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Generalised DOPs with Consideration of the Influence Function of Signal-in-Space Errors

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Integrated navigation using multiple Global Navigation Satellite Systems (GNSS) is beneficial to increase the number of observable satellites, alleviate the effects of systematic errors and improve the accuracy of positioning, navigation and timing (PNT). When multiple constellations and multiple frequency measurements are employed, the functional and stochastic models as well as the estimation principle for PNT may be different. Therefore, the commonly used definition of "dilution of precision (DOP)" based on the least squares (LS) estimation and unified functional and stochastic models will be not applicable anymore. In this paper, three types of generalised DOPs are defined. The first type of generalised DOP is based on the error influence function (IF) of pseudo-ranges that reflects the geometry strength of the measurements, error magnitude and the estimation risk criteria. When the least squares estimation is used, the first type of generalised DOP is identical to the one commonly used. In order to define the first type of generalised DOP, an IF of signal–in-space (SIS) errors on the parameter estimates of PNT is derived. The second type of generalised DOP is defined based on the functional model with additional systematic parameters induced by the compatibility and interoperability problems among different GNSS systems. The third type of generalised DOP is defined based on Bayesian estimation in which the *a priori* information of the model parameters is taken into account. This is suitable for evaluating the precision of kinematic positioning or navigation. Different types of generalised DOPs are suitable for different PNT scenarios and an example for the calculation of these DOPs for multi-GNSS systems including GPS, GLONASS, Compass and Galileo is given. New observation equations of Compass and GLONASS that may contain additional parameters for interoperability are specifically investigated. It shows that if the interoperability of multi-GNSS is not fulfilled, the increased number of satellites will not significantly reduce the generalised DOP value. Furthermore, the outlying measurements will not change the original DOP, but will change the first type of generalised DOP which includes a robust error IF. *A priori* information of the model parameters will also reduce the DOP.

KEY WORDS

1. Generalised dilution of precision. 2. Influence function. 3. Systematic error. 4. Robust estimation. 5. Bayesian estimation.

1. INTRODUCTION.

The term DOP has been well designed and widely used (Parkinson 1996) and the DOP value is often used as a major index to gauge the contribution of satellites. Various types of DOPs were defined such as the geometric (GDOP), positional (PDOP), horizontal (HDOP), vertical (VDOP) and time (TDOP) (Langley 1999). Relationships among the vertical error of navigation, VDOP and the pseudo-range error in GPS were discussed by Leva (1994) and a closed-form expression for DOP was given by Doong (2009). A particular type of DOP called an ambiguity DOP (ADOP) was designed by Teunissen (1997) and its corresponding properties were also discussed. The ADOP in a closed-form expression for a hierarchy of multi-frequency single-baseline GNSS models was given (Odijk and Teunissen 2008).

The commonly used DOP in PNT relies mainly on the geometry of measurements. It is useful when the least squares estimation and only sole satellite constellations are used for the solution. The observation geometric strength represented by the DOP is usually determined by the design matrix of the model parameters such as the parameters of position and time offset (Teunissen and Kleusberg 1999, p 449–451). Generally, the stronger the observation geometry, the less the DOP value and the more precise the PNT solutions. When additional observations/satellites are used in the solution, the increase of the number of observations can always reduce the GDOP value (Yarlagadda et al. 2000), regardless of whether the geometric structure is enhanced by the additional observations/satellites. The precision of positioning can then be improved.

The Chinese Compass/BeiDou and European Galileo systems have experienced quick developments in recent years and nine Compass satellites have been successfully launched (as of July 2011). Additional phase and code measurements from these new multi-GNSS signals have become available and will contribute to the improvement of the phase bias estimation or ambiguity resolutions (Feng 2008; Feng and Li 2008; Feng and Rizos 2009). In addition, Different combinations of measurements may also cause changes in the DOP values, for example, non-differential measurements and differential measurements may have different DOP values (Park and Kim 2000).

Some problems exist when the commonly used DOP definition is applied to measure the possible precision of PNT solutions. These include:

1. It is well known that when multiple frequencies and multiple constellations of GNSS are used in PNT, the number of measurements is increased, which is an advantage. Some additional parameters however may also need to be introduced for the interoperability of the different systems. The DOP values calculated using a fixed functional model may not be applicable anymore;
2. Usually, multiple GNSS constellations may increase the noise floor and may introduce additional outliers in the measurements. DOP computation based on the whole design matrix may not be suitable because the outlying measurements could significantly impact the precision which cannot be reflected by usual DOP;
3. If different stochastic models are used for different satellite constellations, the DOP value from the equal-weight-based approach is not suitable anymore. Instead the weighted least squares estimation (Sairo et al. 2003) or the iterative weighted total least-squares adjustment (Shen and Li 2010) may be applied;

4. When *a priori* information of the model parameters is employed in the parameter estimation, the DOP calculated based on the design matrix of measurement equations does not work;
5. Especially when non-least squares estimation, such as robust estimation is used (Hampel et al. 1986; Yang 1991), the DOP based on least squares principle does not work either;
6. DOP is defined as the square-root of the trace of the normal matrix which neither reflects the correlations among the parameters or the ill-posed problems, nor reflects the systematic error influences. In real time GNSS positioning, the carrier phase ambiguity resolution is often ill-posed or the geometry is nearly collinear as in the case of poor satellite geometry, which may result in unreliable or unsuccessful ambiguity resolution and hence unreliable positioning if no care is taken to mitigate this situation (Li et al. 2010). In this case, the regularisation or ridge regression method (Xu et al. 2006; Xu 2009) is recommended. In addition, calculating GDOP using the ridge regression method was also proposed for improving the accuracy and performance of positioning (Kelly 1990).

To sum up, the DOP values should reflect not only the geometrical strength, but also the other important information related to the PNT. In fact, the precision of a GNSS PNT solution relies on the following four main factors; the magnitude of errors, the geometric strength of measurements, the number of model parameters and the principle used for the parameter estimation. In order to reasonably evaluate and analyse the error influence or DOP of satellite positioning, a suitable DOP statistics that takes into account the above four factors needs to be constructed.

