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# Unsteady flows with a zero acceleration on the free boundary

**E. A. Karabut† and E. N. Zhuravleva**

Lavrentyev Institute of Hydrodynamics, Akademika Lavrentyeva Prospekt 15, Novosibirsk, 630090, Russia

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A new approach to the construction of exact solutions of unsteady equations for plane flows of an ideal incompressible fluid with a free boundary is proposed. It is demonstrated that the problem is significantly simplified and reduces to solving the Hopf equation if the acceleration on the free surface is equal to zero. Some examples of exact solutions are given.

**Key words:** interfacial flows (free surface), waves/free-surface flows

## 1. Introduction

A plane potential unsteady flow of an ideal incompressible fluid with a free boundary is considered. There are no external and capillary forces.

Exact solutions for equations of this kind are quite rare. One class of motions discovered by Dirichlet (1860) is known. These are flows with a linear velocity field. The so-called Dirichlet ellipsoids describe various motions of a gravitating mass of the fluid, which has the form of an ellipsoid deformed with time. Riemann (1860) derived differential equations relating only the geometric characteristics of the second-order surface bounding the fluid. A brief derivation of the theory can be found in Lamb (1932, § 382). The full classification of solutions in the three-dimensional case was performed by Ovsyannikov (1967). The results were described in a lecture course (Nalimov & Pukhnachov 1975). Some individual cases of ellipsoid motion were studied by Lavrent'eva (1980, 1984).

In the two-dimensional case of these motions, the free surface is a second-order curve. Elliptic motion was studied by Taylor (1960), who used it for modelling the structure of the jet escaping from an elliptical orifice. The complete classification of solutions for the two-dimensional case was performed by Longuet-Higgins (1972). Ellipse evolution is studied in Pukhnachov (1978). The hyperbolic case was used by Longuet-Higgins (1980) to mode the phenomenon of gravity waves breaking on a fluid surface.

More recently, Karabut (2013) found a family of exact solutions of a self-similar problem of motion of a fluid with a free boundary. Deformation of a fluid wedge with an apex angle $\alpha$ with a quadratic initial velocity field was studied. A technique successfully applied previously in the problem of gravitational waves on a fluid

† Email address for correspondence: eakarabut@gmail.com

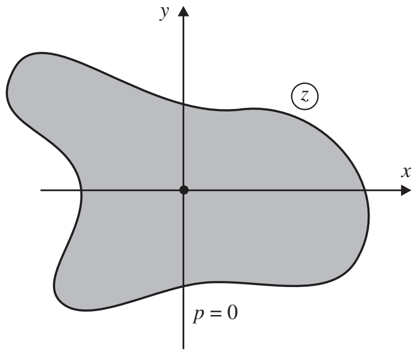


FIGURE 1. The domain occupied by the fluid. The domain boundary is a free boundary within the entire flow duration. The fluid motion is purely inertial. The time evolution of the domain is determined by the initial velocity field.

surface by Karabut (1996, 1998) was used. The essence of this technique is the analytical continuation of the unknown function beyond the domain of its definition. After multiple tracking around the wedge apex, various branches of the sought function are related, generally speaking, by an infinite system of ordinary differential equations. It was shown that the system acquires a finite form if $\alpha/\pi$ is a rational number. Explicit solutions of this system were found for some values of $\alpha$.

It turned out that the acceleration on the free boundary has a zero value for some of the found flows. Moreover, it was found that there is a limit as $\alpha \to 0$. In other words, a solution exists for an infinitely thin fluid wedge.

These two observations are of principal importance for constructing a new class of exact solutions. In this paper, we propose a method of constructing flows with a zero acceleration on the free surface.

## 2. Formulation of the problem

Let the fluid at the initial time $t=0$ occupy a certain domain in the plane of the complex variable $z=x+\mathrm{i}y$, where $x$ and $y$ are the Cartesian coordinates. Let also the initial velocity field of the fluid be given. We have to find the shape of the domain and the velocity field at subsequent time instants $t>0$ if the entire domain boundary is a free surface on which a zero pressure $p=0$ is maintained. The flow pattern is schematically shown in figure 1, where the grey region is occupied by the fluid, as well as in subsequent figures. The motion is purely inertial, i.e. it is induced by the specified initial velocity field. There are also other formulations of the problem, where the motion can be induced by sources, dipoles, or other specific hydrodynamic singularities with a prescribed intensity.

An example of the initial conditions defining the flow with a zero acceleration on the free boundary is demonstrated in figure 2(*a*). At the initial time $t=0$, the fluid occupies a half-plane $x>0$. On the free boundary $x=0$, the initial velocity of the fluid at $t=0$ is directed horizontally from the fluid and is a quadratic function of $y$. As will be shown below, each fluid particle on the free boundary in this problem moves with the same velocity with which it moved at the beginning of the process. As a result, the free boundary is a parabola. It should be noted that the problem solution is unknown if the initial velocity vector on the free boundary is oppositely directed (i.e. inwards into the fluid, as shown in figure 2*b*). In this case, the acceleration is not

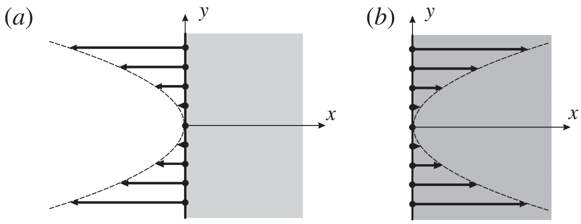


FIGURE 2. Example of initial conditions. At $t=0$, the fluid occupies the right-hand half-plane $x>0$; on the free surface $x=0$, the velocity is directed normal to the free surface and is a quadratic function of $y$. (*a*) The initial velocity is directed outwards from the fluid. (*b*) The initial velocity is directed inwards into the fluid. In case (*a*), we have a flow with a zero acceleration on the free boundary. The solution of this problem can be easily found. In case (*b*), we have a flow with a non-zero acceleration on the free boundary. The solution is unknown.

equal to zero, and the trajectories of fluid particles located on the free boundary are not straight lines.

The unknown function to be found is the complex velocity

$$U(z, t) = u(x, y, t) - \mathrm{i}v(x, y, t), \tag{2.1}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the components of the velocity vector. The complex velocity is an analytical function of the complex variable $z$. It is a derivative with respect to $z$ of the complex potential

$$U = \Phi_z. \tag{2.2}$$

The complex potential is described by the formula

$$\Phi(z, t) = \varphi(x, y, t) + \mathrm{i}\psi(x, y, t), \tag{2.3}$$

where $\varphi(x, y, t)$ and $\psi(x, y, t)$ are the velocity potential and stream function, respectively.

The function $U(z, t)$ is found from a nonlinear boundary-value problem, which has to be solved in the domain occupied by the fluid. Moreover, the time-dependent domain is unknown in advance and should be found in the course of solving the problem. The function $U(z, t)$ should be holomorphic in this domain. Two boundary conditions (kinematic and dynamic) are imposed on the free surface.

The kinematic condition is formulated as follows. At all times $t$, the free boundary consists of the same fluid particles. If the free surface is defined by the equation

$$h(x, y, t) = 0, \tag{2.4}$$

then the kinematic condition can be written as

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \quad \text{at } h(x, y, t) = 0. \tag{2.5}$$

Let us clarify where (2.5) comes from. This equation contains the total derivative with respect to time. This derivative characterizes the change in the function with time in a fixed fluid particle. Let the fluid particle be located on the free surface at the time $t_0$; therefore, the function $h$ at the point where this particle is located has a zero value. At the time $t_0 + \Delta t$, both the free boundary and the fluid particle are shifted. Condition (2.5) means that the function $h$ remains unchanged in a material fluid particle with a small change in time $\Delta t$, i.e. it still has a zero value, which means that the considered material point remains on the free boundary at the time $t_0 + \Delta t$.

