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GROUPS WHOSE IRREDUCIBLE REPRESENTATIONS HAVE FINITE DEGREE II

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If F is a (commutative) field let $\mathfrak{X}_F$ denote the class of all groups G such that every irreducible FG-module has finite dimension over F. The introduction to [7] contains motivation for considering these classes $\mathfrak{X}_F$ and surveys some of the results to date concerning them. In [7] for every field F we determined the finitely generated soluble groups in $\mathfrak{X}_F$. Here, for fields F of characteristic zero, we determine, at least in principle, the soluble groups in $\mathfrak{X}_F$. Our main result is the following.

Theorem 1. *Let G be a soluble $\mathfrak{X}_F$-group where F is any field of characteristic zero. Then G is abelian-by-finite.*

Farkas [1] (see top p. 587) claims that R. Snider has proved Theorem 1 in the special case where F is the complex numbers. Paragraph 3 of [7] enables one to compute $\mathfrak{A}\mathfrak{F} \cap \mathfrak{X}_F$ for any field F. (See below for notation and definitions.) Thus by Theorem 1 we can determine $\mathfrak{S}\mathfrak{F} \cap \mathfrak{X}_F$ for any field F of characteristic zero. In particular we have the following corollaries.

Corollary 1. *Let F be a depleted field of characteristic zero (e.g. $F = \mathbb{Q}$). Then*

$$(\acute{P}_n L\mathfrak{N})\mathfrak{F} \cap \mathfrak{X}_F = (\mathfrak{A} \cap \mathfrak{E}\mathfrak{G})\mathfrak{F} \subseteq \mathfrak{X}.$$

Corollary 2. *Let F be a field of characteristic zero that is either algebraically closed or real closed. Then $\mathfrak{S}\mathfrak{F} \cap \mathfrak{X}_F$ is the class of all groups G with an abelian normal subgroup of finite index and torsion-free rank less than $|F|$.*

Corollary 3. *Let F be a meagre field of characteristic zero. Then*

$$(\acute{P}_n \mathfrak{L}\mathfrak{N})\mathfrak{F} \cap \mathfrak{X}_F$$

is contained in the class of all groups G with an abelian normal subgroup of finite index and finite torsion-free rank.

The terms "depleted" and "meagre" are defined in [7]. The basic example of a depleted field is the rationals. By a theorem of Artin and Schreier ([2], p. 316) the only fields that are *not* meagre are the uncountable fields that are either algebraically closed or real closed. Thus Corollaries 2 and 3 cover all fields of characteristic zero.

$\acute{P}_n L\mathfrak{N}$ is the class of radical groups in the sense of Plotkin. Also $\mathfrak{A}$, $\mathfrak{S}$, $\mathfrak{F}$ and $\mathfrak{G}$ denote respectively the classes of abelian, soluble, finite and finitely generated groups and $\mathfrak{E}$ is the class of groups of finite exponent. $\mathfrak{X}=\bigcap_F \mathfrak{X}_F$, the torsion-free rank of an abelian group A is $\dim_{\mathbb{Q}}(\mathbb{Q}\oplus_{\mathbb{Z}} A)$. For any group G the maximum periodic normal subgroup of G we denote by $\tau(G)$.

Proof of the corollaries

In Corollary 3 the group $\tau(G)$ is abelian-by-finite by a result of B. Hartley ([**5**], 12.4.16) and $G/\tau(G)$ is abelian-by-finite by ([7], 5.3). Thus G is soluble-by-finite and Corollary 3 follows from Theorem 1 and [7], 3.4b). Corollary 1 follows from Corollary 3 and [7], 3.3 and 2.3. Corollary 2 follows from Theorem 1 and [7], 3.1, 3.2 and 2.3.

Almost all of our proof of Theorem 1, with suitable modifications, works for any characteristic and we present it in this generality. If G is any group and p a prime, then $O_p(G)$ denotes the maximum normal p-subgroup of G, and we set $O_0(G)=\langle 1\rangle$. By definition the trivial group is the only O-group. We prove the following.

