

Thompson’s group F is not strongly amenable

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Abstract. We show that Thompson’s group F has a topological action on a compact metric space that is proximal and has no fixed points.

1. Introduction

In his book ‘Proximal Flows’ [7, §II.3, p. 19], Glasner defined the notion of a *strongly amenable group*: A group is strongly amenable if each of its proximal actions on a compact space has a fixed point. A continuous action $G \curvearrowright X$ of a topological group on a compact Hausdorff space is proximal if for every $x, y \in X$ there exists a net $\{g_n\}$ of elements of G such that $\lim_n g_n x = \lim_n g_n y$.

Glasner showed that virtually nilpotent groups are strongly amenable and that non-amenable groups are not strongly amenable. He also gave examples of amenable—in fact, solvable—groups that are not strongly amenable. Glasner and Weiss [9] constructed proximal minimal actions of the group of permutations of the integers, and Glasner constructed proximal flows of Lie groups [8]. To the best of our knowledge there are no other such examples known. Furthermore, there are no other known examples of minimal proximal actions that are not also *strongly proximal*. An action $G \curvearrowright X$ is strongly proximal if the orbit closure of every Borel probability measure on G contains a point mass measure. This notion, as well as that of the related Furstenberg boundary [4–6], have been the object of a much larger research effort, in particular because a group is amenable if and only if all of its strongly proximal actions on compact spaces have fixed points.

Richard Thompson’s group F has been alternatively ‘proved’ to be amenable and non-amenable (see, e.g., [2]), and the question of its amenability is currently unresolved. In this paper we pursue the less ambitious goal of showing that it is not strongly amenable, and do so by directly constructing a proximal action that has no fixed points. This action does admit an invariant measure, and thus does not provide any information about the amenability of F . It is a new example of a proximal action which is not strongly proximal.

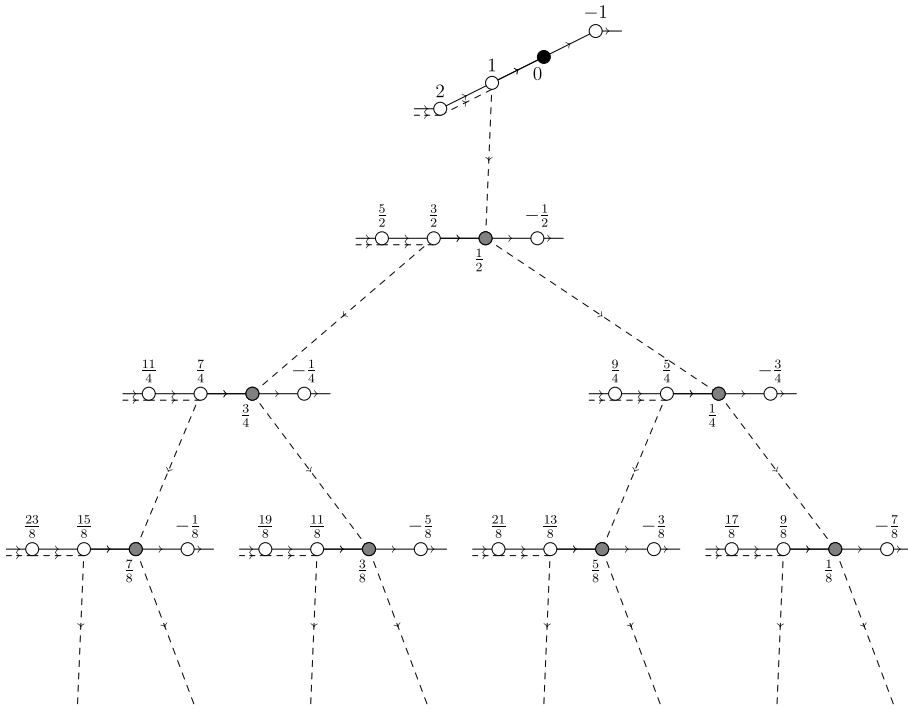


FIGURE 1. The action of F on Γ .

