Probability in the Engineering and Informational Sciences, **15**, 2001, 327–334. Printed in the U.S.A.

EFFECTIVE UPPER BOUNDS FOR EXPECTED CYCLE TIMES IN TANDEM QUEUES WITH COMMUNICATION BLOCKING

Koichi Nakade
Department of Systems Engineering
Nagoya Institute of Technology
Gokiso-cho, Showa-ku, Nagoya 466-8555, Japan
E-mail: nakade@system.nitech.ac.jp

Tandem queues with finite buffers have been widely discussed as basic models of communication and manufacturing systems. The cycle time is the important measure in such systems. In this article, we consider finite tandem queues with communication blocking and general service-time distributions. We introduce an order on pairs of random variable sets to give effective upper bounds for the expected cycle times.

1. INTRODUCTION

The approximate formula and bounds for expected cycle times of tandem queues with blocking have been proposed in the literature (see [1–4, 7], and references therein), because of its difficulties in analysis. Recently, Nakade [5] derived effective lower bounds for cycle times of tandem queues with production and communication blocking. He also obtained simple upper bounds by considering synchronous systems for tandem queues with communication blocking, but he did not discuss which synchronous system gives the most effective upper bound.

In this article, we define tandem queues with communication blocking and consider the synchronous systems to derive upper bounds for the cycle time of the original system. Then, we introduce an order on pairs of random variable sets. We use the order to obtain effective upper bounds for the expected cycle times in tandem queues with communication blocking.

 0269-9648/01 $12.50

https://doi.org/10.1017/S0269964801153039 Published online by Cambridge University Press

The organization of this article is as follows. In Section 2, we define tandem queues with communication blocking and consider the synchronous systems to derive the upper bounds for the cycle time of the original system. To make the bound effective, we introduce an order on pairs of random variable sets in Section 3. In Section 4, we use the order to show effective upper bounds for the expected cycle times in tandem queues with communication blocking. Concluding remarks are given in Section 5.

2. TANDEM QUEUES WITH COMMUNICATION BLOCKING

We consider a tandem queuing system with K stations, numbered $1,2,\ldots,K$ in the sequence, and communication blocking, in which each station has a single buffer for a job in process. Communication blocking means that a station begins processing a new job only if it has already arrived at the station from the preceding one and the buffer at the next station is empty. Note that under communication blocking, the job at station $k \in \{1,2,\ldots,K-1\}$ can move to station $k+1$ just after the end of its processing at station k, because station $k+1$ must be empty before the beginning of the processing. If station k has zero processing time, then it is a waiting buffer in front of station $k+1$. Thus, the model includes a system in which each station has two or more buffers. We, however, assume that stations 1 and K have positive processing times. We also assume that new jobs always exist in front of station 1, and the processed job leaves the system immediately after the completion of the processing at station K.

We assume that, at each station, processing times are mutually independent and identically distributed, and the sequences of processing times are mutually independent among stations. We denote a generic random variable representing the processing time at station k by I_k. Let the expected cycle time in steady state, which is a reciprocal of throughput, be denoted by $E[C]$.

To derive an upper bound for the expected cycle time, we consider the following synchronous system. Let the set of stations whose processing times are positive be $\Omega = \{i_1, i_2, \ldots, i_m\}(1 = i_1 < i_2 < \cdots < i_m = K)$. We divide it into two groups (A, B), under a constraint that if $i_{k+1} = i_k + 1$ (i.e., both stations i_k and $i_k + 1$ have positive processing times), then these stations must belong to different groups. If $i_{k+1} > i_k + 1$, then stations i_k and i_{k+1} may belong to either the same group or different groups. Let a set of feasible pairs of groups be

$$\Phi = \{(A,B); A \cap B = \phi, A \cup B = \Omega, \text{ and if } i_{k+1} = i_k + 1, \text{ then}$$
$$\text{either } i_k \in B, i_{k+1} \in A \text{ or } i_k \in A, i_{k+1} \in B\}.$$

For example, if there are 11 stations, and stations 1, 3, 5, 6, 8, 10, and 11 have positive processing times and stations 2, 4, 7, and 9 have zero processing times, then stations 5 and 6 (and stations 10 and 11) must belong to the different groups.

When the division of stations into two groups is given, the synchronous system is operated in the following way. The system first processes jobs at stations only in

https://doi.org/10.1017/S0269964801153039 Published online by Cambridge University Press

group A. At the time just after all stations in group A complete their processing, only stations in group B begin the processing. After the end of processing of jobs at all stations in group B, stations in group A begin processing. The above division of stations ensures that jobs in the synchronous system can move without blocking. Then, the expected cycle time in this synchronous system, which is the expected time interval of successive outputs of jobs, is an upper bound for the expected cycle time in the original system.

We denote the expected cycle time in the synchronous system with groups (A, B) by $C_U(A, B)$. Then, we have the following lemma [5].