Theorem 2. *Let F be a field of characteristic $u\geqq 0$ that is not locally finite and suppose that for every periodic soluble $\mathfrak{X}_F$-group H the group $H/O_u(H)$ is abelian-by-finite. If G is a soluble $\mathfrak{X}_F$-group then $G/O_u(G)$ is abelian-by-finite.*

If $u=0$ in Theorem 2 and if H is a periodic soluble $\mathfrak{X}_F$-group, then H is abelian-by-finite by Hartley's theorem [**5**], 12.4.16. Thus Theorem 1 is a consequence of Theorem 2.

Lemma 1. *Let G be a group and u zero or a prime such that $H/O_u(H)$ is abelian-by-finite for every countable subgroup H of G. Then $G/O_u(G)$ is abelian-by-finite.*

Proof. Suppose that for $r=1,2,\ldots$ there exists a finitely generated subgroup X_r of G such that $X_r/O_u(X_r)$ does not have an abelian normal subgroup of index at most r. By hypothesis $H=\langle X_r : r\geqq 1\rangle$ has a normal subgroup A of finite index with $A'\subseteq O_u(H)$. If $(H:A)=r$ then $(A\cap X_r)O_u(X_r)/O_u(X_r)$ is an abelian normal subgroup of $X_r/O_u(X_r)$ of index at most r. This contradiction shows that there exists $r\geqq 1$ such that every finitely generated subgroup X of G contains a normal subgroup A_X of index at most r such that A'_X is a u-group. The lemma follows by the usual inverse limit argument, see [3], 1.K.2.

Lemma 2. *Let A be an abelian group of infinite exponent. Then there exists a subgroup B of A such that A/B is infinite but of rank* 1.

(A group has finite rank at most r if each of its finitely generated subgroups can be generated by r elements.)

Proof. If A is torsion-free let X be a basis of A and pick $x\in X$. Now put $B=A\cap\langle X\backslash\{x\}\rangle^{\mathbb{Q}}$; A/B is torsion-free of rank 1. Now we may assume that A is periodic. If every primary component of A has finite exponent then A is a direct sum of cyclic groups by Prüfer's First Theorem ([**4**], p. 173) and involves infinitely many primes. Thus A has a subgroup B such that A/B is the direct product of infinitely many cyclic groups of distinct prime order. Hence we may assume that A is a p-group for some prime p.

Suppose that A contains a finite subgroup X of exponent $p^n > 1$ such that if $a \in A$ has order p^{n+1} then $X \cap \langle a \rangle \neq \langle 1 \rangle$. Now

$$\Omega_n A = \{a \in A : |a| \leqq p^n\}$$

is a direct product of cyclic groups by Prüfer's First Theorem, so $\Omega_n A = Y \times Z$ for some subgroups Y and Z with $X \subseteqq Y$ and Y finite. Let $a \mapsto \bar{a}$ be the natural projection of A onto A/Z and consider $a \in A$ with $a^p \in Z$. Then $|a| \leqq p^{n+1}$. If $|a| = p^{n+1}$ then $\Omega_1 \langle a \rangle \subseteqq X \subseteqq Y$ and $|\bar{a}| = |a| > p$. Thus $|a| \leqq p^n$, so $a \in YZ$ and $\bar{a} \in \bar{Y}$. Therefore $\Omega_1 \bar{A} \subseteqq \bar{Y}$, and in particular is finite. Hence any direct decomposition of $\bar{A}$ has only a finite number of factors and so $\bar{A}$ is the direct product of a finite number of directly indecomposable groups, each of which is cyclic or a Prüfer p^∞-group ([4], p. 181). But A and so $\bar{A}$ has infinite exponent. Therefore $\bar{A}$, and consequently A, has a Prüfer p^∞-image.

