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Periodic orbits of continuous mappings of the circle without fixed points

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Abstract. Let f be a continuous map of the circle to itself. Let $P(f)$ denote the set of periods of the periodic points. In this paper the set $P(f)$ is studied for functions without fixed points, so $1 \notin P(f)$. In particular, it is shown that if s, t are the two smallest integers in $P(f)$ and s and t are relatively prime then $\alpha s + \beta t \in P(f)$ for any positive integers α and β.

1. *Introduction*

Let f be a continuous map of the circle and let $P(f)$ denote the set of positive integers n such that f has a periodic point of period n. Block [**2**] has studied the structure of $P(f)$ when $1 \in P(f)$ and $n \in P(f)$. In the particular case when n is odd it is shown that, for every integer $m > n$, $m \in P(f)$ (see [**1**]).

In this paper the case $1 \notin P(f)$ is studied. It is shown that, if s, t are the two smallest elements of $P(f)$, and s and t are coprime, then

$$\alpha s + \beta t \in P(f)$$

for all positive integers α and β.

It is easily seen that any continuous map of the circle that does not have a fixed point is of degree one. Newhouse–Palis–Takens [**4**] have shown how to assign a rotation set to such a map; this is also done in [**3**]. Suppose that the rotation interval is $[a, b] \subset \mathbb{R}$. Then, in [**4**] and in [**3**, theorem 3.7], it is shown that, for any rational number $m/n \in [a, b]$, with m and n coprime, n belongs to $P(f)$.

The following example illustrates the difference between this result and theorem 1.

Consider a map that has a rotation interval of $[\frac{1}{3}, \frac{1}{2}]$. By the above result, 2 and 3 are contained in $P(f)$. Theorem 1 shows that $10 \in P(f)$. However, this conclusion cannot be drawn from the above result as none of $\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$ is contained in $[\frac{1}{3}, \frac{1}{2}]$.

THEOREM 1. *Let $f \in C^0(S, S)$. Suppose $1 \notin P(f)$. Let t, s be the two smallest elements of $P(f)$. Suppose that t and s are coprime. Then for any positive integers α, β,*

$$\alpha t + \beta s \in P(f).$$

2. *Preliminary definitions and results*

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{N}$ the positive integers and $S = \mathbb{R}/\mathbb{Z}$

the circle. Let $\pi:\mathbb{R}\to S$ denote the canonical projection. Let $f\in C^0(S,S)$ be a map that has no fixed points.

Choose the lift $\bar{f}:\mathbb{R}\to\mathbb{R}$ such that $\bar{f}$ is continuous, $0\le\bar{f}(0)<1$ and $\pi\bar{f}=f\pi$. Since f has no fixed point it is clear that

$$x<\bar{f}(x)<x+1$$

for any $x\in\mathbb{R}$. Thus

$$\bar{f}(x+1)=\bar{f}(x)+1,$$

so f is a map of degree one.

For any $n\in\mathbb{N}$ we define $\bar{f}^n$ and f^n inductively by

$$f^1=f,\quad \bar{f}^1=\bar{f}$$

and

$$f^n=f\circ f^{n-1},\qquad \bar{f}^n=\bar{f}\circ\bar{f}^{n-1}.$$

Let s, t be as in the statement of theorem 1. Choose $z_t\in S$ such that $f^t(z_t)=z_t$. Let $P=\{p_i\in\mathbb{R}\}$ such that

$$0\le p_0<\cdots<p_{t-1}<1$$

and

$$\pi(p_i)\in\{f^i(z_t)\mid i\in\mathbb{N}\}.$$

Given $k\in\mathbb{N}$ let

$$p_{kt+i}=p_i+k.$$

Choose $z_s\in S$ such that $f^s(z_s)=z_s$. Let $Q=\{q_i\in\mathbb{R}\}$ such that

$$0\le q_0<\cdots<q_{s-1}<1$$

and

$$\pi(q_i)\in\{f^i(z_s)\mid i\in\mathbb{N}\}.$$

Given $k\in\mathbb{N}$ let

$$q_{ks+i}=q_i+k.$$

Definition. Let $j(p_i)$ be the *jump* of p_i, defined by

$$j(p_i)=k-i,$$

where $\bar{f}(p_i)=p_k$. Similarly,

$$j(q_i)=k-i$$

where $\bar{f}(q_i)=q_k$.

Notes

(1) It is easily checked that the *jump* is well-defined.

(2) Since $x<\bar{f}(x)<1+x$, it is clear that $j(p_i)$ and $j(q_i)$ are positive integers.

(3) For any $k\in\mathbb{N}$ one has $j(p_i)=j(p_{kt+i})$ and $j(q_i)=j(q_{ks+i})$.

3. *Proof of theorem* 1

LEMMA 2. *The following are true*:

(1) $f^t(\pi[p_i,p_{i+1}])\supset\pi([p_i,p_{i+1}])$.

(2) $f^s(\pi[q_i,q_{i+1}])\supset\pi([q_i,q_{i+1}])$.

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Proof. The first statement will be proved; the second can be proved in a similar manner.

As $\pi(p_i)$ and $\pi(p_{i+1})$ are periodic, of period t, there exist positive integers k and l such that

$$\bar{f}^t(p_i) = k + p_i \quad \text{and} \quad \bar{f}^t(p_{i+1}) = l + p_i.$$

To prove the first statement it is enough to show that $l = k$.