In the particular case where the free surface is defined by the equation $y = f(x, t)$, the kinematic condition (2.5) takes the form

$$f_t + f_x u = v \quad \text{at } y = f(x, t). \tag{2.6}$$

If the free surface is defined by another equation $x = g(y, t)$, then the kinematic condition (2.5) is written in another form as

$$g_t + g_y v = u \quad \text{at } x = g(y, t). \tag{2.7}$$

The dynamic condition in the general case means that a certain pressure $p$ is prescribed on the free surface. In this work, we assume that the pressure applied on the free boundary is a function of time only:

$$p = p_0(t). \tag{2.8}$$

The function $p_0(t)$ may be arbitrary because the fluid velocity and the shape of the free boundary are independent of the form of this function. Indeed, from the Euler equation

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p, \tag{2.9}$$

the pressure is always determined only with accuracy to a term equal to an arbitrary function of time $t$. Therefore, without loss of generality, we can assume that the dynamic condition on the free boundary has the form

$$p = 0. \tag{2.10}$$

Let us draw a tangent line to the free boundary at a certain point of the free boundary and project the acceleration vector evaluated at the same point onto the tangent line. Such a projection is called the projection of the acceleration onto the free boundary.

It follows from (2.9) that, at each time instant, the acceleration is in the direction perpendicular to lines of constant pressure throughout the flow. In particular, this condition is valid for the free boundary on which condition (2.10) is satisfied. Thus, if there is a free boundary, then the projection of the acceleration onto the free boundary is equal to zero.

Therefore, the dynamic condition can be reformulated as follows. The projection of the acceleration onto the free boundary should be equal to zero at each point of the free boundary. Indeed, if this statement is satisfied, then it follows from (2.9) that the projection $\nabla p$ onto the free boundary is also equal to zero. Therefore, the derivative of pressure in the direction along the free boundary is also equal to zero at each point of the free boundary, whence (2.8) follows.

## 3. Zero acceleration lines

An arbitrary analytical function $U(z, t)$ always describes a certain fluid flow, though possibly in a multisheet domain, because the real and imaginary parts of the analytical function always satisfy the Laplace equation, i.e. the condition of incompressible, irrotational motion in two dimensions. However, this flow is not necessarily a flow with a free boundary. For a given function $U(z, t)$, the Bernoulli integral always yields lines on which the pressure is constant (or is a function of time). It is also always possible to find material lines, i.e. lines consisting of the same fluid particles. It is always possible to identify material lines along contours of constant pressure at one instant in time. It is not obvious, however, that any pair of these instantaneous pressure contours and material lines will continue to coincide as time proceeds.

In this paper, we attempt to find flows that satisfy the equations of motion, the free boundary conditions and two additional arbitrary constraints. The first constraint is that the acceleration of the fluid particles on the free surface be zero. The second additional constraint is that $U(z, t)$ must also satisfy the complex Hopf equation

$$U_t + UU_z = 0. \tag{3.1}$$

For such flows, it was noted that lines with a zero acceleration vector are simultaneously isobars and material lines. Such zero acceleration lines moving and deforming with time are found by solving the system of equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = 0. \tag{3.2$a$,$b$}$$

In most cases, however, for an arbitrary function $U(z, t)$, system (3.2) either is incompatible or its solution is several points in the $x$–$y$ plane rather than a line. Indeed, each of the equations of this system yields a curve, and the system solution is the points of intersection of these curves.

In the complex form, the acceleration is found by the formula

$$\frac{\mathrm{d}U}{\mathrm{d}t} = U_t + \overline{U}U_z. \tag{3.3}$$

Therefore, to find lines of zero acceleration for solutions that also satisfy the Hopf equation, we have to solve the system

$$\begin{cases} U_t + UU_z = 0, \\ U_t + \overline{U}U_z = 0. \end{cases} \tag{3.4}$$

Subtracting the second equation from the first one, we obtain the corollary

$$vU_z = 0. \tag{3.5}$$

The second factor cannot be equal to zero because $U \neq \text{const}$.

Thus, we obtain the following statement: if the complex velocity $U(z, t)$ satisfies the Hopf equation (3.1), then the equation

$$v(x, y, t) = 0 \tag{3.6}$$

yields lines on which the acceleration is equal to zero.

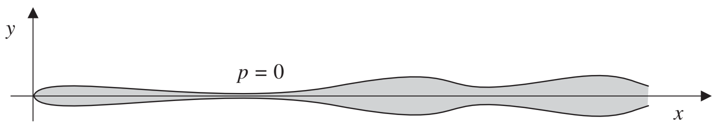


FIGURE 3. Auxiliary problem with a free boundary. The thin strip filled by the fluid is aligned along the $x$ axis. The initial velocity of the fluid at $t = 0$ is directed along the $x$ axis. The strip boundary is the free surface. The fluid velocity at subsequent times $t > 0$ can be found for an infinitely thin strip.

Let us demonstrate that line (3.6) is a free boundary. The dynamic condition in the new formulation given above is satisfied because the projection is equal to zero if the acceleration vector is equal to zero. The kinematic condition (2.5) is rewritten as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \quad \text{at } v(x, y, t) = 0. \tag{3.7}$$

It follows from the second of equations (3.2) that condition (3.7) is satisfied.

Each fluid particle on the free boundary (3.6) moves with a constant velocity parallel to the $x$ axis.

Thus, the algorithm for constructing a flow with a free boundary looks as follows.

*Step* 1. Take $U(z, t)$, which is an arbitrary solution of the complex Hopf equation.

*Step* 2. Find the imaginary part $v(x, y, t)$ of this solution.

*Step* 3. Find the free boundaries from the equation $v(x, y, t) = 0$.

*Step* 4. Check on which side of the free boundary the fluid should be located. For this purpose, we analyse the singular points of the function $U(z, t)$ and make them lie outside the fluid.

## 4. Auxiliary flow

To find a physically meaningful solution of the Hopf equation, we consider the following auxiliary hydrodynamic problem. Let the fluid occupy a thin strip located symmetrically along the positive part of the $x$ axis (figure 3). At the initial time $t = 0$, the components of the velocity vector on the $x$ axis have the form

$$u(x, 0, 0) = F(x), \qquad v(x, 0, 0) = 0. \tag{4.1a,b}$$

The boundary of this strip is considered as a free surface where a zero pressure is maintained. We have to find the time evolution of this strip, i.e. to find the shape of the boundaries and the fluid velocity at $t > 0$.

Such a problem with a free boundary can be solved for a very thin strip. Let the strip be so thin that the pressure across the strip remains unchanged. Then the pressure inside the fluid is the same as that on the free boundary. Therefore, the pressure gradient along the strip is equal to zero and, as a consequence, the acceleration also has a zero value. Thus, the real Hopf equation is satisfied on the $x$ axis:

$$u_t + uu_x = 0. \tag{4.2}$$

Each fluid particle in an infinitely thin strip moves with the velocity it had at the initial time. The general solution of (4.2) is

$$u = F(x - ut). \tag{4.3}$$

Let us perform an analytical continuation for both (4.2) and its solution (4.3) from the real axis to the entire complex plane $z$. Replacing $u$ and $x$ by $U$ and $z$, we find that the complex velocity described by the equation

$$U = F(z - Ut) \tag{4.4}$$

satisfies the complex Hopf equation (3.1). It should be noted that other examples of exact solutions appear in addition to the initial flow of the fluid strip in the case of such an analytical continuation.

Assuming that $t = 0$ in (4.4), we obtain

$$U(z, 0) = F(z). \tag{4.5}$$

Thus, the function $F(z)$ defines the initial data. Choosing arbitrary functions $F(z)$, we can obtain many different flows. In this paper, we consider the following cases:

$$F(z) = z^2/4, \quad F(z) = z, \quad F(z) = k_1 z + k_2 z^2, \quad F(z) = \sqrt{z}. \tag{4.6a–d}$$

## 5. Evolution of the half-plane

The first case

$$F(z) = z^2/4 \tag{5.1}$$

yields flows with initial data presented in figure 2. Let us consider this case in more detail than the other cases. Using the above-proposed algorithm, we construct flows with free boundaries and then check that all boundary conditions for these flows are satisfied.

Let us first find the solution of the auxiliary problem. Equation (4.3) has the form

$$u = (x - ut)^2/4. \tag{5.2}$$

Solving this equation, we find

$$u(x, 0, t) = \left(\frac{\sqrt{1 + xt} - 1}{t}\right)^2. \tag{5.3}$$

Thus, if the velocity at the initial time $t = 0$ is given by (5.1) and each fluid particle moves with a constant velocity, then the velocity profile at subsequent times is given by (5.3).