2. Proofs

Let F denote Thompson’s group F . In the representation of F as a group of piecewise linear transformations of \mathbb{R} (see, e.g., [10, §2.C]), it is generated by a and b , which are given by

$$a(x) = x - 1,$$

$$b(x) = \begin{cases} x, & x \leq 0, \\ x/2, & 0 \leq x \leq 2, \\ x - 1, & 2 \leq x. \end{cases}$$

The set of dyadic rationals $\Gamma = \mathbb{Z}[\frac{1}{2}]$ is the orbit of 0. The Schreier graph of the action $G \curvearrowright \Gamma$ with respect to the generating set $\{a, b\}$ is shown in Figure 1 (see [10, §5.A, Figure 6]). The solid lines denote the a action and the dotted lines denote the b action; self-loops (i.e. points stabilized by a generator) are omitted. This graph consists of a tree-like structure with infinite chains attached to each node.

Equipped with the product topology, $\{-1, 1\}^\Gamma$ is a compact space on which F acts continuously by shifts:

$$[fx](\gamma) = x(f^{-1}\gamma). \tag{2.1}$$

PROPOSITION 2.1. *Let $c_{-1}, c_{+1} \in \{-1, 1\}^\Gamma$ be the constant functions. Then for any $x \in \{-1, 1\}^\Gamma$ it holds that at least one of c_{-1}, c_{+1} is in the orbit closure \overline{Fx} .*

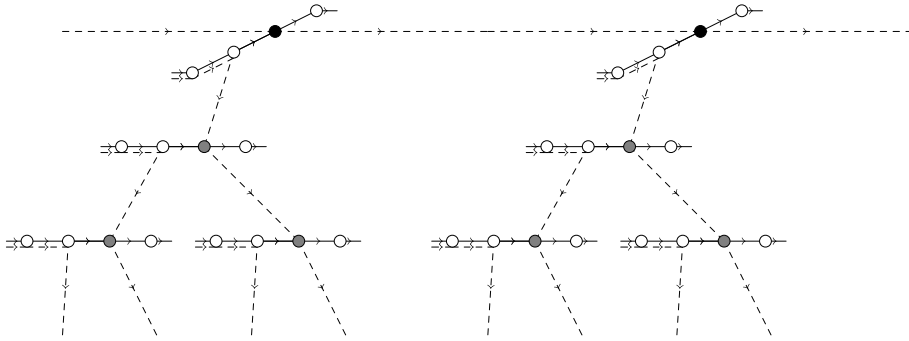


FIGURE 2. The action of F on Λ .

Proof. It is known that the action $F \curvearrowright \Gamma$ is highly transitive [3, Lemma 4.2], i.e. for every finite $V, W \subset \Gamma$ of the same size there exists an $f \in F$ such that $f(V) = W$. Let $x \in \{-1, 1\}^\Gamma$. There is at least one of -1 and 1 , say α , for which we have infinitely many $\gamma \in \Gamma$ with $x(\gamma) = \alpha$. Given a finite $W \subset \Gamma$, choose a $V \subset \Gamma$ of the same size and such that $x(\gamma) = \alpha$ for all $\gamma \in V$. Then there is some $f \in F$ with $f(V) = W$, and so fx takes the value α on W . Since W is arbitrary, we have that c_α is in the orbit closure of x . \square

Given $x_1, x_2 \in \{-1, 1\}^\Gamma$, let d be their pointwise product, given by $d(\gamma) = x_1(\gamma) \cdot x_2(\gamma)$. By Proposition 2.1, there exists a sequence $\{f_n\}$ of elements in F such that either $\lim_n f_n d = c_{+1}$ or $\lim_n f_n d = c_{-1}$. In the first case $\lim_n f_n x_1 = \lim_n f_n x_2$, while in the second case $\lim_n f_n x_1 = -\lim_n f_n x_2$, and so this action resembles a proximal action. In fact, by identifying each $x \in \{-1, 1\}^\Gamma$ with $-x$ one attains a proximal action, and indeed we do this below. However, this action has a fixed point—the constant functions—and therefore does not suffice to prove our result. We spend the remainder of this paper in deriving a new action from this one. The new action retains proximality but does not have fixed points.

