J. Fluid Mech. (2012), vol. 706, pp. 597–606.
doi:10.1017/jfm.2012.292

On boundary-layer flows induced by the motion of stretching surfaces

Talal T. Al-Housseiny[1] **and Howard A. Stone**[2]†

[1] Department of Chemical and Biological Engineering, Princeton University, Princeton, NJ 08544, USA
[2] Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08544, USA

(Received 1 May 2012; revised 5 June 2012; accepted 11 June 2012; first published online 18 July 2012)

We investigate laminar boundary-layer flows due to translating, stretching, incompressible sheets. Unlike the classical problem in the literature where the mechanics of the sheet are neglected, and kinematics are prescribed, the dynamics of both the fluid and the sheet are herein coupled. Two types of stretching sheets are considered: an elastic sheet that obeys linear elasticity and a sheet that deforms as a viscous Newtonian fluid. In both cases, we find self-similar solutions to the coupled fluid/sheet system. These self-similar solutions are only valid under limiting conditions.

Key words: boundary layers, flow-structure interactions

1. Introduction

Drawing a sheet of material through a fluid medium at a high velocity occurs in many industrial settings such as extrusion processes. The sheet stretches in the direction of drawing, and consequently the velocity along the sheet increases. In this process, the sheet induces a flow in the surrounding fluid. The characteristics of the flow around the stretching sheet, which in the existing literature is mostly treated as a flat surface, have been studied extensively by means of boundary-layer theory, which typically allows for self-similar solutions for the velocity distribution in the surrounding fluid. The first solution of this type was given by Sakiadis (1961) who introduced boundary-layer flows driven by a flat surface moving at a constant speed U_0. In this case, the laminar boundary-layer equations reduce to the well-known Blasius equation, but with a different set of boundary conditions. The work of Sakiadis was then extended to flat stretching sheets.

Over the years, there have been many studies of boundary-layer flows induced by impermeable stretching sheets with prescribed kinematics, as summarized in table 1. Specifically, when the sheet is impermeable and its velocity increases linearly in the direction of stretching, which is also referred to as linear stretching, the surrounding fluid flow admits both an exact analytical solution to the Navier–Stokes equations (Crane 1970) and a similarity solution to the Navier–Stokes equations (Wang 1984; Surma Devi, Takhar & Nath 1986). The first comprehensive investigation of boundary-layer flows driven by impermeable stretching sheets was contributed by Banks (1983). Recently, Weidman & Magyari (2009) have shown how to extend Crane's solution to permeable sheets and, by that, have recovered other known

† Email address for correspondence: hastone@princeton.edu

Year	Authors	Sheet velocity	Outer fluid	State
1961	Sakiadis	Uniform $U(x) = U_0$	Newtonian	Steady-state
1969	Fox, Erickson & Fan	Uniform	Power-law	Steady-state
1970	Crane	Linear $U(x) \propto x$	Newtonian	Steady-state
1977	Vleggaar	Power-law $U(x) \propto x^{\lambda}$	Newtonian	Steady-state
1983	Banks	Power-law	Newtonian	Steady-state
1984	Rajagopal, Na & Gupta	Linear	Viscoelastic	Steady-state
1984	Wang	Linear	Newtonian	Steady-state
1986	Surma Devi *et al.*	Linear	Newtonian	Time-dependent
1995	Gorla, Dakappagari & Pop	Linear	Power-law	Steady-state
1996	Andersson *et al.*	Linear	Power-law	Time-dependent
1999	Magyari & Keller	Exponential $U(x) \propto \mathrm{e}^{x}$	Newtonian	Steady-state
2002	Wang and Andersson	Linear (with slip)	Newtonian	Steady-state
2006	Liao	Power-law	Newtonian	Eime-dependent
2006	Andersson & Kumaran	Power-law	Power-law	Steady-state

TABLE 1. A chronological representative summary of the literature on boundary-layer flows due to stretching impermeable sheets. In all cases, the kinematics of the sheet are prescribed.

results (Gupta & Gupta 1977; Lage & Bejan 1990; Kumaran & Ramanaiah 1996). In addition, Magyari (2010) generalized and extended well-known solutions to boundary-layer flows due to stretching, impermeable surfaces (e.g. Crane's solution, Bickley's solution, etc.)

