Proceedings of the Edinburgh Mathematical Society (2018) **61**, 305–327
doi:10.1017/S0013091517000207

FREE AND PROPERLY DISCONTINUOUS ACTIONS OF GROUPS ON HOMOTOPY 2N-SPHERES

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(Received 22 September 2016)

Abstract Let G be a group acting freely, properly discontinuously and cellularly on some finite dimensional CW-complex $\Sigma(2n)$ which has the homotopy type of the $2n$-sphere $\mathbb{S}^{2n}$. Then, that action induces a homomorphism $G \to \mathrm{Aut}(H^{2n}(\Sigma(2n)))$. We classify all pairs (G, φ), where G is a virtually cyclic group and $\varphi : G \to \mathrm{Aut}(\mathbb{Z})$ is a homomorphism, which are realizable in the way above and the homotopy types of all possible orbit spaces as well. Next, we consider the family of all groups which have virtual cohomological dimension one and which act on some $\Sigma(2n)$. Those groups consist of free groups and semi-direct products $F \rtimes \mathbb{Z}_2$ with F a free group. For a group G from the family above and a homomorphism $\varphi : G \to \mathrm{Aut}(\mathbb{Z})$, we present an algebraic criterion equivalent to the realizability of the pair (G, φ). It turns out that any realizable pair can be realized on some $\Sigma(2n)$ with $\dim \Sigma(2n) \leq 2n + 1$.

Keywords: homotopy sphere; orbit space; cohomological dimension; properly discontinuous and cellular action; virtual cohomological dimension; virtually cyclic group

2010 *Mathematics subject classification:* Primary 57S30
Secondary 20F50; 20J06; 57Q91

1. Introduction

The statement of the spherical space form problem in dimension n is: *classify all manifolds with the n-sphere $\mathbb{S}^n$ as the universal cover.* Consequently, manifolds with finite fundamental groups. The development of that motivates classifications of the possible groups (not necessarily finite) which act freely, properly discontinuously and cellularly on an n-homotopy sphere $\Sigma(n)$ (a finite dimensional CW-complex with the homotopy type of the n-sphere $\mathbb{S}^n$). Further, this development began to accelerate with the discovery by Milnor [**14**] that some periodic groups could not act freely on any sphere. Then, Swan [**23**]

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showed that every periodic finite group acts freely on a finite CW-complex $\Sigma(km-1)$ for some k, where m is the period of the group. Further, he has also shown in [23] that a periodic finite group acts freely on a CW-complex $\Sigma(m)$ of dimension m, where $m+1$ is the period of the group. The finite periodic groups have been fully classified by Suzuki–Zassenhaus, see e.g., [1, Chapter IV, Theorem 6.15].

A free action of a discrete (finite or infinite) group G on $\Sigma(n)$ induces a homomorphism $G \to \mathrm{Aut}(H^n(\Sigma(n),\mathbb{Z}))$. Following [3, Chapter VII, Proposition 10.2], for any action of a finite group G on $\Sigma(2n+1)$, the induced homomorphism $G \to \mathrm{Aut}(H^{2n+1}(\Sigma(2n+1),\mathbb{Z}))$ is trivial. On the other hand, in view of [24], the only finite groups acting freely on $\Sigma(2n)$ are, up to isomorphism, trivial or $\mathbb{Z}_2$ and the induced homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n),\mathbb{Z}))$ is non-trivial. If the group G is infinite there are more possibilities for the induced homomorphism $G \to H^n(\Sigma(n),\mathbb{Z})$ than in the finite case, and a characterization of those homomorphisms is part of the problem we are going to study. Actions of infinite discrete groups on $\Sigma(n)$ together with the induced homomorphisms described above have not been too much explored. We will study such a classification which takes into account more geometric aspects of the action.

We state below some relevant results about actions of infinite groups. Recall that the Farrell cohomology, $\hat{H}^*(-,\mathbb{Z})$ is a contravariant functor from groups to $\mathbb{Z}$-graded algebras, whenever the virtual cohomological dimension $\mathrm{vcd}\, G < \infty$. If $\alpha \in \hat{H}^q(G,\mathbb{Z})$, an integer $q > 0$, and the map $\alpha \cup - : \hat{H}^i(G,\mathbb{Z}) \longrightarrow \hat{H}^{i+q}(G,\mathbb{Z})$ is an isomorphism for all i we say that the group G has periodic Farell cohomology with period q.

In 1979 Wall [31, p. 518] raised the following problem: *whether any countable group with periodic Farell cohomology can act freely and properly on some product* $\mathbb{R}^m \times \mathbb{S}^n$? The Wall's problem was solved by Connolly and Prassidis (1989). Namely, [5, Corollary 1.4.] says: *a discrete group G with* $\mathrm{vcd}\, G < \infty$ *acts freely and properly on* $\mathbb{R}^m \times \mathbb{S}^n$ *for some m, n if and only if G is countable and the Farrell cohomology $\hat{H}^*(G,\mathbb{Z})$ is periodic.*

In view of [2] a discrete group G has *periodic cohomology* after d-steps with $d \geq 0$ if there is an integer $q > 0$ and $\alpha \in H^q(G,\mathbb{Z})$ such that the cup product map $\alpha \cup - :$ $H^i(G,M) \longrightarrow H^{i+q}(G,M)$ is an isomorphism for every G-module M and $i > d$.

Wall's question is about actions of groups with finite virtual cohomological dimension. The result [2, Corollary 1.3] characterizes groups which act freely and properly discontinuously on $\mathbb{R}^m \times \mathbb{S}^n$, without assuming that the group has finite virtual cohomological dimension, and states: *A discrete group G acts freely and properly on $\mathbb{R}^m \times \mathbb{S}^n$ for some $m, n > 0$ if and only if G is a countable group with periodic cohomology.* Furthermore, the result of Johnson [12, Theorem on p. 387] states:
Let G be a group. Then the following are equivalent:

(i) there is a manifold M of type $K(G,1)$;

(ii) there is a covering action of G on $\mathbb{R}^m$ for some m;

(iii) G is countable and has finite cohomological dimension.

Consequently, such a group G acts freely and properly discontinuously on $\mathbb{R}^m \times \mathbb{S}^n$ for any $n > 0$. For more about this subject, we refer the reader to the papers [2, 5, 15, 18, 30].

By [26], a group G is said to have *periodic cohomology* after d-steps with $d \geq 0$ if there is a positive integer q such that the functors $H^i(G,-)$ and $H^{i+q}(G,-)$ are naturally equivalent for $i > d$. It is a conjecture by Talelli [27] that the two notions above of periodicity are the same.

It is not clear how to apply most of the results and techniques which appear in [2, 5] for the cases $n = 1$ and n even. The study of properly discontinuous and cellular actions of discrete groups on a homotopy circle $\Sigma(1)$ was done in [9] using different methods than those in the papers mentioned above.

The purpose of this paper is to study free, properly discontinuous and cellular actions of infinite groups G on $\Sigma(2n)$. This also takes into account the induced homomorphism $G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$.

By virtue of [13, Proposition 7.1]: *the group G is torsion-free or $G \cong G_0 \rtimes \mathbb{Z}_2$, where G_0 is a torsion-free group, provided G acts freely and properly discontinuously on $\mathbb{R}^m \times \mathbb{S}^{2n}$.*

Let G be a group and $\varphi : G \to \mathrm{Aut}\,(\mathbb{Z})$ a homomorphism. We say that the pair (G, φ) is *realizable* if there is an action $G \times \Sigma(2n) \to \Sigma(2n)$ that the induced homomorphism $G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$ coincides with $\varphi : G \to \mathrm{Aut}\,(\mathbb{Z})$.

For actions of virtually cyclic groups we show the following.

Proposition 1.1. *Let $G \times \Sigma(2n) \to \Sigma(2n)$ be an action of a non-trivial virtually cyclic group G on any $\Sigma(2n)$ and $\varphi : G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$ the induced homomorphism. Then:*

(1) *G is isomorphic to one of the groups: $\mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$;*

(2) *any of the groups above admits an action on some $\Sigma(2n)$ and the pair (G, φ) is realizable if and only if:*
   - (i) *$G \cong \mathbb{Z}_2$ and φ is non-trivial. The homotopy type of $\Sigma(2n)/G$ is $\mathbb{R}P^{2n}$;*
   - (ii) *$G \cong \mathbb{Z}$ and φ is any homomorphism. The homotopy type of $\Sigma(2n)/G$ is $\mathbb{S}^1 \times \mathbb{S}^{2n}$ in case φ is trivial and it is $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$ (the only non-trivial $\mathbb{S}^{2n}$-bundle over $\mathbb{S}^1$ being the mapping torus of the antipodal map $\mathbb{S}^{2n} \to \mathbb{S}^{2n}$) otherwise;*
   - (iii) *$G \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and, for a suitable automorphism of $\mathbb{Z} \oplus \mathbb{Z}_2$, the restriction $\varphi|_{\mathbb{Z}}$ is trivial and $\varphi|_{\mathbb{Z}_2}$ non-trivial. The homotopy type of $\Sigma(2n)/G$ is $\mathbb{S}^1 \times \mathbb{R}P^{2n}$;*
   - (iv) *$G \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, the restriction $\varphi|_{\mathbb{Z}}$ is trivial and $\varphi|_{\mathbb{Z}_2}$ are non-trivial. The homotopy type of $\Sigma(2n)/G$ is $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$.*

Let F be a free group. Given homomorphisms $\theta : \mathbb{Z}_2 \to \mathrm{Aut}\,(F)$ and $\varphi : F \rtimes_\theta \mathbb{Z}_2 \to \mathrm{Aut}\,(\mathbb{Z}) \cong \mathbb{Z}_2$ with $\varphi|_{\mathbb{Z}_2} = \mathrm{id}_{\mathbb{Z}_2}$, we say that the pair (θ, φ) is *realizable* if $(F \rtimes_\theta \mathbb{Z}_2, \varphi)$ is realizable. The key Lemma 4.1 states a necessary and sufficient conditions for a pair (θ, φ) to be realizable.

For a free group F_m of finite rank $m \geq 1$, we define $m \times m$-matrices $A(k, r, s)$ over the integers which satisfy $A(k, r, s)^2 = I_m$ and are given by k matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the identity matrix I_r and $-I_s$ on the diagonal for $m = 2k + r + s$. Then, we make use of the well-known representation $\rho_m : \mathrm{Aut}\,(F_m) \to GL_m(\mathbb{Z})$ to prove the following:

Theorem 1.2. *Let $F_m = \langle x_1, \ldots, x_m \rangle$ be a free group with $m \geq 1$, $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(F_m)$ and $\varphi : F_m \rtimes_\theta \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ be homomorphisms such that $\rho_m(\theta(1_2)) = A(k, r, s)$ and*

$\varphi|_{\mathbb{Z}_2} = \mathrm{id}_{\mathbb{Z}_2}$. *Then the pair* (θ, φ) *is realizable if and only if* $\varphi(x_l, 0) = 0$ *for* $l = 2k + r + 1, \ldots, 2k + r + s$.