The influence function (IF) defined in statistics (Hampel et al. 1986; Yang 1991, 1997, 2002) is another important criterion in the quality assessment of measurements and the parameter estimates. Both DOP and IF can be used as the criteria for evaluating an observations' contribution to the parameter estimates, thus they should be intrinsically related. To establish the relationship between the DOP and IF, a new generalised DOP (G-DOP) based on different backgrounds such as the principles of a robust estimation and the Bayesian estimation need to be developed. If the robust principle is employed, the first type of G-DOP, expressed as G-DOP$_{\mathrm{I}}$ in this paper, is proposed. If additional parameters for the interoperability of different navigation systems are considered in the functional model, then the second type of G-DOP, expressed as G-DOP$_{\mathrm{II}}$, is defined. If *a priori* weights on the model parameters are considered, the third G-DOP expressed as G-DOP$_{\mathrm{III}}$ is recommended. It is obvious that more information can be included in the calculation of G-DOP values if some particular analysis or computation is needed.

All the aforementioned three types of generalised DOPs are defined by the square root of the variance covariance matrix (VCM) of the estimated parameters, which is the same as that of the ones that have been used. The only differences are in the forms of expression, which are related to the estimation principle, the parameter characteristic as well as whether the *a priori* information is used.

2. INFLUENCE FUNCTION OF SIGNAL-IN-SPACE ERROR.

The influence of the signal-in-space error (SISE) can be expressed by an IF. The IF is related to the geometric structure of observations, which in turn is related to the DOP value. Assuming that the pseudo-range measurement of the receiver to the satellite is L_i, the measurement equation is (Langley 1999);

$$L_i = \rho_i + c(dt_u - dt_i) + dI_i + dT_i + e_i \tag{1}$$

where ρ_i is the geometric range between the antenna phase centres of both receiver and satellite, dt_u and dt_i denote the clock offsets of the receiver and satellite respectively, dI_i and dT_i are the ionospheric and tropospheric delays respectively, e_i is the sum of the measurement noise and un-modelled observation errors such as multipath effects and c denotes the speed of light in vacuum. Equation (1) can be written in the following error equation form:

$$\mathbf{V} = \mathbf{A}\hat{\mathbf{X}} - \mathbf{L} \tag{2}$$

where $\mathbf{V}$ denotes the residual vector, $\mathbf{A}$ is the design matrix, $\hat{\mathbf{X}}$ denotes the unknown parameter vector which is generally the vector of corrections to the model parameters' approximate values and $\mathbf{L}$ denotes the observed-minus-computed observations.

Assume that the contaminated distribution function of L_i is G_i, which is composed of two terms (Hampel et al. 1986):

$$G_i = (1 - \varepsilon)F_i + \varepsilon\delta l_i \tag{3}$$

where δl_i is the contaminated symmetric distribution and ε is the contaminated ratio with $0 \leqslant \varepsilon < 1$ and F_i is the normal distribution function of L_i.

Assuming that the measurements are independent, a general risk function based on the maximum likelihood principle (M-estimation) can be expressed as:

$$\int \rho(L_i; \hat{\mathbf{X}})dF(L_i) = \min \tag{4}$$

where $\rho(\cdot)$ is a convex function, which is continuous and differentiable with the scalar derivative as $\psi(L_i; \hat{\mathbf{X}}) = \rho'(L_i; \hat{\mathbf{X}})$, $\hat{\mathbf{X}}$ is the estimated parameter vector. The solution to the risk function (4) must meet the following requirements:

$$\int \psi(L_i; \hat{\mathbf{X}})dF(L_i) = 0 \tag{5}$$

Based on the estimation principle (4) and the estimation equation (5), an IF is defined by Hampel (Hampel et al. 1986, Yang 1991):

$$\mathbf{IF}(L_i; \hat{\mathbf{X}}, F_i) = \lim_{\varepsilon \to 0} \frac{\hat{\mathbf{X}}[(1-\varepsilon)F_i + \varepsilon\delta l_i] - \hat{\mathbf{X}}(F_i)}{\varepsilon} \tag{6}$$

where $\hat{\mathbf{X}}[(1-\varepsilon)F_i + \varepsilon\delta l_i]$ and $\hat{\mathbf{X}}(F_i)$ are the estimated vectors based on the contaminated and normal distributions respectively.

Considering the error equation (2) and the M-estimation equation (5), we have:

$$\int \mathbf{a}_i^{\mathrm{T}} p_i \psi(L_i; \hat{\mathbf{X}})dF_i = 0 \tag{7}$$

where $\mathbf{a}_i$ is the i^{th} row vector of $\mathbf{A}$, p_i is the weight of L_i and F_i represents the marginal distribution of L_i.

Substituting (7) into (6), we obtain the following derivative with respect to ε (at $\varepsilon=0$):

$$\int p_i\psi(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i d(-F_i + \delta l_i) + \sum_{i=1}^{n}\int p_i\psi'(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i\mathbf{a}_i^{\mathrm{T}} dG(L_i)\frac{\partial\hat{\mathbf{X}}}{\partial\varepsilon}|_{\varepsilon=0} = 0 \tag{8}$$

where $\frac{\partial\hat{\mathbf{X}}}{\partial\varepsilon} = \mathbf{IF}$. Given the following equations (Hampel et al. 1986; Yang 1991, 1997):

$$\int p_i\psi(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i d(-F_i) = 0 \tag{9}$$

$$\int p_i\psi(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i d(\delta l_i) = p_i\psi(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i \tag{10}$$

$$\sum_{i=1}^{n}\int p_i\psi'(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i^{\mathrm{T}}\mathbf{a}_i dG(L_i) = \sum_{i=1}^{n} p_i E\psi'(\mathbf{a}_i\hat{\mathbf{X}} - L_i)\mathbf{a}_i^{\mathrm{T}}\mathbf{a}_i \tag{11}$$

where E denotes expectation, we obtain:

$$\mathbf{IF}(L_i;\hat{\mathbf{X}}, F) = -\mathbf{M}^{-1}\mathbf{a}_i^{\mathrm{T}} p_i\psi(\mathbf{a}_i\hat{\mathbf{X}} - L_i) = -\mathbf{M}^{-1}\mathbf{a}_i^{\mathrm{T}} p_i\psi(e_i) \tag{12}$$

