Ergod. Th. & Dynam. Sys. (2008), **28**, 229–244
doi: 10.1017/S0143385707000399

The accessibility property of expansive geodesic flows without conjugate points

RAFAEL OSWALDO RUGGIERO

Pontificia Universidade Católica do Rio de Janeiro, PUC-Rio, Dep. de Matemática, Rua Marqués de São Vicente 225, Gávea, Rio de Janeiro, Brazil
(e-mail: rorr@mat.puc-rio.br)

(*Received* 15 *September* 2005 *and accepted in revised form* 17 *April* 2007)

Abstract. Let (M, g) be a compact, smooth Riemannian manifold without conjugate points whose geodesic flow is expansive. We show that the geodesic flow of (M, g) has the accessibility property, namely, given two points θ_1, θ_2 in the unit tangent bundle, there exists a continuous path joining θ_1, θ_2 formed by the union of a finite number of continuous curves, each of which is contained either in a strong stable set or in a strong unstable set of the dynamics. Since expansive geodesic flows of compact surfaces have no conjugate points, the accessibility property holds for every two-dimensional expansive geodesic flow.

0. *Introduction*

The accessibility property of dynamical systems, one of the main subjects of control theory, is closely related to contact structures and, more generally, to non-integrable distributions. The classical definition of accessibility refers to a smooth plane field defined in a differentiable manifold. In control theory, it is said that a point p is accessible from a point q if there exists a continuous path formed by a finite union of differentiable curves each of which is tangent to the distribution. This classical notion of accessibility of points defines an equivalence relation, and it is said that the manifold has the accessibility property with respect to the plane field if there is only one equivalence class. Brin [**2**] introduced a dynamical notion of accessibility, closely related to the classical one, well suited for the study of geodesic flows of negative curvature and partially hyperbolic diffeomorphisms: a point p is accessible from q if there exists a *us-path* connecting p and q. A *us-path*, or *admissible path*, is a continuous curve formed by the union of a finite number of curves each of which is contained in some strong invariant submanifold of the dynamics, either some strong stable or some strong unstable one. Accessibility of points is again an equivalence relation, and the dynamical system has the accessibility property if there is only one equivalence class. Brin's idea was used to study the ergodicity of skew products [**2**] and frame flows of manifolds of negative curvature [**4–6**]. After the work of Grayson, Pugh and Shub [**16**], the idea of stable accessibility turned out to be crucial to the study of stable ergodicity of partially hyperbolic dynamics, one of the main fields of research in dynamics in the last ten years. The interest in this latter dynamical notion of accessibility gave rise to intense research on the subject in the context of stable ergodicity of partially hyperbolic systems. Some of the main results are: the stable accessibility of contact Anosov flows proved by Katok and Kononenko [**13**]; the generalization of this result obtained by Burns *et al* [**7**] assuming that the strong stable and unstable foliations of an Anosov flow are not jointly integrable; results for algebraic systems by Pugh and Shub [**22**]; and stable accessibility in the category of skew products by Shub and Wilkinson [**26**], Burns and Wilkinson [**9**] and others [**12, 19**]. We recommend to look at [**8**] for further references and a good survey about stable ergodicity of partially hyperbolic systems.

In the present paper we deal with expansive geodesic flows in manifolds without conjugate points. Given a C^∞ Riemannian manifold (N, g), a differentiable flow $f_t : N \to N$ without singularities is said to be *expansive* if there exists $\epsilon > 0$ such that the following holds: Let $p \in N$, and suppose that there exist $q \in N$, and a continuous, surjective reparametrization $\rho : R \to R$, with $\rho(0) = 0$, of the orbit of q such that $d(f_t(p), f_{\rho(t)}(q)) \leq \epsilon$ for every $t \in R$; then q belongs to the orbit of p. It is easy to construct examples of expansive geodesic flows which are not Anosov. Expansive geodesic flows in manifolds without conjugate points have invariant, continuous foliations by strong stable and strong unstable sets satisfying a local product structure [**24**] (for the definition and a precise statement see §1). The strong invariant sets in general might not be smooth, they are just C^0, locally rectifiable submanifolds of the unit tangent bundle. So the methods used to study the accessibility of partially hyperbolic systems might not be applied to expansive systems; we cannot assume the existence of continuous, invariant subbundles tangent to the strong invariant sets. Our main result is the following.

THEOREM 1. *Let (M, g) be a compact, smooth Riemannian manifold without conjugate points whose geodesic flow is expansive. Then the geodesic flow has the accessibility property, namely, given two points θ_1, θ_2 in the unit tangent bundle, there exists a continuous path joining θ_1, θ_2 formed by the union of a finite number of continuous curves, each of which is contained either in a strong stable set or in a strong unstable set of the dynamics.*

We would like to point out that, in the case of surfaces, the expansiveness of the geodesic flow already implies the absence of conjugate points, as proved by Paternain [**20**]. So Theorem 1 holds for surfaces whose geodesic flows are expansive, without the need of the no conjugate points assumption. As far as we know, Theorem 1 is the first result about accessibility from the dynamical point of view of a class of systems which include non-partially hyperbolic systems (namely, the time 1 map of expansive geodesic flows which are not Anosov). The proof of Theorem 1 is completely topological, since we are not allowed to assume any C^1 regularity of the strong invariant sets.

The main idea of the proof of Theorem 1 is the following. The local accessibility in the unit tangent bundle is closely related to the fact that the curves formed by a us-path followed by an su-path do not give closed loops. This leads naturally to the consideration of the set of points in the universal covering where a horosphere is simultaneously tangent

https://doi.org/10.1017/S0143385707000399 Published online by Cambridge University Press

to two given horospheres. So to examine the accessibility, a local problem *a priori*, we are led to study the global geometry of the universal covering and the configurations of horospheres which are simultaneously tangent to two given ones. To understand the global picture of horospheres we use the fundamental fact that the universal covering is a Gromov hyperbolic space [**23**]. This line of reasoning allows us to avoid C^1 considerations concerning the strong invariant sets, curiously focusing on the global geometry of the universal covering to get information about a local problem.

1. *Expansive geodesic flows and manifolds without conjugate points*

Throughout the paper, (M, g) will be a C^∞, compact Riemannian manifold without conjugate points, $(\tilde{M}, \tilde{g})$ will be the universal covering of M endowed with the pullback of the metric g by the covering map $\pi : \tilde{M} \to M$, and $(T_1 M, g)$ will be the unit tangent bundle of M endowed with the Sasaki metric induced by g (which we call also by g to simplify the notation). All the geodesics will be parametrized by arc length, and given $\theta = (p, v) \in T_1 M$, the geodesic γ_θ will denote the geodesic such that $\gamma_\theta(0) = p$, $\gamma'_\theta(0) = v$. A very special property of manifolds with no conjugate points is the existence of the so-called *Busemann functions*. Given $\theta = (p, v) \in T_1\tilde{M}$ the *Busemann function* $b^\theta : \tilde{M} \to R$ associated to θ is defined by

$$b^\theta(x) = \lim_{t\to+\infty} (d(x, \gamma_\theta(t)) - t).$$

The level sets of b^θ are the *horospheres* $H_\theta(t)$, where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$. We have that $\gamma_\theta(t)$ intersects each level set of b^θ perpendicularly at only one point in $H_\theta(t)$, and that $b^\theta(H_\theta(t)) = -t$ for every $t \in R$. Next, we list some basic properties of horospheres and Busemann functions that will be needed in the forthcoming sections (see [**11, 21**], for instance, for details).

LEMMA 1.1. *The following hold.*

(1) *The Busemann function b^θ is a C^1 function for every θ.*

(2) *The gradient ∇b^θ has norm equal to one at every point.*

(3) *Every horosphere is a C^{1+K}, embedded submanifold of dimension $n-1$ (C^{1+K} means K-Lipschitz normal vector field), where K is a constant depending on curvature bounds.*

(4) *The orbits of the integral flow of $-\nabla b^\theta$, $\psi^\theta_t : \tilde{M} \to \tilde{M}$, are geodesics which are everywhere perpendicular to the horospheres H_θ. In particular, the geodesic γ_θ is an orbit of this flow and we have that*

$$\psi^\theta_t(H_\theta(s)) = H_\theta(s+t)$$

for every $t, s \in R$.