The result [7, Theorem 3] shows that given a free group F_m and a homomorphism $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(F_m)$, it is always possible to find a basis $\{x_1, \ldots, x_m\}$ for F_m which satisfies $\rho_m(\theta) = A(k, r, s)$.

This paper is organized into four sections, in addition to this Introduction.

In § 2 basic facts on actions of groups on $\Sigma(2n)$ are presented.

Section 3 aims to determine all realizable pairs (G, φ), where G is a virtually cyclic group and $\varphi : G \to \mathrm{Aut}\,(\mathbb{Z})$ is a homomorphism, and homotopy types of orbit spaces. The main result is Proposition 3.4 and then Corollary 3.5 classifies orbit spaces of free and properly discontinuous actions of finite groups on certain manifolds having universal covering $\mathbb{R} \times \mathbb{S}^{2n}$.

Section 4 determines all realizable pairs (G, φ), where G is either a free group or isomorphic to $F \rtimes_\theta \mathbb{Z}_2$, where F is a free group. We prove Theorem 4.4 on the realisability of the pair (θ, φ), where $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(F_m)$ and $\varphi : F_m \rtimes_\theta \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ are homomorphisms. Then, we make some comments in Corollary 3.5 about a version of this theorem for F a free group of infinite rank.

Finally, in § 5 we discuss actions of groups G on $\Sigma(n)$, where G has infinite virtual cohomological dimension, or closely related to this case. Then, we pose Question 5.3 on the virtual cohomological dimension of a group acting on $\mathbb{R}^m \times \mathbb{S}^{2n}$.

2. Preliminaries

A CW-complex $\Sigma(n)$ is said to be an *n-homotopy sphere*, if $\dim \Sigma(n) < \infty$ and there is a homotopy equivalence $\Sigma(n) \simeq \mathbb{S}^n$ for the n-sphere $\mathbb{S}^n$ with $n \geq 1$.

From now on, we assume that any action $G \times \Sigma(n) \to \Sigma(n)$ of a group G on a $\Sigma(n)$ is free, properly discontinuous and cellular. In the beginning of [9, Section 1], we have stated the following.

Remark 2.1. Notice that $n \leq \dim \Sigma(n)$ and for an action $G \times \Sigma(n) \to \Sigma(n)$ there is a fibration

$$\Sigma(n) \to \Sigma(n)/G \to K(G, 1).$$

Consequently, there are isomorphisms $\pi_k(\Sigma(n)) \cong \pi_k(\Sigma(n)/G)$ for $k > 1$ and $n \geq 1$, $\pi_1(\Sigma(n)/G) \cong G$ for $n > 1$ and there is an extension

$$e \to \mathbb{Z} \to \pi_1(\Sigma(1)/G) \to G \to e$$

of groups.

Write $\mathrm{cd}\, G$ (respectively $\mathrm{vcd}\, G$) for cohomological (respectively virtual cohomological) dimension of a group G [3, Chapter VIII].

Given an action $G \times \Sigma(2n) \to \Sigma(2n)$, we consider the induced homomorphism

$$G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z})) \cong \mathbb{Z}_2,$$

which we call from now on the *orientation* of the G-action.

Then, we make use of [**13**, Proposition 7.1] and [**24**] to show the following.

Proposition 2.2. *Let $G \times \Sigma(2n) \to \Sigma(2n)$ be an action of a group G on $\Sigma(2n)$. Then:*

(1) *$G \cong \mathbb{Z}_2$ or $G = E$, where E is the trivial group, provided G is finite. Further, $\mathbb{Z}_2 \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$ is non-trivial;*

(2) *G is torsion-free or $G \cong G_0 \rtimes \mathbb{Z}_2$ for some torsion-free subgroup group G_0 of G.*

Proof. (1) If G is finite, then by [**24**, Theorem 4.8], $G \cong \mathbb{Z}_2$ or $G = E$. Suppose that $\mathbb{Z}_2 \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$ is trivial. Then the Leray–Serre spectral sequence $E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(\Sigma(2n), \mathbb{Z}))$ determined by the fibration

$$\Sigma(2n) \to \Sigma(2n)/\mathbb{Z}_2 \to K(\mathbb{Z}_2, 1)$$

collapses. Hence, the group $H^*(\Sigma(2n)/\mathbb{Z}_2, \mathbb{Z})$ does not vanish for infinite many values of $*$, which contradicts the fact that $\dim \Sigma(2n)/\mathbb{Z}_2 < \infty$.

(2) Suppose G is not torsion-free. Then, in view of (1), the induced action $G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n)), \mathbb{Z})$ is onto and $G_0 = \mathrm{Ker}\,(G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z})) \cong \mathbb{Z}_2)$ is torsion-free. Further, the extension

$$e \to G_0 \to G \to \mathbb{Z}_2 \to e$$

splits. Consequently, there is an isomorphism $G \cong G_0 \rtimes \mathbb{Z}_2$. □

Notice that from Proposition 2.2 it follows: if

$$\varphi : G \cong G_0 \rtimes \mathbb{Z}_2 \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z})) \cong \mathbb{Z}_2$$

is the induced action then the restriction $\varphi|_{\mathbb{Z}_2} = \mathrm{id}_{\mathbb{Z}_2}$.

Let G be a group and $\varphi : G \to \mathrm{Aut}\,(\mathbb{Z})$ a homomorphism. We say that the pair (G, φ) is *realizable* if there is an action $G \times \Sigma(2n) \to \Sigma(2n)$ that the induced homomorphism $G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$ coincides with $\varphi : G \to \mathrm{Aut}\,(\mathbb{Z})$.

In [**9**, Proposition 1.7] (see also [**16**, Proposition 2.5] and [**17**, Theorem 4.2] for more generality), we have shown the following.

Proposition 2.3. *If $\mathrm{vcd}\, G < \infty$ and there is an action $G \times \Sigma(n) \to \Sigma(n)$ then $\mathrm{vcd}\, G \leq \dim \Sigma(n) - n$ for $n \geq 1$. In particular, G is finite provided $\dim \Sigma(n) = n$.*

Then, we deduce the following.

Corollary 2.4. *If $G \times \Sigma(2n) \to \Sigma(2n)$ is an action with $\dim \Sigma(2n) \leq m + 2n$ and $\mathrm{vcd}\, G < \infty$ then $\mathrm{cd}\, G \leq m$ or $G \cong G_0 \rtimes \mathbb{Z}_2$ with $\mathrm{cd}\, G_0 \leq m$. In particular, if $m = 1$ then the group G is free or $G \cong F \rtimes \mathbb{Z}_2$ for some free group F.*

Proof. For an action $G \times \Sigma(2n) \to \Sigma(2n)$ with $\dim \Sigma(2n) \leq m + 2n$ and $\mathrm{vcd}\, G < \infty$, Proposition 2.3 yields $\mathrm{vcd}\, G \leq m$. Then, Proposition 2.2 and [**21**] lead to $\mathrm{cd}\, G \leq m$ or $G \cong G_0 \rtimes \mathbb{Z}_2$ with $\mathrm{cd}\, G_0 \leq m$.

https://doi.org/10.1017/S0013091517000207 Published online by Cambridge University Press

If $m=1$ then, by means of the above, [22, 25], the group G is free or $G \cong F \rtimes \mathbb{Z}_2$ for some free group F. □

Observe that the result above is sharp in the following sense: there are groups G such that $\operatorname{cd} G = m$ which acts on $\Sigma(2n)$ with $\dim \Sigma(2n) = m + 2n$. Namely, given a homomorphism $\theta : \mathbb{Z}_2 \to \operatorname{Aut}(\mathbb{Z}^m) \cong GL_m(\mathbb{Z})$, there is an action

$$(\mathbb{Z}^m \rtimes_\theta \mathbb{Z}_2) \times (\mathbb{R}^m \times \mathbb{S}^{2n}) \longrightarrow \mathbb{R}^m \times \mathbb{S}^{2n}$$

determined by $(g,0)(t,x) = (g+t,x)$ and $(0,1_2)(t,x) = (\theta(1_2)t,-x)$ for $g \in \mathbb{Z}^m$ and $(t,x) \in \mathbb{R}^m \times \mathbb{S}^{2n}$, where $\mathbb{Z}_2 = \langle 1_2 \rangle$. Consequently, the free abelian group $\mathbb{Z}^m$, which has $\operatorname{cd} \mathbb{Z}^m = m$, acts on $\Sigma(2n) = \mathbb{R}^m \times \mathbb{S}^{2n}$ with $\dim \Sigma(2n) = m + 2n$.

Now, we show that the family of groups $F \rtimes \mathbb{Z}_2$ for a free group F is closed with respect to free products, which has interest in its own right.

Proposition 2.5. *If F_i are free groups for $i \in I$ then there is an isomorphism*

$$*_{i\in I}(F_i \rtimes \mathbb{Z}_2) \cong F \rtimes \mathbb{Z}_2$$

for some free group F.

Proof. Given $\mathbb{Z}_2 = \langle a_i \rangle$ for $i \in I$, write $\tilde{F} = \langle x_i | \, i \in I \backslash \{i_0\} \rangle$ for the free group generated by the set $\{x_i | \, i \in I \backslash \{i_0\}\}$ for a chosen $i_0 \in I$. Further, consider the homomorphism $\theta : \mathbb{Z}_2 = \langle b \rangle \to \operatorname{Aut}(\tilde{F})$ such that $\theta(b)(x_i) = x_i^{-1}$ for $i \in I \backslash \{i_0\}$. Then, the map

$$\varphi : \tilde{F} \rtimes_\theta \mathbb{Z}_2 \longrightarrow *_{i\in I}\mathbb{Z}_2$$

given by $\varphi(x_i,0) = a_i * a_{i_0}$ for $i \in I \backslash \{i_0\}$, and $\varphi(e,b) = a_{i_0}$ leads to an isomorphism $\tilde{F} \rtimes_\theta \mathbb{Z}_2 \xrightarrow{\cong} *_{i\in I}\mathbb{Z}_2$. So, the group $\tilde{F}$ can be regarded as a subgroup of the group $*_{i\in I}(F_i \rtimes \mathbb{Z}_2)$ via this isomorphism.

Next, consider the split epimorphism

$$p : *_{i\in I}(F_i \rtimes \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2,$$

where $p|_{F_i \rtimes \mathbb{Z}_2} : F_i \rtimes \mathbb{Z}_2 \to \mathbb{Z}_2$ is the projection map for all $i \in I$.