By substituting equations (9)–(11) into Equation (8), with:

$$\mathbf{M} = \mathbf{A}^{\mathrm{T}}\mathbf{Z}\mathbf{A} \tag{13}$$

and $\mathbf{Z}$ the diagonal matrix with elements:

$$Z_{ii} = p_i E\psi'(e_i) \tag{14}$$

The **IF** can also be expressed in the equivalent weight matrix $\overline{\mathbf{P}}$ which has the diagonal elements of $\overline{p}_i = p_i\frac{\psi(e_i)}{e_i}$ (Yang 1991, 1997):

$$\mathbf{IF}(L_i;\hat{\mathbf{X}}, F) = -(\mathbf{A}^{\mathrm{T}}\overline{\mathbf{P}}\mathbf{A})^{-1}\mathbf{a}_i^{\mathrm{T}}\overline{p}_i e_i$$

In the least squares estimation, $\rho(e_i)=e_i^2$, $\psi(e_i)=\rho'(e_i)=2e_i$, $\psi'(e_i)=2$ and

$$Z_{ii} = p_i \tag{15}$$

Hence;

$$\mathbf{M} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} \tag{16}$$

Then the **IF** becomes;

$$\mathbf{IF}_{\mathrm{LS}}(L_i;\hat{\mathbf{X}}, F) = -(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\mathbf{a}_i^{\mathrm{T}} p_i e_i \tag{17}$$

where $\mathbf{IF}_{\mathrm{LS}}$ represents the influence function under the LS estimation.

According to the **IF** expressed in Equation (12), the effects of the SISE on the model parameters mainly depend on ψ. Usually, ψ is a descending function which decreases with the increase of the error, or, it is a re-descending function which decreases to zero when the error is larger than the given threshold value. In other words, the influence of SISE on the model parameters decreases with the increase of the magnitude of the SISE. On the other hand, the influence of SISE e_i is related to the observation geometric structure: the stronger the geometric structure, the smaller the value of $\mathbf{M}^{-1}$ and the less the effects of the SISE.

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3. THE FIRST TYPE OF G-DOP: CONSIDERING A ROBUST RISK FUNCTION. Usually, the DOP is defined based on the cofactor matrix of the parameter estimates. If the principle of maximum likelihood estimation is applied, the integrated IF of the SISE of the measurements L_1, L_2,…L_n can be expressed as:

$$\underset{m\times 1}{\mathbf{IF}}(\mathbf{L};\hat{\mathbf{X}}) = \begin{bmatrix} IF(L_1, L_2, \cdots L_n, \hat{X}_1) \\ IF(L_1, L_2, \cdots L_n, \hat{X}_2) \\ \vdots \\ IF(L_1, L_2, \cdots L_n, \hat{X}_m) \end{bmatrix} = -\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\big[\psi(e)\big] \tag{18}$$

where $[\psi(e)]$ is the vector composed of the elements of $\psi(e_i)$. The posterior covariance matrix of the model parameters is then defined as (Yang 1997):

$$\boldsymbol{\Sigma}_{\hat{\mathbf{X}}} = E(\mathbf{IF}\cdot\mathbf{IF}^{\mathrm{T}}) \tag{19}$$

Based on Equation (12), the approximate estimate of $\boldsymbol{\Sigma}_{\hat{\mathbf{X}}}$ can be expressed as (Yang 1997):

$$\boldsymbol{\Sigma}_{\hat{\mathbf{X}}} = \mathbf{M}^{-1}\hat{\sigma}_0^2 \tag{20}$$

$\boldsymbol{\Sigma}_{\hat{\mathbf{X}}}$ can also be written as the following expression if the equivalent weight matrix $\bar{\mathbf{P}}$ is used:

$$\boldsymbol{\Sigma}_{\hat{\mathbf{X}}} = (\mathbf{A}^{\mathrm{T}}\bar{\mathbf{P}}\mathbf{A})^{-1}\hat{\sigma}_0^2 \tag{21}$$

with the elements of the equivalent weight matrix calculated by the residual as:

$$\bar{p}_i = p_i\frac{\psi(v_i)}{v_i} \tag{22}$$

where v_i denotes the residuals of L_i.

If Huber's ψ function (Huber 1981) is applied to Equation (22) then:

$$\bar{p}_i = \begin{cases} p_i & |v_i'| = \left|\dfrac{v_i}{\sigma_{v_i}}\right| \leqslant c \\ p_i\dfrac{c\sigma_{v_i}}{|v_i|} & |v_i'| > c \end{cases} \tag{23}$$

$$\hat{\sigma}_0^2 = \frac{\mathbf{V}^{\mathrm{T}}\bar{\mathbf{P}}\mathbf{V}}{n-m-m_0} \tag{24}$$

where m_0 denotes the number of $\bar{p}_i$ that have zero value.

In the LS estimation; $E([\psi(e)][\psi(e)]^{\mathrm{T}}) = E(\mathbf{e}\mathbf{e}^{\mathrm{T}}) = \boldsymbol{\Sigma}_{\mathbf{e}} = \mathbf{P}^{-1}\sigma_0^2$ and $\mathbf{M} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}$, thus:

$$\boldsymbol{\Sigma}_{\hat{\mathbf{X}}} = (\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\hat{\sigma}_{0\mathrm{LS}}^2 \tag{25}$$

where $\hat{\sigma}_{0\mathrm{LS}}^2$ is the LS estimate of variance scale.

Similar to the usual GDOP definition, we define:

$$\text{G-DOP}_{\mathrm{I}} = [\mathrm{tr}(\mathbf{M}^{-1})]^{\frac{1}{2}} \tag{26}$$

as the first type of G-DOP.

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If the equivalent weight matrix is used, Equation (26) can be also simplified as:

$$\text{G-DOP}_{\text{I}} = [\text{tr}(\mathbf{A}^{\text{T}}\overline{\mathbf{P}}\mathbf{A})^{-1}]^{\frac{1}{2}} \tag{27}$$

From the analysis of the G-DOP_{I} expression, it is found that:

(1) The first type of G-DOP, as the commonly defined/used GDOP, reflects the observation geometric configuration. The stronger the observation geometry, the smaller the G-DOP_{I} value.
(2) The combined effects of measurement errors and the risk function are reflected in $\mathbf{M}^{-1}$. If there exist outliers, the derivative of ψ, $\psi'(e_i)$ will be reduced whereas $\mathbf{M}^{-1}$ will be increased. It means that when outliers exist in some observations, the contribution of those observations will be removed so that they will not affect the precision of the parameter estimates. This is more reasonable because when $|e_i|$ increases, $\psi(e_i)$ decreases and the corresponding $\overline{p}_i$ of L_i will also decrease. In this case, G-DOP_{I} should be increased. The usual DOP in the LS estimation will not help us to estimate the actual and reasonable variation of the precision, when there exist outliers which do not make any contribution to the model parameter estimates.
(3) Both DOP and IF are related to the inverse of the normal matrix $\mathbf{M}^{-1}$. Another expression of G-DOP can be easily obtained from (26);

$$\text{G-DOP}_{\text{I}} = \left\{\text{tr}[E(\mathbf{IF}\cdot\mathbf{IF}^{\text{T}}]\right\}^{\frac{1}{2}}/\sigma_0 \tag{28}$$

where σ_0 is the standard deviation.

