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Invariant Measures and Natural Extensions

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Abstract. We study ergodic properties of a family of interval maps that are given as the fractional parts of certain real Möbius transformations. Included are the maps that are exactly n-to-1, the classical Gauss map and the Renyi or backward continued fraction map. A new approach is presented for deriving explicit realizations of natural automorphic extensions and their invariant measures.

1 Introduction

The focus of this paper is the family of all piecewise Möbius maps of the unit interval that are finite on $(0, 1)$ and are of the form $T(x) = \langle B(x)\rangle$ where B is a Möbius transformation for which 0 and 1 take values in $\mathbb{Z} \cup \{\infty\}$. Here $\langle\alpha\rangle$ denotes the fractional part of the number α. The restriction to a maximal open subinterval on which the map is continuous is always a homeomorphism onto $(0, 1)$.

The work is made easier by carefully normalizing the transformations. Consider the family of Möbius transformations

$$A_{m,k}(x) = \frac{-mkx}{(m-1)x + 1 - k - m}.$$

For values $m, k \neq 0$ and $k \neq m - 1$, $A_{m,k}$ takes 0 to 0 and 1 to m. The picture is completed by considering three additional one-parameter families of mappings whose behaviors resemble the $A_{m,k}$ in an essential way. Define $A_{\infty,k}(x) = \frac{kx}{1-x}$, $A_{-\infty,k}(x) = \frac{k(1-x)}{x}$ and $A_{m,\infty}(x) = mx$. In general we shall take m to be $\pm\infty$ or an integer $|m| > 1$. The reason for this choice will shortly become clear.

As usual, $[\alpha]$ shall denote the greatest integer less than or equal to α and $\langle\alpha\rangle = \alpha - [\alpha]$ the fractional part of α. Then for $x \in [0, 1]$ define

$$T_{m,k}(x) = \langle A_{m,k}(x)\rangle$$

where we modify this rule by assigning $T_{m,k}(1) = 1$ when $2 \leq m < \infty$, $T_{m,k}(0) = 1$ when $-\infty < m \leq -2$.

If k lies within an appropriate parameter space, it is possible to classify the dynamic and ergodic properties of the map $T_{m,k}$ and, where meaningful, to construct an explicit realization of its natural automorphic extension [3]. The parameter space L_m for a given m is chosen so that $(0, 1)$ maps onto the interval with endpoints 0 and m.

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This is exactly the set of values k for which $A_{m,k}$ is non-constant and $A_{m,k}^{-1}(\infty) \notin (0, 1)$. Let $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. Then we have

$$L_m = \begin{cases} \hat{\mathbb{R}} \setminus [1-m, 0] & \text{if } m \geq 2 \\ \hat{\mathbb{R}} \setminus [0, 1-m] & \text{if } m \leq -2 \\ \mathbb{R} \setminus \{0\} & \text{if } m = \pm\infty. \end{cases}$$

It is easily seen that for a piecewise Möbius map $T(x) = \langle B(x) \rangle$ of the type we are interested in there is precisely one pair of values m, k so that $T(x) = T_{m,k}(x)$ for all $x \in (0, 1)$. In particular, $B = A_{m,k} + u$ for some unique $m, u \in \mathbb{Z}$ and $k \in L_m$.

For certain parameter values the map $T_{m,k}$ possesses a natural automorphic extension $\tilde{T}_{m,k}$ which is defined on a rectangle in the plane. Each $\tilde{T}_{m,k}$ has a unique invariant probability measure, absolutely continuous with respect to Lebesgue measure, with a density of the form $\rho(x, y) = \frac{C}{(x-y)^2}$ for some normalizing constant C, depending on m and k. The construction that produces these densities is motivated by a technique of Adler and Flatto [1], later simplified by Series [16]. In their approach the classical Gauss measure $\frac{1}{\log 2} \frac{dx}{1+x}$ is derived by viewing the Gauss transformation $\langle 1/x \rangle$ as a factor of a section of the geodesic flow on the unit tangent bundle of the Modular surface. These methods were generalized to other transformations and surfaces in [5] and provided a vague inspiration for the results in [6] and [7].

By applying theorems of Rychlik [15] and Thaler [17] it is shown that on a large subset of the parameter space the extension is Bernoulli; while for certain limiting cases it is a K-automorphism with infinite invariant measure. We also show that for the particular family with $m = \infty$ and $k > 0$ every possible finite entropy occurs exactly once. Then by a famous theorem of Ornstein [9] the family provides a model for all finite entropy Bernoulli automorphisms. Finally, by more direct methods, it is possible to exhibit explicit absolutely continuous invariant measures (possibly infinite or signed) for all of the interval maps $T_{m,k}$ considered.