A geodesic β is *asymptotic* to a geodesic γ in $\tilde{M}$ if there exists a constant $C > 0$ such that $d(\beta(t), \gamma(t)) \leq C$ for every $t \geq 0$. We shall denote by *Busemann asymptotes* of γ_θ the orbits of the flow ψ^θ_t. Busemann asymptotes of γ_θ might not be asymptotic to γ_θ, so the relation between geodesics given by

'$\gamma \mathbf{R} \beta$ if and only if β is a Busemann asymptote of γ'

might not be an equivalence relation. Observe that in all known examples of manifolds without conjugate points (non-positive curvature, no focal points, metrics on surfaces without conjugate points), the relation **R** is an equivalence relation. Lemma 1.1(4) implies that the horospheres $H_\theta(t)$ are equidistant, i.e. given any point $p \in H_\theta(s)$, then the distance $d(p, H_\theta(t))$ is equal to $|t - s|$. The canonical lift in $T_1\tilde{M}$ of $H_\theta(0)$ is the set

$$\tilde{\mathcal{F}}^s(\theta) = \{(p, -\nabla_p b^\theta), \ p \in H_\theta(0)\},$$

and the canonical lift $\mathcal{F}^s(\bar{\pi}(\theta))$ of $H_\theta(0)$ in T_1M is the projection of $\tilde{\mathcal{F}}^s(\theta)$ in T_1M by the natural covering map $\bar{\pi} : T_1\tilde{M} \to T_1M$ induced by the covering map $\pi : \tilde{M} \to M$. The canonical lifts $\mathcal{F}^s(\theta)$, $\tilde{\mathcal{F}}^s(\eta)$ are often called *stable horospheres* of $\theta \in T_1M$, $\eta \in T_1\tilde{M}$ respectively, although the behavior of Busemann asymptotes might be very different from the hyperbolic behavior. The set

$$\tilde{\mathcal{F}}^u((p, v)) = \{(p, \nabla_p b^{(p,-v)}), \ p \in H_{(p,-v)}(0)\}$$

is called the *unstable horosphere* of $\theta = (p, v) \in T_1\tilde{M}$, and its projection $\mathcal{F}^u(\bar{\pi}(\theta))$ in T_1M by the map $\bar{\pi}$ is called the *unstable horosphere* of $\bar{\pi}(\theta) \in T_1M$. The topological dynamics of expansive geodesic flows in compact manifolds without conjugate points is well understood. We give next a survey of results contained in [**20, 23–25**], which show essentially that the topological dynamics of such flows is practically the same as in the case of Anosov geodesic flows.

THEOREM 1.1. [**23, 24**] *The geodesic flow of (M, g) is expansive if and only if for every pair of geodesics γ, β in $(\tilde{M}, g)$ with $d(\gamma, \beta) \leq D$ we have that $\gamma = \beta$. Moreover, two geodesics are Busemann asymptotic in $\tilde{M}$ if and only if they are asymptotic, and two horospheres $H_{(p,v)}(t)$, $H_{(p,-v)}(s)$ have points of tangency if and only if $s = -t$ and the only point of tangency is $\gamma_{(p,v)}(t)$. The sets $\mathcal{F}^s(\theta)$, $\mathcal{F}^u(\theta)$ are the stable and unstable sets of $\theta \in T_1M$ according to the usual notion: if $\psi \in \mathcal{F}^s(\theta)$ then*

$$\lim_{t \to +\infty} d(\phi_t(\theta), \phi_t(\psi)) = 0,$$

and if $\psi \in \mathcal{F}^u(\theta)$ then

$$\lim_{t \to -\infty} d(\phi_t(\theta), \phi_t(\psi)) = 0.$$

THEOREM 1.2. [**23, 24**] *If the geodesic flow of (M, g) is expansive then the fundamental group of M is a Gromov hyperbolic group and the universal covering endowed with the pullback of the metric g is a visibility manifold.*

For the definitions and basic facts of Gromov hyperbolic groups and visibility manifolds we refer to [**14, 15, 17**]. If $\tilde{M}$ is a visibility manifold, geodesic rays diverge uniformly (see [**14, 24**] for a precise definition), and this implies that large spheres approach horospheres uniformly on compact sets in $\tilde{M}$, as well as the continuity of $H_{(p,v)}(0)$ with respect to (p, v) in the compact-open topology. The next result is found in [**24**].

LEMMA 1.2. *Suppose that the geodesic rays in $\tilde{M}$ diverge uniformly. Given a compact ball $B \subset \tilde{M}$, and $\epsilon > 0$, there exist $\delta > 0$ and $R > 0$ such that if $(q, w), (p, v) \in T_1\tilde{M}$, where $p, q \in B$, satisfy $d((p, v), (q, w)) \leq \delta$, then the distance in the C^1 topology between $H_{(q,w)}(0) \cap B$ and the spheres $S_r(\gamma_{(p,v)}(r))$ centered at $\gamma_{(p,v)}(r)$ is less than ϵ for every $r \geq R$ in the set B.*

Given a foliation $\mathcal{F}$ in a manifold N let us call by $\mathcal{F}(p)$ the leaf of $\mathcal{F}$ containing the point $p \in N$. A pair of ϕ_t-invariant foliations F_1, F_2 in $T_1 M$ has a *local product structure* if there exists an atlas $\{\Phi_j : U_j \subset T_1 M \to \mathbb{R}^{2n-1}\}$ of $T_1 M$ such that the following hold.

(1) Every Φ_j is continuous.

(2) Each local chart is of the form $\Phi_j = (x_i, y_i, t)$, $t \in (-\epsilon, \epsilon)$, where $\Sigma = \Phi_j^{-1}\{(x_i, y_i, 0), (x_i, y_i) \in \mathbb{R}^{2n-2}\}$ is a local transversal section of the flow, and the coordinate sets $x^i = c$, $y^i = c$ are respectively the preimage by Φ_j of the connected component of

$$\phi_{|t|<\epsilon}(F_1(\Phi_j(c, 0, 0))) \cap \Sigma,$$

containing the point $\Phi_j(c, 0, 0)$, and the preimage by Φ_j of the connected component of

$$\phi_{|t|<\epsilon}(F_2(\Phi_j(0, c, 0))) \cap \Sigma,$$

containing the point $\Phi_j(0, c, 0)$.

THEOREM 1.3. [**24, 25**] *The geodesic flow of* (M, g) *is expansive if and only if the collections*

$$\mathcal{F}^s = \bigcup_{\theta \in T_1 M} \mathcal{F}^s(\theta),$$
$$\mathcal{F}^u = \bigcup_{\theta \in T_1 M} \mathcal{F}^u(\theta)$$

are continuous foliations, invariant by the geodesic flow, with a local product structure.

THEOREM 1.4. [**20**] *A compact surface whose geodesic flow is expansive has no conjugate points.*

2. *Horospheres in visibility manifolds*

The purpose of this section is to show that certain geometric properties of horospheres in the hyperbolic space generalize in a natural way to horospheres in the universal covering of compact manifolds without conjugate points and expansive geodesic flows. Actually, the main results of the section can be shown in the case of surfaces of genus greater than two using the ideas of Morse [**18**] (see also Eberlein [**14**]). In order to deal with manifolds of dimension $n \geq 3$ we shall profit from the hyperbolicity of the global geometry of $\tilde{M}$ granted by Theorem 1.2.

By Theorem 1.2, if the geodesic flow of (M, g) is expansive the universal covering $\tilde{M}$ endowed with the pullback of the metric g is a visibility manifold, so we begin the section by recalling some basic notions concerning such manifolds (for the details we refer to [**1, 14**]). First of all, the universal covering $\tilde{M}$ admits a compactification $\tilde{M}(\infty)$ whose points are the points of $\tilde{M}$ together with the asymptotic classes of geodesics in $\tilde{M}$. There is a natural topology in $\tilde{M}(\infty)$, called the *cone topology*, such that $\tilde{M}(\infty)$ is homeomorphic to the closed n-ball of radius one in $\mathbb{R}^n$. An asymptotic class ω represents a point in the boundary $\partial\tilde{M}(\infty)$ of the compact space $\tilde{M}(\infty)$; given $p \in T_1\tilde{M}$ there is a unique geodesic ray starting at p whose ω-limit is ω; and each pair of different points ω, α in $\partial\tilde{M}(\infty)$ determines a geodesic $\gamma \in \tilde{M}$ whose α-limit is α and whose ω-limit is ω. The points

$\alpha(\theta)$ and $\omega(\theta)$ will be called the *endpoints* of the geodesic γ_θ at infinity. Notice that, in the case of expansive geodesic flows, each pair of points at infinity determines a unique geodesic. Given $\theta \in T_1\tilde{M}$, the compactification of the horosphere $H_\theta(0)$ is the sphere $H_\theta(0) \cup \{\omega(\theta)\}$, where $\omega(\theta)$ is the asymptotic class of the geodesic γ_θ. We gather some basic features of the global geometry of horospheres in the next lemma; such features will be helpful in the forthcoming sections.