It should be noted that the **IF** is the integrated influence vector that contains the effects of the errors e_1, e_2,…, e_n. Furthermore, Equation (28) can be expressed in the form of **IF** components:

$$\text{G-DOP}_{\text{I}} = \left\{\sum_{j=1}^{m} E[IF^2(\mathbf{L}, \hat{X}_i)]\right\}^{\frac{1}{2}} \Big/ \sigma_0 \tag{29}$$

where $IF(\mathbf{L}, \hat{X}_i)$ is a scalar quantity and it can be expressed as:

$$IF(\mathbf{L}, \hat{X}_i) = \left[M_{j1}^{-1}, M_{j2}^{-1}, \cdots, M_{jm}^{-1}\right]\begin{bmatrix} \sum_{i=1}^{n} a_{i1}p_i\psi(e_i) \\ \sum_{i=1}^{n} a_{i2}p_i\psi(e_i) \\ \vdots \\ \sum_{i=1}^{n} a_{im}p_i\psi(e_i) \end{bmatrix} = \sum_{k=1}^{m} M_{jk}^{-1}\cdot\sum_{i=1}^{n} a_{ik}p_i\psi(e_i) \tag{30}$$

In the LS estimation:

$$\text{G-DOP} = (Q_{X_{11}} + Q_{X_{22}} + \cdots Q_{X_{mm}})^{\frac{1}{2}} = \left(\sum_{j=1}^{m} \sigma_{x_j}^2\right)^{\frac{1}{2}} \Big/ \sigma_0 \tag{31}$$

Equation (31) is the same as that of the usual DOP.

It should be pointed out that if the outliers are detected and identified by the DIA method (Teunissen 1990), the G-DOP_{I} is nearly equivalent to the commonly used DOP.

4. THE SECOND TYPE OF G-DOP: CONSIDERING ADDITIONAL MODEL PARAMETERS.

The issues of compatibility and interoperability need to be addressed when multi-GNSS constellations are used. One feasible approach is to add extra model parameters to the functional model for compensating the systematic offsets/errors such as the coordinate system offset, the systematic time offsets and the satellite orbit errors etc. In this case, the observation equation matrix becomes:

$$L_i = \mathbf{a}_i \mathbf{X} + \mathbf{b}_i \mathbf{S}_j + \Delta_i \tag{32}$$

where Δ_i is the random error vector, $\mathbf{S}_j$ denotes the systematic error vector that only includes the additional model parameters, $\mathbf{a}_i$ is the i^{th} row of the design matrix and $\mathbf{b}_i$ is the coefficient vector of the additional model parameters. For a single navigation system, $\mathbf{S}_j$ may be the receiver time offset parameter dt (Milbert 2008, 2009) and $\mathbf{S}_j$ can be merged into the parameter vector $\mathbf{X}$. If each satellite system has its own systematic parameters $\mathbf{S}_j$, then they can be estimated under the support of multi-GNSS. At least the effects of $\mathbf{S}_j$ can be compensated by estimating their values.

Equation (32) can be written in the form of the following error equation:

$$\mathbf{V}_J = \mathbf{A}_J \hat{\mathbf{X}} + \mathbf{B}_J \hat{\mathbf{S}}_J - \mathbf{L}_J \tag{33}$$

where the subscript "J" (J = 1, 2, …, N), represents different satellite systems, such as GPS, GLONASS, COMPASS etc. The parameter estimates and the IF from a robust estimator could be;

$$\begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{\text{T}}\bar{\mathbf{P}}\mathbf{A} & \mathbf{A}^{\text{T}}\bar{\mathbf{P}}\mathbf{B} \\ \mathbf{B}^{\text{T}}\bar{\mathbf{P}}\mathbf{A} & \mathbf{B}^{\text{T}}\bar{\mathbf{P}}\mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}^{\text{T}}\bar{\mathbf{P}}\mathbf{L} \\ \mathbf{B}^{\text{T}}\bar{\mathbf{P}}\mathbf{L} \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} IF(\Delta_J, \hat{\mathbf{X}}) \\ IF(\Delta_J, \hat{\mathbf{S}}) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{\text{T}}\bar{\mathbf{P}}\mathbf{A} & \mathbf{A}^{\text{T}}\bar{\mathbf{P}}\mathbf{B} \\ \mathbf{B}^{\text{T}}\bar{\mathbf{P}}\mathbf{A} & \mathbf{B}^{\text{T}}\bar{\mathbf{P}}\mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_J^{\text{T}}\bar{\mathbf{P}}_J\Delta_J \\ \mathbf{B}_J^{\text{T}}\bar{\mathbf{P}}_J\Delta_J \end{bmatrix} \tag{35}$$

From the estimator (34), it is easy to obtain the effects of the systematic error vector $\mathbf{S}_J$ on the common model parameter vector $\hat{\mathbf{X}}$ by:

$$\mathbf{IF}(\mathbf{S}, \hat{\mathbf{X}}; F_1 \cdots F_N) = \left(\sum_{J=1}^{N} \mathbf{M}_J^{-1}\right)^{-1} \sum_{J=1}^{N} \mathbf{A}_J^{\text{T}}\bar{\mathbf{P}}_J\mathbf{B}_J\mathbf{S}_J \tag{36}$$

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Similarly, the IF of the random errors $\Delta_1, \ldots, \Delta_N$ on the model parameters can be expressed as:

$$\mathbf{IF}(\Delta, \hat{\mathbf{X}}; F_1 \cdots F_N) = \left(\sum_{J=1}^{N} \mathbf{M}_J^{-1}\right)^{-1} \sum_{J=1}^{N} \mathbf{A}_J^{\mathrm{T}} \overline{\mathbf{P}}_J \Delta_J \tag{37}$$