The ergodic properties of this family of mappings was studied by Rudolfer [11] and in that paper he derived the invariant measures for the family with $m = \infty$ and $k > 0$. The question of their Bernoulliness was addressed by Rudolfer and Wilkinson [14] for positive m. Another important approach for constructing an explicit realization of the natural extension for certain interval maps was given by Nakada [8] and has found wide application. See for example [2].

As was evident in the definition of L_m, there are several cases to consider. The first important distinction is made with regard to whether $A_{m,k}$ is an increasing or a decreasing function on $(0, 1)$. When $k \in L_m$ and $A_{k,m}$ is decreasing the iterate $T^2 = T \circ T$ is an expanding map of the interval, that is, for some $\lambda > 1$, $|(T^2)'(x)| > \lambda$ for all $x \in [0, 1]$ at which the derivative (possibly from the left or the right) makes sense. This is proved in Lemma 1. For such maps the natural extension is shown to be a Bernoulli automorphism and the invariant measure is computed by analogy to the invariant measure for the geodesic flow on the hyperbolic plane. The classic Gauss map $A_{-\infty,1} = \langle \frac{1}{x} \rangle$ [3] belongs to this category.

More generally, a map for which some iterate has derivative of modulus bounded below by a constant larger than one is called *expanding*. Define E_m as the set of k for which $T_{m,k}$ is an expanding map.

When A is increasing the dynamical behavior of T shows more variability. The subset of $k \in L_m$ for which $A_{m,k}$ is increasing is partitioned into disjoint sets C_m, E_m^* and P_m. C_m is an open set of values k for which the map T has either 0 or 1 as an attracting fixed point. The fixed point has a large basin of attraction and, while the map is neither ergodic nor conservative, there does always exist an infinite T-invariant signed measure which is finite away from the repelling fixed point. $E_m^* = E_m$ is the set of expanding maps when m is finite. When $m = \pm\infty$, E_m^* is the set of $k \in E_m$ for which $A_{m,k}$ is increasing. It is important to keep in mind the distinction between E_m^* and E_m for infinite m. As in the case when A is decreasing, if $k \in E_m^*$ then T naturally extends to a Bernoulli automorphism with an explicitly computable invariant measure. Finally, for $k \in P_m$ either 0 or 1 is an indifferent(derivative one) fixed point of T and the map is expanding on the rest of the unit interval. The natural extension of T possesses an infinite invariant measure and is a K-automorphism. The backward continued fraction or Renyi map $A_{\infty,1} = \langle \frac{1}{1-x} \rangle$ [1] [5] belongs to this last category.

2 Basic Properties

For all $x \in \mathbb{R} \setminus A^{-1}(\infty)$ the sign of the derivative $A'_{m,k}(x)$ is in agreement with the sign of $-mk(1-k-m)$, k or $-k$ for m respectively finite, $+\infty$ or $-\infty$. Consequently, $A'(x) < 0$ on $(0,1)$ for precisely the following parameter values: $-\infty < m \leq -2$ and $k \in L_m$, $m = \infty$ and $k < 0$ or $m = -\infty$ and $k > 0$. The first lemma treats the case of decreasing T. Note that since $T'(x) = A'(x)$, the one-sided derivatives at the discontinuities of T will always agree. Thus it makes sense to consider the derivative $T'(x)$ at all $x \in [0,1]$ except where one of the points 0 or 1 maps to ∞.

Lemma 1 *Suppose m, k are as above with $A'(x) < 0$ on $(0,1)$. Then there is a number $\lambda_{m,k} > 1$ so that $|(T^2_{m,k})'(x)| > \lambda_{m,k}$ for all $x \in [0,1]$ at which the derivative is defined. As a consequence $k \in E_m$.*

Proof We argue when $-\infty < m \leq -2$. The two other cases are similar. Let $I_l = \big(A^{-1}(l+1), A^{-1}(l)\big]$ where $-l = 1, \dots, |m|$. I_l is the subinterval of $(0,1]$ on which $A(x)$ has integer part l. Also, if $x \in I_l$ then $T^2(x) = \big\langle A\big(A(x) - l\big)\big\rangle$. On I_l define $B_l(x) = A\big(A(x) - l\big)$. Then $(T^2)'(x) = B_l'(x) = A'\big(A(x) - l\big)A'(x) > 0$ for $x \in I_l$. B_l is an increasing Möbius transformation on I_l and, since the second derivative changes sign at $B^{-1}(\infty) \notin [0,1]$, the second derivative $B_l''(x)$ is either positive or negative on all of I_l. Therefore the maximum and minimum values of the derivative B_l' are attained at the endpoints of $\overline{I_l}$. The lemma is proved by analyzing B_l' at the endpoints.