LEMMA 2.1. *Assume that the geodesic flow of* (M, g) *is expansive. Let* $\theta, \eta \in T_1\tilde{M}$ *such that* $\omega(\theta) \neq \omega(\eta)$. *Then the following assertions hold.*

(1) *There exists* $t(\theta\eta) \in \mathbb{R}$ *such that* $H_\theta(0)$ *and* $H_\eta(t(\theta\eta))$ *are tangent at a single point* $p(\theta\eta)$. *Moreover, the sets*

$$H_{\theta\eta}(t) = H_\theta(0) \cap H_\eta(t)$$

are non-empty for every $t \leq t(\theta\eta)$.

(2) *The set* $H_{\theta\eta}(0) = H_\theta(0) \cap H_\eta(0)$ *is a compact, connected set. When this set is not empty, it is either a single point* z *(and* $H_\theta(0)$, $H_\eta(0)$ *must be tangent at* z*), or we have the following cases:*
- *if* $\dim M = 2$, $H_{\theta\eta}(0)$ *consists of two points; and*
- *if* $\dim M > 2$, $H_{\theta\eta}(0)$ *is an* $(n-2)$ *submanifold diffeomorphic to a sphere.*

In any case, the complement of $H_{\theta\eta}(0)$ *in* $H_\theta(0)$ *consists of two* $(n-1)$*-open, connected subsets* A^+_θ, A^-_θ *of* $H_\theta(0)$, *characterized by* $b^\eta(p) > 0$ *for every* $p \in A^+_\theta$, *and* $b^\eta(p) < 0$ *for every* $p \in A^-_\theta$. *Moreover, the set* A^-_θ *has compact closure.*

Proof. Let us begin with the proof of item (1). Since the universal covering $\tilde{M}$ is a visibility manifold, the points at infinity $\omega(\theta)$ and $\omega(\eta)$ determine a geodesic $\gamma_\sigma \subset \tilde{M}$, for some $\sigma = (q, w) \in T_1\tilde{M}$, whose ω-limit is $\omega(\theta)$ and whose α-limit is $\alpha(\eta)$. The horosphere $H_\theta(0)$ coincides with one of the horospheres $H_\sigma(r)$ for some $r \in \mathbb{R}$, and the horosphere $H_\eta(0)$ is one of the unstable horospheres of γ_σ, $H_\eta(0) = H_{(q,-w)}(s)$ for some $s \in \mathbb{R}$. Since the flow is expansive, the geodesic γ_σ is unique by Theorem 1.2. Now, two horospheres $H_{(q,w)}(r)$, $H_{(q,-w)}(s)$ intersect each other if and only if $s \leq -r$. At $s = -r$ the intersection consists of points of tangency, and every point of $H_{(q,w)}(r) \cap H_{(q,-w)}(-r)$ is contained in a geodesic that is bi-asymptotic to γ_σ, according to Theorem 1.2. Hence, the set $H_{(q,w)}(r) \cap H_{(q,-w)}(-r)$ consists of only one point, namely $\gamma_\sigma(r)$. This clearly proves the existence of the number $t(\theta\eta)$ and the point $p(\theta\eta)$. If $s > -r$, the intersection between these horospheres is obviously empty, thus finishing the proof of item (1).

Let us proceed to show item (2). According to Theorem 1.2, the intersections $H_{(q,w)}(r) \cap H_{(q,-w)}(s)$ for $s < -r$ are transversal. So by the implicit function theorem the set $H_{(q,w)}(r) \cap H_{(q,-w)}(s)$ is an $(n-2)$ smooth submanifold of both $H_{(q,w)}(r)$ and $H_{(q,-w)}(s)$. Let us assume without loss of generality that the intersecting horospheres are $H_{(q,w)}(0) \cap H_{(q,-w)}(s)$, where $s < 0$. Let us denote by the *interior* of the horosphere $H_\theta(0)$ the set

$$\{p \in \tilde{M}, b^\theta(p) < 0\}.$$

The horosphere $H_{(q,-w)}(s)$ has some subset A^- in the interior of $H_{(q,w)}(0)$ bounded by the connected components of the intersection $H_{(q,w)}(0) \cap H_{(q,-w)}(s)$. Let

$$H_{(q,-w)}(s) = (H_{(q,w)}(0) \cap H_{(q,-w)}(s)) \cup A^+ \cup A^-$$

be a disjoint union where $b^{(q,w)}(p) < 0$ for every $p \in A^-$, $b^{(q,w)}(p) > 0$ for every $p \in A^+$. Clearly, the closure of the set A^- is compact in $\tilde{M}$. In fact, there exists an open neighborhood V of $\omega((q, -w)) = \alpha((q, w))$ in the cone topology such that

$$V \cap H_{(q,w)}(0) = \emptyset.$$

Since the complement V^c of V in $H_{(q,-w)}(s)$ is a compact subset of $\tilde{M}$, and V^c contains the closure of A^-, we conclude that the latter set is compact as claimed.

To finish the proof of item (2) it remains to describe the topology of the set $H_{\theta,\eta}(t)$. Since the dimension of $H_{\theta,\eta}(t)$ is $n-2$ where n is the dimension of M, in the case of surfaces this set is a finite collection of points. The fact that in dimension two $H_{\theta,\eta}(t)$ consists of either one or two points, and the existence of a diffeomorphism between $H_{\theta,\eta}(t)$ and an $(n-2)$-dimensional sphere, are consequences of basic properties of the Busemann functions. We give a sketch of the argument and leave the details as an exercise. The gradient of the restriction of b^η to $H_\theta(0)$ gives a differentiable, complete flow $\psi_t : H_\theta(0) \to H_\theta(0)$ preserving the sets $H_{\theta\eta}(t)$, i.e. $\psi_s(H_{\theta\eta}(t)) = H_{\theta\eta}(t(s))$ for every $s \in \mathbb{R}$. This flow has a single attracting singularity, namely $p(\theta, \eta)$, so the sets $H_{\theta,\eta}(t)$ are deformation retracts of $H_\theta(0) - p(\theta, \eta)$ for every $t < t(\theta\eta)$. This easily finishes the proof of item (2). □

3. *Horospheres which are simultaneously tangent to two given horospheres*

The purpose of this section is the study of the collection of horospheres which are tangent to two given ones. The picture that we should have in mind is the configuration of such horospheres in the hyperbolic space. From this section we shall assume that (M, g) is a compact Riemannian manifold without conjugate points whose geodesic flow is expansive. Given a horosphere $H_\theta(0)$, we shall denote by the *center at infinity* of $H_\theta(0)$ the ω-limit $\omega(\theta)$ of the geodesic γ_θ. We shall often use the notation $[\omega_1, \omega_2]$ to designate the (unique) geodesic in $\tilde{M}(\infty)$ whose endpoints are $\omega_1 \in \tilde{M}(\infty)$ and $\omega_2 \in \tilde{M}(\infty)$. In the case when $\omega_1, \omega_2 \in \partial\tilde{M}(\infty)$, we shall often employ the notation (ω_1, ω_2) to designate $[\omega_1, \omega_2] \cap \tilde{M}$. Given θ, η in $T_1\tilde{M}$ with $\omega(\theta) \neq \omega(\eta)$, let $\Omega_{\theta\eta}$ be the set of points ω in $\partial\tilde{M}(\infty)$ such that there exists a horosphere H centered at ω that is simultaneously tangent to $H_\theta(0)$ and $H_\eta(0)$. In the hyperbolic space, the set $\Omega_{\theta\eta}$ is diffeomorphic to an $(n-2)$ sphere in $\partial\tilde{M}(\infty)$ that is the limit set of a totally geodesic submanifold of $\tilde{M}$ orthogonal to the geodesic $[\omega(\theta), \omega(\eta)]$. However, without assumptions on the local geometry of M it is not clear that $\Omega_{\theta\eta}$ enjoys such regularity. Our goal in this section is to identify a special submanifold of $\tilde{M}$ whose limit set is just $\Omega_{\theta\eta}$. This natural submanifold might not be a totally geodesic one, but certain features about its geometry provide some mild regularity to $\Omega_{\theta\eta}$. We shall follow the notation of the previous section. The main result of the section is the following.