In multi-GNSS, the influence of the systematic error of a single constellation on the estimates of PNT can be analysed using the total influence function (TIF) by:

$$\mathbf{TIF}(\mathbf{L}, \hat{\mathbf{X}}; F_1 \cdots F_N) = \left(\sum_{J=1}^{N} \mathbf{M}_J^{-1}\right)^{-1} \sum_{J=1}^{N} (\mathbf{A}_J^{\mathrm{T}} \overline{\mathbf{P}}_J \mathbf{B}_J \mathbf{S}_J + \mathbf{A}_J^{\mathrm{T}} \overline{\mathbf{P}}_J \Delta_J) \tag{38}$$

Since the additional model parameters will degrade the precision, so the DOP values will be enlarged. The second type of G-DOP (G-DOP$_{\mathrm{II}}$) that takes into account the additional model parameters can be defined as:

$$\text{G-DOP}_{\mathrm{II}} = \left\{ \mathrm{tr} \begin{bmatrix} \mathbf{A}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{A} & \mathbf{A}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{A} & \mathbf{B}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{B} \end{bmatrix}^{-1} \right\}^{\frac{1}{2}} \tag{39}$$

Using the matrix equivalent equation (Koch 1987), the G-DOP$_{\mathrm{II}}$ can be obtained as:

$$\text{G-DOP}_{\mathrm{II}} = \{\mathrm{tr}[\mathbf{N}_{\mathbf{X}}^{-1} + \mathbf{N}_{\mathbf{X}}^{-1} \mathbf{N}_{\mathbf{XS}} (\mathbf{N}_{\mathbf{S}} - \mathbf{N}_{\mathbf{SX}} \mathbf{N}_{\mathbf{X}}^{-1} \mathbf{N}_{\mathbf{XS}})^{-1} \mathbf{N}_{\mathbf{SX}} \mathbf{N}_{\mathbf{X}}^{-1} + (\mathbf{N}_{\mathbf{S}} - \mathbf{N}_{\mathbf{SX}} \mathbf{N}_{\mathbf{X}}^{-1} \mathbf{N}_{\mathbf{XS}})^{-1}]\}^{\frac{1}{2}} \tag{40}$$

where $\mathbf{N}_{\mathbf{X}} = \mathbf{A}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{A}$, $\mathbf{N}_{\mathbf{S}} = \mathbf{B}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{B}$ and $\mathbf{N}_{\mathbf{XS}} = \mathbf{A}^{\mathrm{T}} \overline{\mathbf{P}} \mathbf{B} = \mathbf{N}_{\mathbf{SX}}^{\mathrm{T}}$.

From the analysis of the G-DOP$_{\mathrm{II}}$ and TIF, the following findings can be obtained:

1. Generally, multi-GNSS will improve the G-DOP, especially when multi-GNSS meet the requirements of compatibility and interoperability, no additional parameters need to be introduced to the functional model for compensating the systematic error effects. The G-DOP$_{\mathrm{II}}$ can then be changed into:

$$\text{G-DOP}_{\mathrm{R}} = \left\{ \mathrm{tr} \left[\sum_{J=1}^{N} \mathbf{A}_J^{\mathrm{T}} \overline{\mathbf{P}}_J \mathbf{A}_J \right]^{-1} \right\}^{\frac{1}{2}} \tag{41}$$

   where G-DOP$_{\mathrm{R}}$ denotes the GDOP based on a robust estimation. It is obvious that the more satellites available, the smaller the G-DOP$_{\mathrm{R}}$ value.
2. If the requirement of interoperability is not met, additional parameters for individual constellations need to be added to compensate for their systematic errors in the integrated navigation or positioning solutions. The G-DOP becomes larger and the level of improvement will be reduced. From Equation (39), we can obtain:

$$\{\mathrm{tr}[\mathbf{N}_{\mathbf{X}}^{-1} + \mathbf{N}_{\mathbf{X}}^{-1} \mathbf{N}_{\mathbf{XS}} (\mathbf{N}_{\mathbf{S}} - \mathbf{N}_{\mathbf{SX}} \mathbf{N}_{\mathbf{X}}^{-1} \mathbf{N}_{\mathbf{XS}})^{-1} \mathbf{N}_{\mathbf{SX}} \mathbf{N}_{\mathbf{X}}^{-1} + (\mathbf{N}_{\mathbf{S}} - \mathbf{N}_{\mathbf{SX}} \mathbf{N}_{\mathbf{X}}^{-1} \mathbf{N}_{\mathbf{XS}})^{-1}]\}^{\frac{1}{2}} \geqslant [\mathrm{tr}(\mathbf{N}_{\mathbf{X}}^{-1})]^{\frac{1}{2}}$$

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which means that $\text{G-DOP}_{II} \geqslant \text{GDOP}$. Therefore, the compatibility and interoperability of multi-GNSS is important for improving the precision of multi-GNSS positioning.

3. If $\mathbf{S}_J$ (J = 1,…, N) have different signs, meaning that they have random characteristics, then $\sum_{J=1}^{N}(\mathbf{A}_J^{\mathrm{T}}\bar{\mathbf{P}}_J\mathbf{B}_J)\mathbf{S}_J$ will have the nature of cumulative decay. As an extreme case: when N approaches infinity, $\sum_{J=1}^{N}(\mathbf{A}_J^{\mathrm{T}}\bar{\mathbf{P}}_J\mathbf{B}_J)\mathbf{S}_J$ will be close to zero. This means that when multi-GNSS is available the systematic errors among different systems in the integrated positioning may be cancelled out or compensated.
4. A large $\mathbf{S}_J$ will result in a descending $\psi(\mathbf{e}_J)$ since $\mathbf{e}_J$ includes both $\boldsymbol{\Delta}_J$ and $\mathbf{S}_J$. Therefore the G-DOP_{II} reflects the impacts of the systematic errors as well.

It should be mentioned that if no additional parameters are to be estimated in the data processing, the G-DOP_{II} will be equal to the commonly used GDOP.