Notice that

$$\operatorname{Ker} p = \{(x_{i_1}, \varepsilon_{i_1} a_{i_1}) \cdots (x_{i_n}, \varepsilon_{i_n} a_{i_n});\ x_{i_k} \in F_{i_k} \text{ and } \varepsilon_{i_k} = 0 \text{ or } 1 \text{ for } k = 1, \ldots, n\}$$

with any $n \geq 1$, where the number $\sharp\{k;\ \varepsilon_{i_k} = 1\}$ is even. This implies that $\operatorname{Ker} p$ is generated by the subgroups $\langle (e_i,a_i)(e_{i_0},a_{i_0}) \rangle_{i\in I}, F_i \times \{0\} \leq *_{i\in I}(F_i \rtimes \mathbb{Z}_2)$, where $e_i \in F_i$ is the identity element of F_i for all $i \in I$. After some straight computations identifying elements of $\operatorname{Ker} p$ with those of $\tilde{F} * (*_{i\in I}(F_i \times \{0\}))$, it is not difficult to derive an isomorphism $\operatorname{Ker} p \cong \tilde{F} * (*_{i\in I}(F_i \times \{0\}))$ and the proof is complete. □

3. Virtually cyclic groups acting on $\Sigma(2n)$

Recall that a *virtually cyclic* group is a group that has a cyclic subgroup of finite index. In this section we classify all pairs (G, φ) which are realizable, where G is a virtually

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cyclic group. Further, homotopy types of the orbit spaces for all possible action of G on an arbitrary homotopy sphere $\Sigma(2n)$ are studied as well.

The main part of the following criterion, namely that (1) is equivalent to (2) and (3), is due to Wall [30, Lemma 4.1] (see also Scott and Wall [20]).

Theorem 3.1. *Let G be a finitely generated group. Then, the following are equivalent:*

(1) *G is a group with two ends;*

(2) *G has an infinite cyclic group of finite index;*

(3) *G has a finite normal subgroup $F \trianglelefteq G$ with the quotient $G/F \cong \mathbb{Z}$ or $\mathbb{Z}_2 \star \mathbb{Z}_2 \cong D_\infty$, the infinite dihedral group.*

Equivalently, G is of the form:

(1) *a semi-direct product $F \rtimes \mathbb{Z}$ with F finite, or*

(2) *$G_1 \star_F G_2$ with F finite, where $[G_i : F] = 2$ for $i = 1, 2$.*

Given an action $G \times \Sigma(n) \to \Sigma(n)$, we follow [19, Section 4] to consider the Moore–Postnikov system

$$\Sigma(n)/G \to \cdots \to X_{m+1} \to X_m \to \cdots \to X_2 \to X_1 = K(G,1)$$

of the fibration $\Sigma(n) \to \Sigma(n)/G \to K(G,1)$, where the fibration $X_{m+1} \to X_m$ has fiber $K(\pi_m(\Sigma(n)), m)$ and characteristic class $k^{m+1}(\Sigma(n)/G) \in H^{m+1}(X_m, \breve{\pi}_m(\Sigma(n)))$ (called the *Postnikov-* or *k-invariant*) for $\breve{\pi}_m(\Sigma(n))$ as $\pi_m(\Sigma(n))$ with the induced G-module structure for $m \geq 1$.

For a G-module π and $n \geq 1$, consider the twisted Eilenberg-MacLane space $\hat{K}_G(\pi, n) = \widetilde{K(G,1)} \times_G K(\pi, n)$, where $\widetilde{K(G,1)}$ is the universal covering of $K(G,1)$. Then, the Postnikov invariant $k^{m+1}(\Sigma(n)/G)$ is the homotopy class of a map

$$X_m \longrightarrow \hat{K}_G(\breve{\pi}_m(\Sigma(n)), m+1).$$

Because the space $\Sigma(n)$ is $(n-1)$-connected, we get

$$k^2(\Sigma(n)/G) = \cdots = k^n(\Sigma(n)/G) = 0$$

and consequently,

$$K(G,1) = X_1 = X_2 = \cdots = X_n.$$

Further, $k^{n+1}(\Sigma(n)/G) \in H^{n+1}(X_n, \breve{\pi}_n(\Sigma(n))) = H^{n+1}(G, \breve{\mathbb{Z}})$ is the first possibly non-trivial Postnikov invariant of the orbit space $\Sigma(n)/G$.

In the sequel we need the following.

Lemma 3.2. *Let a discrete group G act on $\Sigma_1(n)$ and $\Sigma_2(n)$ with $\dim \Sigma_1(n)/G \leq n+1$ for $n \geq 2$, and $\dim \Sigma_2(n)/G$ arbitrary.*

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The orbit spaces $\Sigma_1(n)/G$ and $\Sigma_2(n)/G$ have the same homotopy type if and only if there is an automorphism $\varphi \in \mathrm{Aut}(G)$ with $\varphi^*(k^{n+1}(\Sigma_2(n)/G)) = k^{n+1}(\Sigma_1(n)/G)$.

Proof. If the orbit spaces $\Sigma_1(n)/G$ and $\Sigma_2(n)/G$ have the same homotopy type, then certainly there is $\varphi \in \mathrm{Aut}(G)$ with $\varphi^*(k^{n+1}(\Sigma_2(n)/G)) = k^{n+1}(\Sigma_1(n)/G)$.

Now, suppose that there is $\varphi \in \mathrm{Aut}(G)$ with $\varphi^*(k^{n+1}(\Sigma_2(n)/G)) = k^{n+1}(\Sigma_1(n)/G)$. Write $\Sigma_1(n)/G \to \cdots \to X_{m+1} \to X_m \to \cdots \to X_2 \to X_1 = K(G,1)$ and $\Sigma_2(n)/G \to \cdots \to Y_{m+1} \to Y_m \to \cdots \to Y_2 \to Y_1 = K(G,1)$ for the the Moore–Postnikov systems of $\Sigma_1(n)/G$ and $\Sigma_2(n)/G$, respectively.

Because $k^{n+1}(\Sigma_1(n)/G) : K(G,1) \to \hat{K}_G(\breve{\mathbb{Z}}, n+1)$ and $k^{n+1}(\Sigma_2(n)/G) : K(G,1) \to \hat{K}_G(\breve{\mathbb{Z}}, n+1)$ are characteristic classes of the fibrations $X_{n+1} \to X_n = K(G,1)$ and $Y_{n+1} \to Y_n = K(G,1)$, respectively, we derive that the commutative square

$$\begin{array}{ccc} X_n = K(G,1) & \xrightarrow{\bar{\varphi}} & Y_n = K(G,1) \\ \downarrow & & \downarrow \\ \hat{K}_G(\breve{\mathbb{Z}}, n+1) & \overset{\mathrm{id}_{\hat{K}_G(\breve{\mathbb{Z}},n+1)}}{=\!=\!=} & \hat{K}_G(\breve{\mathbb{Z}}, n+1) \end{array}$$

leads to a homotopy equivalence

$$f_{n+1} : X_{n+1} \longrightarrow Y_{n+1},$$

where $\bar{\varphi}$ is the induced map by $\varphi \in \mathrm{Aut}(G)$ of the Eilenberg-MacLane space $K(G,1)$.

But $\dim \Sigma_1(n)/G \leq n+1$ and $k^{n+2}(\Sigma_2(n)/G) : Y_{n+1} \to \hat{K}_G(\breve{\pi}_{n+2}(\Sigma_2(n)), n+2)$ is characteristic class of the fibration $Y_{n+2} \to Y_{n+1}$, so there is no obstruction to a lifting

$$\begin{array}{ccc} \Sigma_1(n)/G & \dashrightarrow & Y_{n+2} \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1}. \end{array}$$

Repeating the argument above *mutatis mutandis* we get a map

$$f : \Sigma_1(n)/G \longrightarrow \Sigma_2(n)/G$$

with the commutative square

$$\begin{array}{ccc} \Sigma_1(n)/G & \overset{f}{\dashrightarrow} & \Sigma_2(n)/G \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1}. \end{array}$$

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Then, the lifting $\tilde{f} : \Sigma_1(n) \dashrightarrow \Sigma_2(n)$ of $f : \Sigma_1(n)/G \to \Sigma_2(n)/G$ leads to a map of fibrations

$$\begin{array}{ccccc} \Sigma_1(n) & \longrightarrow & \Sigma_1(n)/G & \longrightarrow & K(G,1) \\ {\scriptstyle \tilde{f}} \downarrow & & {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \bar{\varphi}} \\ \Sigma_2(n) & \longrightarrow & \Sigma_2(n)/G & \longrightarrow & K(G,1). \end{array}$$

But the commutative diagram

$$\begin{array}{ccc} \pi_n(\Sigma_1(n)/G) & \xrightarrow{\pi_n(f)} & \pi_n(\Sigma_2(n)/G) \\ \downarrow & & \downarrow \\ \pi_n(X_{n+1}) & \xrightarrow{\pi_n(f_{n+1})} & \pi_n(Y_{n+1}) \end{array}$$

implies an isomorphism

$$\pi_n(f) : \pi_n(\Sigma_1(n)/G) \longrightarrow \pi_n(\Sigma_2(n)/G)$$

which yields an isomorphism

$$\pi_n(\tilde{f}) : \pi_n(\Sigma_1(n)) \to \pi_n(\Sigma_2(n)).$$

Since the spaces $\Sigma_1(n)$ and $\Sigma_2(n)$ have the homotopy type of the n-sphere, we have isomorphisms $H_m(\tilde{f}) : H_m(\Sigma_1(n)) \to H_m(\Sigma_2(n))$ of homology which lead to isomorphisms $\pi_m(\tilde{f}) : \pi_m(\Sigma_1(n)) \to \pi_m(\Sigma_2(n))$ for $m \geq 0$. Therefore the map of fibrations above yields isomorphisms

$$\pi_m(f) : \pi_m(\Sigma_1(n)/G) \longrightarrow \pi_m(\Sigma_2(n)/G)$$

for $m \geq 0$. Consequently, $f : \Sigma_1(n)/G \to \Sigma_2(n)/G$ is a homotopy equivalence and the proof is complete. □

Now, consider an arbitrary free action $\mathbb{Z}_2 \times \Sigma(2n) \to \Sigma(2n)$. In the next proposition, the main result of this section, we need to know that the Postnikov invariant $k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2) \neq 0$. To aim at that take the $2n$-Postnikov–Moore stage $X_{2n} \simeq K(\mathbb{Z}_2, 1)$ and fibration $q_{2n} : \Sigma(2n)/\mathbb{Z}_2 \to X_{2n}$ associated with the fibration

$$\Sigma(2n) \longrightarrow \Sigma(2n)/\mathbb{Z}_2 \longrightarrow K(\mathbb{Z}_2, 1),$$

and apply [19, Theorem 2.2] with $\pi = \pi_1(\Sigma(2n)/\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\kappa = \mathrm{id}_{\mathbb{Z}_2}$. There results a map of fibrations

$$\begin{array}{ccc} \Sigma(2n)/\mathbb{Z}_2 & \xrightarrow{\psi} & P\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n+1) \\ {\scriptstyle q_{2n}} \downarrow & & \downarrow \\ X_{2n} & \xrightarrow{\phi} & \hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, n+1), \end{array}$$

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where $P\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n+1)$ is the path-space on $\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n+1)$. Then, the pullback construction yields fibration $p_{2n+1} : X_{2n+1} \to X_{2n}$ with fibre $K(\mathbb{Z}, 2n)$ and $q_{2n} : \Sigma(2n)/\mathbb{Z}_2 \to X_{2n}$ factorises into

$$\Sigma(2n)/\mathbb{Z}_2 \overset{q_{2n+1}}{\longrightarrow} X_{2n+1} \overset{p_{2n+1}}{\longrightarrow} X_{2n}.$$

The map $\phi : X_{2n} \to \hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, n+1)$ represents the characteristic class $k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2) \in H^{2n+1}(X_{2n}, \breve{\mathbb{Z}})$ of $X_{2n+1} \to X_{2n}$ which is sent by the map $q_{2n} : \Sigma(2n)/\mathbb{Z}_2 \to X_{2n} \simeq K(\mathbb{Z}_2, 1)$ into the trivial class in $H^{2n+1}(\Sigma(2n)/\mathbb{Z}_2, \breve{\mathbb{Z}})$.