PROPOSITION 3.1. *Let $\theta, \eta \in T_1\tilde{M}$ such that $\omega(\theta) \neq \omega(\eta)$. Then the following assertions hold.*

(1) *The set of points which are equidistant from $H_\theta(0)$ and $H_\eta(0)$ is a complete, $(n-1)$ smooth submanifold of $\tilde{M}$.*

(2) *The boundary of this set in $\partial\tilde{M}(\infty)$ is closed, non-empty and coincides with $\Omega_{\theta\eta}$.*

(3) *Given* $t \in \mathbb{R}$, *the set* $\Omega_{\theta\eta}$ *is the set of centers of horospheres which are simultaneously tangent to* $H_\theta(t)$ *and* $H_\eta(t)$.

(4) *There exist homeomorphisms (in the cone topology)*

$$\tau_{\theta\eta} : \Omega_{\theta\eta} \to H_\theta(0), \quad \tau_{\eta\theta} : \Omega_{\theta\eta} \to H_\eta(0),$$

such that $\tau_{\theta\eta}(\omega)$ *and* $\tau_{\eta\theta}(\omega)$ *are tangent to a horosphere whose center at infinity is* $\omega \in \Omega_{\theta\eta}$. *Moreover, the sets* $\tau_{\eta\theta}(\Omega_{\theta\eta})$ *and* $\tau_{\theta\eta}(\Omega_{\theta\eta})$ *are compact sets.*

(5) *The geodesics* $(\omega(\theta), \omega)$, *where* $\omega \in \Omega_{\theta\eta}$, *do not intersect the geodesics* $(\omega(\eta), \bar{\omega})$, *where* $\bar{\omega} \in \Omega_{\theta\eta}$. *In particular, the interior of* $\Omega_{\theta\eta}$ *in* $\partial\tilde{M}(\infty)$ *is empty, and if* $\dim M = 2$, $\Omega_{\theta\eta}$ *consists only of two points.*

Proof. We shall prove Proposition 3.1 in several steps. Given $\theta, \eta \in T_1\tilde{M}$ such that $\omega(\theta) \neq \omega(\eta)$, let us define the function

$$f_{\theta\eta} : \tilde{M} \to \mathbb{R},$$
$$f_{\theta\eta}(p) = b^\theta(p) - b^\eta(p).$$

The key idea of the proof of Proposition 3.1 is the relationship between the set $\Omega_{\theta\eta}$ and the level sets of the function $f_{\theta\eta}$.

LEMMA 3.1. *Let* $\theta, \eta \in T_1\tilde{M}$ *such that* $\omega(\theta) \neq \omega(\eta)$. *Then the following assertions hold.*

(1) *The function* $f_{\theta\eta}$ *is differentiable and its gradient* $\nabla f_{\theta\eta}$ *is everywhere non-zero.*

(2) *The level set* $f_{\theta\eta}^{-1}(s)$ *is an* $(n-1)$ *submanifold of class* C^1 *that is foliated by the intersections* $H_\theta(-1/2s + t) \cap H_\eta(1/2s + t)$, *for* $t \in \mathbb{R}$, *with just one singular leaf.*

(3) *The set* $f_{\theta\eta}^{-1}(s)$ *is the set of points which are equidistant to the horospheres* $H_\theta(-1/2s)$ *and* $H_\eta(1/2s)$.

Proof. According to Lemma 2.1, the horospheres $H_\theta(0)$ and $H_\eta(0)$ are the stable and the unstable horospheres respectively of a geodesic γ_σ, with $\sigma = (q, w)$, whose endpoints at infinity are $\omega(\sigma) = \omega(\theta)$ and $\alpha(\sigma) = \omega(\eta)$. Up to reparametrization, we can assume that

$$H_\theta(0) = H_\sigma(r),$$
$$H_\eta(0) = H_{(q,-w)}(r).$$

In this way, we have that $b^\theta(q) = b^\eta(q)$, and that $b^\theta = b^{(q,w)} - r$, $b^\eta = b^{(q,-w)} - r$. So $q \in f_{\theta\eta}^{-1}(0)$, and

$$f_{\theta\eta}(x) = f_{(q,w)(q,-w)}(x),$$

for every $x \in \tilde{M}$. Notice then that $f_{\theta\eta}^{-1}(0) - \{q\}$ is in the complement of the interiors of $H_{(q,w)}(0)$ and $H_{(q,-w)}(0)$, i.e. $b^{(q,w)}(x) > 0$ and $b^{(q,-w)}(x) > 0$ for every $x \in f_{\theta\eta}^{-1}(0) - \{q\}$. Moreover, the set $f_{\theta\eta}^{-1}(0)$ is a C^1 submanifold by the implicit function theorem: the gradient $\nabla_x(b^\theta - b^\eta)$ vanishes if and only if $\nabla_x b^\theta = \nabla_x b^\eta$, which would imply that there exists a geodesic β in $\tilde{M}$ with $\beta(0) = x$ and whose ω-limit at infinity is $\omega(\theta) = \omega(\eta)$, contradicting the assumption. This shows item (1).

Now, since $H_\theta(0) = H_{(q,w)}(r)$, we have that $H_\theta(t) = H_{(q,w)}(r+t)$ and hence $b^{(q,w)}(H_\theta(t)) = -r - t$. Analogously, $b^{(q,-w)}(H_\eta(t)) = -r - t$. Thus we get that

$$\Sigma_{\theta\eta}(t) = H_\theta(t) \cap H_\eta(t) \subset f_{(q,w)(q,-w)}^{-1}(0),$$

for every $t \in \mathbb{R}$. By Lemma 2.1, the sets $\Sigma_{\theta\eta}(t)$ are $(n-2)$-dimensional, C^1 submanifolds for every $t < -r$, and $\Sigma_{\theta\eta}(-r) = q$. These sets constitute a foliation of $f_{\theta\eta}^{-1}(0) - \{q\}$ if $\dim M > 2$ because the horospheres $H_\theta(t)$ foliate $\tilde{M}$ and

$$\Sigma_{\theta\eta}(t) = f_{\theta\eta}^{-1}(0) \cap H_\theta(t).$$

This proves item (2) for $s = 0$. To show item (2) for any s, observe that $\gamma_{(q,w)}(s)$ is the only point in the set $H_{(q,w)}(s) \cap H_{(q,-w)}(-s)$, and hence

$$f_{\theta\eta}(\gamma_{(q,w)}(s)) = f_{(q,w)(q,-w)}(\gamma_{(q,w)}(s)) = -2s.$$

Moreover, $f_{(q,w)(q,-w)}(H_{(q,w)}(s+t) \cap H_{(q,-w)}(-s+t)) = -2s$ as well, which clearly implies item (2).

Item (3) is a straightforward consequence of item (2), because the distance between the horospheres $H_\theta(-1/2s + t)$ and $H_\theta(-1/2s)$ is $|t|$, as well as the distance between the horospheres $H_\eta(-1/2s + t)$ and $H_\eta(-1/2s)$. □

Lemma 3.1 implies item (1) of Proposition 3.1. The above lemma leads to the following result.

LEMMA 3.2. *Let $\theta, \eta \in T_1\tilde{M}$ such that $\omega(\theta) \neq \omega(\eta)$. Then the boundary at infinity (i.e. the ω-limit set) $\partial f_{\theta\eta}^{-1}(0)$ of $f_{\theta\eta}^{-1}(0)$ is the set $\Omega_{\theta\eta}$.*

Proof. Let us first show that the set $\partial f_{\theta\eta}^{-1}(0)$ is contained in $\Omega_{\theta\eta}$. Let $\omega \in \partial f_{\theta\eta}^{-1}(0)$, and let $x_n \in \tilde{M}$ be a sequence of points in $f_{\theta\eta}^{-1}(0)$ converging to ω in the cone topology. By Lemma 3.1, the points x_n are equidistant to $H_\theta(0)$ and $H_\eta(0)$, so if d_n is the distance from x_n to one of these horospheres, we have that the sphere $S_{d_n}(x_n)$ of radius d_n centered at x_n is tangent to both $H_\theta(0)$ and $H_\eta(0)$. Let $[\omega(\theta), x_n]$ be the geodesic whose endpoints are $\omega(\theta)$ and x_n, and let $[\omega(\eta), x_n]$ be the geodesic whose endpoints are $\omega(\eta)$ and x_n. It is clear that

$$a_n = H_\theta(0) \cap S_{d_n}(x_n)$$

coincides with $(\omega(\theta), x_n] \cap H_\theta(0)$, and that

$$b_n = H_\eta(0) \cap S_{d_n}(x_n)$$

coincides with $[\omega(\eta), x_n] \cap H_\eta(0)$. Since the universal covering is a visibility manifold, the sequences $[\omega(\theta), x_n]$ and $[\omega(\eta), x_n]$ converge to the geodesics $[\omega(\theta), \omega]$ and $(\omega(\eta), \omega)]$ respectively, which are uniquely defined by their endpoints due to the expansiveness of the geodesic flow. Hence, we have that

$$\lim_{n\to+\infty} a_n = [\omega(\theta), \omega] \cap H_\theta(0)$$

and

$$\lim_{n\to+\infty} b_n = [\omega(\eta), \omega] \cap H_\eta(0).$$

Hence, all the spheres $S_{d_n}(x_n)$ meet a compact ball in $\tilde{M}$ containing the point $[\omega(\theta), \omega] \cap H_\theta(0)$, and by Lemma 1.4 the sequence $S_{d_n}(x_n)$ converges uniformly in the C^1 topology to the horosphere $H(\omega)$ centered at ω. By the construction, this horosphere is tangent to both $H_\theta(0)$ and $H_\eta(0)$, showing that $\omega \in \Omega_{\theta\eta}$.