Now assume that no such X exists. We choose $x_1 \in A$ of order p. Suppose we have found $X = \langle x_1 \rangle \times \ldots \times \langle x_n \rangle \subseteqq A$ where $|x_i| = p^i$ for each i. By the above there exists $x_{n+1} \in A$ of order p^{n+1} with $X \cap \langle x_{n+1} \rangle = \langle 1 \rangle$. Thus by induction we can construct $D = \times_{i=1}^{\infty} \langle x_i \rangle \subseteq A$ with each $|x_i| = p^i$. Let

$$E = \langle x_i x_{i+1}^{-p} : i = 1, 2, \ldots \rangle.$$

Then D/E is a Prüfer p^∞-group. As such it is $\mathbb{Z}$-injective, so A/E splits over D/E and again A has a Prüfer p^∞-image.

Lemma 3. *Let A be an abelian normal subgroup of the completely reducible, soluble subgroup G of $GL(n, F)$; here F is any field. If μ is any function satisfying Mal'cev's Theorem* ([**6**], 3.6), *then there exists an abelian normal subgroup B of G containing A with* $(G:B) \leqq n! \cdot \mu(n)$.

Proof. By [**6**], 1.22 we may assume that F is algebraically closed. By Clifford's Theorem A is also completely reducible, so [**6**], 1.12 yields that $(G:C_G(A))$ divides $n!$. Now G has an abelian normal subgroup D of finite index at most $\mu(n)$ by Mal'cev's Theorem. Now set $B = A \cdot C_D(A)$.

We have no interest here in the bound of Lemma 2, merely in the finiteness of $(G:B)$. Now in Lemma 3 necessarily $\tau(A)$ has finite rank. Thus the qualitive part of that lemma is a special case of the following, whose proof we leave to the reader.

Lemma 3'. *Let A be an abelian group. Then $A \lhd G$ implies that A lies in an abelian normal subgroup of G of finite index for ALL abelian-by-finite groups G if and only if for every prime p either A has no subgroup of index p or A contains no infinite elementary abelian p-subgroup.*

Lemma 4. *Let F be a field of characteristic $u \geqq 0$ that is not locally finite and let $G = \langle x \rangle [A$ (split extension) be a group where A is abelian normal of finite torsion-free rank and $\langle x \rangle$ is infinite. If $G \in \mathfrak{X}_F$ and if $A \setminus \langle 1 \rangle$ contains no elements of order u, then $C_{\langle x \rangle}(A) \neq \langle 1 \rangle$.*

Proof. Assume we have a counter example to the lemma. We prove first that A contains a subgroup $X = \times_{i=1}^{\infty} \langle a_i^G \rangle$ such that $[a_i, x^i] \neq 1$ for each i. Suppose $a_1, \ldots, a_{i-1}$ have been constructed and set $Y = \langle a_1^G, \ldots, a_{i-1}^G \rangle$. If $y \in A$ then $\langle x, y \rangle$ is abelian-by-finite by the main theorem of [7] and $\langle y^{\langle x \rangle} \rangle = \langle y^G \rangle$ is finitely generated. Thus Y is finitely generated. Pick a normal subgroup $N \subseteq A$ of G maximal subject to $Y \cap N = \langle 1 \rangle$. If $C_{\langle x \rangle}(N) = 1$ there exists $a_i \in N$ such that $[a_i, x^i] \neq 1$ and clearly $a_i^G \subseteq N$. Thus the construction of X can proceed inductively.

We have to eliminate the possibility that $C_{\langle x \rangle}(N) \neq \langle 1 \rangle$, so assume that this is so. Since F is not locally finite and since Y contains no non-trivial elements of order u there exists a faithful, finite-dimensional, completely reducible representation of Y over F. Hence by Hall's Lemma ([7], 2.1) and the hypothesis $G \in \mathfrak{X}_F$ there exists a finite-dimensional, completely reducible representation ρ of G over F such that $N \subseteq \ker \rho$ and $Y \cap \ker \rho = \langle 1 \rangle$. By the choice of N we have $A \bigcap \ker \rho = N$, By [6], 1.22, 1.12 and Clifford's Theorem there exists $r > 0$ with $[A\rho, x^r\rho] = \langle 1 \rangle$; that is with $[A, x^r] \subseteq N$, and we choose r large enough so that also $[N, x^r] = \langle 1 \rangle$.