In the previous studies, the sheet dynamics are ignored; rather, the stretching sheet velocity $U(x)$ is prescribed as a function of position in the stretching direction, e.g. linear stretching $U(x) \propto x$, power-law stretching $U(x) \propto x^{\lambda}$, etc. Unlike the standard problem of this type considered in the literature, our motivation in this paper is to include the sheet mechanics via the stress balance in the sheet. Thus, due to the coupling of the momentum equations in both the fluid and the sheet, we obtain as part of the solution to the problem the velocity of the stretching sheet, which decreases in thickness along the direction of stretching. When the sheet mechanics are included, the self-similar solutions of the system only exist under restrictive conditions that we identify.

In § 2, we formulate the dynamics of the sheet and the surrounding flow. We consider two cases for the nature of the stretching sheet. Sections 3.1 and 3.2 cover, respectively, the results for a sheet that obeys linear elasticity and for a viscous sheet that deforms as a viscous Newtonian fluid.

2. Problem formulation and governing equations

We analyse a process where a stretching sheet is supplied at a velocity U_0 at $x = 0$ and is collected at a velocity $U_L > U_0$ at $x = L$: see figure 1. An example configuration is a system where an elastic sheet immersed in a fluid is supplied and collected by two wind-up rollers operating at different speeds. The stretching sheet of density ρ_s is assumed incompressible and is supplied with half-thickness h_0 and at a constant volumetric flow rate per unit width q. The surrounding fluid has a density ρ, viscosity μ and kinematic viscosity $\nu = \mu/\rho$.

Since the width W of the sheet in the lateral direction is assumed large such that $W \gg h_0$, we treat the two-dimensional dynamics of the system. We consider the system (the sheet and the surrounding fluid) to be at steady state. Assuming that the

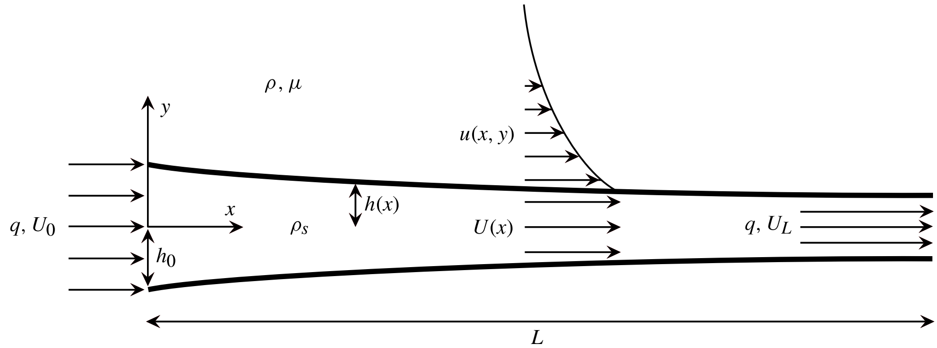

FIGURE 1. Schematic of a stretching sheet immersed in a fluid. The thin sheet is assumed to have a large width $W \gg h_0$ in the lateral, or spanwise, direction and so only the two-dimensional, steady dynamics are analysed.

half-thickness $h(x)$ of the sheet varies slowly, $|\mathrm{d}h/\mathrm{d}x| \ll 1$, the velocity profile in the sheet is nearly uniform in the y-direction, i.e. the horizontal velocity of the sheet is denoted $U(x)$. Hence, the Lagrangian acceleration of an element of the sheet is $U(\mathrm{d}U/\mathrm{d}x)$. Finally, we assume that the fluid is at rest far away from the stretching sheet. We next give the governing equations for the sheet and the surrounding fluid, respectively.

2.1. *Momentum equation for the sheet*

We assume that the sheet is incompressible and therefore has a constant rate of supply q such that

$$2h(x)U(x) = 2h_0U_0 = q. \tag{2.1}$$

The steady-state momentum balance in the sheet is written in the x-direction as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \rho_s U \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{2.2}$$

where σ_{ij} are the components of the stress tensor in the sheet. Assuming σ_{xx} is a function of x only since $|\mathrm{d}h/\mathrm{d}x| \ll 1$, we integrate (2.2) through the thickness of the sheet. Using the Leibniz rule and taking advantage of the symmetry of the system, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(h\sigma_{xx}) + \tau_{yx} = \rho_s hU \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{2.3}$$