The converse of the above statement, namely, $\Omega_{\theta\eta} \subset \partial f_{\theta\eta}^{-1}(0)$, has a similar proof. Let $\omega \in \Omega_{\theta\eta}$, and let $H(\omega)$ be the horosphere centered at ω that is tangent to $H_\theta(0)$ and $H_\eta(0)$. Let us parametrize the geodesic $(\omega(\theta), \omega)$ by $\gamma_\sigma(t)$, where $\sigma = (p, v)$ and $p = H(\omega) \cap H_\theta(0)$. Analogously, let us parametrize $(\omega(\eta), \omega)$ by $\gamma_\rho(t)$, where $\rho = (q, w)$ is such that $q = H(\omega) \cap H_\eta(0)$. The sphere $S_r(\gamma_\sigma(r))$ of radius r with center at $\gamma_\sigma(r)$ is tangent to $H_\theta(0)$ and $H(\omega)$ at p. Since the set $B_r(\gamma_\sigma(r)) - \{p\}$, where $B_r(\gamma_\sigma(r))$ is the (closed) ball of radius r centered at $\gamma_\sigma(r)$, is contained in the interior of $H(\omega)$, we get that

$$d(\gamma_\sigma(r), H_\eta(0)) > r.$$

In the same way, we conclude that

$$d(\gamma_\rho(r), H_\theta(0)) > r.$$

Let $c : [0, 1] \to \tilde{M}$ be the geodesic with endpoints $c(0) = \gamma_\sigma(r)$ and $c(1) = \gamma_\rho(r)$. Then the function $f(t) = d(c(t), H_\theta(0)) - d(c(t), H_\eta(0))$ is a continuous function that changes sign in $[0, 1]$, and therefore there exists $t_r \in (0, 1)$ such that $f(t_r) = 0$. But now, the point $c(t_r)$ is equidistant to $H_\theta(0)$ and $H_\eta(0)$, which implies that $c(t_r) \in f_{\theta\eta}^{-1}(0)$ by Lemma 3.1(3). Finally, observe that $\lim_{r\to\infty} d(\gamma_\sigma(r), \gamma_\rho(r)) = 0$ by the expansiveness of the geodesic flow, and hence the set of points $x_n = c(t_n)$ is a sequence of points in $f_{\theta\eta}^{-1}(0)$ converging to ω in the cone topology. □

Lemmas 3.1(2) and 3.2 prove item (2) in Proposition 3.1. The next result implies item (3) in Proposition 3.1.

COROLLARY 3.1. *In the hypothesis of Proposition 3.1, given $t \in \mathbb{R}$, the set $\Omega_{\theta\eta}$ is the set of centers at infinity of the horospheres which are simultaneously tangent to $H_\theta(t)$ and $H_\eta(t)$.*

Proof. Indeed, each point $\omega \in \Omega_{\theta\eta}$ gives rise to two geodesics: the unique geodesic γ_{θ_1}, where $\theta_1 \in \mathcal{F}^s(\theta)$, with endpoints at infinity $\omega(\gamma_{\theta_1}) = \omega(\theta)$ and $\alpha(\gamma_{\theta_1}) = \omega$; and the unique geodesic γ_{η_1}, where $\eta_1 \in \mathcal{F}^s(\eta)$, whose endpoints at infinity are $\omega(\gamma_{\eta_1}) = \omega(\eta)$ and $\alpha(\gamma_{\eta_1}) = \omega$. Letting $H(\omega)$ be the horosphere centered at ω, we have that $H_\theta(0)$ is the stable horosphere of γ_{θ_1}, and $H(\omega)$ is the unstable horosphere of both γ_{θ_1} and γ_{θ_2}. By the choice of θ_1, η_1, $H_{\theta_1}(0) = H_\theta(0)$ and $H_{\eta_1}(0) = H_\eta(0)$; besides, if $\theta_1 = (q_1, w_1)$, $\eta_1 = (p_1, v_1)$ then $H_{(q_1,-w_1)}(0) = H(\omega)$, $H_{(p_1,-v_1)}(0) = H(\omega)$. But now observe that the horospheres $H_\theta(t) = H_{(q_1,w_1)}(t)$ and $H_{(q_1,-w_1)}(-t)$ are tangent for every $t \in \mathbb{R}$ at the point $\gamma_{\theta_1}(t)$; as well as the horospheres $H_\eta(t) = H_{(p_1,v_1)}(t)$ and $H_{(p_1,-v_1)}(-t)$ are tangent at the point $\gamma_{\eta_1}(t)$. Since $H(\omega) = H_{(q_1,-w_1)}(0) = H_{(p_1,-v_1)}(0)$, we have that $H_{(q_1,-w_1)}(t) = H_{(p_1,-v_1)}(t)$ for every $t \in \mathbb{R}$ and hence the horosphere $H_{(q_1,-w_1)}(t) = H_{(p_1,-v_1)}(t)$ is simultaneously tangent to $H_\theta(t)$ and $H_\eta(t)$. The corollary follows from the fact that the horospheres $H_{(q_1,-w_1)}(t) = H_{(p_1,-v_1)}(t)$ are centered at infinity at the point $\omega \in \Omega_{\theta\eta}$. □

The proof of item (4) in Proposition 3.1 is straightforward from the previous arguments. In fact, given $\omega \in \Omega_{\theta\eta}$, let $\tau_{\theta\eta}(\omega) \in H_\theta(0)$ be the point of intersection between $H_\theta(0)$ and the geodesic $[\omega(\theta), \omega]$. Since the geodesic $[\omega(\theta), \omega]$ is unique, the map

$$\tau_{\theta\eta} : \Omega_{\theta\eta} \to H_\theta(0)$$

https://doi.org/10.1017/S0143385707000399 Published online by Cambridge University Press

is well defined and injective, and it is easy to show that it is continuous in the cone topology. In particular, $\tau_{\theta\eta}(\Omega_{\theta\eta})$ is a compact subset of $H_\theta(0)$ since it is included in the complement of a neighborhood at infinity of $\omega(\theta)$ relative to $H_\theta(0)$. It remains to show item (5) of Proposition 3.1. Let us begin with the following remark.

COROLLARY 3.2. *Let $\theta, \eta \in T_1\tilde{M}$ such that $\omega(\theta) \neq \omega(\eta)$. Then, the set*

$$\Lambda_\theta = \bigcup_{\omega \in \Omega_{\theta\eta}} (\omega(\theta), \omega)$$

is contained in the set $f_{\theta\eta}^{-1}((-\infty, 0))$, and

$$\Lambda_\eta = \bigcup_{\omega \in \Omega_{\theta\eta}} (\omega(\eta), \omega)$$

is contained in $f_{\theta\eta}^{-1}((0, \infty))$. In particular, $\Lambda_\theta \cap \Lambda_\eta = \emptyset$.