Set $B = [A, x^r]$. Then B is a homomorphic image of $A/N \cong A\rho$. The torsion subgroup of $A\rho$ has finite rank (cf. [5], 2.2) and A and hence $A\rho$ has finite torsion-free rank. Therefore B has finite rank. But then there exists a faithful direct sum of a finite number of irreducible representations of B over F and hence there exists a finite-dimensional, completely reducible representation σ of G over F with $B \cap \ker \sigma = \langle 1 \rangle$. By [6], 1.22 and 1.12 again there exists $s > 0$ with $[A, x^s] \subseteq \ker \sigma$. But then

$$[A, x^{rs}] \subseteq [A, x^r] \cap [A, x^s] \subseteq B \cap \ker \sigma = \langle 1 \rangle,$$

which contradicts our assumption that we are considering a counter example to the lemma. Therefore $C_{\langle x \rangle}(N) = \langle 1 \rangle$ and this completes the construction of X.

We complete the proof of the lemma by constructing an infinite-dimensional, irreducible FG-module. Let $\bar{F}$ be an algebraic closure of F. Since $\langle a_i^G \rangle$ is finitely generated and abelian there exists a homomorphism ϕ_i of $\langle a_i^G \rangle$ into $\bar{F}^*$ with $[a_i, x^i]\phi_i \neq 1$. Since $\bar{F}^*$ is $\mathbb{Z}$-injective there exists a homomorphism $\phi: A \to \bar{F}^*$ such that $[a_i, x^i]\phi = [a_i, x^i]\phi_i$ for each i. Thus $a_i\phi \neq a_i^{x^i}\phi$ for each $i \geq 1$. Set $K = F(A\phi) \subseteq \bar{F}$ and $V = \oplus_{i \in \mathbb{Z}} v_i K$. Make V into a KG-module by defining

$$v_i x = v_{i-1} \quad \text{and} \quad v_i a = v_i(a^{x^i}\phi)$$

for each $i \in \mathbb{Z}$ and $a \in A$.

In particular V becomes an FG-module of infinite dimension. Let U be a non-zero KG-submodule of V. Pick $v = \sum_{i=r}^{s} v_i\alpha_i \in U \backslash \{0\}$ where each $\alpha_i \in K$ and $s - r$ is minimal. Replacing v by vx^r we may choose v with $r = 0$. Suppose $s > 0$. Then U also contains

$$va_s - v(a_s\phi) = \sum_{i=1}^{s} v_i\alpha_i(a_s^{x^i}\phi - a_s\phi).$$

By construction $\alpha_s(a_s^{x^s}\phi - a_s\phi) \neq 0$. This contradicts the choice of v. Thus $s = 0$ and so $v_0 \in U$. But clearly $v_0 FG = V$. Consequently $U = V$ and V is irreducible as KG-module.

Since V is FG-cyclic there exists a maximal FG-submodule W of V. Suppose $\dim_F(V/W)$ is finite. Let $\mathfrak{a}$ be the annihilator of V/W in FG. Then $\dim_F(FG/\mathfrak{a})$ is finite. But $V\mathfrak{a}$ is a KG-submodule of V and consequently by the above is $\{0\}$. Thus V is an image of $FG/\mathfrak{a}$ and as such is finite dimensional. This contradiction proves that V/W is an infinite-dimensional, irreducible FG-module, and completes the proof of the lemma.

The argument of the previous paragraph can be used to prove the following, which should be compared with [7], 2.3.

Proposition. *Let $F \subseteq K$ be fields with $(K:F)$ finite. Then $\mathfrak{X}_F = \mathfrak{X}_K$.*

If K is an arbitrary extension field of F then 3.1 (or alternatively 3.2) shows that sometimes $\mathfrak{X}_K \not\subseteq \mathfrak{X}_F$ and the work of P. Hall and Roseblade shows that sometimes $\mathfrak{X}_F \not\subseteq \mathfrak{X}_K$.