Let us apply an analytical continuation of the solution of (5.3) from the real axis to the entire complex plane $z$. We obtain a complex velocity

$$U(z, t) = \left(\frac{\sqrt{1 + zt} - 1}{t}\right)^2, \tag{5.4}$$

which is the solution of the complex Hopf equation (3.1).

In (5.4), we take the square-root branch for which $\sqrt{1 + zt}$ at $z = 0$ is equal to unity. It follows from here that $U(0, t) = 0$. We can show that the acceleration and all subsequent higher-order derivatives of velocity with respect to time at the point $z = 0$

are also equal to zero. This means that a fluid particle initially located at the origin of the coordinate system stays there all the time.

The free surface of the infinitely thin strip whose evolution was considered above passes through the origin. Therefore, we can naturally assume that the pressure in the flow described by (3.1) is equal to zero at the origin:

$$p|_{z=0} = 0. \tag{5.5}$$

Solution (5.4) contains the branching point

$$z^* = -1/t. \tag{5.6}$$

This singular point should be outside the fluid.

### 5.1. *Construction of the free surface*

To construct the free surface, we have to take the imaginary part in (5.4) and equate it to zero. It is difficult to make it directly because it contains a radical. Let us eliminate the radical by introducing a complex parameter

$$\zeta = \sqrt{1+zt} - 1. \tag{5.7}$$

Now, instead of the complex plane $z$, we have the complex plane $\zeta$. Formula (5.7) ensures correspondence between the planes, and the point $\zeta = 0$ corresponds to the point $z = 0$.

Instead of solution (5.4) consisting of one formula, we obtain a solution that does not contain radicals, but consists of two formulae:

$$U(\zeta, t) = \left(\frac{\zeta}{t}\right)^2, \tag{5.8}$$

$$z(\zeta, t) = \frac{\zeta^2 + 2\zeta}{t}. \tag{5.9}$$

Substituting $\zeta = \xi + \mathrm{i}\eta$ into (5.8) and taking the real and imaginary parts, we obtain

$$u = \frac{\zeta^2 - \eta^2}{t^2}, \quad v = -\frac{2\xi\eta}{t^2}. \tag{5.10a,b}$$

Therefore, (3.6) is satisfied in the plane $\zeta$ on two straight lines

$$\eta = 0 \tag{5.11}$$

or

$$\xi = 0. \tag{5.12}$$

Let us find which curves in the plane $z$ correspond to these straight lines. It follows from (5.9) that

$$x = \frac{\xi^2 - \eta^2 + 2\xi}{t}, \quad y = \frac{2\xi\eta + 2\eta}{t}. \tag{5.13a,b}$$

Substituting (5.11) here, we obtain the first free surface in the form of a ray:

$$y = 0 \quad (x \geqslant -1/t). \tag{5.14}$$

Substituting (5.12), we obtain

$$x = -\eta^2/t, \quad y = 2\eta/t. \tag{5.15a,b}$$

Thus, the second free surface has the form of a parabola:

$$x = -ty^2/4. \tag{5.16}$$

### 5.2. *Verification of the dynamic condition*

Let us find the pressure on both free boundaries (5.14) and (5.16). Integrating (2.2) and assuming that $\varPhi=0$ at the origin, we obtain an expression for the complex potential:

$$\varPhi(z,t)=\frac{2z}{t^2}+\frac{z^2}{2t}-\frac{4}{3t^3}[(1+zt)^{3/2}-1]. \tag{5.17}$$

To find the pressure, we have to use the Bernoulli integral

$$\mathrm{Re}\,\varPhi_t+\frac{1}{2}|U|^2+\frac{p}{\rho}=c(t). \tag{5.18}$$

From condition (5.5), we find $c(t)=0$. It is easier to perform calculations if we pass from the variable $z$ to the variable $\zeta$. In this case, the potential

$$\varPhi(\zeta,t)=\frac{3\zeta^4+4\zeta^3}{6t^3} \tag{5.19}$$

should be substituted into the Bernoulli integral

$$\mathrm{Re}(\varPhi_t+\varPhi_\zeta\zeta_t)+\frac{1}{2}|U|^2+\frac{p}{\rho}=0. \tag{5.20}$$

We find the derivative $\zeta_t$ in (5.20) by differentiating (5.9) with respect to $t$ and taking into account that $z_t=0$:

$$\zeta_t=\frac{1}{2t}\frac{\zeta^2+2\zeta}{\zeta+1}. \tag{5.21}$$

Substituting all these functions into the Bernoulli integral (5.20), we obtain

$$\frac{p}{\rho}=\frac{\mathrm{Re}\,\zeta^4-|\zeta|^4}{2t^4}. \tag{5.22}$$

From here, we see that the condition $p=0$ is satisfied on lines (5.11) and (5.12). Thus, lines (5.14) and (5.16) are isobars.

It is seen from (5.22) that an example with a negative pressure inside the fluid is constructed. This is not typical, but such situations are observed in hydrodynamics. For instance, if the fluid contains a source, a sink, a dipole, or a point vortex, then the pressure in the vicinity of these hydrodynamic singularities is negative.

### 5.3. *Verification of the kinematic condition*

Condition (2.6) for isobar (5.14) is satisfied because $f=0$ and $v=0$. Thus, ray (5.14) is a free boundary.

Let us check whether another kinematic condition (2.7) is satisfied on isobar (5.16). First, substituting $\xi=0$ into (5.10), we find the velocity vector components

$$v=0,\quad u=-\eta^2/t^2. \tag{5.23a,b}$$

Then, using the second formula in (5.15), we obtain $u=-y^2/4$. Taking into account that $g=-ty^2/4$, we verify that the kinematic condition is satisfied. Parabola (5.16) is a free boundary.

### 5.4. *Calculation of the acceleration*

The free surface (5.14) coincides with the thin strip of the fluid in the auxiliary problem. Therefore, the acceleration on this free surface is equal to zero. Let us verify that the acceleration on the second free boundary (5.16) is also equal to zero.

In the complex form, the acceleration is found by (3.3). Replacing here the variable $z$ by the variable $\zeta$, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}t} = U_t + U_\zeta \zeta_t + \overline{U}(U_\zeta \zeta_z). \tag{5.24}$$

Let us substitute the complex velocity (5.8), the previously calculated derivative (5.21) and the derivative

$$\zeta_z = \frac{t}{2(\zeta+1)} \tag{5.25}$$

obtained from (5.9) into (5.24). As a result, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\overline{\zeta^2}\zeta - \zeta^3}{t^3(\zeta+1)}. \tag{5.26}$$

It is seen that $\mathrm{d}U/\mathrm{d}t = 0$ at $\zeta = \mathrm{i}\eta$. Thus, the acceleration on the free surface (5.16) is equal to zero. Each fluid particle on this surface moves parallel to the $x$ axis with a constant velocity.

### 5.5. *Evolution of the free surface*

Deformation of the free surface (5.16) with time is illustrated in figure 4. The marker dot shows the location of the singularity $z^*$. The fluid should be located on the right of the parabola because the fluid should not contain singular points. An unsteady problem with a free boundary based on the initial data shown in figure 2(*a*) is solved. The fluid initially occupying a half-plane finally transforms to a plane with a straight-line cut. The singularity reaches the free surface within an infinite time.

If the fluid in figure 4 is located on the left of the free boundary, then the singularity $z^*$ is located at infinity at the initial time $t = 0$, and the initial velocity of the fluid on the free surface is directed inwards into the fluid. Thus, we obtain the initial conditions shown in figure 2(*b*). However, the found solution does not fit the problem with such initial conditions because it contains a singular point in the fluid.

The $x$ axis in figure 4 is not only the line of symmetry, but also the free surface (5.14). Therefore, considering the upper half of the flow in figure 4, we obtain the third flow. It is illustrated in figure 5. We have found an interesting example of the flow where some part of the free surface remains a straight line during the entire flow phenomenon.