Proof. The proof follows straightforwardly from Lemma 3.2. Indeed, since $\partial f_{\theta\eta}^{-1}(0) = \Omega_{\theta\eta}$, we have that the geodesics in Λ_θ are limits of geodesics contained in $f_{\theta\eta}^{-1}((-\infty, 0))$, and the geodesics in Λ_η are limits of geodesics contained in $f_{\theta\eta}^{-1}((0, \infty))$. □

So to get item (5) of Proposition 3.1 it is enough to show that the interior of $\partial f_{\theta\eta}^{-1}(0)$ in $\partial\tilde{M}(\infty)$ is empty in the cone topology. Otherwise, we would get a point $p \in \tilde{M}$ and an open cone of geodesic rays starting at p whose endpoints at infinity contain an open subset V of $\partial f_{\theta\eta}^{-1}(0)$. By the continuity of the endpoints with respect to the initial conditions of geodesics, given $\omega \in V$ there exists an open neighborhood $W \subset \tilde{M}(\infty)$ containing ω such that every $q \in W \cap \tilde{M}$ satisfies the following property: the geodesics $[\omega(\theta), q]$ and $[\omega(\eta), q]$ have their ω-limits in V (recall that a basis of open neighborhoods of ω in the compactification $\tilde{M}(\infty)$ can be obtained by taking the complements of open cones of geodesics starting at $p \in \tilde{M}$ with respect to large spheres centered at p). But this contradicts Corollary 3.2, since two geodesics of the form $[\omega(\theta), \omega_1]$ and $[\omega(\eta), \omega_2]$ where $\omega_i \in \Omega_{\theta\eta}$ cannot meet in $\tilde{M}$. In the case when M is a surface, the set $f_{\theta\eta}^{-1}(0)$ is a curve, which has two endpoints at infinity due to Corollary 3.2; therefore $\Omega_{\theta\eta}$ consists only of two points as claimed.

This completes the proof of Proposition 3.1. □

Combining Corollary 3.2 and Proposition 3.1, we get the following corollary.

COROLLARY 3.3. *Let $\theta, \eta \in T_1\tilde{M}$ such that $\omega(\theta) \neq \omega(\eta)$. Then the sets*

$$\tau_{\theta\eta}(\Omega_{\theta\eta}) \subset H_\theta(0), \qquad \tau_{\eta\theta}(\Omega_{\theta\eta}) \subset H_\eta(0)$$

of points where $H_\theta(0)$ and $H_\eta(0)$ are simultaneously tangent to some horosphere centered at a point in $\Omega_{\theta\eta}$ have empty interior relative to $H_\theta(0)$ and $H_\eta(0)$ respectively. If dim $M = 2$*, then $\tau_{\theta\eta}(\Omega_{\theta\eta})$ consists of only two points, as well as $\tau_{\eta\theta}(\Omega_{\theta\eta})$.*

https://doi.org/10.1017/S0143385707000399 Published online by Cambridge University Press

4. *The accessibility property*

In this section we finish the proof of Theorem 1. The main step toward its proof is to show that, given $\theta \in T_1 M$, there exists an open neighborhood V of θ where every point $\eta \in V$ is accessible from θ. The accessibility in our context means the existence of a continuous path joining θ and η that is contained in a finite union of leaves of the invariant foliations $\mathcal{F}^s$, $\mathcal{F}^u$.

Given $\theta \in T_1 M$, let $\mathcal{F}^s(\theta)$, $\mathcal{F}^u(\theta)$ be respectively the (strong) stable and the (strong) unstable set of $\theta \in T_1 M$. Let $U \subset T_1 M$ be an open product neighborhood of θ, namely, there exists a local continuous chart

$$\Phi : U \subset T_1 M \to (-1, 1)^{n-1} \times (-1, 1)^{n-1} \times (-\epsilon, \epsilon),$$

$\Phi(\eta) = (x^i(\eta), y^i(\eta), t)$, where we can suppose without loss of generality that $\Phi(\theta) = (0, 0, 0)$, which trivializes the central foliations according to §1. The set $\Sigma = \Phi_i^{-1}\{(x_i, y_i, 0), (x_i, y_i) \in \mathbb{R}^{2n-2}\}$ is a local transversal section of the flow, and the level sets $x^i =$ constant, $y^i =$ constant are the connected components of the intersections of the central stable and unstable foliations respectively, with Σ (like in Theorem 1.3). Given $\eta \in U$, let $W^s(\eta)$ be the connected component of $\mathcal{F}^s(\eta) \cap U$ containing η, and let $W^u(\eta)$ be the connected component of $\mathcal{F}^u(\eta) \cap U$ containing η. Let

$$\Gamma^{su}(\theta) = \bigcup_{\eta \in W^s(\theta)} W^u(\eta), \qquad \Gamma^{us}(\theta) = \bigcup_{\eta \in W^u(\theta)} W^s(\eta).$$

The sets $\Gamma^{su}(\theta)$ and $\Gamma^{us}(\theta)$ are continuous $2n - 2$ submanifolds of $T_1 M$, and the sets

$$B^{su} = \bigcup_{|t|<\epsilon} \phi_t(\Gamma^{su}(\theta)), \qquad B^{us} = \bigcup_{|t|<\epsilon} \phi_t(\Gamma^{us}(\theta))$$

are open subsets of $T_1 M$ by Brouwer's invariance of domain theorem (see for instance [**24**]). Let us assume without loss of generality that our ambient manifold is $T_1 \tilde{M}$.

Let us start with some preliminaries concerning the accessibility of points in $\Gamma^{su}(\theta)$ and $\Gamma^{us}(\theta)$. Given a smooth manifold N, the composition $c_2 \circ c_1 : [0, 1] \to N$ of two continuous curves $c_1 : [0, 1] \to N$, $c_2 : [0, 1] \to N$ satisfying $c_1(1) = c_2(0)$ is given by the usual formula: $c_2 \circ c_1(t) = c_1(2t)$ for every $t \in [0, 1/2]$, and $c_2 \circ c_1(t) = c_2(2t - 1)$ for every $t \in [1/2, 1]$. Given an admissible path $c : [0, 1] \to T_1 \tilde{M}$, we say that c is elementary if c is obtained by either a composition of the form $c_s \circ c_u$, or a composition of the form $c_u \circ c_s$, where $c_s \subset W^s(c(0))$, $c_u \subset W^u(c(1))$ are two continuous curves. Let us denote by c_{su} an elementary path obtained as $c_{su} = c_s \circ c_u$, and by c_{us} an elementary path obtained as $c_{us} = c_u \circ c_s$. We say that two elementary paths c_{su}, c_{us} with $c_{su}(0) = c_{us}(0) = \theta$ *commute* if $c_{su}(1) = c_{us}(1)$. Inspired by contact geometry, we look for non-commuting elementary paths if we want to show the accessibility of an open neighborhood from a point. From the definition of the sets $\Gamma^{su}(\theta)$ and $\Gamma^{us}(\theta)$ it is easy to check that the following result holds.

LEMMA 4.1. *If two elementary paths $c_{su} : [0, 1] \to T_1 \tilde{M}$, $c_{us} : [0, 1] \to T_1 \tilde{M}$, with $c_{su}(0) = c_{us}(0) = \theta$, commute, then the point $c_{su}(1) = c_{us}(1) = \eta \in T_1 \tilde{M}$ is an element of $\Gamma^{su}(\theta) \cap \Gamma^{us}(\theta)$.*

The following result is fundamental for the proof of the main theorem.

https://doi.org/10.1017/S0143385707000399 Published online by Cambridge University Press

LEMMA 4.2. *Let $\theta \in T_1\tilde{M}$, and consider $\eta \in W^u(\theta)$, $\eta \neq \theta$. Suppose that there exists a continuous curve $C_\eta : [-\epsilon, \epsilon] \to W^s(\eta)$ such that:*

(1) $C_\eta(0) = \eta = C_\eta \cap \Gamma^{su}(\theta)$*; and*

(2) $C_\eta(t)$ *crosses topologically the set* $\Gamma^{su}(\theta)$*, namely,* $C_\eta(\epsilon) \in \Gamma^{su}(\phi_\epsilon(\theta))$ *and* $C_\eta(-\epsilon) \in \Gamma^{su}(\phi_{-\epsilon}(\theta))$.

Then there exists an open neighborhood $V(\theta)$ of θ in $T_1\tilde{M}$ whose points are accessible from θ by admissible paths.