Proof of Theorem 2

This we break into a number of pieces. Let F and G be as in the theorem and assume that $G/O_u(G)$ is not abelian-by-finite. Let $\bar{F}$ be an algebraic closure of F.

1. *G contains a countable subgroup H_1 such that $H = H_1/O_u(H_1)$ has an abelian normal subgroup A containing H' with H' and H/A periodic, and such that H is not abelian-by-finite.*

Proof. By Lemma 1 we may assume that G is countable. By hypothesis $\tau(G)$ contains a normal subgroup T of finite index with $T' \subseteq O_u(G)$. Hall's Lemma applied to irreducible $F(\tau(G)/O_u(G))$-modules shows that there exists a finite-dimensional (from $G \in \mathfrak{X}_F$), completely reducible representation ρ of G over F such that $\tau(G) \cap \ker \rho$ has its derived group in $O_u(G)$. By [7], 5.1 the group $G/\tau(G)$ is abelian-by-finite, and also $G\rho$ is abelian-by-finite ([6], 3.5). Hence G contains a normal subgroup H_1 of finite index with $H'_1 \subseteq \tau(G) \cap \ker \rho$. Set $H = H_1/O_u(H_1)$. Then H' is periodic and abelian and H is not abelian-by-finite.

Since H is countable there exist elements $x_1, x_2, \ldots$ of H such that $H/\langle x_i : i \geqq 1 \rangle H'$ is periodic. Suppose we have constructed $r_1, \ldots, r_{i-1} > 0$ such that $A_i = \langle x_j^{r_j} : j < i \rangle H'$ is abelian. Since A_i is normal in H, there are no elements of order u in $A_i \backslash \langle 1 \rangle$ and Lemma 4 yields that there exists an integer $r_i > 0$ with $[A_i, x_i^{r_i}] = 1$ (if $|x_i| < \infty$ set $r_i = |x_i|$). Then $A_{i+1} = A_i \langle x_i^{r_i} \rangle$ is abelian. By induction we construct an abelian normal subgroup. $A = \bigcup_{i \geqq 1} A_i \supseteq H'$ of H with H/A periodic.

Let L denote the Fitting subgroup of H.

2. *We may choose H and A such that L/A is finite. We may also choose A maximal, that is with $A = C_H(A)$.*

Proof. Initially let H and A be as in 1. Necessarily $A \subseteq L$. Suppose L' has infinite exponent. Clearly $L' \backslash \langle 1 \rangle$ contains no elements of order u. By Lemma 2 there exists a homomorphism of L' into $\bar{F}^*$ with infinite image. By Hall's Lemma there exists an irreducible representation ρ of L with $L'\rho$ infinite. But $L\rho$ is nilpotent ([6], 8.2ii), so $L'\rho$ is finite ([6], 3.13). Consequently L' has finite exponent m say.

Let Q/L' be a maximal torsion-free subgroup of A/L'. Then Q^m is normal in L and L/Q^m is periodic. Let $P/Q^m = O_u(L/Q^m)$. By hypothesis L/Q^m contains a normal subgroup M/Q^m of finite index with $M' \subseteqq P$. But $L' \cap Q^m = \langle 1 \rangle$, so

$$M' \subseteqq L' \cap P \subseteqq O_u(L') = \langle 1 \rangle$$

and M is an abelian subgroup of L of finite index, n say.

H/L^n is periodic and so contains a normal subgroup N/L^n of finite index such that $N'L^n/L^n$ is a u-group. But $N' \subseteqq H'$ and so has no proper u-images. Consequently $N' \subseteqq L^n \subseteqq M$. Since $(H:N)$ is finite L contains the Fitting subgroup of N and so $M \cap N$ has finite index in this Fitting subgroup. Now replace H by N and A by any maximal abelian subgroup of N containing $M \cap N$.