## 6. Replacement of variables

We did not construct all the flows described by (5.4). Obvious inequality of the method is worth noting: we used (3.6), but we did not use the similar equation

$$u(x, y, t) = 0. \tag{6.1}$$

Let us now find the lines of zero acceleration not among the solutions of the Hopf equation (3.1), but among the solutions of the affine equation

$$U_t - UU_z = 0. \tag{6.2}$$

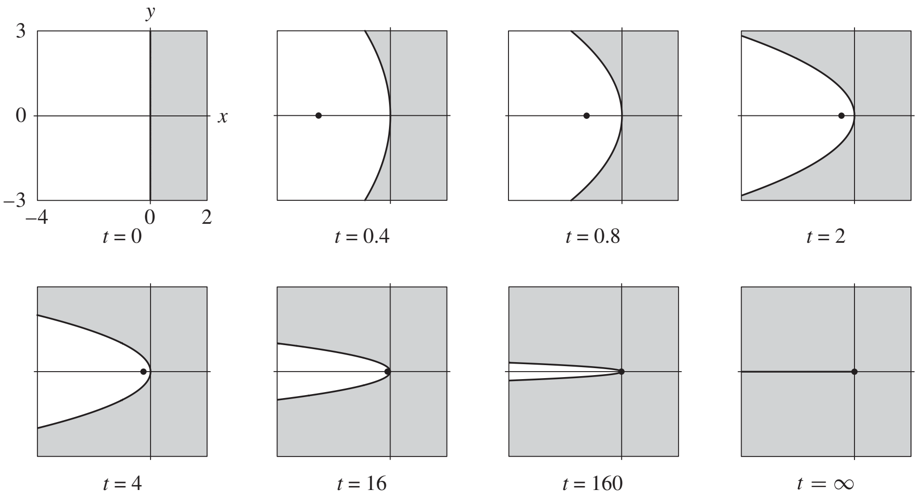


FIGURE 4. Evolution of the free surface with the initial data presented in figure 2($a$). At the initial time $t=0$, the fluid occupies the right-hand half-plane $x>0$. The initial velocity field is $U(z,0)=z^2/4$. The free surface is a parabola $x=-ty^2/4$. Within an infinite time, the half-plane transforms to a plane with a cut. The marker dot shows the virtual singularity $z^*=-1/t$.

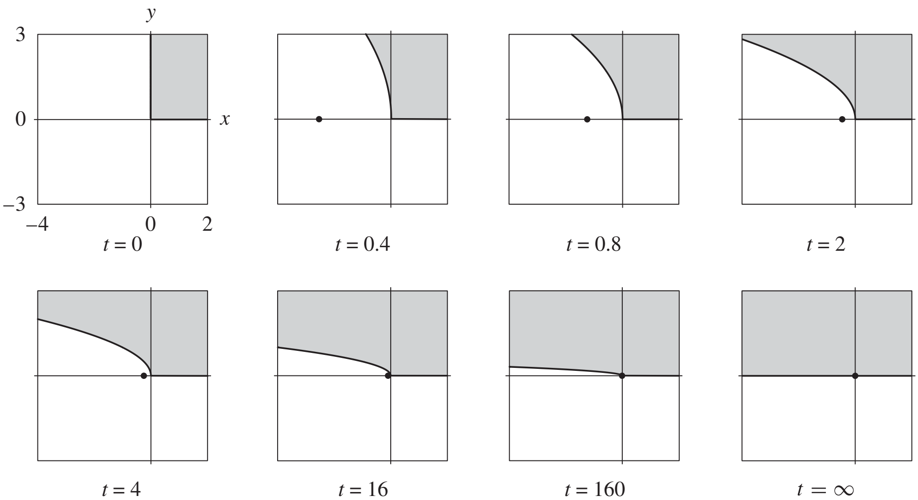


FIGURE 5. Example of an exact solution with a free boundary. One half of the flow presented in figure 4 is shown here. The domain initially occupied by the fluid is a wedge with a right angle at the apex. This wedge transforms to a half-plane with time. The free surface consists of a deforming parabola and a stationary ray. The marker dot shows the solution singularity.

For this purpose, we have to solve the system of equations

$$\begin{cases} U_t - UU_z = 0, \\ U_t + \overline{U}U_z = 0, \end{cases} \tag{6.3}$$

which has a corollary (6.1).

There are four replacements of dependent and independent variables that transform the solution of the Hopf equation to the solution of (6.2):

(i) $t \to -t$;
(ii) $z \to -z$;
(iii) $U \to -U$;
(iv) $t \to -t$, $z \to -z$, $U \to -U$.

Therefore, to find new exact solutions with a free boundary, we have to apply one of these replacements in the solution of the Hopf equation, identify the real part of the complex velocity, and equate it to zero. This procedure yields the equation of the free surface.

It should be noted that other new exact solutions can be obtained by passing from the Hopf equation not to (6.2), but to the more general equation

$$U_t + e^{i\theta} U U_z = 0, \tag{6.4}$$

where $\theta$ is an arbitrary real parameter.

Let us apply the following replacements in solution (5.4):

$$U \to -U, \quad z \to -z, \quad t \to -t. \tag{6.5$a$–$c$}$$

Then we obtain the complex velocity

$$U(z, t) = -\left(\frac{\sqrt{1+zt}-1}{t}\right)^2, \tag{6.6}$$

which is the solution of (6.2). Using the complex parameter $\zeta$ still defined by (5.7), we obtain the solution with no radicals:

$$U(\zeta, t) = -\left(\frac{\zeta}{t}\right)^2, \tag{6.7}$$

$$z(\zeta, \tau) = \frac{\zeta^2 + 2\zeta}{t}. \tag{6.8}$$

Now we have to identify the real part of the complex velocity. Substituting $\zeta = \xi + i\eta$ into (6.7), we obtain

$$u = -\frac{-\xi^2 + \eta^2}{t^2}, \quad v = \frac{2\xi\eta}{t^2}. \tag{6.9$a$,$b$}$$

Thus, (6.1) in the plane $\zeta$ is satisfied along two straight lines:

$$\xi = \eta \tag{6.10}$$

or

$$\xi = -\eta. \tag{6.11}$$

Let us check how these straight lines look in the plane $z$. From (6.8), we have

$$x = \frac{\xi^2 - \eta^2 + 2\xi}{t}, \quad y = \frac{2\xi\eta + 2\eta}{t}. \tag{6.12$a$,$b$}$$

Substituting (6.10) into these formulae, we obtain the first free surface

$$x = \frac{2\eta}{t}, \quad y = \frac{2\eta^2 + 2\eta}{t} \tag{6.13$a$,$b$}$$

or

$$y = x + \frac{tx^2}{2}. \tag{6.14}$$

Substituting (6.11), we obtain the second free surface

$$x = -\frac{2\eta}{t}, \quad y = \frac{-2\eta^2 + 2\eta}{t} \tag{6.15$a$,$b$}$$

or

$$y = -x - \frac{tx^2}{2}. \tag{6.16}$$

From the Bernoulli integral, we obtain the following formula instead of (5.22):

$$\frac{p}{\rho} = -\frac{\mathrm{Re}\,\zeta^4 + |\zeta|^4}{2t^4}. \tag{6.17}$$

Therefore, lines (6.10) and (6.11) (and also lines (6.14) and (6.16)) are isobars.

Let us now demonstrate that they are also material lines. Substituting the first straight line (6.10) into (6.9), we obtain the velocity components on isobar (6.14): $u = 0$, $v = 2\eta^2/t^2$. Further, using the first formula in (6.13), we obtain $v = x^2/2$. Substituting the found velocity components and the function $f = x + tx^2/2$ into the kinematic condition (2.6), we verify that it is satisfied. Thus, parabola (6.14) is a free surface. The fact that the other parabola (6.16) is also a free surface is proved in a similar manner.

The formula for the fluid acceleration differs from (5.26):

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\overline{\zeta^2}\zeta + \zeta^3}{t^3(\zeta + 1)}. \tag{6.18}$$

Using (6.18), we can easily show that the acceleration is equal to zero on both free surfaces. Each fluid particle on the free surface moves with a constant velocity parallel to the $y$ axis.

Figure 6($a$) shows the two parabolae (6.14) and (6.16). An analysis of the singular points and branches of the solution shows that three different flows are possible here.

The first flow with a free boundary located at $x > 0$ is shown in figure 7. The marker dot shows the singularity $z^*$. Being initially located at infinity, it finally moves to the free surface. A flow in which a fluid wedge with a right angle at the apex transforms to a half-plane is found.

One more fluid flow located between the two free surfaces (6.14) and (6.16) is illustrated in figure 8. The fluid wedge with a right angle whose bisector is directed along the $y$ axis transforms to a half-plane with time.

