $t_0 = [m]/[n]$ and $t_1 = [m+2]/[n]$ in (2.1), where $0 < q \le 1$, we see that, if f is convex,

$$f_{m+1} \le \frac{q}{1+q} f_m + \frac{1}{1+q} f_{m+2}$$

from which we deduce that

$$f_{m+2} - (1+q)f_{m+1} + qf_m = \Delta^2 f_m \ge 0.$$

Thus the second q-differences of a convex function are non-negative, generalizing the well known result for ordinary differences (where $q = 1$).

For any fixed natural number k we now construct a set of $n-k+1$ piecewise polynomials whose kth q-differences take the value 1 at a given knot, say $([m]/[n])$, and the value 0 at all the other knots. For $0 \le m \le n-k$ define

$$g^{k,m}(x) = \begin{cases} 0 & 0 \le x \le [m+k-1]/[n], \\ \gamma^{k,m}(x), & [m+k-1]/[n] < x \le 1, \end{cases} \tag{2.2}$$

where

$$\gamma^{k,m}(x) = \prod_{r=m+1}^{m+k-1} \left(\frac{[n]x - [r]}{[2r-m] - [r]} \right). \tag{2.3}$$

When $k = 1$ in (2.3) the empty product denotes 1 and then (2.2) is the piecewise constant function

$$g^{1,m}(x) = \begin{cases} 0, & 0 \le x \le [m]/[n], \\ 1, & [m]/[n] < x \le 1, \end{cases} \tag{2.4}$$

for $0 \le m \le n-1$. For a general value of k, the values of these piecewise polynomials at the knots are given by

$$g_j^{k,m} = g^{k,m}([j]/[n]) = \begin{cases} 0, & 0 \le j \le m+k-1, \\ \begin{bmatrix} j-m-1 \\ k-1 \end{bmatrix}, & m+k \le j \le n. \end{cases} \tag{2.5}$$

Since the "polynomial part" of $g^{k,m}(x)$ is of degree $k-1$, the kth q-differences involving knots from that part of the domain are zero. From this and (1.8), and noting where $g^{k,m}(x)$ is zero, we see that

$$\Delta^k g_j^{k,m} = \begin{cases} 1, & j = m, \\ 0, & j \neq m, 0 \leq j \leq n-k. \end{cases} \tag{2.6}$$

We can use the functions $g^{k,m}(x), 0 \leq m \leq n-k$, and the monomials $1, x, \ldots, x^{k-1}$ as a basis for the space of functions whose kth q-differences are non-negative on the knots $([j]/[n]), 0 \leq j \leq n$. Let $p_{k-1} \in P_{k-1}$ denote the polynomial which interpolates f on the first k of these knots, $([j]/[n]), 0 \leq j \leq k-1$, and let us write

$$\tilde{f}(x) = p_{k-1}(x) + \sum_{m=0}^{n-k} \Delta^k f_m g^{k,m}(x). \tag{2.7}$$

This is a piecewise polynomial of degree $k-1$ with respect to the knots. On the interval $[0, [k-1]/[n]]$, all of the $n-k+1$ functions $g^{k,m}(x)$ are zero and thus

$$\tilde{f}([j]/[n]) = p_{k-1}([j]/[n]) = f([j]/[n]), 0 \leq j \leq k-1, \tag{2.8}$$

so that

$$\Delta^r \tilde{f}_0 = \Delta^r f_0, 0 \leq r \leq k-1. \tag{2.9}$$

Also, we deduce from (2.7) and (2.6) that

$$\Delta^k \tilde{f}_m = \Delta^k f_m, 0 \leq m \leq n-k,$$

and so

$$\Delta^r \tilde{f}_0 = \Delta^r f_0, k \leq r \leq n. \tag{2.10}$$

Combining (2.9) and (2.10), we deduce that

$$\tilde{f}([j]/[n]) = f([j]/[n]), 0 \leq j \leq n. \tag{2.11}$$

Thus the function $\tilde{f}$, a piecewise polynomial of degree $k-1$, takes the same values as f on all $n+1$ knots. When $k=1$, $\tilde{f}$ is a step function which interpolates f on all $n+1$ knots and, when $k=2$, the function $\tilde{f}$ is the linear spline which interpolates f. For a general value of k, we deduce that

$$B_n(\tilde{f}; x) = B_n(f; x) \tag{2.12}$$

and thus, from (2.7) and the linearity of the Bernstein operator B_n,

$$B_n(f; x) = B_n(p_{k-1}; x) + \sum_{m=0}^{n-k} \Delta^k f_m C_{k,m}(x) \tag{2.13}$$

say, where

$$C_{k,m}(x) = B_n(g^{k,m}; x). \tag{2.14}$$

We now state:

Theorem 2.1. *The kth derivatives of the generalized Bernstein polynomials of order n are non-negative on* $[0, 1]$ *for all functions f whose kth q-differences are non-negative if and only if the kth derivatives of the generalized Bernstein polynomials of the* $n-k+1$ *functions* $g^{k,m}(x)$, $0 \le m \le n-k$, *are all non-negative.*

Proof. This follows from (2.13) and (2.14). □

We will find it useful to derive an alternative expression for the kth derivative of $B_n(g^{k,m}; x)$. We begin by expressing higher order q-differences (of order not less than k) in terms of the kth q-differences. For $0 \le s \le n-k$, we may write

$$\Delta^{s+k} f_i = \sum_{t=0}^{s} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} s \\ t \end{bmatrix} \Delta^k f_{s+i-t}. \tag{2.15}$$

This is easily verified by induction on s, using the recurrence relation for q-differences (1.7) and the Pascal identities. We now write the q-difference form of the generalized Bernstein polynomial (1.9) as

$$B_n(f; x) = \sum_{r=0}^{k-1} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r + \sum_{s=0}^{n-k} \begin{bmatrix} n \\ s+k \end{bmatrix} \Delta^{s+k} f_0 x^{s+k}.$$

Using (2.15) to replace the higher order differences in the second summation and rearranging the resulting double summation, we obtain

$$B_n(f; x) = \sum_{r=0}^{k-1} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r + \sum_{m=0}^{n-k} \Delta^k f_m D_{k,m}(x) \tag{2.16}$$

say, where

$$D_{k,m}(x) = \sum_{t=0}^{n-m-k} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n \\ m+t+k \end{bmatrix} \begin{bmatrix} m+t \\ t \end{bmatrix} x^{m+t+k}. \tag{2.17}$$

On comparing (2.13) and (2.16), which hold for all functions f, we deduce that

$$\frac{d^k}{dx^k} C_{k,m}(x) = \frac{d^k}{dx^k} D_{k,m}(x). \tag{2.18}$$

Thus, given that we are interested only in their kth derivatives, the sets of polynomials $C_{k,m}$ and $D_{k,m}$ are equivalent.

It is well known (see Davis [3]) that, with $q = 1$, the kth derivatives of $D_{k,m}$ are non-negative. This is easily verified from (2.17) since with $q = 1$ we have

$$\frac{d^k}{dx^k} D_{k,m}(x) = \frac{n!}{m!(n-m-k)!} x^m \sum_{t=0}^{n-m-k} (-1)^t \binom{n-m-k}{t} x^t,$$

so that, mindful of (2.18),

$$\frac{d^k}{dx^k} D_{k,m}(x) = \frac{d^k}{dx^k} C_{k,m}(x) \frac{n!}{m!(n-m-k)!} x^m (1-x)^{n-m-k} \geq 0$$

for $0 \leq x \leq 1$. From (2.17) we can also see that, as q tends to zero from above, each q-integer tends to 1 and we have the limiting form

$$D_{k,m}(x) = x^{m+k}$$

and so its kth derivative is non-negative. We conjecture that the kth derivative of each $D_{k,m}$ is non-negative for $0 < q < 1$, but have not found a proof, except for certain values of m which we will mention below.