First consider the value of the derivative at the right endpoint of I_l, that is, $B_l'\big(A^{-1}(l)\big) = A'(0)A'\big(A^{-1}(l)\big)$. Suppose $k > 0$. Since $A''(0) = \frac{2mk(m-1)}{(1-k-m)^2} > 0$, A' is negative and increasing. Therefore $\big|A'\big(A^{-1}(l)\big)\big| \geq |A'(1)|$. It follows that $B_l'(A^{-1}(l) \geq A'(0)A'(1) = m^2 \geq 4$. If $k < 0$ then $A''(x) < 0$ on $[0,1]$ and $\big|A'\big(A^{-1}(l)\big)\big| \geq \big|A'\big(A^{-1}(-1)\big)\big|$. As a result

$$B_l'\big(A^{-1}(l)\big) \geq A'(0)A'\big(A^{-1}(-1)\big) = \left(\frac{m-1-mk}{1-k-m}\right)^2 > 1.$$

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At the left endpoint of I_l, $B_l'(A^{-1}(l+1) = A'(1)A'\big(A^{-1}(l+1)\big)$. Then for $k > 0$ we have

$$B_l'(A^{-1}(l+1) \geq A'(1)A'\big(A^{-1}(m+1)\big) = \Big(\frac{1-m^2-mk}{k}\Big)^2 > 1$$

and for $k < 0$, $B_l'(A^{-1}(l+1) \geq A'(0)A'(1) = m^2 \geq 4$. ■

The next lemma gathers together some properties of the increasing maps.

Lemma 2 *1. Suppose $2 \leq m < \infty$.*

a) *$A'(x) > 1$ for all $x \in [0,1]$ if and only if $k \in (-\infty, -m) \cup (1, \infty)$. Thus $E_m = (-\infty, -m) \cup (1, \infty)$.*
b) *If $k = 1$ then $A'(0) = 1$ and $A'(x) > 1$ for $x \in (0,1]$ and if $k = -m$ then $A'(1) = 1$ and $A'(x) > 1$ for $x \in [0,1)$. Thus, $P_m = \{1, -m\}$.*
c) *If $0 < k < 1$ then $A'(0) < 1$ and $A'(1) > 1$. If $-m < k < 1 - m$ then $A'(0) > 1$ and $A'(1) < 1$. Thus $C_m = (-m, 1-m) \cup (0,1)$.*

2. Suppose $m = \pm\infty$

a) *$|A'(x)| > 1$ (where $A'(x)$ is defined in $[0,1]$) if and only if $|k| > 1$.*
b) *If $m = \infty$ and $k = 1$ then $A'(0) = 1$ and $A'(x) > 1$ for $x \in (0,1)$. If $m = -\infty$ and $k = -1$ then $A'(1) = 1$ and $A'(x) > 1$ for $x \in (0,1)$.*
c) *When $k \in (0,1)$, $A'(x) < 1$ for $x \in [0, 1-\sqrt{k})$ and $A'(x) > 1$ for $x \in (1-\sqrt{k}, 1]$. Thus $C_\infty = (0,1)$. Similarly, when $k \in (-1,0)$, $A'(x) < 1$ for $x \in (\sqrt{k}, 1]$ and $A'(x) > 1$ for $x \in [0, \sqrt{k})$. Thus $C_{-\infty} = (-1, 0)$.*

The proof is by elementary computation in the various cases and we leave it to the curious reader. One keeps in mind that when m is finite A is concave up or down depending on whether $k > 0$ or $k < 0$. Consequently, for all cases it suffices to check $A'(x)$ at either 0 or 1.

Remark 1 Note that $E_m = L_m$ for $m \leq -2$, $E_\infty = (-\infty, 0) \cup (1, \infty)$, $E_{-\infty} = (-\infty, -1) \cup (0, \infty)$, $P_\infty = \{1\}$ and $P_{-\infty} = \{-1\}$.