It should be noted that an interesting feature of the two flows shown in figures 7 and 8 is the identical shape of the free surface. Therefore, one more flow can be constructed by matching the upper part of the flow in figure 7 to the flow in figure 8. Such a combined flow is shown in figure 9. To avoid a multisheet phenomenon, a solid wall is placed along the $x$ axis. The dotted curve shows some part of parabola (6.14). In this situation, it can be called an internal free boundary because this isobar consisting of the same particles is located inside the fluid. We have found a flow for which a wedge of fluid with initial angle $3\pi/4$ evolves towards a wedge of final angle $3\pi/2$.

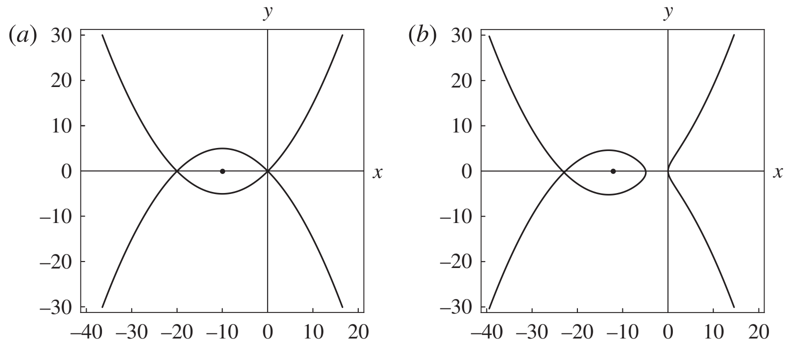


FIGURE 6. Lines that are simultaneously isobars and material lines. (*a*) Parabolae $y=\pm(x+tx^2/2)$ and singularity $z^*=-1/t$ at $t=0.1$. The panel illustrates the flow defined by (6.6). (*b*) Lines defined by the equation $y^2=x(x+4)(xt/2+1+t)^2$ and singularity $z^*=-(1+t)^2/t$ at $t=0.1$. The panel illustrates the flow defined by (8.25).

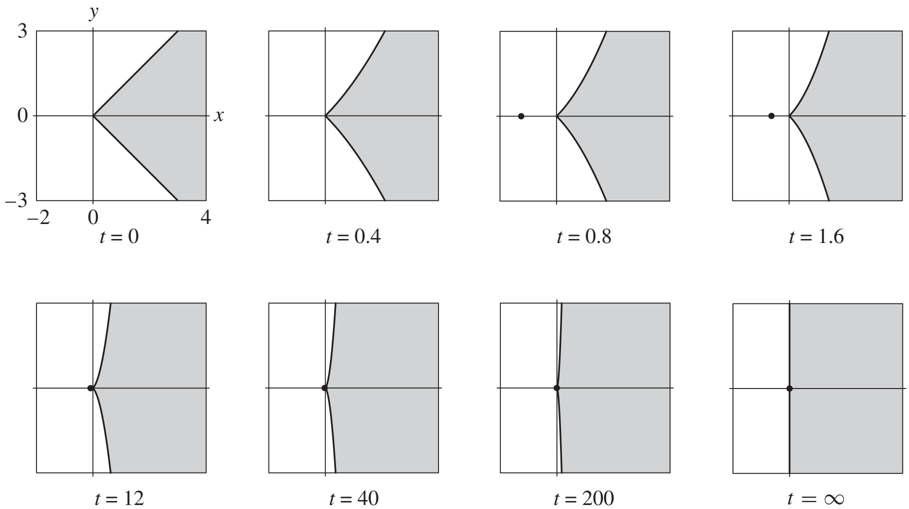


FIGURE 7. Evolution of the fluid domain initially shaped as a wedge with a right angle at the apex. The initial velocity field is $U(z,0)=-z^2/4$. The free surface consists of two parabolae $y=\pm(x+tx^2/2)$ $(x\geqslant 0)$. The free surface at the origin has a sharpening point with a right angle, which remains unchanged within the entire flow duration. The marker dot shows the singularity $z^*=-1/t$.

## 7. Flow with a linear velocity field

We considered the case with a quadratic velocity field along a thin strip for the auxiliary flow. Let us also consider the case with a linear velocity field. Let the initial complex velocity be

$$F(z)=z. \tag{7.1}$$

Then we find from (4.4) that

$$U(z,t)=\frac{z}{1+t} \tag{7.2}$$

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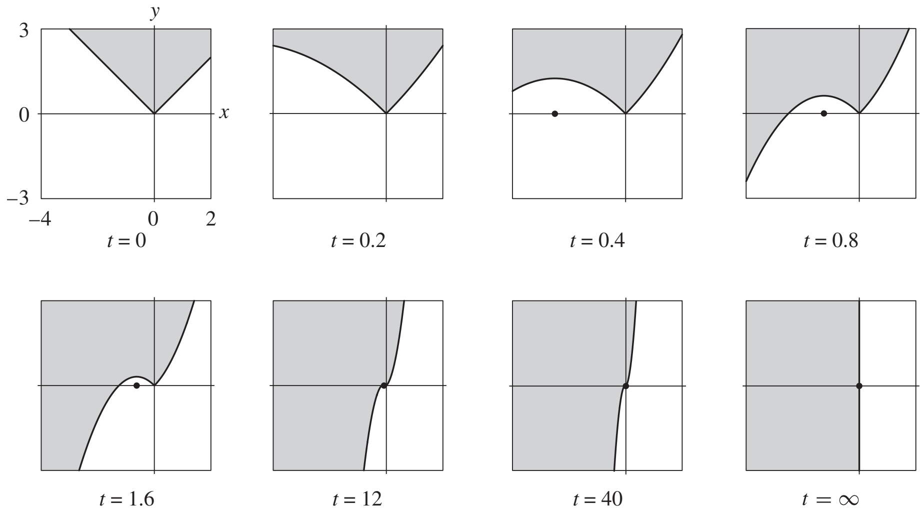


FIGURE 8. Evolution of the fluid domain initially shaped as a wedge with a right angle at the apex. The initial velocity field is $U(z, 0) = -z^2/4$. The free surface consists of two parabolae $y = x + tx^2/2$ $(x \geqslant 0)$ and $y = -x - tx^2/2$ $(x \leqslant 0)$. At the origin, the free surface has a sharpening point with a right angle, which does not change with time. The marker dot shows the singularity $x^* = -1/t$.

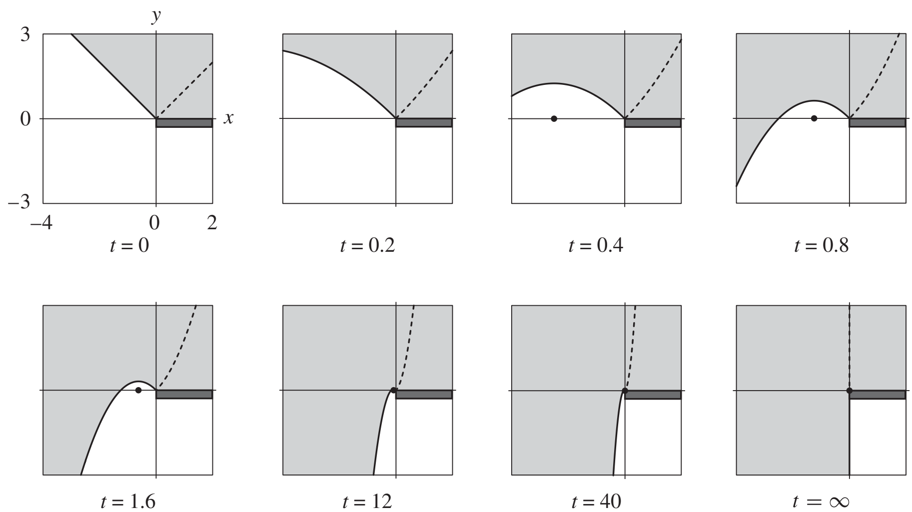


FIGURE 9. Example of a flow with a free boundary obtained by matching two flows from figures 7 and 8. The fluid initially occupies a wedge with an angle $3\pi/4$. One boundary of the wedge is a free boundary; the other boundary of the wedge is a motionless solid wall. The angle between the free boundary and the solid wall at the origin remains unchanged within the entire flow duration. The initial velocity field is $U(z, 0) = -z^2/4$. The dotted curve is the line inside the fluid on which the condition of a constant pressure and the kinematic condition are satisfied. The marker dot shows the singularity $z^* = -1/t$.

at subsequent times. Taking the real and imaginary parts, we obtain

$$u(x, y, t) = \frac{x}{1+t}, \quad v(x, y, t) = -\frac{y}{1+t}. \tag{7.3a,b}$$

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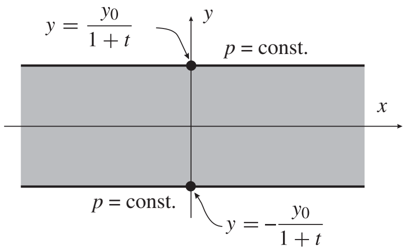


FIGURE 10. Example of a flow with a free boundary. If the fluid occupies a strip of width $2y_0$ at the initial time $t=0$ and the initial velocity is $U(z,0)=z$, then the fluid domain retains the shape of a strip at subsequent times $t>0$. The complex velocity is $U(z,t)=z/(1+t)$. The fluid motion at the $x$ axis occurs with a zero acceleration.