Consider the path $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, 1/2^n, \dots)$ in the Schreier graph of Γ (Figure 1); it starts in the top gray node and follows the dotted edges through the gray nodes on the rightmost branch of the tree. The pointed Gromov–Hausdorff limit of this sequence of rooted graphs \dagger is given in Figure 2, and hence is also a Schreier graph of some transitive F -action $F \curvearrowright F/K$. In terms of the topology on the space $\text{Sub}_F \subset \{0, 1\}^F$ of the subgroups of F , the subgroup K is the limit of the subgroups K_n , where K_n is the stabilizer of $1/2^n$. It is easy to verify that K is the subgroup of F consisting of the transformations that stabilize 0 and have right derivative 1 at 0 (although this fact will not be important). Let $\Lambda = F/K$.

We can naturally identify with \mathbb{Z} the chain of black nodes at the top of Λ (see Figure 2). Let Λ' be the subgraph of Λ in which the dotted edges connecting the black nodes have been removed. Given a black node $n \in \mathbb{Z}$, denote by T_n the connected component of n in Λ' ; this includes the black node n , the chain that can be reached from it using solid edges, and the entire tree that hangs from it. Each graph T_n is isomorphic to the Schreier graph

\dagger The limit of a sequence of rooted graphs (G_n, v_n) is a rooted graph (G, v) if each ball of radius r around v_n in G_n is, for n large enough, isomorphic to the ball of radius r around v in G (see, e.g., [1, p. 1460]).

of Γ , and so the graph Λ is a covering graph of Γ (in the category of Schreier graphs). Let

$$\Psi : \Lambda \rightarrow \Gamma$$

be the covering map. That is, Ψ is a graph isomorphism when restricted to each T_n , with the black nodes in Λ mapped to the black node $0 \in \Gamma$.

Using the map Ψ , we give names to the nodes in Λ . Denote the nodes in T_0 as $\{(0, \gamma) : \gamma \in \Gamma\}$, so that $\Psi(0, \gamma) = \gamma$. Likewise, in each T_n denote by (n, γ) the unique node in T_n that Ψ maps to γ . Hence, we identify Λ with

$$\mathbb{Z} \times \Gamma = \{(n, \gamma) : n \in \mathbb{Z}, \gamma \in \Gamma\}$$

and the F -action is given by

$$a(n, \gamma) = (n, a\gamma), \tag{2.2}$$

$$b(n, \gamma) = \begin{cases} (n, b\gamma) & \text{if } \gamma \neq 0, \\ (n + 1, 0) & \text{if } \gamma = 0. \end{cases} \tag{2.3}$$

Equip $\{-1, 1\}^\Lambda$ with the product topology to get a compact space. As usual, the F -action on Λ (given explicitly in 2.2 and 2.3) defines a continuous action on $\{-1, 1\}^\Lambda$.

Consider $\pi : \{-1, 1\}^\Gamma \rightarrow \{-1, 1\}^\Lambda$, given by $\pi(x)(n, \gamma) = (-1)^n x(\gamma)$. Let $Y = \pi(\{-1, 1\}^\Gamma) \subseteq \{-1, 1\}^\Lambda$.

CLAIM 2.2. *Y is compact and F-invariant.*

Proof. π is injective and continuous, so $Y = \pi(\{-1, 1\}^\Gamma) \subseteq \{-1, 1\}^\Lambda$ is compact and isomorphic to $\{-1, 1\}^\Gamma$. Moreover, Y is invariant to the action of F , because $a^{\pm 1}\pi(x) = \pi(a^{\pm 1}x)$ and $b^{\pm 1}\pi(x) = \pi(b^{\pm 1}\bar{x})$, where $\bar{x}(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \neq 0, \\ -x(\gamma) & \text{if } \gamma = 0. \end{cases}$ □

The last F -space we define is Z , the set of pairs of mirror image configurations in Y :

$$Z = \{ \{y, -y\} : y \in Y \}. \tag{2.4}$$

Now it is clear that equipped with the quotient topology, Z is a compact and Hausdorff F -space. Furthermore, we now observe that Z admits an invariant measure. Consider the independent and identically distributed Bernoulli 1/2 measure on $\{-1, 1\}^\Gamma$, i.e. the unique Borel measure on $\{-1, 1\}^\Gamma$, for which

$$X_\gamma : \{-1, 1\}^\Gamma \rightarrow \{0, 1\}, \quad x \mapsto \frac{x(\gamma) + 1}{2}$$

are independent Bernoulli 1/2 random variables for all $\gamma \in \Gamma$. Clearly, it is an invariant measure and hence it is pushed forward to an invariant measure on Y , and then on Z . In particular, this shows that Z is not strongly proximal.