where τ_{yx} is the fluid shear stress at the upper surface of the sheet $y = h(x)$. For a Newtonian fluid surrounding the sheet, and consistent with $|\mathrm{d}h/\mathrm{d}x| \ll 1$, $\tau_{yx} \simeq \mu\, (\partial u/\partial y)|_{y=h(x)}$, the momentum balance for the sheet becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(h\sigma_{xx}) + \mu \left.\frac{\partial u}{\partial y}\right|_{y=h(x)} = \rho_s hU \frac{\mathrm{d}U}{\mathrm{d}x}. \tag{2.4}$$

2.2. *An elastic sheet*

We assume that the elastic sheet is incompressible with Poisson ratio equal to 1/2. We next need to evaluate σ_{xx} for the sheet. We start from the known time variation

https://doi.org/10.1017/jfm.2012.292 Published online by Cambridge University Press

of strain in terms of the velocity distribution for unidirectional motion (Malvern 1969, p. 150):

$$\frac{\mathrm{d}\varepsilon_{xx}}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}X}, \tag{2.5}$$

where $\varepsilon_{xx}(x)$ is the strain along the sheet and X denotes the material points. Assuming small deformations, $\partial/\partial X \simeq \partial/\partial x$, and steady-state motion, $\mathrm{d}/\mathrm{d}t = U(\mathrm{d}/\mathrm{d}x)$, we find

$$U\frac{\mathrm{d}\varepsilon_{xx}}{\mathrm{d}x} = \frac{\mathrm{d}U}{\mathrm{d}x}. \tag{2.6}$$

The integral of (2.6) yields $\varepsilon_{xx} = \ln(U/U_0)$ for a sheet with zero strain at $x = 0$. For small variations in speed, which is also consistent with small sheet deformations, we approximate the strain–velocity relation as

$$\varepsilon_{xx} = \frac{U(x)}{U_0} - 1. \tag{2.7}$$

The sheet is assumed linearly elastic and thus obeys Hooke's law for unidirectional plane stress $\sigma_{xx} \simeq E\varepsilon_{xx}$, where E is the effective modulus of the sheet. We rearrange the sheet momentum balance (2.4) and we use (2.1) and (2.7) to obtain

$$\left(\frac{E}{\rho_s U^2} - 1\right)\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\mu}{\rho_s h_0 U_0}\left.\frac{\partial u}{\partial y}\right|_{y=h(x)} = 0. \tag{2.8}$$

Next, we consider the limit where the elastic forces in the sheet are much larger than the inertial forces $E/\rho_s U^2 \gg 1$. To check the validity of this assumption, a soft synthetic rubber sheet, for instance, has $E \approx 10^7$ Pa and $\rho_s \approx 1200$ kg m^{-3}, so that for stretching with an end velocity, $U_L = 20$ m s^{-1}, then $E/\rho_s U^2 \approx 20$. In this limit, the momentum balance for an elastic sheet reduces to

$$\frac{E h_0 U_0}{\mu U^2}\frac{\mathrm{d}U}{\mathrm{d}x} + \left.\frac{\partial u}{\partial y}\right|_{y=h(x)} = 0. \tag{2.9}$$

2.3. *A viscous sheet*

As an alternative description, we assume the stretching sheet to be a viscous Newtonian fluid (e.g. melts), in which case the stress is proportional to the strain rate: $\sigma_{xx} = \mu_{sh}(\mathrm{d}U/\mathrm{d}x)$, where μ_{sh} is the constant extensional viscosity, also known as the Trouton viscosity, and in two dimensions is equal to four times the actual viscosity of the sheet. The viscosity of the surrounding fluid is assumed to be much smaller than the viscosity of the sheet: $\mu \ll \mu_{sh}$. For high velocities, we expect that the effects of surface tension are negligible. To check, we can estimate a capillary number $Ca = \mu_{sh}U/\gamma$, which measures the relative importance of surface tension γ with respect to viscous forces. For example, for fluids with $\mu_{sh} = 100\,\mu_{H_2O}$ and $\gamma \approx 10 \times 10^{-3}$ N m^{-1} (common for viscous oils in water), and $U_L = 20$ m s^{-1} as above, we get $Ca \approx 200 \gg 1$. Furthermore, we note that the process of drawing a viscous sheet is susceptible to the draw resonance instability (Denn 1980; Scheid *et al.* 2009) that occurs above a critical draw ratio U_L/U_0. In this paper, the system is assumed to be operating below the critical draw ratio for instability.