This is a known flow with a linear velocity field, which describes the evolution of the strip. It is illustrated in figure 10.

Equation (3.6) yields only one free surface $y=0$. It is on this surface that the acceleration equals zero. There are also other free surfaces

$$y = \pm \frac{y_0}{1+t}, \tag{7.4}$$

but our method does not find them because the acceleration on these surfaces differs from zero. The pressure is given by the formula

$$\frac{p}{\rho} = -\frac{y^2}{(1+t)^2} + \frac{{y_0}^2}{(1+t)^4}. \tag{7.5}$$

It is seen that the pressure at $y=0$ is a function of time.

## 8. Non-self-similar flows

If we assume that

$$F(z) = k_1 z \tag{8.1}$$

or

$$F(z) = k_2 z^2, \tag{8.2}$$

then the flows with such initial data contain only one dimensional constant: $k_1$ or $k_2$. However, regardless of the choice of these real constants, we obtain the same solutions that were obtained above for $k_1 = 1$ and $k_2 = 1/4$. Indeed, the solution written in the dimensionless form is independent of $|k_1|$ and $|k_2|$. Moreover, we can show that the change in the sign of $k_1$ and $k_2$ does not yield new solutions.

Problem (8.2) is self-similar and can be solved by a method proposed in Karabut (2013). Let us consider a superposition of (8.1) and (8.2). Let the initial velocity be

$$F(z) = k_1 z + k_2 z^2, \tag{8.3}$$

where both numbers $k_1$ and $k_2$ differ from zero. This problem is no longer self-similar and cannot be solved by the method proposed in Karabut (2013).

### 8.1. *Galileo transform*

Identifying the perfect square in (8.3), we can rewrite the initial velocity in the form

$$U+\frac{k_1^2}{4k_2}=k_2\left(z+\frac{k_1}{2k_2}\right)^2. \tag{8.4}$$

Let us pass to a new coordinate system moving with a constant velocity with respect to the initial coordinate system. Let the Galileo transform have the form

$$z=z'-\frac{k_1(2+k_1t)}{4k_2},\quad U=U'-\frac{k_1^2}{4k_2}. \tag{8.5$a,b$}$$

Then the initial velocity in the new coordinate system can be written as

$$U'=k_2(z')^2, \tag{8.6}$$

which coincides with (8.2). Thus, we have reduced the problem to that studied previously. Therefore, we have grounds to suspect that new solutions cannot be obtained by using the initial data (8.3). Let us show that this suspicion is only partly valid.

Solving (4.4), which has the form

$$U=k_1(z-Ut)+k_2(z-Ut)^2 \tag{8.7}$$

for (8.3), we obtain

$$U(z,t)=\frac{z}{t}-\frac{\sqrt{(1+k_1t)^2+4zk_2t}-(1+k_1t)}{2k_2t^2}. \tag{8.8}$$

Identifying here the imaginary part and equating it to zero, we find the equation of the free boundary:

$$x=-k_2ty^2-\frac{k_1(2+k_1t)}{4k_2}. \tag{8.9}$$

Substituting the Galileo transform (8.5) into (8.9) and (8.8), we obtain the equation of the free boundary and the complex velocity in the moving coordinate system:

$$x'=-k_2t(y')^2,\quad U'=\frac{z'}{t}-\frac{\sqrt{1+4z'k_2t}-1}{2k_2t^2}. \tag{8.10$a,b$}$$

If we assume here that $k_2=1/4$, then the formulae take the familiar form:

$$x'=-t(y')^2/4,\quad U'=\left(\frac{\sqrt{1+z't}-1}{t}\right)^2, \tag{8.11$a,b$}$$

which coincide with (5.16) and (5.4). Thus, we have obtained the previously found flow, but it is written in a different coordinate system.

### 8.2. *Replacement of variables*

New flows with a free boundary arise if we apply the following replacements of variables in (8.8):

$$U \to -U, \quad z \to -z, \quad t \to -t. \tag{8.12a–c}$$

As a result, we have a complex velocity

$$U = -\frac{z}{t} + \frac{\sqrt{(1-k_1 t)^2 + 4zk_2 t} - (1-k_1 t)}{2k_2 t^2}, \tag{8.13}$$

which satisfies (6.2). New flows arise because (6.2), in contrast to the Hopf equation, is not invariant with respect to the Galileo transform.

Introducing the notation

$$\zeta = \sqrt{(1-k_1 t)^2 + 4zk_2 t} - (1-k_1 t), \tag{8.14}$$

we then obtain two formulae instead of (8.13). These two formulae do not contain radicals:

$$z = \frac{\zeta^2 + 2\zeta(1-k_1 t)}{4k_2 t}, \tag{8.15}$$

$$U = \frac{2\zeta k_1 t - \zeta^2}{4k_2 t^2}. \tag{8.16}$$

In the complex velocity, we have to identify the real part and equate it to zero. From (8.16), we obtain

$$u = \frac{2\xi k_1 t - \xi^2 + \eta^2}{4k_2 t^2}, \quad v = \frac{\eta(\xi - k_1 t)}{2k_2 t^2}. \tag{8.17a,b}$$

It is seen from here that (6.1) has the following solution:

$$\eta^2 = \xi^2 - 2\xi k_1 t. \tag{8.18}$$

If we substitute this solution into the formulae

$$x = \frac{\xi^2 - \eta^2 + 2\xi(1-k_1 t)}{4k_2 t}, \quad y = \frac{2\xi\eta + 2\eta(1-k_1 t)}{4k_2 t} \tag{8.19a,b}$$

derived from (8.15), we obtain a parametric representation of the free surface:

$$x = \frac{\xi}{2k_2 t}, \quad y^2 = \frac{(\xi^2 - 2\xi k_1 t)(\xi + 1 - k_1 t)^2}{4k_2^2 t^2}. \tag{8.20a,b}$$

Eliminating $\xi$, we obtain the free-surface equation

$$y^2 = x(x - k_1/k_2)(2xk_2 t + 1 - k_1 t)^2. \tag{8.21}$$

### 8.3. *Number of solutions*

We obtained a two-parameter family of solutions defined by (8.13) and (8.21), which contain two parameters $k_1$ and $k_2$. Let us demonstrate that the solution is actually unique.

Without loss of generality, $k_1$ and $k_2$ can have arbitrary absolute values. Indeed, $|k_1|$ defines the time scale, and we can assume that $|k_2|$ defines the length scale. We can assume that these scales are equal to unity; then the quantities $|k_1|$ and $|k_2|$ are not involved in the solution after normalization.

We have to demonstrate that the solution is independent of the signs of $k_1$ and $k_2$. Applying the replacement $k_1 \rightarrow -k_1$ in (8.13), we obtain

$$U = -\frac{z}{t} + \frac{\sqrt{(1+k_1t)^2 + 4zk_2t} - (1+k_1t)}{2k_2t^2}. \tag{8.22}$$

We can easily see that the replacement $z \rightarrow z - k_1/k_2$ transforms solution (8.22) to solution (8.13). Thus, the two solutions (8.13) and (8.22), which differ by the sign of $k_1$, are actually one solution written in two different stationary coordinate systems shifted with respect to each other by the distance $k_1/k_2$.