3. *H/A is reduced.*

Proof. Let D/A be the divisible part of H/A. Let ρ be any irreducible representation of D over F. Necessarily ρ is finite dimensional ([7], 2.2). Also $D\rho/A\rho$ has no proper subgroup of finite index. Therefore $D\rho$ is abelian by Lemma 3 and thus

$$D' \subseteqq \bigcap \ker \rho \subseteqq O_u(D) = \langle 1 \rangle$$

by [7], 2.5. Thus $D = A$ by the maximal choice of A.

Let $K = \{x \in H : [A, x] \text{ is finite}\}$. It is easily seen that K is a subgroup of H containing A.

4. *$[A, K]$ has finite exponent.*

Proof. Suppose not. By Lemma 2 there exists a homomorphism of $[A, K]$ into $\bar{F}^*$ with its image infinite and of rank 1. In particular this image has infinite exponent. By Hall's Lemma these exists an irreducible representation ρ of H over F such that $[A, K]\rho$ has infinite exponent. Let r be the index in $A\rho$ of its Zariski connected component containing 1. If $k \in K$ then $|[A, k]\rho| = |k^A \rho|$, and the latter, being finite, divides r ([**6**], 5.3, 5.4, cf. 5.5). Thus the abelian group $[A, K]\rho$ has exponent dividing r. This contradiction confirms 4.

5. *K/A is finite.*

Proof. Let $Q/[A, K]$ be a maximal torsion-free subgroup of $A/[A, K]$. By 4, there exists $m > 0$ with $[A, K]^m = \langle 1 \rangle$. Now Q is normal in K and K/Q^m is periodic. Set $P/Q^m = O_u(K/Q^m)$. By hypothesis there exists a normal subgroup M of K of finite index such that $M' \subseteqq P$. But $[A, K] \cap Q^m \langle 1 \rangle$, so

$$[A, K] \cap P \subseteqq O_u[A, K] = \langle 1 \rangle \quad \text{and} \quad [A \cap M, M] = \langle 1 \rangle.$$

Also $M' \subseteqq H' \cap M \subseteqq A \cap M$ and M is nilpotent. Consequently as K/M is finite we have $M \subseteqq L$, and so AM/A is finite by 2. But again K/M is finite, so 5 follows.

6. *The final contradiction; H/K is finite.*

Proof. Suppose otherwise. By 3 and 5 there exists an infinite subgroup X/A of H/A with $K \cap X = A$. Let $\{1\} \cup Y = \{1, y_1, y_2, \ldots\}$ be a transversal of A to X (recall that H is countable). Suppose we have constructed $a_1, \ldots, a_{i-1} \in A$ and a homomorphism $\phi_i: A_i = \langle a_j: j < i \rangle \to \bar{F}^*$ such that for each $j < i$, $a_j \in [A, y_j] \backslash \langle 1 \rangle$ and $|a_j \phi_i| = |a_j|$. Now A_i is finite, being a finitely generated subgroup of H', and $[A, y_i]$ is infinite since $y_i \notin K$, so there exists $a_i \in [A, y_i] \backslash A_i$. Let r be the order of a_i modulo A_i and let α be a primitive r-th root of $a_i^r \phi_i$ in $\bar{F}$. There exists a homomorphism ϕ_{i+1} of $A_i \langle a_i \rangle$ into $\bar{F}^*$ such that $a\phi_{i+1} = a\phi_i$ for all $a \in A_i$ and $a_i \phi_{i+1} = \alpha$.

Thus inductively we can construct $a_i \in [A, y_i]$ for $i = 1, 2, \ldots$ and a *homomorphism* ϕ *of* $\langle a_i: i \geqq 1 \rangle$ into $\bar{F}^*$ such that each $a_i \phi \neq 1$. By Hall's Lemma there exists an irreducible representation ρ of H over F with $a_i \rho \neq 1$ for each i. Now by Lemma 3 there exists an abelian subgroup of $X\rho$ of finite index containing $A\rho$. Since X/A is infinite there exists i with $\langle A, y_i \rangle \rho$ abelian. But then $a_i \in [A, y_i] \subseteqq \ker \rho$, which is false. This contradiction yields 6 and completes the proof of the theorem.

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