Now

$$\bar{f}^t(p_i) - p_i = \sum_{j=1}^{t} [\bar{f}(\bar{f}^{j-1}(p_i)) - \bar{f}^{j-1}(p_i)].$$

Since

$$\bar{f}(p_{jt+i}) - p_{jt+i} = \bar{f}(p_i) - p_i,$$

one obtains

$$\bar{f}^t(p_i) - p_i = \sum_{j=1}^{t} [\bar{f}(p_j) - p_j].$$

Similarly,

$$\bar{f}^t(p_{i+1}) - p_{i+1} = \sum_{j=1}^{t} [\bar{f}(p_j) - p_j].$$

Thus $l = k$. □

In the rest of the paper it will be assumed, without loss of generality, that $t < s$.

In the proofs that follow simple use will be made of Markov graphs; for more information see [**3**] or [**5**].

LEMMA 3. *For any $i \in \mathbb{N}$ the jump of p_i is equal to the jump of p_0.*

Proof. For $0 \leq j < t$ let

$$I_j = \pi([p_{kt+j}, p_{kt+j+1}]).$$

Construct a directed graph (Markov graph), with vertices I_j and an edge $I_j \to I_k$ if and only if $f(I_j) \supset I_k$.

Using lemma 2 it can be seen that there is a loop starting and ending at I_0 of length t. There cannot be a shorter loop, as this would imply the existence of a periodic point with period less than t (see [**1**], [**2**] or [**3**]). Thus, there exists a permutation, σ, on $\{0, \ldots, t-1\}$ such that the Markov graph contains the graph shown in figure 1.

Suppose that the lemma is not true, then some interval, I_k, must be mapped onto at least two intervals. This is a contradiction, as it would imply the existence of a shorter loop. □

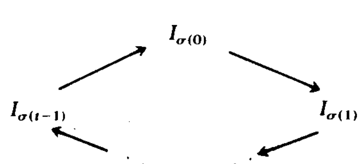

FIGURE 1

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Lemma 4. *If there exist two positive integers i, j such that $j(q_i) \neq j(q_j)$, then for any positive integers α, β the number $\alpha s + \beta t$ belongs to $P(f)$.*

Proof. For $0 \leq j < t$ let

$$I_j = \pi([q_{kt+j}, q_{kt+j+1}]).$$

As in the previous lemma, some interval, I_j, is mapped onto at least two intervals. This implies that there exists an interval, I_k, such that

$$f^r(I_k) \supset I_k, \quad \text{where } 0 < r < s.$$

Clearly, $r = t$ since t and s are the two smallest elements of $P(f)$.

By lemma 2,

$$f^s(I_k) \supset I_k.$$

So the Markov graph has two loops starting and ending at I_k, one of length s and the other of length t.

Let α and β be as in the statement of the lemma. Then there exists a periodic point of period $\alpha s + \beta t$ that corresponds to travelling around the 's length' loop α times and then travelling around the 't length' loop β times. □

The following lemma completes the proof of theorem 1.

Lemma 5. *Suppose for all i that $j(p_i) = u$ and $j(q_i) = v$. Then $\alpha s + \beta t \in P(f)$, for any positive integers α and β.*

Proof. Relabel the points in $P \cup Q$ by

$$m_0, m_1, \ldots, m_{s+t-1},$$

where

$$0 \leq m_0 < m_1 < \cdots < m_{s+t-1} < 1.$$

For $k \in \mathbb{N}$ let

$$m_{k(s+t)+i} = m_i + k.$$

Define $F : \mathbb{N} \to \mathbb{N}$ by $F(i) = k$, where $f(m_i) = m_k$.

Suppose that $m_i \in P$. Then

$$\bar{f}^t(m_i) - m_i = u$$

and, for any $r \in \mathbb{N}$, one has

$$\left[\frac{ru}{t}\right]\downarrow \leq \bar{f}^r(m_i) - m_i \leq \left[\frac{ru}{t}\right]\uparrow,$$

where [ ]↓ means round down to the nearest integer and [ ]↑ means round up to the nearest integer. Then one obtains

$$ru + \left[\frac{ru}{t}\right]\downarrow s \leq F^r(i) - i \leq ru + \left[\frac{ru}{t}\right]\uparrow s, \tag{1}$$

since after r iterates m_i has 'jumped' ru elements of P and between $[ru/t]\downarrow s$ and $[ru/t]\uparrow s$ elements of Q. Similarly, if $m_j \in Q$ one obtains

$$tv + \left[\frac{tv}{s}\right]\downarrow t \leq F^t(j) - j \leq tv + \left[\frac{tv}{s}\right]\uparrow t. \tag{2}$$

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Suppose that $u/t \le v/s$. Then choose an integer k with $0 \le k \le t+s-1$ such that $m_k \in P$ and $m_{k+1} \in Q$. Inequality (1) gives

$$F^s(k)-k \le su + \left[\frac{su}{t}\right]\uparrow s.$$

Since $u/t \le v/s$ one has

$$F^s(k)-k \le su+vs \le v(t+s).$$

However,

$$F^s(k+1)-(k+1) = v(t+s),$$

so

$$f^s\pi([m_k, m_{k+1}]) \supset [m_k, m_{k+1}].$$

Using the second inequality, one can show similarly that

$$f^t\pi([m_k, m_{k+1}]) \supset [m_k, m_{k+1}].$$

For $0 \le i < t+s$ let

$$I_i = \pi[m_{j(t+s)+i}, m_{j(t+s)+i+1}],$$

where j is any positive integer. From the above, the associated Markov graph has two loops starting at I_k, one of length t and the other of length s. Thus f has periodic points of the form $\alpha s + \beta t$ for any positive integers α and β.

The proof for the case $v/s < u/t$ is similar after choosing an integer $0 \le k \le t+s-1$ such that $m_k \in Q$ and $m_{k+1} \in P$. □

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