The same is valid for the free surface (8.21). If we replace the variables in this equation as $k_1 \rightarrow -k_1$, it transforms to

$$y^2 = x(x + k_1/k_2)(2xk_2t + 1 + k_1t)^2. \tag{8.23}$$

We can easily see that the replacement $x \rightarrow x - k_1/k_2$ transforms (8.23) into (8.21).

Similarly, we note that the free surface (8.21) is invariant with respect to the transform $k_2 \rightarrow -k_2$, $x \rightarrow -x$. The complex velocity (8.8) is invariant with respect to the replacements $U \rightarrow -U$, $k_2 \rightarrow -k_2$, $z \rightarrow -z$. Thus, for all these reasons, the solution is mirror-reflected with respect to the $y$ axis if the sign of $k_2$ changes.

Thus, $k_1$ and $k_2$ may be arbitrary numbers. In what follows, we assume that

$$k_1 = -1, \quad k_2 = 1/4. \tag{8.24$a,b$}$$

In this case, the complex velocity (8.13) acquires the form

$$U(z, t) = -\frac{z}{t} + 2\frac{\sqrt{(1+t)^2 + zt} - (1+t)}{t^2}. \tag{8.25}$$

The free surface is described by the formula

$$y^2 = x(x+4)(xt/2 + 1 + t)^2. \tag{8.26}$$

Equation (8.25) has a singularity

$$z^* = -\frac{(1+t)^2}{t}, \tag{8.27}$$

which should be outside the fluid.

### 8.4. *Evolution of the free boundary*

The lines described by (8.26) are shown in figure 6(*b*). Where and on which side of these lines should the fluid be placed? We answer this question by considering figure 6(*a*). Acting by analogy, we can obtain the flows shown in figures 11–13. Each of these flows is an analogue of the flows shown in figures 7, 8 and 9, respectively.

An important feature of the flows in figures 12 and 13 is the fact that the singularity reaches the free surface within a finite time. It is not clear how the solution at $t > 1$ should be constructed.

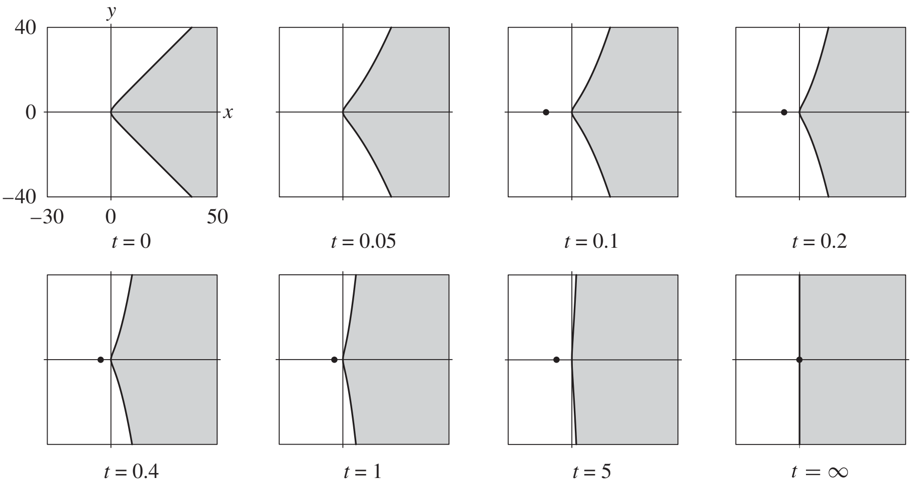


FIGURE 11. Evolution of the fluid domain initially located on the right of the hyperbola $x \geqslant \sqrt{4+y^2}-2$. The initial velocity field is $U(z, 0)=-z-z^2/4$. The free-surface shape and the complex velocity are given by (8.26) and (8.25). The marker dot shows the singularity $z^*=-(1+t)^2/t$.

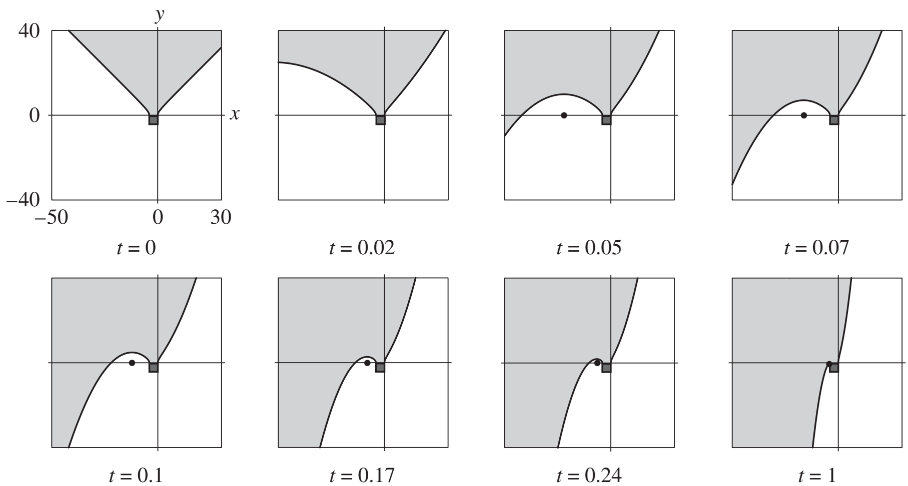


FIGURE 12. Example of a flow with a free boundary. The fluid domain is initially located between two branches of the hyperbolae $y^2=x(x+4)$, which are free surfaces, and is bounded by a motionless solid wall $y=0$, $-4\leqslant x\leqslant 0$. The initial velocity field is $U(z, 0)=-z-z^2/4$. The free-surface shape and the complex velocity are given by (8.26) and (8.25). The marker dot shows the singularity $z^*=-(1+t)^2/t$.

## 9. Crack in the fluid

Let us consider the case where the initial velocity has a singularity in the form of a quadratic root at the origin:

$$F(z)=\sqrt{z}. \tag{9.1}$$

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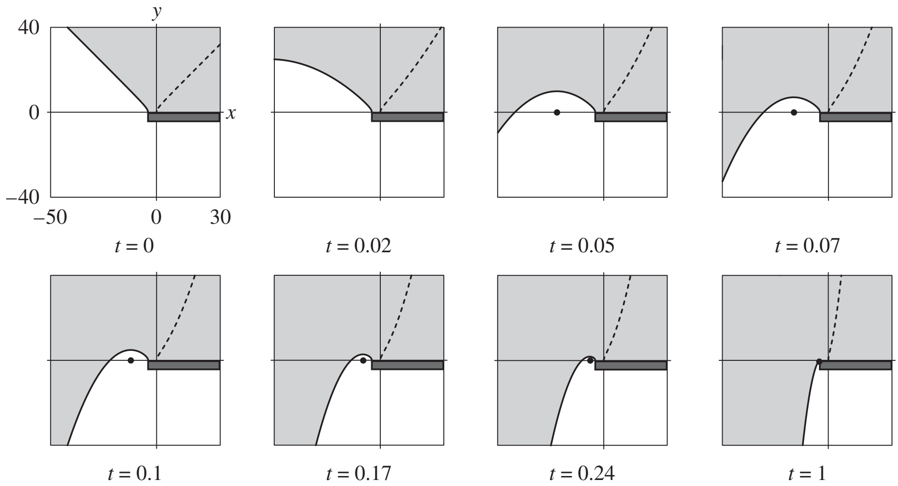


FIGURE 13. Example of a flow with a free boundary. The flow is obtained by matching the flow in figure 12 and one half of the flow in figure 11. The fluid is initially located on the right of the hyperbola $x \geqslant -\sqrt{4-y^2}-2$, $y \geqslant 0$ and is bounded by a motionless solid wall $y=0$, $x \geqslant -4$. The initial velocity field is $U(z,0)=-z-z^2/4$. The free-surface shape and the complex velocity are given by (8.26) and (8.25). The dotted curve is the line located inside the fluid, which is simultaneously an isobar and a material line. The marker dot shows the singularity $z^*=-(1+t)^2/t$.