We will now work with $C_{k,m}$ rather than $D_{k,m}$. From (2.14), (1.1) and (2.5) we have

$$C_{k,m}(x) = \sum_{r=m+k}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r-m-1 \\ k-1 \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \tag{2.19}$$

for $0 \leq m \leq n-k$. With $m = n-k$, we have

$$C_{k,n-k}(x) = x^n,$$

whose kth derivative is clearly non-negative on $[0, 1]$. With $m = n-k-1$, we obtain from (2.19) that

$$C_{k,n-k-1}(x) = [n] x^{n-1}(1-x) + [k] x^n$$

and, with a little work, we find that the kth derivative of the latter polynomial is also non-negative on $[0, 1]$.

We can express $C_{k,m}(x)$ in another way, as follows. Since B_n is a linear operator, we may write

$$C_{k,m}(x) = B_n(g^{k,m}; x) = B_n(\gamma^{k,m}; x) + B_n(g^{k,m} - \gamma^{k,m}; x), \tag{2.20}$$

where $\gamma^{k,m}$ is defined in (2.3). Let

$$B_n(\gamma^{k,m}; x) = p_{k,m}(x)$$

say, where $p_{k,m}(x) \in P_{k-1}$. Then we obtain from (2.20) that

$$C_{k,m}(x) = p_{k,m}(x) + q^{-(2m+k)(k-1)/2}(-1)^k S_{k,m} \tag{2.21}$$

say, where

$$S_{k,m} = \sum_{r=0}^{m} q^{r(k-1)} \begin{bmatrix} m+k-1-r \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-1-r} (1 - q^s x). \tag{2.22}$$

In particular, (2.21) gives

$$C_{k,0}(x) = p_{k,0}(x) + q^{-k(k-1)/2}(-1)^k \prod_{s=0}^{n-1}(1 - q^s x).$$

Since, for $0 < q < 1$, the zeros of the function $(-1)^k \prod_{s=0}^{n-1}(1 - q^s x)$ are all greater than unity, the repeated application of Rolle's theorem shows that this is true of each of its first n derivatives. Also, Euler's identity (1.11) shows that its kth derivative is positive at $x = 0$ and so is positive on [0, 1]. Since $p_{k,0}(x) \in P_{k-1}$ it follows that the kth derivative of $C_{k,0}$ is also positive on [0, 1].

3. Monotonicity for convex functions

It is well known (see Davis [3]) that, when the function f is convex on [0, 1], its Bernstein polynomials are monotonic decreasing, in the sense that

$$B_{n-1}(f; x) \geq B_n(f; x), \qquad n = 2, 3, \ldots, \qquad 0 \leq x \leq 1.$$

We now show that this result extends to the generalized Bernstein polynomials, for $0 < q \leq 1$. In Figure 1, which illustrates this monotonicity, the function is concave rather than convex and thus the Bernstein polynomials are monotonic increasing. Figure 1 here is modelled on Fig. 6.3.1 in Davis [3], which relates to the classical Bernstein polynomials. The function is the linear spline which joins up the points (0, 0), (0.2, 0.6), (0.6, 0.8), (0.9, 0.7) and (1, 0) and the Bernstein polynomials are those of degrees 2, 4 and 10, with $q = 0.8$ in place of $q = 1$ in [3].

Theorem 3.1. *Let f be convex on* [0, 1]. *Then, for $0 < q \leq 1$, $B_{n-1}(f; x) \geq B_n(f; x)$ for $0 \leq x \leq 1$ and all $n \geq 2$. If $f \in C[0, 1]$ the inequality holds strictly for $0 < x < 1$ unless*

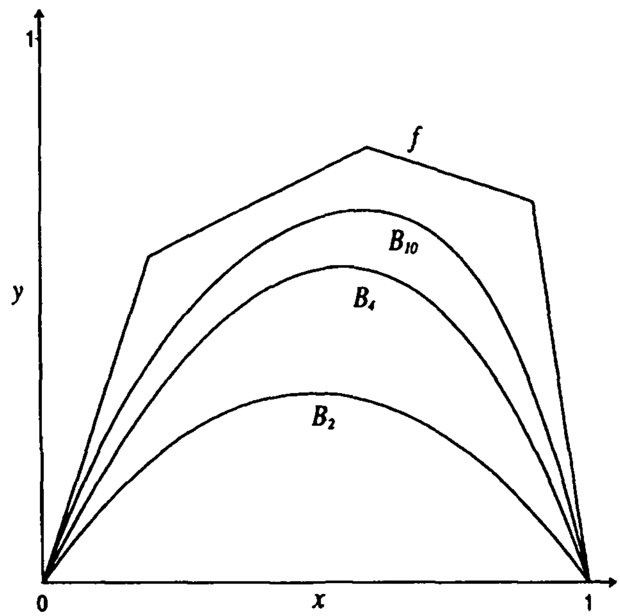

FIGURE 1: Monotonicity of generalized Bernstein polynomials for a concave function. The polynomials are B_2, B_4 and B_{10}, with $q = 0.8$