5. THE THIRD TYPE OF G-DOP: CONSIDERING THE *A PRIORI* WEIGHTS OF SOME PARAMETERS.

Different types of systematic errors and offsets in multi-GNSS may exist and in many cases these errors are treated in different ways, depending on their characteristics. For example, some of the systematic errors are treated as constants, e.g. the coordinate system errors are usually expressed as; $\Delta\mathbf{X}_0 = [\Delta x_0 \quad \Delta y_0 \quad \Delta z_0]^{\mathrm{T}}$. Some other errors are time-varying, which may be modelled by a polynomial function or an autoregressive process. For example, the linear polynomial function $\Delta t = dt_0 + (t - t_0)dt_1$ is commonly used for systematic time errors. These systematic errors, or the values (or even their approximate values) of the additional model parameters at the first observation epoch, expressed as Δx_0, Δy_0, Δz_0, Δt_0 and Δt_1, are normally unknown. After the first epoch's data processing, a set of approximate values along with the variance-covariance matrix of the parameter estimates can be obtained. The variance-covariance matrix is usually called an *a priori* weight matrix (for the next epoch's data processing). Thus, the DOP value should reflect the a priori information of the additional model parameters.

If $\Delta\mathbf{X}_0$ and Δt are merged into the additional model parameter vector $\mathbf{S}$, with *a priori* estimates of $\bar{\mathbf{S}}$ and $\boldsymbol{\Sigma}_{\bar{\mathbf{S}}}$, then the error equation can be expressed as:

$$\mathbf{V}_J = \mathbf{A}_J\hat{\mathbf{X}} + \mathbf{B}_J\hat{\mathbf{S}} - \mathbf{L}_J \tag{42}$$

where $\hat{\mathbf{S}}$ denotes the estimated $\mathbf{S}$ vector.

To conveniently express the estimator under the Bayesian rule (Koch 1990), the weight matrix $\mathbf{P}_{\bar{\mathbf{S}}} = \boldsymbol{\Sigma}_{\bar{\mathbf{S}}}^{-1}$ for parameter $\bar{\mathbf{S}}$ is used. The Bayesian estimator of model parameters including the additional parameters can be then expressed as:

$$\begin{bmatrix}\hat{\mathbf{X}}\\ \hat{\mathbf{S}}\end{bmatrix} = \begin{bmatrix}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} & \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{B}\\ \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{A} & \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{B} + \mathbf{P}_{\bar{\mathbf{S}}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}^{\mathrm{T}}\bar{\mathbf{P}}\mathbf{L}\\ \mathbf{B}^{\mathrm{T}}\bar{\mathbf{P}}\mathbf{L} + \mathbf{P}_{\bar{\mathbf{S}}}\bar{\mathbf{S}}\end{bmatrix} \tag{43}$$

Therefore, the third type of G-DOP (G-DOP_{III}) which takes into account the *a priori* information of some model parameters is defined as:

$$\text{G-DOP}_{III} = \left\{\mathrm{tr}\begin{bmatrix}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} & \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{B}\\ \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{A} & \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{B} + \mathbf{P}_{\bar{\mathbf{S}}}\end{bmatrix}^{-1}\right\}^{\frac{1}{2}} \tag{44}$$

Table 1. Orbital Parameters of Compass and Galileo Satellites.

	Compass	Galileo
a	27878·1 km	29601·297 km
i	55°	56°
e	0	0
ω	0°	0°
$\mathbf{\Omega}$	0°, 120°, 240°	60°, 180°, 300°
M_0	At the starting time of the 1st satellite in each orbit, the mean anomaly are $M_0=0°$, 15° and 30° respectively, others plus 45° for each satellite. In addition, the mean anomalies for the standby satellites in the three orbits are 10°, 55° and 105° respectively.	At the starting time of the 1st satellite in each orbit, the mean anomaly are $M_0=0°$, 15° and 30° respectively, others plus 40° for each satellite.

In positioning and navigation with multi-GNSS, the G-DOP_{III} will be improved even in the case where additional systematic parameters need to be introduced for addressing the interoperability problem of different systems if the *a priori* information is used in the estimation process.

It should be pointed out that G-DOP_{III} is not new and that it depends on the data processing procedure. If the *a priori* information is employed in the data processing the G-DOP_{III} is valid. In the case where there is no *a priori* information available, the usual DOP is still used. G-DOP_{III} should be used if some estimated information can be obtained during the data processing.

6. SIMULATED EXAMPLES.

A simulation was carried out in this research. The time period of the simulation is 24 hours starting from 0000 hours March 28, 2010, in GPS time. The sampling interval is 300 s. The mask elevation angle is 5 degrees. For GPS and GLONASS satellites, their orbit parameters are obtained from broadcast ephemerides. In the time period, 30 GPS and 21 GLONASS satellites were involved. Circular orbits for Galileo and Compass were used. 27 Galileo satellites and 27 Compass MEO satellites distributed evenly in three orbital planes were simulated respectively. The simulated Keplerian orbit parameters for these satellites are listed in Table 1.

In Table 1, a denotes the major semi-axis, i is the inclination angle, e is the orbit eccentricity, ω is the argument of perigee and M_0 is the mean anomaly.

The five Compass GEO satellites are located at 58·75°E, 80°E, 110·5°E, 140°E and 160°E respectively. The inclinations of the three IGSO (Inclined Geo-Synchronous Orbit) satellites are all 55° and their right ascension of the ascending nodes are $\mathbf{\Omega}=0°$, 120° and 240° respectively. The cross node is at 118°E.

At the simulated station (34·8°N, 113·7°E, 110·4 m), the number of satellites in view is shown in Figure 1.

The flowing three scenarios are used for the simulation:

Case 1: 5%, 10% and 20% outliers were simulated in the pseudo-range measurements. The GDOPs in the two cases: with and without the outliers were compared, as shown in Table 2.

Case 2: In the integrated positioning system of the four GNSS constellations, some additional parameters are added for compensating the systematic errors. Three

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Table 2. The first type of G-DOP values with different proportion of outliers.