Further, let $\mathcal{E}$ be a path-connected space such that $\pi_1(\mathcal{E}) \cong \mathbb{Z}_2$, $\pi_{2n}(\mathcal{E}) \cong \mathbb{Z}$ and $\pi_m(\mathcal{E}) = 0$ for $m \neq 1, 2n$ with a non-trivial action of $\pi_1(\mathcal{E})$ on $\pi_{2n}(\mathcal{E})$. A space $\mathcal{E}$ with the properties above is called a *two-stage Postnikov system*.

Lemma 3.3. *There are two homotopy types of such two-stage Postnikov systems: one represented by* $\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n)$ *with* $k^{2n+1}(\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n)) = 0$ *and the other by the* $(2n+1)$*-Postnikov–Moore stage* X_{2n+1} *with* $k^{2n+1}(X_{2n+1}) = k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2) \neq 0$.

Proof. The first possibly non-trivial Postnikov invariant of such a two-stage Postnikov system $\mathcal{E}$ lies in $H^{2n+1}(\mathbb{Z}_2, \breve{\mathbb{Z}}) \cong \mathbb{Z}_2$. Thus, we have two Postnikov invariants $k^{2n+1}(\mathcal{E})$ which provide two-stage Postnikov systems with different homotopy types.

Certainly, the spaces $\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n)$ and X_{2n+1} are two-stage Postnikov systems with the properties above. The characteristic class of the fibration $\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n) \to K(\mathbb{Z}_2, 1)$ is determined by the trivial class in $H^{2n+1}(\mathbb{Z}_2, \breve{\mathbb{Z}})$, because its pullback, according to [**19**, Section 4], should have the cohomology of the path-space $P\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n)$. Consequently, we can state that $k^{2n+1}(\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n)) = 0$.

For $k^{2n+1}(X_{2n+1}) = 0$, the space X_{2n+1} has the homotopy type of $\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n)$. Because of the sectioned fibration $K(\mathbb{Z}, 2n) \to \hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n) \to K(\mathbb{Z}_2, 1)$, the cohomology $H^1(\hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n), \mathbb{Z}_2)$ contains a class such that its cup $(2n+1)$-power is non-trivial. But, this cup $(2n+1)$-power is sent by the fibrations $X_m \to X_{2n+1}$ for $m > 2n+1$ into a non-trivial class in $H^{2n+1}(X_m, \mathbb{Z}_2)$ which is a contradiction, because of the cohomology algebra $H^*(\Sigma(2n)/\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[t]/(t^{2n+1})$ and the result follows. □

Now, we are in a position to show the following.

Proposition 3.4. *Let* $G \times \Sigma(2n) \to \Sigma(2n)$ *be an action of a non-trivial virtually cyclic group* G *on any* $\Sigma(2n)$ *and* $\varphi : G \to \mathrm{Aut}\,(H^{2n}(\Sigma(2n), \mathbb{Z}))$ *the induced homomorphism. Then:*

(1) G *is isomorphic to one of the groups:* $\mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_2$ *or* $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$*;*

(2) *any of the groups above admits an action on some* $\Sigma(2n)$ *and the pair* (G, φ) *is realizable if and only if:*
   (i) $G \cong \mathbb{Z}_2$ *and* φ *is non-trivial. The homotopy type of* $\Sigma(2n)/G$ *is* $\mathbb{R}P^{2n}$*;*

   (ii) $G \cong \mathbb{Z}$ *and* φ *is any homomorphism. The homotopy type of* $\Sigma(2n)/G$ *is* $\mathbb{S}^1 \times \mathbb{S}^{2n}$ *in case* φ *is trivial and it is* $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$ *(the only non-trivial* $\mathbb{S}^{2n}$*-bundle over* $\mathbb{S}^1$ *being the mapping torus of the antipodal map* $\mathbb{S}^{2n} \to \mathbb{S}^{2n}$*) otherwise;*

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(iii) *$G \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and, for a suitable automorphism of $\mathbb{Z} \oplus \mathbb{Z}_2$, the restriction $\varphi|_{\mathbb{Z}}$ is trivial and $\varphi|_{\mathbb{Z}_2}$ non-trivial. The homotopy type of $\Sigma(2n)/G$ is $\mathbb{S}^1 \times \mathbb{R}P^{2n}$;*

(iv) *$G \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, the restriction $\varphi|_{\mathbb{Z}}$ is trivial and $\varphi|_{\mathbb{Z}_2}$ are non-trivial. The homotopy type of $\Sigma(2n)/G$ is $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$.*

Proof. (1) Follows immediately from Proposition 2.2 and Theorem 3.1.

(2) Write $\breve{\mathbb{Z}}$ for the G-module structure on $H^{2n}(\Sigma(2n), \mathbb{Z}) \cong \mathbb{Z}$ for G being one of the groups from (1). Now, we show simultaneously the items of (2).

(i) $G \cong \mathbb{Z}_2$. By [**24**], the orientation of any $\mathbb{Z}_2$-action is non-trivial. Certainly, the antipodal action $\mathbb{Z}_2 \times \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ yields $\mathbb{S}^{2n}/\mathbb{Z}_2 = \mathbb{R}P^{2n}$ for any $n \geq 1$. In view of Lemma 3.3, the Postnikov invariant $k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2) \neq 0$. Because $k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2) = k^{2n+1}(\mathbb{R}P^{2n})$, Lemma 3.2 yields a homotopy equivalence $\Sigma(2n)/\mathbb{Z}_2 \simeq \mathbb{R}P^{2n}$.

We point out that the above yields a new proof of the result [**11**, Lemma 2.5] in the even case.

(ii) $G \cong \mathbb{Z}$. There are only two possible orientations of any $\mathbb{Z}$-action and any of them can be realized. Namely, consider the $\mathbb{Z}$-actions:

$$\circ, \bar{\circ} : \mathbb{Z} \times (\mathbb{R} \times \mathbb{S}^{2n}) \to \mathbb{R} \times \mathbb{S}^{2n}$$

given by $m \circ (t, x) = (t + m, x)$ and $m \bar{\circ} (t, x) = (t + m, (-1)^m x)$, respectively for $m \in \mathbb{Z}$ and $(t, x) \in \mathbb{R} \times \mathbb{S}^{2n}$. The corresponding orbit spaces are $\mathbb{S}^1 \times \mathbb{S}^{2n}$ or $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$, respectively.

To study the homotopy type of the orbit spaces for an arbitrary action, we first observe that $H^{2n+1}(\mathbb{Z}, \breve{\mathbb{Z}}) = H^{2n+1}(\mathbb{Z}, \mathbb{Z}) = 0$. Therefore, we can apply Lemma 3.2 using one of the two explicit actions provided above, since $\Sigma(2n) = \mathbb{R} \times \mathbb{S}^{2n}$ is a homotopy sphere with $\dim \Sigma(2n) = 2n + 1$ and the result follows.

(iii) $G \cong \mathbb{Z} \oplus \mathbb{Z}_2$. By Proposition 2.2 the action of $\mathbb{Z} \oplus \mathbb{Z}_2$ on the cohomology $H^{2n}(\Sigma(2n), \mathbb{Z}) \cong \mathbb{Z}$ restricted to $\mathbb{Z}_2$ is non-trivial, so $\mathbb{Z} \oplus \mathbb{Z}_2$ acts non-trivially on $H^{2n}(\Sigma(2n), \mathbb{Z}) \cong \mathbb{Z}$. Hence, we have an epimorphism

$$\varphi : \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(H^{2n}(\Sigma(2n), \mathbb{Z})) \cong \mathbb{Z}_2$$

such that $\varphi(0, 1_2) = 1_2$ and $\varphi(1, 0) = 0$ or $\varphi(1, 0) = 1_2$. Since there is an automorphism $\mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}_2$ given by: $(1, 0) \mapsto (1, 1_2)$ and $(0, 1_2) \mapsto (0, 1_2)$, we can assume that the homomorphism φ is as in the statement of (iii).

There is an action

$$\circ : (\mathbb{Z} \oplus \mathbb{Z}_2) \times (\mathbb{R} \times \mathbb{S}^{2n}) \longrightarrow \mathbb{R} \times \mathbb{S}^{2n}$$

given by $(1, 0) \circ (t, x) = (t + 1, x)$ and $(0, 1_2) \circ (t, x) = (t, -x)$ for $(t, x) \in \mathbb{R} \times \mathbb{S}^{2n}$ with the corresponding quotient space homeomorphic to $\mathbb{S}^1 \times \mathbb{R}P^{2n}$, where $\mathbb{Z} = \langle 1 \rangle$. This action provides the orientation of a $\mathbb{Z} \oplus \mathbb{Z}_2$-action stated in (iii).