After substituting (2.1) in (2.4) and rearranging, the momentum balance for a viscous sheet becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mu_{sh}}{U}\frac{\mathrm{d}U}{\mathrm{d}x} - \rho_s U\right) + \frac{\mu}{h_0 U_0}\left.\frac{\partial u}{\partial y}\right|_{y=h(x)} = 0. \tag{2.10}$$

2.4. *The dynamics of the surrounding fluid*

The sheet is immersed in a surrounding fluid whose velocity vector is $\boldsymbol{u} = (u(x, y), v(x, y))$. Within laminar boundary-layer theory, which assumes that a suitably defined Reynolds number is large and $|\partial^2 u/\partial x^2| \ll |\partial^2 u/\partial y^2|$, and neglecting the pressure gradient term since the far-field velocity of the fluid is zero, the continuity and momentum equations for the steady, incompressible flow of a Newtonian fluid surrounding the stretching sheet are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.11a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{2.11b}$$

We note that here we choose to use the (x, y) coordinate system not only for the mechanics of the sheet, but also for the boundary-layer description of the surrounding fluid. For sufficiently slow shape changes $|\mathrm{d}h/\mathrm{d}x| \ll 1$, the solution procedure we present next is equivalent to the familiar boundary-layer approach of using coordinates parallel and normal to the slowly varying surface $h(x)$ (e.g. Acrivos, Shah & Petersen 1960; Rosenhead 1963). The discrepancy between our Cartesian formulation and the traditional curvilinear formulation is of the order of $(\mathrm{d}h/\mathrm{d}x)^2$.

Since the system is symmetric, we only consider its upper half $y \geqslant 0$. The no-slip and kinematic boundary conditions ensure that the velocity of the fluid at the surface of the sheet, $y = h(x)$, is equal to the sheet velocity. In addition, as stated previously, the fluid is assumed to be at rest far away from the sheet, so that we have the conditions

$$u(x, h(x)) = U(x), \quad v(x, h(x)) = U(x)\frac{\mathrm{d}h}{\mathrm{d}x}, \quad u(x, y \to \infty) = 0. \tag{2.12}$$

To avoid confusion, observe that the stretching sheet is impermeable and the non-homogeneous kinematic boundary condition on the transverse velocity v is due to the Lagrangian thinning of the sheet. We note that Fang, Zhang & Zhong (2012) studied boundary-layer flows induced by impermeable stretching sheets that vary in thickness, yet again with prescribed kinematics; moreover, their formulation fails to account for the Lagrangian acceleration of the sheet.

Returning to (2.11), we seek a self-similar solution by using a transformation of the form

$$u(x, y) = U(x)f'(\eta) \quad \text{where } \eta = \frac{y - h(x)}{h_0 g(x)}, \tag{2.13a}$$

$$v(x, y) = -h_0\frac{\mathrm{d}(gU)}{\mathrm{d}x}f(\eta) + U(x)\left(\frac{\mathrm{d}h}{\mathrm{d}x} + h_0\eta\frac{\mathrm{d}g}{\mathrm{d}x}\right)f'(\eta). \tag{2.13b}$$

This similarity ansatz, which identically satisfies the continuity equation in (2.11*a*), naturally accounts for the unknown shape $h(x)$ of the sheet. The velocity of the sheet $U(x)$ and the dimensionless function $g(x)$ are to be determined from coupling the mechanics of the sheet and the surrounding fluid. Substituting the transformation (2.13) into the fluid momentum balance (2.11), the latter reduces to the standard equation that results from wall-driven boundary-layer flows in an otherwise quiescent fluid (e.g. Rosenhead 1963; Banks 1983),

$$f''' + \left(\frac{h_0^2 g^2}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{h_0^2 U g}{\nu}\frac{\mathrm{d}g}{\mathrm{d}x}\right) ff'' - \frac{h_0^2 g^2}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x}(f')^2 = 0. \tag{2.14}$$