The solution of the Hopf equation with such initial data is found from the equation

$$U=\sqrt{z-Ut}, \tag{9.2}$$

solving which we obtain

$$U=\frac{\sqrt{t^2+4z}-t}{2}. \tag{9.3}$$

Introducing a parameter

$$\zeta=\sqrt{t^2+4z}, \tag{9.4}$$

solution (9.3) can be rewritten as

$$U=\frac{\zeta-t}{2}, \tag{9.5}$$

$$z=\frac{\zeta^2-t^2}{4}. \tag{9.6}$$

From (9.5), we obtain

$$u=\frac{\xi-t}{2}, \quad v=-\frac{\eta}{2}. \tag{9.7a,b}$$

Thus, the free boundary (3.6) is satisfied only if $\eta=0$. From the formulae

$$x=\frac{\xi^2-\eta^2-t^2}{4}, \quad y=\frac{\xi\eta}{2} \tag{9.8a,b}$$

derived from (9.6), we find that the only free boundary here is the real axis $y=0$. We obtain a trivial free boundary of the auxiliary flow.

More interesting flows can be found by using the replacement

$$U \rightarrow -U. \tag{9.9}$$

As a result, we obtain the complex velocity

$$U = \frac{t - \sqrt{t^2 + 4z}}{2}. \tag{9.10}$$

The solution has a singularity

$$z^* = -\frac{t^2}{4}, \tag{9.11}$$

which should be outside the fluid. Let us rewrite the solution in the form

$$U = \frac{t - \zeta}{2}, \tag{9.12}$$

$$z = \frac{\zeta^2 - t^2}{4}. \tag{9.13}$$

From (9.12), we obtain

$$u = \frac{t - \xi}{2}, \quad v = \frac{\eta}{2}. \tag{9.14$a$,$b$}$$

Equating the real part of the velocity to zero, we obtain the free-boundary equation

$$\xi = t. \tag{9.15}$$

Substituting (9.15) into (9.13), we obtain

$$z = \frac{-\eta^2 + 2\mathrm{i}t\eta}{4} \tag{9.16}$$

or

$$x = -\frac{\eta^2}{4}, \quad y = \frac{t\eta}{2}. \tag{9.17$a$,$b$}$$

Thus, we have the free boundary

$$x = -\left(\frac{y}{t}\right)^2. \tag{9.18}$$

Initially, the fluid occupies the entire plane with a cut. The cut models a crack in the fluid. On the upper boundary of the cut, the fluid velocity is directed vertically upwards; on the lower boundary of the cut, the fluid velocity is directed vertically downwards. The further evolution of the domain occupied by the fluid is illustrated in figure 14. In this flow, the initial plane with a cut transforms to a half-plane within an infinite time. The singular point is initially located on the free surface, as defined by the initial data. With time, the singularity moves to infinity.

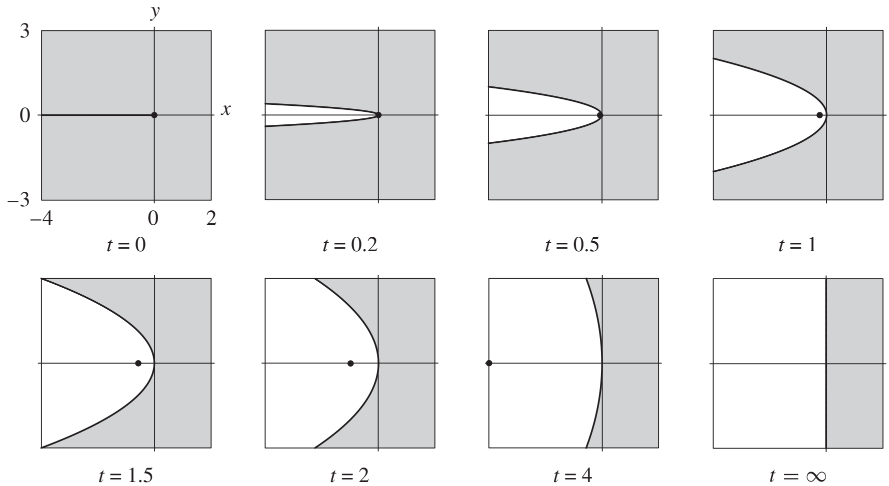


FIGURE 14. Example of a flow with a free boundary. The fluid initially occupies a plane with a cut. The initial velocity field is $U(z, 0) = -\sqrt{z}$. The free-surface shape is a parabola $x = -(y/t)^2$. The complex velocity is given by the formula $U(z, t) = (t - \sqrt{t^2 + 4z})/2$. The marker dot shows the singularity $z^* = -t^2/4$.

## 10. Conclusions

The main advantage of the proposed method of constructing flows with a free boundary is its simplicity. Many new interesting flows can be obtained by using this method. The drawback of this method is the simple kinematics of the flow on the free boundary: each fluid particle moves with a zero acceleration. However, the shape of the free boundary in general can be rather complicated. For instance, the free boundary (8.26) is defined by a fourth-order curve. Concerning the flows with a linear velocity field, the free surface in such flows can be only a hyperbola, ellipse, parabola, or a pair of straight lines in the degenerate case.

Another merit of the method is the possibility of finding and studying singular points of the solution. Normally, such singular points are located outside the fluid. With time, however, these points migrate and can reach the free surface, leading to failure of the solution (Kuznetsov, Spector & Zakharov 1994). The equations of motion of such singular points were derived in Zakharov & Dyachenko (2012). Apparently, the analysis of the behaviour of the singular points plays a key role in the integrability of a plane problem with a free boundary. The outcome of the singular point onto the free surface for gravitational waves with the maximum amplitude was studied theoretically in Grant (1973) and numerically in Lukomsky & Gandzha (2003). The result of this outcome is the formation the sharp crest with an internal angle of 120° on the free surface.

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## REFERENCES

DIRICHLET, G. L. 1860 Untersuchungen über ein Problem der Hydrodynamik. *J. Reine Angew. Math.* **58**, 181–216.

GRANT, M. A. 1973 The singularity at the crest of a finite amplitude progressive Stokes wave. *J. Fluid Mech.* **59**, 257–262.

KARABUT, E. A. 1996 Asymptotic expansions in the problem of a solitary wave. *J. Fluid Mech.* **319**, 109–123.

KARABUT, E. A. 1998 An approximation for the highest gravity waves on water of finite depth. *J. Fluid Mech.* **372**, 45–70.

KARABUT, E. A. 2013 Exact solutions of the problem of free-boundary unsteady flows. *C. R. Méc.* **341**, 533–537.

KUZNETSOV, E. A., SPECTOR, M. D. & ZAKHAROV, V. E. 1994 Formation of singularities on the free surface of an ideal fluid. *Phys. Rev.* E **49** (2), 1283–1290.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LAVRENT'EVA, O. M. 1980 Motion of a fluid ellipsoid. *Dokl. Akad. Nauk SSSR* **253** (4), 828–831.

LAVRENT'EVA, O. M. 1984 One class of motions of a fluid ellipsoid. *J. Appl. Mech. Tech. Phys.* **25** (4), 642–645.

LONGUET-HIGGINS, M. S. 1972 A class of exact, time-dependent, free-surface flows. *J. Fluid Mech.* **55**, 529–543.

LONGUET-HIGGINS, M. S. 1980 On the forming of sharp corners at a free surface. *Proc. R. Soc. Lond.* A **371**, 453–478.

LUKOMSKY, V. P. & GANDZHA, I. S. 2003 Fractional Fourier approximations for potential gravity waves on deep water. *Nonlinear Process. Geophys.* **10**, 599–614.

NALIMOV, V. I. & PUKHNACHOV, V. V. 1975 Unsteady motions of an ideal fluid with a free boundary. Report. Novosibirsk State University, Novosibirsk.

OVSYANNIKOV, L. V. 1967 General equations and examples. In *Problem of Unsteady Motion of a Fluid with a Free Boundary*, pp. 5–75. Nauka.

PUKHNACHOV, V. V. 1978 *Motion of a Fluid Ellipse*, Dynamics of Continuous Media, vol. 33, pp. 68–75. Institute of Hydrodynamics.

RIEMANN, B. 1860 Ein Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoides. *Abh. d. Köningl. Gesell. der Wiss. zur Göttingen* **9**, 3–36.

TAYLOR, G. I. 1960 Formation of thin flat sheets of water. *Proc. R. Soc. Lond.* A **259**, 1–17.

ZAKHAROV, V. E. & DYACHENKO, A. I. 2012 Free-surface hydrodynamics in the conformal variables. arXiv:1206.2046v1 [physics.flu-dyn].