Now, we show that $H^m(\mathbb{Z}\oplus\mathbb{Z}_2,\breve{\mathbb{Z}})\cong\mathbb{Z}_2$. For this purpose, we can use either the action of $\mathbb{Z}\oplus\mathbb{Z}_2$ on $\mathbb{Z}$ given by φ above or by

$$\varphi' : \mathbb{Z}\oplus\mathbb{Z}_2 \longrightarrow \mathrm{Aut}(H^{2n}(\Sigma(2n),\mathbb{Z}))\cong\mathbb{Z}_2$$

defined by: $\varphi'(0,1_2)=1_2$ and $\varphi'(1,0)=1_2$.
We use $\varphi' : \mathbb{Z}\oplus\mathbb{Z}_2 \to \mathrm{Aut}(H^{2n}(\Sigma(2n),\mathbb{Z}))\cong\mathbb{Z}_2$ to simplify the calculation of the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q}=H^p(\mathbb{Z}_2,H^q(\mathbb{Z},\breve{\mathbb{Z}}))\Rightarrow H^{p+q}(\mathbb{Z}\oplus\mathbb{Z}_2,\breve{\mathbb{Z}})$$

corresponding to the extension

$$0\to\mathbb{Z}\longrightarrow\mathbb{Z}\oplus\mathbb{Z}_2\longrightarrow\mathbb{Z}_2\to 0.$$

Since $H^q(\mathbb{Z},\breve{\mathbb{Z}})\cong\begin{cases}\mathbb{Z}_2, & \text{if } q=1,\\ 0, & \text{otherwise,}\end{cases}$ this is a one line spectral sequence which implies that $H^p(\mathbb{Z}_2,\mathbb{Z}_2)\cong H^{p+1}(\mathbb{Z}\oplus\mathbb{Z}_2,\breve{\mathbb{Z}})$ and so $H^m(\mathbb{Z}\oplus\mathbb{Z}_2,\breve{\mathbb{Z}})\cong\mathbb{Z}_2$ for $m>0$. In particular, $H^{2n+1}(\mathbb{Z}\oplus\mathbb{Z}_2,\breve{\mathbb{Z}})\cong\mathbb{Z}_2$ and there are two possible values for the Postnikov invariant

$$k^{2n+1}(\Sigma(2n)/(\mathbb{Z}\oplus\mathbb{Z}_2)) : K(\mathbb{Z}\oplus\mathbb{Z}_2,1)\longrightarrow\widehat{K}_{\mathbb{Z}\oplus\mathbb{Z}_2}(\breve{\mathbb{Z}},2n+1).$$

Next, we study the homotopy type of orbit spaces. Given an action

$$(\mathbb{Z}\oplus\mathbb{Z}_2)\times\Sigma(2n)\longrightarrow\Sigma(2n),$$

its restriction to $\mathbb{Z}_2$ leads to the non-trivial (in view of Lemma 3.3) Postnikov invariant $k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2)$. Recall that the orientation of a $\mathbb{Z}\oplus\mathbb{Z}_2$-action can be given by the map $\varphi : \mathbb{Z}\oplus\mathbb{Z}_2\to\mathbb{Z}_2$ defined above. Then, the commutative diagram

$$\begin{array}{ccc} K(\mathbb{Z}_2,1) & \longrightarrow & \hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}},2n+1) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}\oplus\mathbb{Z}_2,1) & \longrightarrow & \hat{K}_{\mathbb{Z}\oplus\mathbb{Z}_2}(\breve{\mathbb{Z}},2n+1) \end{array}$$

with horizontal arrows determined by appropriate Postnikov invariants shows that $k^{2n+1}(\Sigma(2n)/(\mathbb{Z}\oplus\mathbb{Z}_2))$ is non-trivial. Now, we are in a position to apply Lemma 3.2, as in the proof of (ii), to conclude that there is only one homotopy type $\mathbb{S}^1\times\mathbb{R}P^{2n}$ of the orbit space $\Sigma(2n)/(\mathbb{Z}\oplus\mathbb{Z}_2)$.

(iv) $G\cong\mathbb{Z}\rtimes\mathbb{Z}_2\cong\mathbb{Z}_2*\mathbb{Z}_2$. Given an orientation $\varphi:\mathbb{Z}\rtimes\mathbb{Z}_2\to\mathbb{Z}_2$ of a $\mathbb{Z}\rtimes\mathbb{Z}_2$-action, Proposition 2.2 implies that $\varphi(1,1_2)=\varphi(1,0)+\varphi(0,1_2)=\varphi(1,0)+1_2=1_2$ since the orders of $(0,1_2),(1,1_2)\in\mathbb{Z}\rtimes\mathbb{Z}_2$ are two. Hence, φ restricts to the trivial one on $\mathbb{Z}$ and to the identity map on $\mathbb{Z}_2$. Next, consider $\Sigma(2n) = \mathbb{R} \times \mathbb{S}^{2n}$ with $\dim \Sigma(2n) = 2n + 1$ and the action

$$\circ : (\mathbb{Z} \rtimes \mathbb{Z}_2) \times (\mathbb{R} \times \mathbb{S}^{2n}) \longrightarrow \mathbb{R} \times \mathbb{S}^{2n}$$

given by $(1, 0) \circ (t, x) = (t + 1, x)$ and $(0, 1_2) \circ (t, x) = (-t, -x)$ for $(t, x) \in \mathbb{R} \times \mathbb{S}^{2n}$. Then, the corresponding orbit space is homeomorphic to $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$. Given an action of $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \Sigma(2n) \to \Sigma(2n)$, by Proposition 2.2, the induced action of $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ on the cohomology $H^{2n}(\Sigma(2n), \mathbb{Z}) \cong \mathbb{Z}$ restricts to non-trivial on both copies of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 * \mathbb{Z}_2$. Next, the isomorphism $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ leads to

$$\begin{aligned} H^{2n+1}(\mathbb{Z} \rtimes \mathbb{Z}_2, \breve{\mathbb{Z}}) &\cong H^{2n+1}(\mathbb{Z}_2 * \mathbb{Z}_2, \breve{\mathbb{Z}}) \\ &\cong H^{2n+1}(\mathbb{Z}_2, \breve{\mathbb{Z}}) \oplus H^{2n+1}(\mathbb{Z}_2, \breve{\mathbb{Z}}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

Because the Postnikov invariants $k^{2n+1}(\Sigma(2n)/\mathbb{Z}_2)$ associated with restricted actions $\mathbb{Z}_2 \times \Sigma(2n) \to \Sigma(2n)$ of both copies of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 * \mathbb{Z}_2$ are (in view of Lemma 3.3) non-trivial, the commutative diagram

$$\begin{array}{ccc} K(\mathbb{Z}_2, 1) & \longrightarrow & \hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n+1) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}_2 * \mathbb{Z}_2, 1) & \longrightarrow & \hat{K}_{\mathbb{Z}_2 * \mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n+1) \\ \uparrow & & \uparrow \\ K(\mathbb{Z}_2, 1) & \longrightarrow & \hat{K}_{\mathbb{Z}_2}(\breve{\mathbb{Z}}, 2n+1) \end{array}$$

with horizontal arrows determined by appropriate Postnikov invariants shows that only one class of $H^{2n+1}(\mathbb{Z} \rtimes \mathbb{Z}_2, \breve{\mathbb{Z}}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ corresponds to $k^{2n+1}(\Sigma(2n)/(\mathbb{Z} \rtimes \mathbb{Z}_2))$.

Finally, like in the proof of (iii), we can apply Lemma 3.2, to conclude that there is only one homotopy type $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$ of the orbit space $\Sigma(2n)/(\mathbb{Z} \rtimes \mathbb{Z}_2)$ and the proof follows. □

In view of [**29**, Corollary 2], the classification of all free of finite groups on $\mathbb{S}^1 \times \mathbb{S}^2$ follows from the observation that there exist only four compact 3-manifolds which have $\mathbb{R} \times \mathbb{S}^2$ as a universal covering space.

Now, we deduce below that any manifold with the universal covering space $\mathbb{R} \times \mathbb{S}^{2n}$ has the homotopy type one of the following manifolds: $\mathbb{S}^1 \times \mathbb{S}^{2n}$, $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$, $\mathbb{S}^1 \times \mathbb{R}P^{2n}$ or $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$.

Corollary 3.5. *Suppose that a finite non-trivial group G acts freely on one of the manifolds: $\mathbb{S}^1 \times \mathbb{S}^{2n}$, $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$, $\mathbb{S}^1 \times \mathbb{R}P^{2n}$ or $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$. Let M be the orbit space by G.*

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(1) *If G acts on $\mathbb{S}^1 \times \mathbb{S}^{2n}$ then:*
 (i) $M \simeq \mathbb{S}^1 \times \mathbb{S}^{2n}$ *for* $G \cong \mathbb{Z}_m$ *with* $m \geq 2$*;*

 (ii) $M \simeq \mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$ *for* $G \cong \mathbb{Z}_{2m}$ *with* $m \geq 1$*;*

 (iii) $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$ *for* $G \cong \mathbb{Z}_m \oplus \mathbb{Z}_2$ *with* $m \geq 1$*;*

 (iv) $M \simeq \mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$ *for* $G \cong \mathbb{Z}_m \rtimes \mathbb{Z}_2 = D_m$ *with* $m > 2$*, the dihedral group of order* $2m$*.*

(2) *If G acts on $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$ then:*
 (i) $M \simeq \mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$ *for* $G \cong \mathbb{Z}_m$ *with* $m \geq 2$*;*

 (ii) $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$ *for* $G \cong \mathbb{Z}_m \oplus \mathbb{Z}_2$ *with* $m \geq 1$*.*

(3) *If G acts on $\mathbb{S}^1 \times \mathbb{R}P^{2n}$ then $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$ for $G \cong \mathbb{Z}_m$ with $m \geq 2$.*

(4) *If G acts on $\mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$ then $M \simeq \mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$ for $G \cong \mathbb{Z}_2$.*

Further, in all four cases above, the groups described act on the corresponding manifold.

Proof. First, we point out that $\mathbb{R} \times \mathbb{S}^{2n}$ is the universal covering space of the manifolds listed above and hence the orbit spaces M as well. In virtue of Proposition 3.4, we deduce that M has the homotopy type of one of those manifolds.

(1) If G acts on $\mathbb{S}^1 \times \mathbb{S}^{2n}$ then the covering

$$\mathbb{S}^1 \times \mathbb{S}^{2n} \to \mathbb{S}^1 \times \mathbb{S}^{2n}/G$$

determines an extension of groups

$$e \to \mathbb{Z} \to \pi \to G \to e,$$

where $\pi = \pi_1(\mathbb{S}^1 \times \mathbb{S}^{2n}/G)$.
If $M \simeq \mathbb{S}^1 \times \mathbb{S}^{2n}$ then $G \cong \mathbb{Z}_m$ for some $m \geq 2$.
If $M \simeq \mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}$ then we have a finite covering

$$\mathbb{S}^1 \times \mathbb{S}^{2n} \longrightarrow \mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n}.$$

Since $\mathbb{S}^1 \times \mathbb{S}^{2n}$ is orientable, it follows that the image of the induced homomorphism $\pi_1(\mathbb{S}^1 \times \mathbb{S}^{2n}) \to \pi_1(\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n})$ is determined by orientable loops in $\pi_1(\mathbb{S}^1 \tilde{\times} \mathbb{S}^{2n})$. Therefore, $G \cong \mathbb{Z}_{2m}$ for some $m \geq 1$.
If $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$ then $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_m$ for some $m \geq 1$.
If $M \simeq \mathbb{R}P^{2n+1} \sharp \mathbb{R}P^{2n+1}$ then $\pi \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ and so $G \cong \mathbb{Z}_m \rtimes \mathbb{Z}_2$, the dihedral group of order $2m$ for some $m \geq 2$.