f is linear in each of the intervals between consecutive knots $[r]/[n-1]$, $0 \le r \le n-1$, in which case we have the equality $B_{n-1}(f; x) = B_n(f; x)$.

Proof. The key to the proof in Davis [3] for the case $q = 1$ is to express the difference between the consecutive Bernstein polynomials in terms of powers of $x/(1-x)$. Since the generalized Bernstein polynomials involve the product $\prod_{s=0}^{n-r-1}(1-q^s x)$ rather than $(1-x)^{n-r}$ we need to modify the proof somewhat. For $0 < q < 1$ we begin by writing

$$\prod_{s=0}^{n-1}(1-q^s x)^{-1}(B_{n-1}(f;x) - B_n(f;x))$$
$$= \sum_{r=0}^{n-1} f\left(\frac{[r]}{[n-1]}\right)\begin{bmatrix} n-1 \\ r \end{bmatrix} x^r \prod_{s=n-r-1}^{n-1}(1-q^s x)^{-1}$$
$$- \sum_{r=0}^{n} f\left(\frac{[r]}{[n]}\right)\begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=n-r}^{n-1}(1-q^s x)^{-1}.$$

We now split the first of the above summations into two, writing

$$x^r \prod_{s=n-r-1}^{n-1}(1-q^s x)^{-1} = \psi_r(x) + q^{n-r-1}\psi_{r+1}(x),$$

where

$$\psi_r(x) = x^r \prod_{s=n-r}^{n-1} (1 - q^s x)^{-1}. \tag{3.1}$$

The resulting three summations may be combined to give

$$\prod_{s=0}^{n-1}(1 - q^s x)^{-1}(B_{n-1}(f; x) - B_n(f; x)) = \sum_{r=1}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix} a_r \psi_r(x), \tag{3.2}$$

say, where

$$a_r = \frac{[n-r]}{[n]} f\left(\frac{[r]}{[n-1]}\right) + q^{n-r}\frac{[r]}{[n]} f\left(\frac{[r-1]}{[n-1]}\right) - f\left(\frac{[r]}{[n]}\right). \tag{3.3}$$

From (3.1) it is clear that each $\psi_r(x)$ is non-negative on [0, 1] for $0 \leq q \leq 1$ and thus, in view of (3.2), it suffices to show that each a_r is non-negative. We return to (2.1) and put $t_0 = [r-1]/[n-1]$, $t_1 = [r]/[n-1]$ and $\lambda = q^{n-r}[r]/[n]$. Then $0 \leq t_0 < t_1 \leq 1$ and $0 < \lambda < 1$ for $1 \leq r \leq n-1$ and, comparing (2.1) and (3.3), we deduce that, for $1 \leq r \leq n-1$,

$$a_r = \lambda f(t_0) + (1-\lambda) f(t_1) - f(\lambda t_0 + (1-\lambda)t_1) \geq 0.$$

Thus $B_{n-1}(f; x) \geq B_n(f; x)$. Of course we have equality for $x = 0$ and $x = 1$ since all Bernstein polynomials interpolate f on these end-points. The inequality will be strict for $0 < x < 1$ unless each $a_r = 0$ which can only occur when f is linear in each of the intervals between consecutive knots $[r]/[n-1]$, $0 \leq r \leq n-1$, when we have $B_{n-1}(f; x) = B_n(f; x)$ for $0 \leq x \leq 1$. This completes the proof. □

For a convex function, Goodman, Oruç and Phillips [4] show that the generalized Bernstein polynomials are also monotonic in the parameter q, for $0 < q \leq 1$.

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