3 The Natural Extension

Given an endomorphism T of a measure space, there is a well known construction [3] that gives an automorphism $\tilde{T}$ of a new measure space very naturally related to T and therefore called the *natural automorphic extension* of T. The idea parallels that of extending the shift map on a space of infinite sequences to the shift on an associated space of biinfinite sequences. While the construction is abstract in nature, we shall look at explicit concrete realizations of the natural extension.

The natural extensions of the maps $T_{m,\infty}(x) = mx$ are well understood and are invariant with respect to Lebesgue measure. For example, on $[0,1] \times [0,1]$

$$\tilde{T}_{2,\infty}(x, y) = \big(\langle 2x\rangle, 1/2(y + [2x])\big)$$

Attention will therefore be restricted to the cases $k \neq \infty$.

Throughout this section we suppose that $k \in E_m \cup P_m$. The extension will be constructed for a measure equivalent restriction of $T_{m,k}$ to a subset $I_{m,k}$ of $[0,1]$. For decreasing T more care must be taken. The problems that arise are similar to those with the classical Gauss map $\langle \frac{1}{x} \rangle$ where it is customary to remove the set of rational numbers from the interval.

The set of m,k-rationals is defined as $\mathbb{Q}_{m,k} = \{x \in (0,1) \mid T^n(x) = 0 \text{ for some } n > 0\}$. Then define

$$I_{m,k} = \begin{cases} [0,1) & \text{if } T_{m,k} \text{ is increasing or } m \neq -\infty \\ (0,1) \setminus \mathbb{Q}_{m,k} & \text{if } T_{m,k} \text{ is decreasing or } m = -\infty. \end{cases}$$

The extension of $T_{m,k}$ is defined as a self map of a cartesian product $I_{m,k} \times J_{m,k}$. In the increasing and $m \neq -\infty$ cases set

$$J_{m,k} = \begin{cases} (1-k-m, 1-k] & \text{if } m \geq 2 \\ (-\infty, 1-k] & \text{if } m = \infty, k \leq 1. \end{cases}$$

Again, for decreasing $T_{m,k}$ and $m = -\infty$ the definition of $J_{m,k}$ is more involved. First, define the sets

$$J^*_{m,k} = \begin{cases} (1-k, 1-k-m) & \text{if } m \leq -2 \\ (1-k, \infty) & \text{if } m = \infty, k < 0 \\ (-\infty, -k) & \text{if } m = -\infty, k > 0 \\ (-k, \infty) & \text{if } m = -\infty, k \leq -1. \end{cases}$$

If we let $A_{m,k}$ act on the sets $J^*_{m,k}$ we get

$$K^*_{m,k} = A_{m,k}(J^*_{m,k}) = \begin{cases} (-k, 1-k) & \text{if } m \leq -2 \\ (-k, 1-k) & \text{if } m = \infty, k < 0 \\ (-k-1, -k) & \text{if } m = -\infty, k > 0 \\ (-k-1, -k) & \text{if } m = -\infty, k \leq -1. \end{cases}$$

Let $V_{m,k} = \{v \in \mathbb{Z} \mid K^*_{m,k} - v \subset J^*_{m,k}\}$. The integers in $V_{m,k}$ are exactly the values attained as the integral parts of $A_{m,k}(x)$ as x varies over $(0,1)$. Let α, β denote the endpoints of the interval $K^*_{m,k}$. Set $\Phi_1 = \{\alpha - v, \beta - v \mid v \in V_{m,k}\}$ and inductively define the sets $\Phi_n = \{A_{m,k}(x) - v \mid v \in V_{m,k} \text{ and } x \in \Phi_{n-1}\}$. Then let $\Phi_{m,k} = \bigcup_{n=1}^{\infty} \Phi_n$ and define $J_{m,k} = J^*_{m,k} \setminus \Phi_{m,k}$. Certainly, $J_{m,k}$ differs from the interval $J^*_{m,k}$ by a countable set of points. It also has the property that $A_{m,k}(J_{m,k}) - V_{m,k} = J_{m,k}$, which is important for defining the extension.

For $2 \leq |m| \leq \infty$ and $k \in E_m \cup P_m$ define the map $\tilde{T}_{m,k} \colon I_{m,k} \times J_{m,k} \to I_{m,k} \times J_{m,k}$ by

$$\tilde{T}_{m,k}(x,y) = \big(A_{m,k}(x) - [A_{m,k}(x)], A_{m,k}(y) - [A_{m,k}(x)]\big) \tag{1}$$

As a result of the choices made above, the map is real valued and measurable, and the action on the first coordinate agrees with T on a set of full measure.