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(2) If G acts on $\mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$ then the covering

$$\mathbb{S}^1\tilde{\times}\mathbb{S}^{2n} \to \mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}/G$$

determines an extension of groups

$$e \to \mathbb{Z} \to \pi \to G \to e,$$

where $\pi = \pi_1(\mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}/G)$. Because $\mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$ is non-oriented, $M \simeq \mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$ or $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$.
If $M \simeq \mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$ then $G \cong \mathbb{Z}_m$ for some $m \geq 2$.
If $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$ then $G \cong \mathbb{Z}_m \oplus \mathbb{Z}_2$ for $m \geq 1$.

(3) If G acts on $\mathbb{S}^1 \times \mathbb{R}P^{2n}$ then the covering

$$\mathbb{S}^1 \times \mathbb{R}P^{2n} \to \mathbb{S}^1 \times \mathbb{R}P^{2n}/G$$

determines an extension of groups

$$e \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \pi \to G \to e,$$

where $\pi = \pi_1(\mathbb{S}^1 \times \mathbb{R}P^{2n}/G)$. Because $\mathbb{S}^1 \times \mathbb{R}P^{2n}$ is non-oriented, $M \simeq \mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$ or $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$.
If $M \simeq \mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$ then $\pi = \pi_1(\mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}) \cong \mathbb{Z}$ and any monomorphism $0 \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}$ does not exit.
If $M \simeq \mathbb{S}^1 \times \mathbb{R}P^{2n}$ then any monomorphism $0 \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z} \oplus \mathbb{Z}_2$ restricts to the identity map on $\mathbb{Z}_2$ and consequently $G \cong \mathbb{Z}_m$ for some $m \geq 2$.

(4) If G acts on $\mathbb{R}P^{2n+1}\sharp\mathbb{R}P^{2n+1}$ then the covering

$$\mathbb{R}P^{2n+1}\sharp\mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}\sharp\mathbb{R}P^{2n+1}/G$$

determines an extension of groups

$$e \to \mathbb{Z} \rtimes \mathbb{Z}_2 \to \pi \to G \to e,$$

where $\pi = \pi_1(\mathbb{R}P^{2n+1}\sharp\mathbb{R}P^{2n+1}/G)$. Because $\pi_1(\mathbb{R}P^{2n+1}\sharp\mathbb{R}P^{2n+1}) \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ is non-abelian, we have only to analyse the extension

$$e \to \mathbb{Z} \rtimes \mathbb{Z}_2 \to \mathbb{Z} \rtimes \mathbb{Z}_2 \to G \to e.$$

Since $\mathbb{Z} \rtimes \mathbb{Z}_2$ must be sent to its normal subgroup by the monomorphism $e \to \mathbb{Z} \rtimes \mathbb{Z}_2 \to \mathbb{Z} \rtimes \mathbb{Z}_2$, we deduce that $G \cong \mathbb{Z}_2$ and the proof is complete. □

Let $n \geq 2$ and let τ be a free involution on $\mathbb{S}^1 \times \mathbb{S}^n$. Then, in view of [**11**, Theorem 2.1] the quotient $\mathbb{S}^1 \times \mathbb{S}^n/\tau$ belongs to one of the four homotopy types: $\mathbb{S}^1 \times \mathbb{S}^n$, $\mathbb{S}^1\tilde{\times}\mathbb{S}^n$, $\mathbb{S}^1 \times \mathbb{R}P^n$ and $\mathbb{R}P^{n+1}\sharp\mathbb{R}P^{n+1}$ realized by the standard involutions.

Now, we are in position to conclude the following generalization of the above, provided n is even.

Corollary 3.6. *Let $n \geq 1$ and τ be a free involution on one of the four manifolds: $\mathbb{S}^1 \times \mathbb{S}^{2n}$, $\mathbb{S}^1\tilde{\times}\mathbb{S}^{2n}$, $\mathbb{S}^1 \times \mathbb{R}P^{2n}$ or $\mathbb{R}P^{2n+1}\sharp\mathbb{R}P^{2n+1}$. Then, the corresponding orbit space also belongs to one of their homotopy types.*

https://doi.org/10.1017/S0013091517000207 Published online by Cambridge University Press

4. Other groups acting on $\Sigma(2n)$

Here, we analyse actions $G \times \Sigma(2n) \to \Sigma(2n)$ with $\operatorname{vcd} G = 1$. By Corollary 2.4, the group G is free or $G \cong F \rtimes \mathbb{Z}_2$ for some free group F with an arbitrary rank.

Let F be a free group and $\operatorname{Aut}(F)$ its automorphism group. For a homomorphism $\theta : \mathbb{Z}_2 \to \operatorname{Aut}(F)$ we consider the semidirect product $G \cong F \rtimes_\theta \mathbb{Z}_2$ which is completely determined by θ. Given also $\varphi : F \rtimes_\theta \mathbb{Z}_2 \to \operatorname{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ with $\varphi|_{\mathbb{Z}_2} = \mathrm{id}_{\mathbb{Z}_2}$, we say that the pair (θ, φ) is *realizable* if $(F \rtimes_\theta \mathbb{Z}_2, \varphi)$ is realizable (see Introduction and § 2).

Notice that any free group F acts on the homotopy $2n$-sphere $(\widetilde{\bigvee_{i\in I} \mathbb{S}^1}) \times \mathbb{S}^{2n}$ for any $n \geq 1$, where $\widetilde{\bigvee_{i\in I} \mathbb{S}^1}$ is the universal covering of the wedge $\bigvee_{i\in I} \mathbb{S}^1$ provided $F = \langle x_i;\, i \in I\rangle$. Consequently, for the trivial homomorphism $\theta_0 : \mathbb{Z}_2 \to \operatorname{Aut}(F)$, any pair (θ_0, φ) is realizable by the action

$$\circ : (F \rtimes_{\theta_0} \mathbb{Z}_2) \times \left(\left(\widetilde{\bigvee_{i\in I} \mathbb{S}^1} \right) \times \mathbb{S}^{2n} \right) \to \left(\widetilde{\bigvee_{i\in I} \mathbb{S}^1} \right) \times \mathbb{S}^{2n}$$

given by: $(g, 0) \circ (t, s) = (gt, \operatorname{sgn}(g)s,)$ and $(g, 1_2) \circ (t, s) = (gt, -\operatorname{sgn}(g)s)$ for $g \in F$ and $(t, s) \in (\widetilde{\bigvee_{i\in I} \mathbb{S}^1}) \times \mathbb{S}^{2n}$, where $\operatorname{sgn} : F \to \mathbb{Z}_2 = \{\pm 1\}$ is the homomorphism determined by the restriction of $\varphi : F \rtimes_{\theta_0} \mathbb{Z}_2 \to \operatorname{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ to the group F.

Writing $\mathbb{Z}_2 = \langle 1_2 \rangle$, we show a general fact, as follows.

Lemma 4.1 (fundamental Lemma). *The pair (θ, φ) is realizable if and only if it does not exist $g \in F$ such that*

$$\begin{cases} \theta(1_2)(g) = g^{-1}, \\ \varphi(g, 0) = 1_2. \end{cases}$$

Proof. Let $\theta : \mathbb{Z}_2 \to \operatorname{Aut}(F)$. By the 1-dimensional analog of the Nielsen realization problem [**6**, Theorems 2.1 and 4.1], the automorphism $\theta(1_2) \in \operatorname{Aut}(F)$ can be realized by a homeomorphism $h : \Gamma \to \Gamma$ of a graph Γ with the fundamental group $\pi_1(\Gamma) \cong F$ such that h has a fixed point $x_0 \in \Gamma$ and $h^2 = \mathrm{id}_\Gamma$. Choose a base point $\tilde{x}_0 \in \tilde{\Gamma}$, where $\tilde{\Gamma}$ is the universal covering of Γ. The homeomorphism $h : \Gamma \to \Gamma$ admits several liftings, but we consider the unique lifting $\tilde{h} : \tilde{\Gamma} \to \tilde{\Gamma}$ such that $\tilde{h}(\tilde{x}_0) = \tilde{x}_0$. Certainly, $\tilde{h}^2$ covers $h^2 = \mathrm{id}_\Gamma$ and $\tilde{h}^2(\tilde{x_0}) = \tilde{x_0}$. But $\mathrm{id}_{\tilde{\Gamma}}$ also covers h^2 and fixes the point $\tilde{x_0}$. Therefore, it follows that $\tilde{h}^2 = \mathrm{id}_{\tilde{\Gamma}}$.

Then, we are in a position to consider a map

$$\circ : (F \rtimes_\theta \mathbb{Z}_2) \times (\tilde{\Gamma} \times \mathbb{S}^{2n}) \to \tilde{\Gamma} \times \mathbb{S}^{2n}$$

given by: $(g, 0) \circ (t, s) = ((\theta(1_2)g)t, \operatorname{sgn}(g)s)$ and $(g, 1_2) \circ (t, s) = ((\theta(1_2)g)(\tilde{h}(t)), -\operatorname{sgn}(g)s)$ for $g \in F$ and $(t, s) \in \tilde{\Gamma} \times \mathbb{S}^{2n}$. Now, we prove that the map defined above is an action of the group $F \rtimes_\theta \mathbb{Z}_2$. So for any two elements $w_1, w_2 \in F \rtimes_\theta \mathbb{Z}_2$ and $(t, s) \in \tilde{\Gamma} \times \mathbb{S}^{2n}$ we must show that $w_2(w_1(t, s)) = (w_2 w_1)(t, s)$.