We note that (2.14) involves $U(x)$ and $g(x)$ but not the unknown shape of the sheet $h(x)$. In the case of a sheet moving at constant speed, i.e. $U(x) = U_0$, (2.14) reduces to the familiar Blasius equation with $g(x) \propto \sqrt{x}$. In the case of a stretching sheet, i.e. $\mathrm{d}U/\mathrm{d}x \neq 0$, for a similarity solution to exist and without loss of generality, we must have

$$\frac{h_0^2 g^2}{\nu}\frac{\mathrm{d}U}{\mathrm{d}x} = 1 \quad \text{and} \quad \frac{h_0^2 U g}{\nu}\frac{\mathrm{d}g}{\mathrm{d}x} = c, \tag{2.15}$$

where c is a constant to be determined from coupling both the fluid and the sheet dynamics. The resulting ODE, and the corresponding boundary conditions, are

$$f''' + (1+c)ff'' - (f')^2 = 0, \quad f(0) = 0, \quad f'(0) = 1, \quad f'(\eta \to \infty) = 0. \tag{2.16}$$

Equation (2.15) restricts $U(x)$ and $g(x)$ to power-law functions of the form $(Ax+B)^p$, except when $c = -1/2$, in which case $U(x)$ and $g(x)$ must be exponential functions. For any value of c, $U(x)$ and $g(x)$ are related by $g \propto U^c$. Therefore, if the system admits a self-similar solution, then for $c \neq -1/2$ and to satisfy $U(0) = U_0$,

$$U(x) = U_0\left(-m\frac{x}{L} + 1\right)^{-\alpha}, \tag{2.17a}$$

$$g(x) = Re^{-1/2}\left(-m\frac{x}{L} + 1\right)^{-\alpha c}. \tag{2.17b}$$

By substituting (2.17) into (2.15), the constants α and c are related such that $\alpha = -1/(1+2c)$. The positive parameters m and Re are determined respectively from the end velocity U_L and from substituting (2.17) into (2.15),

$$m = 1 - \left(\frac{U_0}{U_L}\right)^{1/\alpha}, \quad Re = \frac{\alpha m U_0 h_0^2}{\nu L}, \tag{2.18}$$

where $0 < m < 1$ for stretching to take place and Re is the effective Reynolds number of the system. The self-similar velocity profile of the sheet/fluid system is characterized by the constants α and c, and the dimensionless parameters m and Re. Henceforth, the constants and parameters corresponding to the elastic sheet and the viscous sheet are denoted by subscripts e and v respectively.

3. Results

3.1. *Case I: elastic sheet*

Since we are seeking a solution of the self-similar form, we use the transformation in (2.13) to rewrite the elastic sheet momentum balance (2.9) as

$$\frac{Eh_0^2 U_0}{\mu U^3}\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{f''(0)}{g(x)} = 0. \tag{3.1}$$

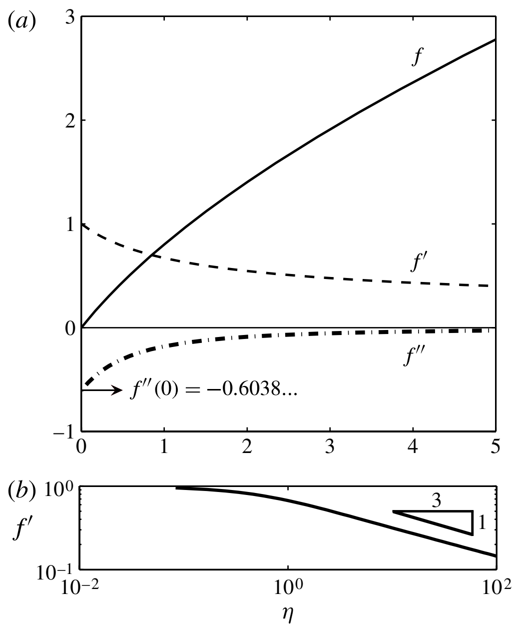

FIGURE 2. (a) Solution of the BVP in (3.3) and (b) highlights the algebraic decay of f' for large η.