LEMMA 1:

$$E[C] \leq \min_{(A,B)\in\Phi} C_U(A, B), \tag{1}$$

where

$$C_U(A, B) = E\left[\max_{i_k\in A} I_{i_k}\right] + E\left[\max_{i_k\in B} I_{i_k}\right].$$

Note that if there is no sequence of two or more successive stations with positive processing times, then it is clear that a division (Ω, ϕ) attains the minimum on the right-hand side of (1). In the remainder, we assume that there is a sequence of two or more successive stations with positive processing times.

Let a set of sequences of indexes of two or more successive stations which have positive processing times be denoted by

$$J = \{(i_k^a, i_k^a + 1, \ldots, i_k^b),\ k = 1, \ldots, m;\ E[I_{i_k^a - 1}] = 0,\ E[I_{i_k^a}] > 0,\ E[I_{i_k^a + 1}] > 0,$$
$$\ldots, E[I_{i_k^b}] > 0,\ E[I_{i_k^b + 1}] = 0,\ 1 < i_k^a < i_k^b \leq K\},$$

where $E[I_0] = E[I_{K+1}] = 0$ and m is the number of such sequences. We define a set of stations whose adjacent stations have zero processing times by

$$J' = \{i_1^c, i_2^c, \ldots, i_l^c;\ E[I_{i_k^c - 1}] = E[I_{i_k^c + 1}] = 0 \text{ and } E[I_{i_k^c}] > 0\},$$

where l is the number of such stations.

3. AN ORDER ON PAIRS OF RANDOM VARIABLE SETS

To derive an effective upper bound (i.e., to find a pair of set $(A, B) \in \Phi$ minimizing the right-hand side in (1)), we introduce an order on pairs of random variable sets. We define a joint cumulative distribution function for a set of random variables $A = \{X_1, \ldots, X_n\}$ by

$$F_A(x) = P\left(\max_{X_i\in A} X_i \leq x\right) \quad \text{for } x \in \mathcal{R},$$

where $\mathcal{R}$ is a set of real numbers. We set $F_\phi(x) = 1$ for all $x \in \mathcal{R}$, where ϕ is an empty set. In the following, we abbreviate $X_i \in A$ by $i \in A$. From the definition, when $A, B \neq \phi$, $F_A(x) \geq F_B(x)$ for all $x \in \mathcal{R}$ implies that $\max_{i\in A} X_i \leq_{\text{st}} \max_{i\in B} X_i$, where $\leq_{\text{st}}$ is a stochastic order (see [6]).

Definition 1: *For four random variable sets A, B, C, and D, we say that a pair (A, B) is smaller than a pair (C, D) in a sense of a sum of joint cumulative distribution functions, which is denoted by $(A, B) \prec_{j,c} (C, D)$, if it holds for all $x \in \mathcal{R}$:*

$$F_A(x) + F_B(x) \geq F_C(x) + F_D(x).$$

We show properties of the order in the following lemma.

Lemma 2: (i) *We assume that A,B,C,D $\subseteq S = \{X_1, X_2, \ldots, X_n\}$ and A,B,C,D $\neq \phi$. If $(A, B) \prec_{j,c} (C, D)$ and $E[\max_{i\in A\cup B\cup C\cup D} |X_i|] < \infty$, then*

$$E\left[\max_{i\in A} X_i + \max_{i\in B} X_i\right] \leq E\left[\max_{i\in C} X_i + \max_{i\in D} X_i\right].$$

(ii) *Let $S = \{X_1, X_2, \ldots, X_n\}$ be a set of mutually independent random variables, and we assume that $A_1, A_2, B_1, B_2, Y, Y' \subseteq S$, $A_1 \cap B_2 = \phi$, $A_2 \cap B_1 = \phi$, and $(A_1 \cup B_2) \cap Y = (A_2 \cup B_1) \cap Y' = \phi$. If it holds that $F_{A_1}(x) \geq F_{A_2}(x)$ and $F_{B_1}(x) \geq F_{B_2}(x)$ for all $x \in \mathcal{R}$, then we have*

$$(A_1 \cup B_1, A_2 \cup B_2 \cup Y \cup Y') \prec_{j,c} (A_1 \cup B_2 \cup Y, A_2 \cup B_1 \cup Y').$$

Proof: Part (i) is easily shown by the fact that when $A \neq \phi$, it holds that

$$E\left[\max_{i\in A} X_i\right] = -\int_{-\infty}^{0} F_A(x)\,dx + \int_0^\infty (1 - F_A(x))\,dx.$$