Proof. We just sketch the proof for the sake of completeness. Since $C_\eta(\epsilon) \in \Gamma^{su}(\phi_\epsilon(\theta))$, $C_\eta(-\epsilon) \in \Gamma^{su}(\phi_{-\epsilon}(\theta))$, and C_η is continuous, the curve C_η must intersect $\Gamma^{su}(\phi_t(\theta))$ for every $t \in (-\epsilon, \epsilon)$ (observe that $\Gamma^{su}(\phi_t(\theta))$ separates the product neighborhood B^{su} defined in §4 for each small t). The set

$$V(\theta) = \bigcup_{|t|<\epsilon} \Gamma^{su}(\phi_t(\theta))$$

is an open neighborhood of θ by the local product structure. The point is that $V(\theta)$ is accessible from θ through admissible paths. Indeed, if $\kappa \in V(\theta)$, then the unstable set $W^u(\kappa)$ is in $V(\theta)$, say $W^u(\kappa) \subset \Gamma^{su}(\phi_{t_0}(\theta))$. Hence, by the local product structure, the sets $W^u(\kappa)$ and $W^u(C_\eta(t_0)) \subset \Gamma^{su}(\phi_{t_0}(\theta))$ intersect the stable set $W^s(\phi_{t_0}(\theta))$. So we can construct a continuous path c_1^u joining κ to a point $\kappa_1 \in W^s(\phi_{t_0}(\theta))$; then a continuous path c_1^s from κ_1 to $\kappa_2 = W^s(\phi_{t_0}(\theta)) \cap W^u(C_\eta(t_0))$; and a continuous path c_2^u from κ_2 to $C_\eta(t_0)$. Let $c_2^s : [0, t_0] \to W^s(\eta)$ be the curve $c_2^s(t) = C_\eta(t_0 - t)$, and let c_3^u be a continuous path in $W^u(\theta)$ joining η and θ. Consider the continuous path joining κ and θ formed by the composition

$$c = c_3^u \circ c_2^s \circ c_2^u \circ c_1^s \circ c_1^u.$$

Since c is admissible and $V(\theta)$ is an open neighborhood of θ, this shows the lemma. □

Let $\tau_{\theta\eta}(\Omega_{\theta\eta}) \in H_\theta(0)$ be the set defined in §3, where $\theta \in T_1\tilde{M}$. When $\omega(\theta) \neq \omega(\eta)$, the set $\tau_{\theta\eta}(\Omega_{\theta\eta})$ is a topological $(n-2)$-dimensional submanifold according to Proposition 3.1.

LEMMA 4.3. *For every $\theta \in T_1\tilde{M}$ and each product neighborhood U of θ we have*

$$\Gamma^{su}(\theta) \cap \Gamma^{us}(\theta) = W^s(\theta) \cup W^u(\theta) \bigcup_{\eta \in W^u(\theta)} \tau_{\eta\theta}(\Omega_{\eta\theta}) \cap U.$$

If (M, g) is a compact surface, then $\Gamma^{su}(\theta) \cap \Gamma^{us}(\theta) = W^s(\theta) \cup W^u(\theta)$.

Proof. We leave the proof to the reader; it is straightforward from the definitions of the sets involved. □

So our goal is to exhibit a continuous family of curves satisfying the hypothesis of Lemma 4.2. In order to do this, we have to look at the points of intersection $W^s(\eta) \cap \Gamma^{su}(\phi_t(\theta))$ for $t \neq 0$. We can assume without loss of generality that the ambient manifold is $T_1\tilde{M}$. The next result improves Proposition 3.1.

LEMMA 4.4. *Let* $\theta, \eta \in T_1\tilde{M}$ *such that* $\omega(\theta) \neq \omega(\eta)$. *Let* $f_{\theta\eta}(p) = b^{\theta}(p) - b^{\eta}(p)$, *and let* $\Omega_{\phi_t(\theta)\eta} = \partial f^{-1}_{\phi_t(\theta)\eta}(0)$. *Then the following all hold:*

(1) $\Omega_{\phi_t(\theta)\eta} = \partial f^{-1}_{\theta\eta}(-t)$, *in particular,* $\Omega_{\phi_t(\theta)\eta} \cap \Omega_{\phi_r(\theta)\eta} = \emptyset$ *if* $t \neq r$;

(2) $\partial\tilde{M}(\infty) = \Omega_{\theta\eta} \cup V^{+} \cup V^{-}$, *where* V^{+} *is an open neighborhood of* $\omega(\theta)$, V^{-} *is an open neighborhood of* $\omega(\eta)$, $V^{+} \cap V^{-} = \emptyset$, *and*

$$V^{+} = \Omega^{+} \cup \{\omega(\theta)\} = \left(\bigcup_{t>0} \Omega_{\phi_t(\theta)\eta}\right) \cup \{\omega(\theta)\},$$

$$V^{-} = \Omega^{-} \cup \{\omega(\eta)\} = \left(\bigcup_{t<0} \Omega_{\phi_t(\theta)\eta}\right) \cup \{\omega(\eta)\};$$

(3) $\bigcup_{t\in\mathbb{R}} \tau_{\eta\phi_t(\theta)}(\Omega_{\eta\phi_t(\theta)}) = H_\eta(0) - \{p(\eta\theta)\}$, *where* $p(\eta\theta) \in H_\eta(0)$ *is the point defined in Lemma 2.1(1);*

(4) *the sets*

$$B^{+} = \bigcup_{t>0} \tau_{\eta\phi_t(\theta)}(\Omega_{\eta\phi_t(\theta)}), \qquad B^{-} = \bigcup_{t<0} \tau_{\eta\phi_t(\theta)}(\Omega_{\eta\phi_t(\theta)})$$

are disjoint, connected open subsets of $H_\eta(0) - \{p(\eta\theta)\}$.

Proof. First of all, observe that $f_{\phi_t(\theta)\eta} = f_{\theta\eta} + t$, which implies that $f^{-1}_{\phi_t(\theta)\eta}(0) = f^{-1}_{\theta\eta}(-t)$. Hence, we can apply Lemmas 3.1 and 3.2 to the sets $f^{-1}_{\phi_t(\theta)\eta}(0)$. In particular, they are C^1, complete, pairwise disjoint submanifolds varying continuously in the C^1 topology uniformly on compact sets. Besides, the sets

$$L^{+} = \bigcup_{t>0} f^{-1}_{\phi_t(\theta)\eta}(0), \qquad L^{-} = \bigcup_{t<0} f^{-1}_{\phi_t(\theta)\eta}(0)$$

are disjoint, open, connected subsets of $\tilde{M}$. It is clear that $\Omega_{\phi_t(\theta)\eta} \cap \Omega_{\phi_r(\theta)\eta} = \emptyset$ if $t \neq r$: a horosphere that is simultaneously tangent to $H_\eta(0)$ and $H_\theta(t)$ cannot be simultaneously tangent to $H_\eta(0)$ and $H_\theta(r)$ if $t < r$ because $H_\theta(r)$ is contained in the interior of $H_\theta(t)$. This yields item (1) in the statement.

To show item (2), let us consider $\omega \in \partial\tilde{M}(\infty)$, with $\omega \neq \omega(\theta)$, $\omega \neq \omega(\eta)$. The geodesic $[\omega(\eta), \omega]$ meets $H_\eta(0)$ at a unique point $q(\omega)$, and there exists a horosphere $H(\omega)$ centered at ω that is tangent to $H_\eta(0)$ at $q(\omega)$. By an analogous reasoning, there exists a horosphere $H_\theta(t_0)$ that is tangent to $H(\omega)$. Therefore, $\omega \in \Omega_{\phi_{t_0}(\theta)\eta}$, and $q(\omega) \in \tau_{\eta\phi_{t_0}(\theta)}(\Omega_{\phi_{t_0}(\theta)\eta})$. Recall that the point $p(\eta\theta)$ defined in Proposition 3.1 is the point of intersection between the geodesic $[\omega(\theta), \omega(\eta)]$ and $H_\eta(0)$, so it is clear that $q(\omega) \neq p(\eta\theta)$ under the assumptions for ω. This shows that

$$\bigcup_{t\in\mathbb{R}} \Omega_{\phi_t(\theta)\eta} = \partial\tilde{M}(\infty) - \{\omega(\theta), \omega(\eta)\}.$$

The set $\Omega_{\theta\eta}$ is closed and connected, and separates $\partial\tilde{M}(\infty)$ into two disjoint open, connected subsets V^{+} and V^{-}. Namely, V^{+} is the union of $\omega(\theta)$ and the set of centers at infinity of horospheres which are tangent to $H_\eta(0)$ and $H_\theta(t)$ for some $t > 0$, and V^{-} is the union of $\omega(\eta)$ and the set of centers at infinity of horospheres which are tangent to both $H_\eta(0)$ and $H_\theta(t)$ for some $t < 0$. So we get

$$V^{+} = \Omega^{+} \cup \{\omega(\theta)\}, \qquad V^{-} = \Omega^{-} \cup \{\omega(\eta)\},$$

thus proving item (2).