Proportion of Outliers	5%	10%	20%
Number of visible satellites	42	42	42
Number of Outliers	2	4	8
GDOP	0·827	0·827	0·827
The first type of G-DOP	0·836	0·843	0·866

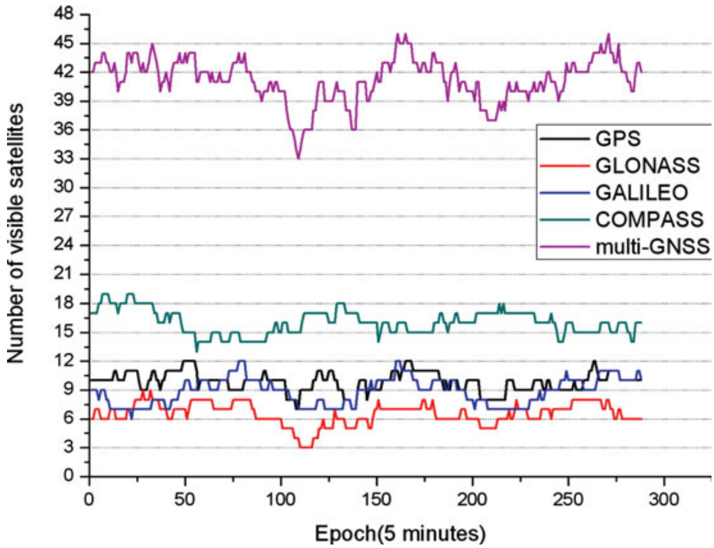

Figure 1. Numbers of visible satellites from the four GNSS and the sum of all the visible satellites.

parameters for systematic time errors and six parameters for systematic time and coordinate errors are used. The G-DOP values with and without additional parameters were calculated and compared in Figure 2 and Table 3.

Case 3: The *a priori* information of the additional parameters for the compensation of time and coordinate errors and offsets were used in the positioning solution. The third type G-DOP was calculated and compared with the cases of with and without the additional parameters but not using the a priori information. The results are shown in Figure 3 and Table 4 respectively.

In addition, the following five schemes were designed for cases 2 and 3:

Scheme 1: Only four parameters were considered in the positioning model: three for position and one for receiver clock. This means that the systematic errors of both time and coordinate systems were not included in the parameter vector.

Scheme 2: Seven parameters were considered in the positioning model in which three additional parameters for coordinate systematic errors were included in addition to the three position and one receiver clock parameters.

Scheme 3: Ten model parameters were included: three position and one receiver clock parameters, three parameters for coordinate systematic errors and three parameters for time systematic errors.

Table 3. Average G-DOP of 24 hours of schemes 1, 2 and 3.

Scheme	Scheme 1	Scheme 2	Scheme 3
Mean G-DOP	0·82	1·21	3·67

(Note that when the number of visible GLONASS satellites is less than four, the corresponding G-DOP is not included in the calculation of the average values).

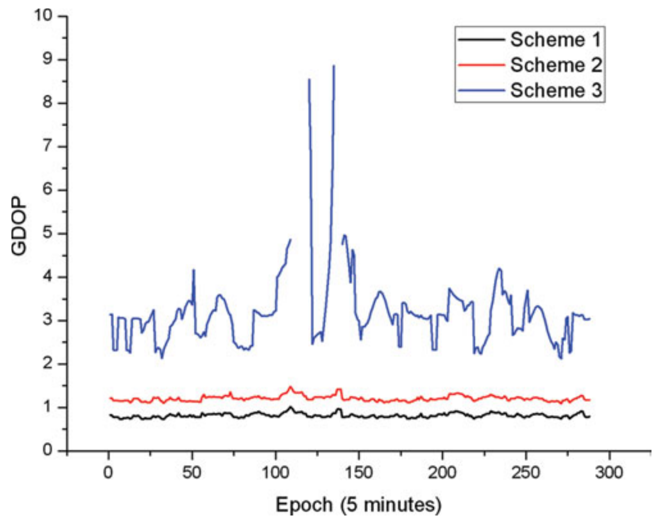

Figure 2. The second type of G-DOP changes with observation epochs.

Scheme 4: Similar to scheme 2 but an *a priori* variance-covariance matrix of the systematic error parameters is used.

Scheme 5: Similar to scheme 3 but an *a priori* variance-covariance matrix of the six systematic parameters is considered.

From the analysis of the above results, it is concluded that:

- From Table 2, it is found that the first type of G-DOP increases with the increase of the number of outliers. This type of G-DOP reflects the outlier impact on the observation geometry based on the robust estimation. It is reasonable that when there are outliers in the measurements the geometry strength should be weakened.
- From Figure 2 and Table 3, it is found that when interoperability problems exist among the GNSS, systematic error parameters should be included in the positioning model. This will result in an increase in the value of the second type of G-DOP. The second type of G-DOP increases with the increase in the number of the systematic error parameters. It is reasonable that when the number of model parameters increases the strength of the corresponding observation geometry will be weakened.
- From Figure 3, it is found that if the *a priori* information of the systematic error parameters is continuously used in the estimation process, the value of the third

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Table 4. Average G-DOP for periods of 10-epochs for schemes 1, 4 and 5.

Epoch period	1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Scheme 1	0·77	0·78	0·77	0·79	0·81	0·81	0·85	0·81	0·85	0·83
Scheme 4	0·90	0·81	0·79	0·81	0·82	0·82	0·86	0·81	0·85	0·83
Scheme 5	1·55	0·99	0·90	0·88	0·88	0·88	0·90	0·85	0·88	0·86

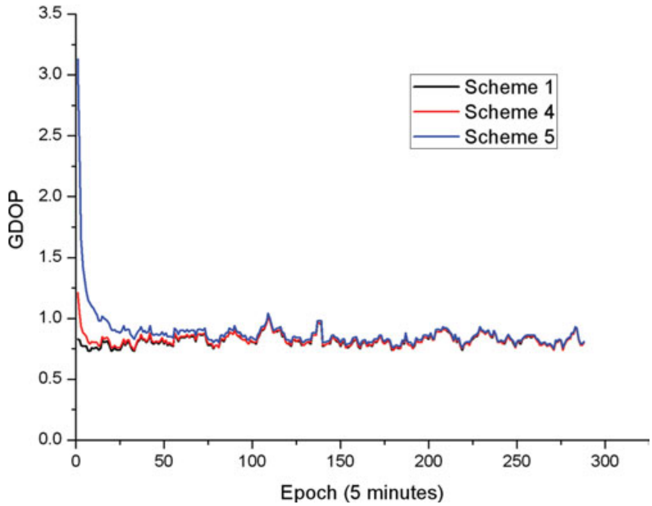

Figure 3. The third type of G-DOP changes with observation epochs.

type of G-DOP will be getting smaller and smaller with the increase in the number of observation epochs. Therefore, the *a priori* information of the systematic error parameters can strengthen the observation structure and weaken the adverse effects of the interoperability parameters on the integrated positioning of multi-GNSS and this can be reflected by the third type of G-DOP. From Table 4, it is concluded that the fewer the number of systematic parameters, the quicker the convergence of the third type of G-DOP.