Substituting the form of (2.17) into (3.1) and using $\alpha_e = -1/(1+2c_e)$, we obtain a system of two equations for the two unknowns α_e and c_e, which yields

$$\alpha_e = \tfrac{1}{5}, \quad c_e = -3. \tag{3.2}$$

We confirm that our system does not admit an exponentially varying sheet velocity by substituting $U(x) \propto \mathrm{e}^x$ and $g(x) \propto \mathrm{e}^{c^\star x}$, where $c^\star$ is a constant, into the elastic sheet stress balance (3.1). It is evident that the resulting expression cannot be balanced for $c^* = -1/2$; our power-law solution is therefore unique. Using the results in (3.2), the ODE corresponding to the self-similar boundary-layer flow over a stretching elastic sheet is given by

$$f''' - 2ff'' - (f')^2 = 0, \quad f(0) = 0, \quad f'(0) = 1, \quad f'(\eta \to \infty) = 0 \tag{3.3}$$

$$\text{subject to } f''(0) = -\left(\frac{E}{\rho U_0^2}\right) Re_e^{1/2} \quad \text{where } Re_e = \frac{m_e U_0 h_0^2}{5\nu L}. \tag{3.4}$$

The numerical solution to the BVP in (3.3), shown in figure 2(a), was obtained using a relaxation method (the MATLAB routine BVP4c), and required a large domain due to the slow algebraic decay of $f'(\eta)$. For $\eta \gg 1$, the dominant terms in (3.3) are ff'' and $(f')^2$, from which we find $f' = O(\eta^{-1/3})$ (see figure 2b).

The condition on $f''(0)$ in (3.4) results from simplifying the momentum equation of the sheet (3.1) using (2.17) and (3.2). It is the necessary condition for which the coupled fluid/sheet system admits a self-similar solution. From the numerical solution of (3.3), we have $f''(0) \simeq -0.6038$. Therefore, a system where an elastic (Hookean) sheet is drawn through a Newtonian fluid only admits a self-similar boundary-layer solution if

$$\left(\frac{E}{\rho U_0^2}\right) Re_e^{1/2} \simeq 0.6038. \tag{3.5}$$

Effectively, (3.5) determines the draw ratio (via m_e) and the sheet dimensions (h_0, L) for which a self-similar solution is possible for both the velocity of the sheet and that of the fluid. We remind the reader that the inertia of the elastic sheet had to be neglected (refer to (2.9)) to allow for a similarity solution.

After solving for the velocity profile, the shape of the sheet $h(x)$ is obtained from the incompressibility condition (2.1),

$$h(x) = h_0\left(-m_e\frac{x}{L}+1\right)^{1/5}. \tag{3.6}$$

We also verify that our obtained results for the elastic sheet hold for small sheet deformations or, equivalently, small variations in speed; this can be checked by examining (2.17*a*) and (3.6).

3.2. *Case II: viscous sheet*

Following the same steps as § 3.1, the constants of the viscous sheet case are determined by substituting (2.17) into the viscous sheet momentum balance (2.10) and using $\alpha_v = -1/(1+2c_v)$. We find

$$\alpha_v = 1, \quad c_v = -1. \tag{3.7}$$

Again, using the same reasoning as previously, we check that our system does not admit an exponentially increasing sheet velocity. It is also important to note that in the case of a viscous sheet, we find a similarity solution without having to neglect the inertia of the sheet. The ODE of the self-similar boundary-layer flow over a stretching viscous sheet, along with the boundary conditions, is given by

$$f''' - (f')^2 = 0, \quad f(0)=0, \quad f'(0)=1, \quad f'(\eta\to\infty)=0 \tag{3.8}$$

$$\text{subject to } f''(0) = -\left(\frac{\rho_s}{\rho}\right)\left(m_v\left(\frac{\mu_{sh}}{\rho_s U_0 L}\right)-1\right)Re_v^{1/2} \quad \text{where } Re_v = \frac{m_v U_0 h_0^2}{\nu L}. \tag{3.9}$$

Using (3.7) and (2.17), we note that the product $g(x)U(x) = U_0 Re_v^{-1/2}$ is constant in the case of a Newtonian sheet. Thus, the velocities in (2.13) are independent of $f(\eta)$. Therefore, it is sufficient to solve (3.8) for $f'(\eta)$, which admits two exact solution branches, one of which is singular. For $\eta \geqslant 0$, the physically relevant solution is

$$f'(\eta) = \frac{6}{\left(\eta+\sqrt{6}\right)^2} \quad \text{with } f''(0) = -\frac{2}{\sqrt{6}}. \tag{3.10}$$