Theorem 1 *$\tilde{T}_{m,k}$ is an automorphism of $I_{m,k} \times J_{m,k}$ with invariant density $\rho(x,y) = \frac{1}{(x-y)^2}$.*

Proof We first show that $\tilde{T}_{m,k}$ is bijective for positive integers m. Define $I_l = \big[A^{-1}(l), A^{-1}(l+1)\big)$ and $K_l = (-k-l, 1-k-l]$ for $l = 0, \dots, m-1$. These are partitions of the intervals $[0,1)$ and $(1-k-m, 1-k]$, respectively. I_l is exactly the subinterval of $[0,1)$ on which $[A_{m,k}(x)] = l$. Since A is increasing, $A(1-k) = 1-k$ and $A(1-k-m) = -k$ it follows that the restriction $\tilde{T}_{m,k} \colon I_l \times (1-k-m, 1-k] \to [0,1) \times K_l$ is a bijection. The sets $I_l \times (1-k-m, 1-k]$ are disjoint for different values of l, as are the sets $[0,1) \times K_l$. It follows that $\tilde{T}$ is an automorphism of $[0,1) \times (1-k-m, 1-k]$.

More generally, fix m, k, let $I = I_{m,k}$, $J = J_{m,k}$, $K = K_{m,k} = A_{m,k}(J_{m,k})$ and $V = V_{m,k} = \{v \in \mathbb{Z} \mid K_{m,k} - v \subset J_{m,k}\}$. Define $I_l = \{x \in I \mid [A(x)] = l\}$ and $K_l = K - l$. The restriction $\tilde{T} \colon I_l \times J \to I \times K_l$ is a bijection. $\bigcup_{l \in V} I_l \times J = I \times J$ and $\bigcup_{l \in V} I \times K_l = I \times J$ are both disjoint unions. As above, it follows that $\tilde{T}$ is an automorphism.

Let D be a measurable subset of $I \times J$. Define the measure μ, absolutely continuous with respect to Lebesgue measure on $I \times J$, by

$$\mu(D) = \mu_{m,k}(D) = \int\int_D \frac{dx\,dy}{(x-y)^2}.$$

Note that μ is finite as long as $k \notin P_m$ and for $k \in P_m$ the measure has a singularity at either $(1,1)$ or $(0,0)$ and is finite off a neighborhood of the singular point.

We need to verify that $\mu(D) = \mu(\tilde{T}^{-1}D)$. There is no loss of generality in assuming that D is a measurable subset of $I \times K_l$ for some $l \in V$. $\tilde{T}$ is invertible on $I \times K_l$ and the inverse is given by $\tilde{T}^{-1}(x,y) = \big(A^{-1}(x+l), A^{-1}(y+l)\big)$. The Jacobian derivative of $\tilde{T}^{-1}$ is then simply $(A^{-1})'(x+l)(A^{-1})'(y+l)$.

We make use of the following well know identity that holds for any Möbius transformation C and for $z, w \in \mathbb{C}$ with $C(z), C(w) \neq \infty$ [7]:

$$\left(\frac{C(z) - C(w)}{z - w}\right)^2 = C'(z)C'(w). \tag{2}$$

Applying the identity with $C = A^{-1}$, $z = x + l$ and $w = y + l$ gives

$$\begin{aligned}
\mu(\tilde{T}^{-1}D) &= \int\int_{\tilde{T}^{-1}D} \frac{du\,dv}{(u-v)^2} \\
&= \int\int_D \frac{(A^{-1})'(x+l)(A^{-1})'(y+l)}{\big(A^{-1}(x+l) - A^{-1}(y+l)\big)^2}\,dx\,dy \\
&= \int\int_D \frac{dx\,dy}{\big((x+l) - (y+l)\big)^2} \\
&= \mu(D).
\end{aligned}$$

■

Remark 2 The second part of the proof of Theorem 1 shows the universal invariance of the density function $\rho(x, y) = \frac{1}{(x-y)^2}$ related to the hyperbolic metric. More generally, we have actually shown that for any Möbius transformation A and measurable sets $I, J \subset \mathbb{R}$ that are sufficiently non-overlapping, if the map $T\colon I \times J \to I \times J$ given by definition (1) is bijective then ρ is an invariant density for T.