7. CONCLUSIONS AND REMARKS. The observation geometry strength is related not only to the design matrix of the observation equation but also to the health of observations, i.e. related to the precision and reliability of the measurements. The first type of G-DOP can reflect the contributions of the outlying measurements to the DOP based on the robust estimation principle. The outlying measurements will not change the original DOP, however it does change the first type of G-DOP.

The observation geometry strength is also related to the number of model parameters. The higher the number of the model parameters to be estimated, the weaker the observation strength. The second type of G-DOP can reflect the effects of the additional systematic error parameters on the positioning precision. It shows that if the interoperability of multi GNSS is not fulfilled, the increased number of satellites will not significantly reduce the DOP value because additional model parameters should be introduced and estimated, which weakens the contribution of the increased number of measurements.

If the *a priori* information of the model parameters is used in the positioning process, the precision of the position will benefit from both *a priori* information of the parameters and new measurements. The third type of G-DOP can reflect both the contribution of the *a priori* information of model parameters and new measurements to the DOP. From this point of view, if the systematic errors change smoothly with time or position, the *a priori* estimates of the systematic parameters and their corresponding variance-covariance matrix should be taken into account in the actual GNSS positioning to reduce the DOP values.

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REFERENCES

Doong, S. H. (2009). A closed-form formula for GPS GDOP computation. *GPS Solutions*, **13**, 183–190.

Feng, Y. (2008). GNSS three carrier ambiguity resolution using ionosphere-reduced virtual signals. *Journal of Geodesy*, **82**(12), 847–862.

Feng, Y. and Li, B. (2008). A benefit of multiple carrier GNSS signals: regional scale network-based RTK with doubled inter-station distances. *Journal of Spatial Sciences*, **53**(1), 135–147.

Feng, Y. and Rizos, C. (2009). Network-based geometry-free three carrier ambiguity resolution and phase bias calibration. *GPS Solutions*, **13**, 43–56.

Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J. and Stahel, W. A. (1986). *Robust Statistics: the Approach Based on Influence Functions*. John Wiley, New York.

Huber, P. J. (1981). *Robust Statistics*. John Wiley, New York.

Kelly, R. J. (1990). Reducing geometric dilution of precision using ridge regression. *IEEE Transactions on Aerospace and Electronic Systems*, **26**(1), 154–168.

Koch, K. R. (1987). *Parameter Estimation and Hypothesis Testing in Linear Models*. Springer, Berlin Heidelberg, New York.

Koch, K. R. (1990). *Bayesian Inference with Geodetic Applications*. Springer, Berlin Heidelberg, New York.

Langley, R. B. (1999). Dilution of precision. *GPS World*, **10**(5), 52–59.

Leva, J. L. (1994). Relationships between navigation vertical error, VDOP, and pseudo-range error in GPS. *IEEE Transactions on Aerospace Electronic Systems*, **30**(4), 1138–1142.

Li, B., Shen, Y. and Feng, Y. (2010). Fast GNSS ambiguity resolution as an ill-posed problem. *Journal of Geodesy*, **84**(11), 683–698.

Milbert, D. (2008). Dilution of precision revisited. *NAVIGATION, Journal of The Institute of Navigation*, **55**(1), 67–81.

Milbert, D. (2009). Improving dilution of precision, a companion measure of systematic effects. *GPS World*, **20**(11), 38–47.

Odijk, D. and Teunissen, P. J. G. (2008). ADOP in closed form for a hierarchy of multi-frequency single-baseline GNSS models. *Journal of Geodesy*, **82**, 473–492.

Park, C. and Kim, I. (2000). Comments on "Relationship between dilution of precision for point positioning and for relative positioning with GPS". *IEEE Transactions on Aerospace Electronic Systems*, **36**(1), 315–316.

Parkinson, B. W. and Spilker, J. J. (1996). *Global Positioning System: Theory and Applications*, vol. II. American Institute of Aeronautics and Astronautics, Inc., Cambridge, MA, USA.

Sairo, H., Akopian, D. and Takala, J. (2003). Weighted dilution of precision as quality measure in satellite positioning. *IEE Proc.-Radar Sonar Navig*, **150**(6), 430–436.

Shen, Y., Li, B. and Chen, Y. (2010). An iterative solution of weighted total least-squares adjustment. *Journal of Geodesy*, **85**(4), 229–238.

Teunissen, P. J. G. (1990). Quality control in integrated navigation systems. *IEEE Aerospace and Electronics Systems*, **5**(7), 35–41.

Teunissen, P. J. G. and Odijk, D. (1997). Ambiguity dilution of precision: definition, properties and application. *In: Proc of ION GPS-1997*, Kansas City, September 16–19, 891–899.

Teunissen, P. J. G. and Kleusberg, A. (1999). *GPS for Geodesy*, 2nd Edition. Springer, Berlin Heidelberg, New York.

Xu, P. L., Fukuda, Y. and Liu, Y. M. (2006). Multiple parameter regularization: numerical solutions and applications to the determination of geopotential from precise satellite orbits. *Journal of Geodesy*, **80**(1), 17–27.

Xu, P. (2009). Iterative generalized cross-validation for fusing heteroscedastic data of inverse ill-posed problems. *Geophysical Journal International*, **179**(1), 182–200.

Yang, Y. (1991). Robust Bayesian Estimation. *Bulletin Géodésique*, **65**(3), 145–150.

Yang, Y. (1997). Estimators of covariance matrix at robust estimation based on influence functions. *Zeitschrift für Vermessungswesen*, **122**(4), 166–174.

Yang, Y., Song, L. and Xu, T. (2002). Robust estimator for correlated observations based on bifactor equivalent weights. *Journal of Geodesy*, **76**, 353–358.

Yarlagadda, R., Ali, I., Al-Dhahire, N. and Hershey, J. (2000). GPS GDOP metric. *IEE Proc.-Radar Sonar Navig*, **147**(5), 259–264.