We note that asymptotically the fluid velocity f' decays away from the sheet as η^{-2}, which is much faster than the $\eta^{-1/3}$ decay obtained in the previous section. From the solution (3.10), given a viscous sheet drawn through a surrounding Newtonian fluid, the coupled system admits a self-similar solution provided that

$$\left(\frac{\rho_s}{\rho}\right)\left(m_v\left(\frac{\mu_{sh}}{\rho_s U_0 L}\right)-1\right)Re_v^{1/2} = \frac{2}{\sqrt{6}}. \tag{3.11}$$

Again, for given fluids and flow parameters, self-similarity for all x, from input to take-up, only occurs for a draw ratio (directly related to m_v) and sheet dimensions (h_0, L)

that satisfy (3.11). We also note that the half-thickness $h(x)$ of the viscous sheet decreases at a constant rate $\mathrm{d}h/\mathrm{d}x = -m_v(h_0/L)$.

Although we considered in this section the case of a viscous sheet, the results can be generalized to a sheet composed of a power-law fluid with a constitutive relation of the form $\sigma_{xx} = \mu_{sh}(\mathrm{d}U/\mathrm{d}x)^n$. However, it can be shown that our chosen case of a viscous Newtonian fluid is the only case that allows for a self-similar solution when the sheet inertia is kept in the stress balance (2.10). For a sheet consisting of an arbitrary power-law fluid, and using the ansatz given in (2.13), the sheet stress balance in (2.10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mu_{sh}}{U}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^n - \rho_s U\right) + \frac{\mu}{U_0 h_0^2}\frac{U(x)}{g(x)}f''(0) = 0. \tag{3.12}$$

After substituting (2.17) into (3.12) and using $\alpha = -1/(1+2c)$, it becomes clear that one cannot simultaneously balance all three terms in the power-law sheet stress balance (3.12) unless $n = 1$. Therefore, for $n \neq 1$, a self-similar solution is only possible if we neglect the inertia of the sheet, i.e. for large $\mu_{sh}/\rho_s U_0^{2-n} L$, which is a quantity that also appears in the similarity condition (3.11) where $n = 1$.

An examination similar to that above reveals that the system in this study does not admit a self-similar solution if the surrounding fluid is Newtonian and the stretching sheet is viscoelastic obeying a Kelvin–Voigt-type model of the form $\sigma_{xx} = E\varepsilon_{xx} + \mu_{sh}(\mathrm{d}U/\mathrm{d}x)^n$ for unidirectional stretching. However, a system consisting of a viscoelastic sheet of a given exponent n immersed in a power-law fluid of a given shear index might allow a solution of the self-similar form since a new degree of freedom is introduced.

4. Concluding remarks

In this study, we revisited the classical boundary-layer flow over a stretching sheet. To the best of our knowledge, all previous studies have prescribed the kinematics of the sheet. Here we treated the coupled fluid/sheet system and showed that it allows for a self-similar solution provided that a certain condition is satisfied. The mechanics of two types of stretching sheets are considered: an elastic sheet and a viscous sheet. The similarity condition is expressed, respectively, in (3.5) and (3.11) for the elastic sheet and the viscous sheet cases. These self-similar solutions could serve as a benchmark for computational studies.

The problem we considered in this paper can be extended to study the heat transfer between a stretching sheet and the surrounding fluid across a thermal boundary layer. Another possible extension would be to study the mass transfer from or to a permeable stretching sheet. However, we believe that the main contribution of our work is that our analysis provides a framework for studying boundary-layer flows over deformable surfaces that are shaped due to the stress induced by the outer fluid. Such problems occur in systems where there is fluid flow over various types of soft surfaces.

Acknowledgements

The authors thank I. C. Christov and the anonymous referees for their valuable input. We also thank the National Science Foundation for partial support under Grant No. CBET-1132835. T.T.A. is supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-0646086.

REFERENCES

ACRIVOS, A., SHAH, M. J. & PETERSEN, E. E. 1960 Momentum and heat transfer in laminar boundary-layer flows of non-Newtonian fluids past external surfaces. *AIChE J.* **6**, 312–317.

ANDERSSON, H. I. 2002 Slip flow past a stretching surface. *Acta Mechanica* **158**, 121–125.