Remark 3 Nakada's approach [8] can also be applied to give an explicit realization of the natural automorphic extension of the interval map $T_{m,k}$. Such an extension, defined on a subset of $[0, 1]\times[0, 1]$ has the form $\tilde{T}(x, y) = \Big(T(x), A^{-1}\big(y+[A(x)]\big)\Big)$ and induces a natural shift action on the associated continued fraction expansion. This approach is analogous to the example given for the map $\tilde{T}_{2,\infty}$ earlier. When $m = \pm\infty$ it can be shown that the function $\frac{1}{(1-xy)^2}$ is an invariant density for the extension, as expected. For finite m Nakada's realization of the extension has an invariant density that is a rational function depending non-trivially on the parameters m and k.

Corollary 1 *For integers m with $2 \le |m| \le \infty$ and $k \in E_m \cup P_m$ the function*

$$\rho_{m,k}(x) = \begin{cases} \operatorname{sgn}(m)(\frac{1}{x+k-1} - \frac{1}{x+m+k-1}) & \text{if } |m| < \infty \\ \operatorname{sgn}(k)(\frac{1}{x+k-1}) & \text{if } m = \infty \\ \operatorname{sgn}(k)(\frac{1}{x+k}) & \text{if } m = -\infty \end{cases}$$

is an invariant density for $T_{m,k}$.

Proof T is a factor of the mapping $\tilde{T}$ and it is possible to induce a T-invariant measure ν from μ. Let $\varphi\colon I \times J \to I$ be the projection $\varphi(x, y) = x$. Then $\varphi \circ \tilde{T} = T \circ \varphi$ and the measure $\nu(D) = \mu(\varphi^{-1}D)$ is T-invariant. Fix m, k and let (α, β) be the smallest open interval containing $J_{m,k}$. Since $J_{m,k}$ is a subset of (α, β) of full measure

$$\begin{aligned} \nu(D) = \mu(D \times J) &= \int_D \left(\int_{J_{m,k}} \frac{dy}{(x-y)^2} \right) dx \\ &= \int_D \left(\int_\alpha^\beta \frac{dy}{(x-y)^2} \right) dx = \int_D \rho(x)\, dx \end{aligned}$$

Considering the various cases gives the corollary. ∎

4 Ergodic Properties

Theorem 2 *For integers m with $2 \le |m| \le \infty$ and $k \in E_m$ the dynamical system $(\tilde{T}_{m,k}, \mu_{m,k})$ is isomorphic to a Bernoulli shift.*

Proof After Rychlik [15] it will suffice for us to verify that T possesses the following two additional properties: 1) for any open $U \subset (0, 1)$ there is some integer $n > 0$ so that $T^n(U) \supset (0, 1)$, and 2) Var $|\frac{1}{T'(x)}| < \infty$.

Suppose that m, k have been fixed. Using a familiar approach, for example see [3] pg.168, we define open intervals $\Delta_{l_1,\dots,l_n}$ where $l_1, \dots, l_n, \in V_{m,k}$. Let $I = (0, 1)$

and let $\Delta_l = \left(A^{-1}(l), A^{-1}(l+1)\right)$, the interior of the interval I_l defined above. Observe that T maps Δ_l bijectively onto I. Suppose $\Delta_{l_1,\dots,l_n}$ has been defined and T maps $\Delta_{l_1,\dots,l_k}$ bijectively onto $\Delta_{l_2,\dots,l_k}$ for each $k \leq n$. Then, in particular, T^n maps $\Delta_{l_1,\dots,l_n}$ bijectively onto I. By taking the restriction of T^n to the interval $\Delta_{l_1,\dots,l_n}$ define $\Delta_{l_1,\dots,l_{n+1}} = (T^n)^{-1}(\Delta_{l_{n+1}})$. It follows from the definition that for any sequence $l_1, \dots, l_k$, T maps $\Delta_{l_1,\dots,l_k}$ bijectively onto $\Delta_{l_2,\dots,l_k}$. It is also true that for fixed n the union of all intervals of the form $\Delta_{l_1,\dots,l_n}$ is the complement of a finite or a countable nowhere dense set of points in I, depending respectively on whether the parameter value m is finite or infinite.

Since in all cases T^2 is expanding on $(0,1)$, $|(T^2)'(x)| > \lambda > 1$ for some $\lambda > 1$ and for all x so that $T^2(x) \neq \infty$. Thus for n even and $n > 2$, $|(T^n)'(x)| > \lambda^{n/2}$ for all $x \in \Delta_{l_1,\dots,l_n}$. It follows that $|\Delta_{l_1,\dots,l_n}| < \lambda^{-n/2}$. Consequently, for any interval $U \subset I$, there is an interval $\Delta_{l_1,\dots,l_n}$ with $U \supset \Delta_{l_1,\dots,l_n}$. The first property follows.