CLAIM 2.3. *The action F ⋈ Z does not have any fixed points.*

Proof. Pick $\hat{y} = \{y, -y\} \in Z$. We have $[by](0, -1) = y(0, -1) \neq -y(0, -1)$, so $by \neq -y$. Similarly, $[by](0, 0) = y(-1, 0) = -y(0, 0) \neq y(0, 0)$, and so $by \neq y$. Hence, $b\hat{y} \neq \hat{y}$. □

PROPOSITION 2.4. *The action $F \curvearrowright Z$ is proximal.*

Proof. Let $\hat{y}_1 = \{y_1, -y_1\}$ and $\hat{y}_2 = \{y_2, -y_2\}$ be two points in Z , and let $y_i = \pi(x_i)$.

Let $x_1 \cdot x_2$ denote the pointwise product of x_1 and x_2 . Now by Proposition 2.1 there is a sequence of elements $\{f_n\}_n$ in F such that $\{f_n(x_1 \cdot x_2)\}_n$ tends to either c_{-1} or c_{+1} in $\{-1, 1\}^\Gamma$. Since Y is compact, we may assume that $\{f_n y_1\}_n$ and $\{f_n y_2\}_n$ have limits, by descending to a subsequence if necessary.

It is straightforward to check that $f_n y_1 \cdot f_n y_2 = f_n \pi(x_1) \cdot f_n \pi(x_2) = \pi(f_n x_1) \cdot \pi(f_n x_2)$. So,

$$\begin{aligned} [f_n y_1 \cdot f_n y_2](n, \gamma) &= [\pi(f_n x_1) \cdot \pi(f_n x_2)](n, \gamma) \\ &= (-1)^{2n} [f_n x_1](\gamma) [f_n x_2](\gamma) \\ &= [f_n x_1 \cdot f_n x_2](\gamma) = [f_n(x_1 \cdot x_2)](\gamma). \end{aligned}$$

So, $\lim_n f_n y_1 = \pm \lim_n f_n y_2$, which implies that $\lim_n f_n \hat{y}_1 = \lim_n f_n \hat{y}_2$. \square

THEOREM 2.5. *Thompson's group F is not strongly amenable.*

Proof. Since the space Z we constructed above is proximal (Proposition 2.4), and has no fixed points (Claim 2.3), we conclude that F has a proximal action with no fixed points, so F is not strongly amenable. \square

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REFERENCES

- [1] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electron. J. Probab.* **12**(54) (2007), 1454–1508.
- [2] J. W. Cannon and W. J. Floyd. What is Thompson's group? *Notices Amer. Math. Soc.* **58**(8) (2011), 1112–1113.
- [3] J. W. Cannon, W. J. Floyd and W. R. Parry. Notes on Richard Thompson's groups F and T . *Enseign. Math. Zbl.* **42**(3) (1996), 215–256.
- [4] A. Furman. On minimal, strongly proximal actions of locally compact groups. *Israel J. Math.* **136**(1) (2003), 173–187.
- [5] H. Furstenberg. A Poisson formula for semi-simple Lie groups. *Ann. of Math. (2)* **77** (1963), 335–386.
- [6] H. Furstenberg. Boundary theory and stochastic processes on homogeneous spaces. *Harmonic Analysis on Homogeneous Spaces (Proceedings of Symposia Pure Mathematics, XXVI, Williams College, Williamstown, MA, 1972)*. American Mathematical Society, Providence, RI, 1973, pp. 193–229.
- [7] S. Glasner. *Proximal Flows (Lecture Notes in Mathematics, 517)*. Springer, Berlin, 1976.
- [8] S. Glasner. Proximal flows of Lie groups. *Israel J. Math.* **45**(2) (1983), 97–99.
- [9] E. Glasner and B. Weiss. Minimal actions of the group $\mathbb{S}(\mathbb{Z})$ of permutations of the integers. *Geom. Funct. Anal.* **12**(5) (2002), 964–988.
- [10] V. A. Kaimanovich. Thompson's group F is not Liouville. *Preprint*, 2016, arXiv:1602.02971.