Notice that $\operatorname{sgn}(g\theta(1_2)g') = \operatorname{sgn}(gg')$ and $(g, 1_2)(s, t) = ((g, 0)(e, 1_2))(t, s) = (g, 0)(\tilde{h}(t), -s) = ((\theta(1_2)g)\tilde{h}(t), -\operatorname{sgn}(g)s)$. In the case, where $w_1 = (g, 0)$ and $w_2 = (g', 0)$ $i = 1, 2$ we have:

(i) $((g, 0)(g', 1_2))(t, s) = (gg', 1_2)(t, s) = ((\theta(1_2)(gg'))\tilde{h}(t), -\operatorname{sgn}(gg')s)$ and $(g, 0)((g', 1_2)(t, s)) = (g, 0) = ((g, 0)(g', 1_2))(t, s) = (g, 0)((\theta(1_2)g')\tilde{h}(t), -\operatorname{sgn}(g')s) = ((\theta(1_2) g)(\theta(1_2)g')\tilde{h}(t), -\operatorname{sgn}(g)\operatorname{sgn}(g')s) = (g, 0)(g', 1_2))(t, s)$;

(ii) $((g, 1_2)(g', 0))(t, s) = (g\theta(1_2)g', 1_2)(t, s) = (\theta(1_2)(g\theta(1_2)g')\tilde{h}(t), -\operatorname{sgn}(g\theta(1_2)g')s) = ((\theta(1_2)g)g'\tilde{h}(t), \operatorname{sgn}(gg')s)$ and $(g, 1_2)((g', 0)(t, s)) = (g, 1_2)((\theta(1_2)g')t, \operatorname{sgn}(g')s) = (\theta(1_2)g\tilde{h}(\theta(1_2)g')(t), -\operatorname{sgn}(g)\operatorname{sgn}(g')s) = ((\theta(1_2)g)g'\tilde{h}(t), -\operatorname{sgn}(g)\operatorname{sgn}(g')s) = ((g, 1_2)(g', 0))(t, s)$;

(iii) $((g, 1_2)(g', 1_2))(t, s) = (g\theta(1_2)g', 0)(t, s) = ((\theta(1_2)g\theta(1_2)g')t, -\operatorname{sgn}(g\theta(1_2)g')s) = (((\theta(1_2)g)g')t, -\operatorname{sgn}(gg')s)$ and $(g, 1_2)((g', 1_2)(t, s)) = (g, 1_2)((\theta(1_2)g')\tilde{h}(t), -\operatorname{sgn}(g')s) = ((\theta(1_2)g)\tilde{h}((\theta(1_2)g')\tilde{h}(t)), -\operatorname{sgn}(g)\operatorname{sgn}(g')s) = (((\theta(1_2)g)g')t, -\operatorname{sgn}(g)\operatorname{sgn}(g')s) = ((g, 1_2)(g', 1_2))(t, s)$.

The case, where $w_i = (g_i, 0)$ for some $g_i \in F$ with $i = 1, 2$ is easier and we leave that to the reader.

Consequently, $\circ : (F \rtimes_\theta \mathbb{Z}_2) \times (\tilde{\Gamma} \times \mathbb{S}^{2n}) \to \tilde{\Gamma} \times \mathbb{S}^{2n}$ is a well-defined action. Because it does not exist $g \in F$ such that $\begin{cases}\theta(1_2)(g) = g^{-1}\\ \varphi(g, 0) = 1_2\end{cases}$, for any $g \in F$, the action $\circ : (F_m \rtimes_\theta \mathbb{Z}_2) \times (\tilde{\Gamma} \times \mathbb{S}^{2n}) \to \tilde{\Gamma} \times \mathbb{S}^{2n}$ is free. Otherwise suppose that $(g, 1_2) \circ (t, s) = (t, s)$. Then we have $(t, s) = ((\theta(1_2)g)(\tilde{h}(t)), -\operatorname{sgn}(g)s) = (\tilde{h}(gt), -\operatorname{sgn}(g)s)$ which implies $\operatorname{sgn}(g) = -1$ and $t = (\theta(1_2)g)(\tilde{h}(t))$. The second equation is equivalent to $\tilde{h}(t) = \tilde{h}^2(gt) = gt = g\theta(1_2)(g)\tilde{h}(t)$ or $g\theta(1_2)(g) = 1$. So the system of equations has a solution which is a contradiction. So we have a free, properly discontinuous and cellular action. Further, the induced homomorphism $\varphi : F \rtimes_\theta \mathbb{Z}_2 \to \operatorname{Aut}(H^{2n}(\tilde{\Gamma} \times \mathbb{S}^{2n}), \mathbb{Z})$ coincides with the given one $\varphi : F \rtimes_\theta \mathbb{Z}_2 \to \operatorname{Aut}(\mathbb{Z})$.

Next, suppose that $\begin{cases}\theta(1_2)(g) = g^{-1}\\ \varphi(g, 0) = 1_2\end{cases}$, for some $g \in F$ and there is an action $(F \rtimes_\theta \mathbb{Z}_2) \times \Sigma(2n) \to \Sigma(2n)$. Then, on one hand we have that $\varphi(g, 1_2) = \varphi(g, 0)\varphi(e, 1_2) = 0$ and on the other hand, Proposition 2.2(1) leads to $\varphi(g, 1_2) = 1_2$, because the order of $(g, 1_2) \in F \rtimes_\theta \mathbb{Z}_2$ is two. This contradiction completes the proof. □

Corollary 4.2. *The group $F \rtimes_\theta \mathbb{Z}_2$ acts on $\Sigma(2n) = \tilde{\Gamma} \times \mathbb{S}^{2n}$ for any $n \geq 1$, where Γ is a graph (a finite graph provided F is of finite rank) with $\pi_1(\Gamma) = F$.*

Proof. Given the group $F \rtimes_\theta \mathbb{Z}_2$, consider the homomorphism $\varphi : F \rtimes_\theta \mathbb{Z}_2 \to \operatorname{Aut}(\mathbb{Z})$ given by the projection map onto the second factor. Then, in view of Lemma 4.1, the pair (θ, φ) is realizable and this leads to an action of $F \rtimes_\theta \mathbb{Z}_2$ on $\Sigma(2n) = \tilde{\Gamma} \times \mathbb{S}^{2n}$ for any $n \geq 1$, where Γ is a graph (a finite one provided F is of finite rank) with $\pi_1(\Gamma) = F$, and the result follows.

Now, let F_m be the free group with finite rank $m \geq 1$. For $\theta : \mathbb{Z}_2 \to \operatorname{Aut}(F_m)$ and $\varphi : F_m \rtimes_\theta \mathbb{Z}_2 \to \operatorname{Aut}(\mathbb{Z})$ with $\varphi|_{\mathbb{Z}_2} = \mathrm{id}_{\mathbb{Z}_2}$, we aim to classify realizable pairs (θ, φ), i.e., in view of Lemma 4.1, pairs (θ, φ) for which it does not exist $g \in F_m$ such that

$$\begin{cases} \theta(1_2)(g) = g^{-1}, \\ \varphi(g, 0) = 1_2. \end{cases}$$

First, we recall a very useful result by Dyer and Scott [7, Theorem 3]. □

Theorem 4.3. *Let F be any free group, $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(F)$ a homomorphism and $F^{\theta(1_2)} < F$ the fixed point subgroup of the automorphism $\theta(1_2)$. Then there is a decomposition*

$$F = F^{\theta(1_2)} * (*_{i \in I} F_i) * (*_{\lambda \in \Lambda} F_\lambda)$$

into the free product, where each factor is $\theta(1_2)$-invariant and:

(1) *for each $i \in I$, $F_i = \langle x_{i,1}, x_{i,2} \rangle$ such that*

$$\theta(1_2)(x_{i,1}) = x_{i,2} \ \text{ and } \ \theta(1_2)(x_{i,2}) = x_{i,1};$$

(2) *for each $\lambda \in \Lambda$, there is a set J_λ with $F_\lambda = \langle x_\lambda, y_j \mid j \in J_\lambda \rangle$ such that*

$$\theta(1_2)(x_\lambda) = x_\lambda^{-1} \text{ and}$$

$$\theta(1_2)(y_j) = x_\lambda^{-1} y_j x_\lambda \text{ for } j \in J_\lambda \text{ and } \lambda \in \Lambda.$$

Based on Lemma 4.1 and Theorem 4.3, we can provide a criterion to decide whether a pair (θ, φ) is realizable or not. For this purpose the following is useful.

A well-known representation of $\mathrm{Aut}\,(F_m)$ is given by

$$\rho_m : \mathrm{Aut}\,(F_m) \to \mathrm{Aut}\,(F_m/F'_m) \cong GL_m(\mathbb{Z}),$$

where F'_m is the commutator subgroup of F_m, $GL_m(\mathbb{Z})$ the group of all invertible $m \times m$-matrices over $\mathbb{Z}$ and $\rho_m(\theta)$ is the automorphism of the free abelian group $F_m/F'_m \cong \mathbb{Z}^m$ induced by $\theta \in \mathrm{Aut}\,(F_m)$.

Write I_m for the identity $m \times m$-matrix and define $m \times m$-matrices:

$$A(k, r, s) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \mathbf{0} & & & & \\ & & \ddots & & & \\ & & \mathbf{0} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \mathbf{0} & \\ & & & \mathbf{0} & I_r & \mathbf{0} \\ & & & & \mathbf{0} & -I_s \end{pmatrix}$$

over integers which satisfy $A(k, r, s)^2 = I_m$ with k matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $m = 2k + r + s$.

Given $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(F_m)$ with $F_m = \langle x_1, \ldots, x_m \rangle$, by $\rho_m(\theta(1_2))$, we denote the matrix of the automorphism of the abelianization $F_m^{ab} \cong \mathbb{Z}^m$ with respect to the basis $\{\bar{x}_1, \ldots, \bar{x}_m\}$, where $\bar{x}_i$ is the projection of x_i onto $F_m^{ab} \cong \mathbb{Z}^m$ for $i = 1, \ldots, m$. Because $\theta^2(1_2) = \mathrm{id}_{F_m}$ we have that $\rho_m(\theta(1_2))^2 = I_m$.

Notice that for any automorphism $\Theta : F_m \to F_m$ with $\Theta^2 = \mathrm{id}_{F_m}$, applying Theorem 4.3, we can always assume that $F = \langle y_1, \ldots, y_m \rangle$ and $\rho_m(\theta) = A(k, r, s)$ with $m = 2k + r + s$.

Given $g \in F_m = \langle x_1, \ldots, x_m \rangle$, write $|g|_{x_i}$ for its x_i-*exponent*, i.e., $|g|_{x_i} = \sum_{k=1}^{t} r_k$ provided $x_i^{r_k}$ appears in g for $i = 1, \ldots, m$, $r_k \neq 0$ and $k = 1, \ldots, t$, and zero otherwise. Now, we are in a position to state the following.