In general $|\frac{1}{T'(x)}| = |\frac{1}{A'(x)}|$ is a bounded monotone function with a smooth extension to $[0,1]$ and is therefore a function of bounded variation. ■

Theorem 3 *Suppose $2 \leq m < \infty$ or $m = \pm\infty$ and $k \in P_m$. Then the dynamical system $(T_{m,k}, \nu_{m,k})$ is exact and the extension $(\tilde{T}_{m,k}, \mu_{m,k})$ is a K-automorphism.*

Proof For certain values of m we shall modify the intervals Δ_l defined above as follows. If m is finite and $k = -m$ let Δ_{m-1} include the endpoint 1, and if $m = -\infty$ let Δ_{-1} include the endpoint 1. To prove exactness we verify the hypothesis of [17] Theorem 1. First, the union of the intervals Δ_l with $l \in V$ is a set of full measure in $[0,1]$. T is C^∞ on each Δ_l. $\overline{T(\Delta_l)} = [0,1]$. Each Δ_l contains exactly one fixed point and one of 0 or 1 is the unique fixed point x_0 with $T'(x_0) = 1$. $T' = A'$ is strictly monotone on $(0,1)$ and $|T'(x)| = |A'(x)| > 1$ at all $x \neq x_0$. Computing in the various cases one sees that $|T''(x)|T'(x)^{-2}$ is bounded on $[0,1]$. That proves exactness.

The natural extension of an exact endomorphism is a K-automorphism [3]. ■

5 Entropy and a Model for Bernoulli automorphisms

We calculate the entropy for the dynamical system $(\tilde{T}_{\infty,k}, \rho)$, show that it is increasing in k and attains every value. From the work of Ornstein [9] it follows that if T is a finite entropy Bernoulli automorphism of a probability space $(X, \mathcal{A}, \mu)$, then for a unique value $k > 1$ the extension $\tilde{T}_{\infty,k}$ is isomorphic to T.

Proposition 1 *For each h in the interval $(0,\infty)$ there is a unique $k \geq 1$ so that the entropy of the dynamical system $(\tilde{T}_{\infty,k}, \rho)$ is h.*

Proof Since the entropy of the natural extension $\tilde{T}_{\infty,k}$ is equal to the entropy of the endomorphism $T_{\infty,k}$ [12], it will suffice to show that there is a unique $k > 1$ so that the entropy of $T_{\infty,k}$ is h.

By Corollary 1, the invariant density for the $T_{\infty,k}$-invariant absolutely continuous probability measure ν_k has the form

$$\rho_k(x) = \left(\log\frac{k}{k-1}\right)^{-1}\frac{1}{x+k-1}.$$

Rohlin's entropy formula [13, 10] is applicable and the entropy $h_k = h(T_{\infty,k})$ is finite, given by

$$h_k = \int_0^1 \log |T'(x)|\, d\nu_k(x).$$

Computing as in the $u = 1/N$ case from [5]

$$h_k = \log k + 2\mathcal{L}_2(\frac{1}{k})\Big(\log \frac{k}{k-1}\Big)^{-1},$$

where $\mathcal{L}_2(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^2}$ is the Euler dilogarithm. Note that the dilogarithm satisfies the equation

$$\mathcal{L}_2'(x) = -\frac{\log(1-x)}{x} \quad \text{for } |x| < 1.$$

By elementary computation

$$\lim_{k\to\infty} h_k = \infty \quad \text{and} \quad \lim_{k\to 1} h_k = 0.$$

It follows that h_k takes on all values in $(0, \infty)$. To complete the proof we show that h_k is an increasing function.

Consider the derivative

$$\frac{dh_k}{dk} = \frac{1}{k(\log \frac{k}{k-1})^2}\left[\frac{2\mathcal{L}_2(\frac{1}{k})}{k-1} - \Big(\log \frac{k}{k-1}\Big)^2\right].$$

In order to show this is positive for $k > 1$ and therefore that h_k is increasing it will suffice to show that

$$g(k) = 2\mathcal{L}_2\Big(\frac{1}{k}\Big) - (k-1)\Big(\log \frac{k}{k-1}\Big)^2$$

is positive for $k > 1$. Computing limits again, we have

$$\lim_{k\to\infty} g(k) = 0 \quad \text{and} \quad \lim_{k\to 1} g(k) = 2\mathcal{L}_2(1) = \frac{\pi^2}{3} > 0.$$