Moreover, if $x \in H_\eta(0)$ is not $p(\eta\theta)$ the ω-limit of the geodesic $[\omega(\eta), x]$ is not $\omega(\theta)$, and the above argument shows that there exists $t(x) \in \mathbb{R}$ such that $x \in \tau_{\eta\phi_{t(x)}(\theta)}(\Omega_{\eta\phi_{t(x)}(\theta)})$. This yields

$$\bigcup_{t\in\mathbb{R}} \tau_{\eta\phi_t(\theta)}(\Omega_{\eta\phi_t(\theta)}) = H_\eta(0) - \{p(\eta\theta)\},$$

as claimed in item (3). Combining items (2) and (3) we get item (4). □

Proof of Theorem 1. Let $\theta = (p, v) \in T_1\tilde{M}$, and consider $\eta = (q, w) \in W^u(\theta)$, with $\eta \neq \theta$. We shall construct a curve C_η satisfying the assumptions of Lemma 4.2.

Given a small open ball $B(q)$ containing q there exist $\epsilon > 0$, and a continuous curve $c_\eta : [-\epsilon, \epsilon] \to H_\eta(0)$ satisfying:

(1) $c_\eta(\epsilon) \in \tau_{\eta\phi_\epsilon(\theta)}(\Omega_{\eta\phi_\epsilon(\theta)})$;

(2) $c_\eta(0) = q \in \tau_{\eta\theta}(\Omega_{\eta\theta})$; and

(3) $c_\eta(-\epsilon) \in \tau_{\eta\phi_{-\epsilon}(\theta)}(\Omega_{\eta\phi_{-\epsilon}(\theta)})$.

This curve exists by Lemma 4.4(4) and the fact that the interior of $\tau_{\eta\theta}(\Omega_{\eta\theta})$ is empty in $H_\eta(0)$. Let b^η be the Busemann function of the geodesic γ_η, and define

$$C_\eta(t) = (c_\eta(t), -\nabla_{c_\eta(t)} b^\eta),$$

for $t \in [-\epsilon, \epsilon]$. For a suitably small ball $B(q)$ the curve C_η is a subset of $W^s(\eta)$. Moreover, C_η crosses $\Gamma^{su}(\theta)$ in the sense of Lemma 4.2. Indeed, by the construction of the map $\tau_{\eta\phi_t(\theta)}$, we have that the α-limits $\alpha(C_\eta(\epsilon))$ and $\alpha(C_\eta(-\epsilon))$ satisfy

$$\alpha(C_\eta(\epsilon)) \in \Omega_{\phi_\epsilon(\theta)\eta}, \quad \alpha(C_\eta(-\epsilon)) \in \Omega_{\phi_{-\epsilon}(\theta)\eta}.$$

Moreover, $\alpha(C_\eta(0)) = \alpha(\eta) = \alpha(\theta)$ by the choice of η. The geodesic $\gamma_{C_\eta(\epsilon)}$ is forward asymptotic to $\omega(\eta)$ and backward asymptotic to a point in $\Omega_{\phi_\epsilon(\theta)\eta}$. Thus, its α-limit is the α-limit of a point $\psi \in W^s(\phi_\epsilon(\theta))$ and therefore $\alpha(C_\eta(\epsilon)) \in W^u(\psi) \subset \Gamma^{su}(\phi_\epsilon(\theta))$. The same claim holds on replacing ϵ by $-\epsilon$.

Therefore, the curve C_η satisfies the hypothesis of Lemma 4.2. If $\pi : \tilde{M} \to M$ is the covering map, we conclude that there exists an open neighborhood $V(\pi(\theta))$ of $\pi(\theta)$ in T_1M where every point is accessible from $\pi(\theta)$ by admissible paths. Since the covering map is surjective, given any point $\sigma \in T_1M$ there exists an accessible open neighborhood $V(\sigma)$. By the compactness of T_1M and the local product structure of expansive systems, we get that the geodesic flow satisfies the accessibility property, thus proving Theorem 1.

Acknowledgement. The author was partially supported by CNPq of the Brazilian Government.

References

[1] W. Ballmann, M. Gromov and V. Schroeder. *Manifolds of Non-positive Curvature*. Birkhäuser, Boston, 1985.

[2] M. Brin. Topological transitivity of one class of dynamical systems and flows of frames on manifolds of negative curvature. *Funct. Anal. Appl.* **9** (1975), 9–19.

[3] M. Brin. The topology of group extensions of C systems. *Mat. Zametki* **18** (1975), 453–465.

[4] M. Brin and M. Gromov. On the ergodicity of frame flows. *Invent. Math.* **60** (1980), 1–7.

[5] M. Brin and H. Karcher. Frame flows on manifolds with pinched negative curvature. *Compositio Math.* **52** (1984), 275–297.

[6] M. Brin and Ya. Pesin. Flows of frames on manifolds of negative curvature. *Uspekhi Mat. Nauk.* **28** (1973), 169–170.

[7] K. Burns, C. Pugh and A. Wilkinson. Stable ergodicity and Anosov flows. *Topology* **39** (2000), 149–159.

[8] K. Burns, C. Pugh, M. Shub and A. Wilkinson. Recent results about stable ergodicity. *Smooth Ergodic Theory and Its Applications (Proc. AMS Summer Research Institute, Seattle, WA, 26 July–13 August 1999) (Proceedings of Symposia in Pure Mathematics, 69)*. Eds. A. Katok *et al*. American Mathematical Society, Providence, RI, 2001, pp. 327–366.

[9] K. Burns and A. Wilkinson. Stable ergodicity of skew products. *Ann. Sci. École Norm. Sup.* **32** (1999), 859–889.

[10] H. Busemann. *The Geometry of Geodesics*. Academic Press, New York, 1955.

[11] C. Croke and V. Schroeder. The fundamental group of manifolds without conjugate points. *Comment. Math. Helv.* **61** (1986), 161–175.

[12] D. Dolgopyat and A. Wilkinson. Stability accessibility in C^1 dense. *Geometric Methods in Dynamics (II). Volume in Honor of J. Palis*. Eds. W. de Melo, M. Viana and J.-C. Yoccoz. *Astérisque* **287** (2003), 33–60. American Mathematical Society, Providence, RI.

[13] A. Katok and A. Kononenko. Cocycles' stability for partially hyperbolic systems. *Math. Res. Lett.* **3** (1996), 191–210.

[14] P. Eberlein. Geodesic flows in certain manifolds with no conjugate points. *Trans. Amer. Math. Soc.* **167** (1972), 151–170.

[15] E. Ghys and P. De La Harpe. *Sur les Groupes Hyperboliques d'après M. Gromov (Progress in Mathematics, 83)*. Birkhäuser, Boston, 1990.

[16] M. Grayson, C. Pugh and M. Shub. Stably ergodic diffeomorphisms. *Ann. of Math.* (2) **140** (1994), 295–329.

[17] M. Gromov. *Hyperbolic Groups (Essays in Group Theory, 8)*. Ed. S. M. Gersten. Springer, Berlin, pp. 75–264 .

[18] J. M. Morse. A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.* **26** (1924), 25–60.

[19] V. Nitică and A. Török. An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one. *Topology* **40** (2001), 259–278.

[20] M. Paternain. Expansive geodesic flows on surfaces. *Ergod. Th. & Dynam. Sys.* **13**(1) (1993), 153–165.

[21] Ja. B. Pesin. Geodesic flows on closed Riemannian manifolds without focal points. *Math. USSR Izvestija* **11**(6) (1977), 1195–1228.

[22] C. Pugh and M. Shub. Stable ergodicity and julienne quasiconformality. *J. Eur. Math. Soc.* **1**(2) (2000), 1–52.

[23] R. Ruggiero. Expansive dynamics and hyperbolic geometry. *Bol. Soc. Brasil. Mat.* **25** (1994), 139–172.

[24] R. Ruggiero. Expansive geodesic flows in manifolds without conjugate points. *Ergod. Th. & Dynam. Sys.* **17** (1997), 211–225.

[25] R. Ruggiero. On the divergence of geodesic rays in manifolds without conjugate points, dynamics of the geodesic flow and global geometry. *Geometric Methods in Dynamics (II). Volume in Honor of Jacob Palis*. Eds. W. de Melo, M. Viana and J.-C. Yoccoz. *Astérisque* **287** (2003), 231–250. American Mathematical Society, Providence, RI.

[26] M. Shub and A. Wilkinson. Stably ergodic approximation: two examples. *Ergod. Th. & Dynam. Sys.* **20**(3) (2000), 875–893.