ANDERSSON, H. I., AARSETH, J. B., BRAUD, N. & DANDAPAT, B. S. 1996 Flow of a power-law fluid film on an unsteady stretching surface. *J. Non-Newtonian Fluid Mech.* **62**, 1–8.

ANDERSSON, H. & KUMARAN, V. 2006 On sheet-driven motion of power-law fluids. *Intl J. Non-Linear Mech.* **41**, 1228–1234.

BANKS, W. H. H. 1983 Similarity solutions of the boundary-layer equations for a stretching wall. *J. Méc. Théor. Appl.* **2**, 375–392.

CRANE, L. J. 1970 Flow past a stretching plate. *J. Appl. Math. Phys. (Z. Angew. Math. Phys.)* **21**, 645–647.

DENN, M. M. 1980 Continuous drawing of liquids to form fibres. *Annu. Rev. Fluid Mech.* **12**, 365–387.

FANG, T., ZHANG, J. & ZHONG, Y. 2012 Boundary layer flow over a stretching sheet with variable thickness. *Appl. Math. Comput.* **218**, 7241–7252.

FOX, V. G., ERICKSON, L. E. & FAN, L. T. 1969 The laminar boundary layer on a moving continuous flat sheet immersed in a non-Newtonian fluid. *AIChE J.* **15**, 327–333.

GORLA, R. S. R., DAKAPPAGARI, V. & POP, I. 1995 Three-dimensional flow of a power-law fluid due to a stretching flat surface. *J. Appl. Math. Mech. (Z. Angew. Math. Mech.)* **75**, 389–394.

GUPTA, P. S. & GUPTA, A. S. 1977 Heat and mass transfer on a stretching sheet with suction or blowing. *Can. J. Chem. Engng* **55**, 744–746.

KUMARAN, V. & RAMANAIAH, G. 1996 A note on the flow over a stretching sheet. *Acta Mechanica* **116**, 229–233.

LAGE, J. L. & BEJAN, A. 1990 Convection from a periodically stretching plane wall. *Trans. ASME: J. Heat Transfer* **112**, 92–99.

LIAO, S. 2006 Series solutions of unsteady boundary-layer flows over a stretching flat plate. *Stud. Appl. Math.* **117**, 239–263.

MAGYARI, E. 2010 Translation groups of the boundary-layer flows induced by continuous moving surfaces. *J. Fluid Mech.* **655**, 327–343.

MAGYARI, E. & KELLER, B. 1999 Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface. *J. Phys.* D **32**, 2876–2881.

MALVERN, L. E. 1969 *Introduction to the Mechanics of a Continuous Medium*. Prentice Hall.

RAJAGOPAL, K. R., NA, T. Y. & GUPTA, A. S. 1984 Flow of a viscoelastic fluid over a stretching sheet. *Rheol. Acta* **23**, 213–215.

ROSENHEAD, L. 1963 *Laminar Boundary Layers*. Oxford University Press.

SAKIADIS, B. C. 1961 Boundary-layer behavior on continuous solid surfaces. Part 2. The boundary layer on a continuous flat surface. *AIChE J.* **7**, 221–225.

SCHEID, B., QUILIGOTTI, S., TRAN, B., GY, R. & STONE, H. A. 2009 On the (de)stabilization of draw resonance due to cooling. *J. Fluid Mech.* **636**, 155–176.

SURMA DEVI, C. D., TAKHAR, H. S. & NATH, G. 1986 Unsteady, three-dimensional, boundary-layer flow due to a stretching surface. *Intl J. Heat Mass Transfer* **29**, 1996–1999.

VLEGGAAR, J. 1977 Laminar boundary layer behavior on continuous, accelerating surfaces. *Chem. Engng Sci.* **32**, 1517–1525.

WANG, C. Y. 1984 The three-dimensional flow due to a stretching flat surface. *Phys. Fluids* **27**, 1915–1917.

WANG, C. Y. 2002 Flow due to a stretching boundary with partial slip: an exact solution of the Navier–Stokes equations. *Chem. Engng Sci.* **57**, 3745–3747.

WEIDMAN, P. D. & MAGYARI, E. 2009 Generalized Crane flow induced by continuous surfaces stretching with arbitrary velocities. *Acta Mechanica* **209**, 353–362.