Since

$$g'(k) = -\Big(\log \frac{k}{k-1}\Big)^2$$

is clearly negative for $k > 1$, g is monotone decreasing for $k > 1$ and we can conclude that $g(k) > 0$ for all $k > 1$ ■

6 The Other Maps

It remains for us to consider the maps $T_{m,k}$ with $k \in C_m$. As was seen in Lemma 2 the derivative of T is larger than 1 for some x and less than 1 for others. When m is finite with $k > 0$ or when $m = \infty$ with $0 < k < 1$, zero is an attracting fixed point of T and $1 - k$ is a repelling fixed point of T. In fact, the interval $[0, 1-k]$ is mapped bijectively onto itself and $\lim_{n\to\infty} T^n(x) = 0$ for all $x \in [0, 1-k)$. When m is finite with $k < 0$, 1 and $m + k - 1$ are respectively attracting and repelling fixed points, T is bijective on $(m + k - 1, 1]$ and all orbits on the interval limit at 1. Finally, when $m = -\infty$ and $-1 < k < 0$, 1 and $-k$ are respectively attracting and repelling fixed points and the above holds.

Observe that each of the densities $\rho_{m,k}$ of Corollary 1 has a unique singularity inside $(0, 1)$ for values $k \in C_m$. Using more direct methods we can reprove and extend Corollary 1 to show that:

Corollary 2 *For integers m with $2 \le |m| \le \infty$ and $k \in L_m$ the function*

$$\rho_{m,k}(x) = \begin{cases} \operatorname{sgn}(m)(\frac{1}{x+k-1} - \frac{1}{x+m+k-1}) & \text{if } |m| < \infty \\ \operatorname{sgn}(k)(\frac{1}{x+k-1}) & \text{if } m = \infty \\ \operatorname{sgn}(k)(\frac{1}{x+k}) & \text{if } m = -\infty \end{cases}$$

is an invariant density for $T_{m,k}$.

Proof In case $2 \le m < \infty$ and $k \in L_m$ we verify that the density ρ is an eigenfunction of eigenvalue 1 of the Perron-Frobenius operator [4]

$$L_T\rho(x) = \sum_{\{y|Ty=x\}} \frac{1}{|T'(y)|}\rho(y).$$

The argument is similar in the other cases.

$$\begin{aligned} &L_T\rho(x) \\ &= \sum_{j=0}^{m-1} (A^{-1})'(x+j)\left(\frac{m}{\big(A^{-1}(x+j)+k-1\big)\big(A^{-1}(x+j)+m+k-1\big)}\right) \\ &= \sum_{j=0}^{m-1} (A^{-1})'(x+j) \\ &\qquad \times \left(\frac{m}{\big[A^{-1}(x+j) - A^{-1}\big(A(1-k)\big)\big]\big[A^{-1}(x+j) - A^{-1}\big(A(1-k-m)\big)\big]}\right). \end{aligned}$$

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Employing the identity (2) and setting $n = k + m$ this can be rewritten as

$$= \sum_{j=0}^{m-1} (A^{-1})'(x+j)$$

$$\times \left(\frac{m}{\big[(A^{-1})'(x+j)(A^{-1})'(A(1-k))\big]^{1/2} \big[(A^{-1})'(x+j)(A^{-1})'\big(A(1-n)\big)\big]^{1/2}} \right)$$

$$\times \left(\frac{1}{[x+j-A(1-k)][x+j-A(1-n)]} \right).$$

$$= \sum_{j=0}^{m-1} \left(\frac{m}{[x+j-A(1-k)][x+j-A(1-n)]} \right)$$

$$\times \left(\frac{1}{\big[(A^{-1})'\big(A(1-k)\big)\,(A^{-1})'\big(A(1-n)\big)\big]^{1/2}} \right).$$

Using the inverse function theorem and identity (2) again the computation continues:

$$= \sum_{j=0}^{m-1} \frac{A(1-k) - A(1-n)}{[x+j-A(1-k)][x+j-A(1-n)]}.$$

Also, since $A(1-k) = 1-k$ and $A(1-n) = -k$ we have

$$L_T\rho(x) = \sum_{j=0}^{m-1} \frac{1}{(x+j-1+k)(x+j+k)}$$

$$= \sum_{j=0}^{m-1} \left(\frac{1}{x+j-1+k} - \frac{1}{x+j+k} \right)$$

$$= \frac{1}{x+k-1} - \frac{1}{x+k+m-1}. \qquad \blacksquare$$

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