Theorem 4.4. *Let $F_m = \langle x_1, \ldots, x_m \rangle$ be a free group with $m \geq 1$, $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(F_m)$ and $\varphi : F_m \rtimes_\theta \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ be homomorphisms such that $\rho_m(\theta(1_2)) = A(k, r, s)$ with $m = 2k + r + s$ and $\varphi|_{\mathbb{Z}_2} = \mathrm{id}_{\mathbb{Z}_2}$. Then the pair (θ, φ) is realizable if and only if $\varphi(x_l, 0) = 0$ for $l = 2k + r + 1, \ldots, 2k + r + s$.*

Proof. Given $g \in G$, write $\bar{g} \in F_m^{ab}$ for its image by the quotient map $F_m \to F_m^{ab} \cong \mathbb{Z}^m$. If $A(k, r, s)\bar{g} = -\bar{g}$ then we get:

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \mathbf{0} & & & \\ & \ddots & & & \\ & \mathbf{0} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \mathbf{0} & \\ & & \mathbf{0} & I_r & \mathbf{0} \\ & & & \mathbf{0} & -I_s \end{pmatrix} \begin{pmatrix} |g|_{x_1} \\ \vdots \\ |g|_{x_{2k}} \\ \vdots \\ |g|_{x_{2k+r}} \\ \vdots \\ |g|_{x_{2k+r+s}} \end{pmatrix} = \begin{pmatrix} -|g|_{x_1} \\ \vdots \\ -|g|_{x_{2k}} \\ \vdots \\ -|g|_{x_{2k+r}} \\ \vdots \\ -|g|_{x_{2k+r+s}} \end{pmatrix}.$$

Consequently, $\begin{cases} |g|_{x_1} = -|g|_{x_2}, \\ \vdots \\ |g|_{x_{2k-1}} = -|g|_{x_{2k}} \end{cases}$ and $\begin{cases} |g|_{x_{2k+1}} = 0, \\ \vdots \\ |g|_{x_{2k+r}} = 0. \end{cases}$

Then, we derive that

$$\bar{g} = \sum_{i=1}^{k} |g|_{x_i} (\bar{x}_{2i-1} - \bar{x}_{2i}) + \sum_{j=1}^{s} |g|_{x_{2k+r+j}} \bar{x}_{2k+r+j}. \tag{4.1}$$

Because $\varphi(x_i, 0) = \varphi(\theta(1_2)(x_i), 0) = \sum_{l=1}^{2n+r} |\theta(1_2)(x_i)|_{x_l} \varphi(x_l, 0)$ for $i = 1, \ldots, 2k$, we get that

$$\varphi(x_i, 0) = \begin{cases} \varphi(x_{i+1}, 0) & \text{if } i \text{ is odd} \\ \varphi(x_{i-1}, 0) & \text{if } i \text{ is even.} \end{cases}$$

Now, we show one implication. Suppose that $\varphi(x_l, 0) = 0$ for $l = 2k + r + 1, \ldots, 2k + r + s$ and assume that $\theta(1_2)(g) = g^{-1}$ for some $g \in F_m$. Then, $A(k, r, s)\bar{g} = -\bar{g}$ and by means of the above, we get $\varphi(g, 0) = \sum_{i=1}^{2k+r+s} |g|_{x_i} \varphi(x_i, 0) = \sum_{i=1}^{2k} |g|_{x_i} \varphi(x_i, 0) = |g|_{x_1} \varphi(x_1, 0) - |g|_{x_1} \varphi(x_1, 0) + \cdots + |g|_{x_{2k-1}} \varphi(x_{2k-1}, 0) - |g|_{x_{2k-1}} \varphi(x_{2k-1}, 0) = 0$. Therefore, the system given by Lemma 4.1 has no solution and the result follows.

To show the converse suppose that the pair (θ, φ) is realizable. Let

$$\{x'_1, x'_2, \ldots, x'_{2k'}, x'_{2k'+1}, \ldots, x'_{2k'+s'}, x'_{2k'+s'+1}, \ldots, x'_{2k'+s'+r'}\}$$

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be a basis of F_m given by Theorem 4.3, where certainly $2k'+r'+s'=m$. The matrix $A'(k',r',s')$ of the induced automorphism $\rho_m(\theta(1_2)): F_m^{ab} \to F_m^{ab}$, with respect to the associated basis of F_m^{ab}, satisfies $A(k,r,s) = B^{-1}A'(k',r',s')B$ for some matrix $B \in GL_m(\mathbb{Z})$. This implies that $k=k'$, $r=r'$ and $s=s'$. Because $\theta(1_2)(x_l') = x_l'^{-1}$ and $A(k,r,s) = B^{-1}A'(k',r',s')B$, we get that $A(k,r,s)\bar{x}_l = -\bar{x}_l$ for $l = 2k+r+1,\dots,2k+r+s$. From (4.1) we get

$$\bar{x}_l = \sum_{i=1}^{k} |x_l|_{x_i'}(\bar{x}'_{2i-1} - \bar{x}'_{2i}) + \sum_{j=1}^{s} |x_l|_{x'_{2k+r+j}} \bar{x}'_{2k+r+j}$$

for $l = 2k+r+1,\dots,2k+r+s$. But, the pair (θ,φ) is realizable, so $\theta(1_2)(x_l') = x_l'^{-1}$, in view of Lemma 4.1, leads to $\varphi(x_l',0) = 0$ for $l = 2k+r+1,\dots,2k+r+s$. Consequently, $\varphi(x'_{2i-1},0) = \varphi(x'_{2i},0)$ for $i = 1,\dots,k$ and the above imply $\varphi(x_l,0) = 0$ for $l = 2k+r+1,\dots,2k+r+s$, and the proof is complete. □

Remark 4.5. Since Theorem 4.3 holds for any free group F, it is not difficult to show that Theorem 4.4, with obvious suitable changes, holds for any such a group as well.

5. Miscellanea

In this section we discuss actions of groups G on $\Sigma(n)$, where G has infinite virtual cohomological dimension, or closely related to this case.

In view of Petrosyan [**16**], a discrete group G has *jump cohomology* over a ring R if there exists an integer $k \geq 0$ such that if $H \leq G$ is any subgroup of G with $\mathrm{cd}_R H < \infty$ then $\mathrm{cd}_R H \leq k$. The bound k is called the *jump height* over R. If $R = \mathbb{Z}$ then it is said that G has jump cohomology and a jump height k. That is related with a problem if the group G acts freely on a certain finite dimensional CW-complex. By the naturality of the cup product it follows that if a group G has periodic cohomology in the sense of [**2**] over R, then G has jump cohomology over R. More precisely, [**16**, Lemma 2.7] states: *If a group G has periodic cohomology over R starting in dimension $k+1$, then G has a jump cohomology of height k over R.* Consequently, jump cohomology is a weaker condition than periodic cohomology in the sense of [**2**].

We recall that by [**15**, Corollary 5.6], the Thompson group

$$F = \left\langle x_0, x_1, \dots \,|\, x_i x_j x_i^{-1} = x_{j+1},\ i<j \right\rangle$$

with $\mathrm{cd}\, F = \mathrm{vcd}\, F = \infty$ does not act freely and properly discontinuously on any $\mathbb{R}^m \times \mathbb{S}^n$. In fact, in view of [**16**, Example 3.8], it is also true that F does not act freely and properly discontinuously on any homotopy sphere $\Sigma(n)$.

By means of [**4**], this countable group F has periodic cohomology in the sense that $H^k(F,\mathbb{Z}) \cong H^{k+2}(F,\mathbb{Z})$ for all $k > 1$. Nevertheless, in view of [**16**, Lemma 2.7 and Example 3.8], the group F cannot have periodic cohomology in the sense defined by Talleli [**26**] neither jump cohomology over any ring, since it has an infinite rank abelian subgroup $\mathbb{Z}^\infty \cong \left\langle x_0x_1^{-1}, x_2x_3^{-1}, \dots \right\rangle \leq F$. The fact that a group not having jump cohomology cannot act on $\Sigma(n)$ it follows from a more general result, namely by [**17**, Theorem 4.2]: ‘a group not having jump cohomology implies that it cannot act freely and properly

https://doi.org/10.1017/S0013091517000207 Published online by Cambridge University Press

discontinuously on a finite dimensional CW-complex whose first top non-trivial integral homology is finitely generated and infinite'.

Prassidis has shown in the paragraph after [**18**, Theorem 10] the following.

Theorem 5.1. *There exist discrete groups G with $\operatorname{vcd} G = \infty$ which act freely and properly on some $\mathbb{R}^m \times \mathbb{S}^n$.*

The action given in [**18**] is free, properly discontinuous but not co-compact. Then Farrell and Stark [**8**, Theorem 1] showed the following.

Theorem 5.2. *For each $m \geq 2$ and $n \geq m(m+1)$, there are smooth closed manifolds with universal covering spaces $\mathbb{R}^m \times \mathbb{S}^{2n-1}$ and fundamental group of infinite virtual cohomological dimension.*

Groups from the results above are torsion with $\operatorname{vcd} G = \infty$ and, in view of Proposition 2.2, they cannot act on any $\Sigma(2n)$, in particular on any $\mathbb{R}^m \times \mathbb{S}^{2n}$. But it is natural to ask: can a torsion-free group G with $\operatorname{cd} G = \infty$ act (possibly co-compactly), freely and properly discontinuously on some $\mathbb{R}^m \times \mathbb{S}^n$? By a private communication with F.X. Connolly and S. Prasidis this question is not settled. Further, the proposed question above is related to a conjecture by O. Talelli which will be described after Question 5.3 below.

Several of the questions and results above can be studied if we restrict ourselves to the family of homotopy spheres $\Sigma(2n)$. Taking into account [**13**, Theorem 5.2], we close this paper with the following.

Question 5.3. Suppose that a group G acts, freely and properly discontinuously (possibly co-compactly) on some $\Sigma(2n)$ with $\dim \Sigma(2n) \leq m + 2n$. Does it follow that $\operatorname{vcd} G \leq m$?

At the end of [**27**] Talelli states: '*if G has periodic cohomology after some steps and is torsion-free, then G has finite cohomological dimension*, which we expect to be true'. Next, this has been transformed by Talelli in [**28**, Conjecture III, p. 304] into a conjecture: 'If G has periodic cohomology after some steps and G is torsion-free then $\operatorname{cd} G < \infty$'.

Therefore, in view of Proposition 2.2(2), Question 5.3 would follow affirmatively from Talelli's conjecture above. Also notice that Proposition 3.4 yields $\operatorname{vcd} G \leq 1$ for any virtually cyclic group G acting on some $\Sigma(2n)$. Further, by Corollary 2.4 and [**16**, Theorem 4.2] the answer to Question 5.3 is affirmative for $m = 0, 1$.

Regarding other aspects of group actions on homotopy spheres, a very good survey can be found in [**10**], where several questions are posed and discussed.

Acknowledgements. We are indebted to F.X. Connolly and S. Prasidis for fruitful discussions on many aspects of this paper, in particular, on the current status of the question stated below Theorem 5.2.

The first and second authors gratefully acknowledge support from Instituto de Matemáticas, UNAM, Oaxaca Branch, where the main part of this paper has been discussed. The authors were partially supported by CONACyT Grant 98697 and the third author was also supported by CONACyT Grant 151338.

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