(ii) Because the random variables are mutually independent, when $A, B \subseteq S$ it follows that

$$F_{A\cup B}(x) = \prod_{i\in A\cup B} P(X_i \leq x) \geq \prod_{i\in A} P(X_i \leq x) \prod_{i\in B} P(X_i \leq x) = F_A(x)F_B(x),$$

where equality in the inequality holds if $A \cap B = \phi$. Therefore, from the assumption, for any $x \in \mathcal{R}$,

$$\begin{aligned}
&F_{A_1\cup B_1}(x) + F_{A_2\cup B_2\cup Y\cup Y'}(x) - (F_{A_1\cup B_2\cup Y}(x) + F_{A_2\cup B_1\cup Y'}(x)) \\
&\geq F_{A_1}(x)F_{B_1}(x) + F_{A_2}(x)F_{B_2}(x)F_Y(x)F_{Y'}(x) \\
&\quad - (F_{A_1}(x)F_{B_2}(x)F_Y(x) + F_{A_2}(x)F_{B_1}(x)F_{Y'}(x)) \\
&= (F_{A_1}(x) - F_{A_2}(x)F_{Y'}(x))(F_{B_1}(x) - F_{B_2}(x)F_Y(x)) \geq 0,
\end{aligned}$$

which proves (ii). ■

https://doi.org/10.1017/S0269964801153039 Published online by Cambridge University Press

Let $S = \{X_1, X_2, \ldots, X_n\}$ denote a set of n mutually independent random variables. We consider a problem for dividing it into two disjoint sets Z and Z^c $(= S \setminus Z)$, as (Z, Z^c) is the smallest in $\prec_{j,c}$, under the following constraint: If $(X_{i_k}, X_{j_k}) \in Q$, where $Q = \{(X_{i_k}, X_{j_k});\ k = 1,2,\ldots,m\}$ satisfies $\{X_{i_k};\ k = 1,\ldots,m\} \cap \{X_{j_k};\ k = 1,\ldots,m\} = \phi$, then X_{i_k} and X_{j_k} must belong to the different sets. We say that the division (Z, Z^c) is feasible if it satisfies the constraint.

Let a set of random variables which are not included in Q be denoted by

$$\{X_{k_1}, X_{k_2}, \ldots, X_{k_l}\} = S \setminus \{X_{i_1}, \ldots, X_{i_m}, X_{j_1}, \ldots, X_{j_m}\}.$$

Proposition 1: *Suppose that $X_{i_k} \leq_{\text{st}} X_{j_k}$ for all $(X_{i_k}, X_{j_k}) \in Q$, and $j_k \neq j_{k'}$ for $k \neq k'$. Then, for any feasible division (Z, Z^c) of S, we have*

$$(\{X_{i_1}, \ldots, X_{i_m}\}, \{X_{j_1}, \ldots, X_{j_m}, X_{k_1}, \ldots, X_{k_l}\}) \prec_{j,c} (Z, Z^c). \tag{2}$$

Proof: For any feasible division (Z, Z^c) without loss of generality, we can set

$$Z = A_1 \cup B_2 \cup Y, \qquad Z^c = A_2 \cup B_1 \cup Y', \qquad A_1 = \{X_{i_1}, \ldots, X_{i_p}\},$$
$$A_2 = \{X_{j_1}, \ldots, X_{j_p}\}, \qquad B_1 = \{X_{i_{p+1}}, \ldots, X_{i_m}\}, \qquad B_2 = \{X_{j_{p+1}}, \ldots, X_{j_m}\},$$
$$Y \cup Y' = \{X_{k_1}, \ldots, X_{k_l}\}, \qquad Y \cap Y' = \phi.$$

Since $j_k \neq j_{k'}$ for $k \neq k'$, by the assumption,

$$F_{A_1}(x) \geq \prod_{k=1}^{p} P(X_{i_k} \leq x) \geq \prod_{k=1}^{p} P(X_{j_k} \leq x) = F_{A_2}(x)$$

and $F_{B_1}(x) \geq F_{B_2}(x)$ for all $x \in \mathcal{R}$. Therefore, we obtain (2) from Lemma 2, because $(\{X_{i_1}, \ldots, X_{i_m}\},\ \{X_{j_1}, \ldots, X_{j_m}, X_{k_1}, \ldots, X_{k_l}\})$ is feasible and $\{X_{i_k};\ k = 1,\ldots,m\} \cap \{X_{j_k}; k = 1,\ldots,m\} = \phi$. ■

4. AN EFFECTIVE UPPER BOUND FOR THE CYCLE TIME

We have the following main theorem from Lemmas 1 and 2 and Proposition 1.

Theorem 1: *For each sequence $(i_k^a, i_k^a + 1, \ldots, i_k^b) \in J$, we define*

$$C_k = \{i_k^a + 2n;\ i_k^a + 2n \leq i_k^b,\ n = 0,1,2,\ldots\}$$

and

$$D_k = \{i_k^a + 2n + 1;\ i_k^a + 2n + 1 \leq i_k^b,\ n = 0,1,2,\ldots\}.$$

We assume that for each $k \in \{1, \ldots, m\}$, either (i)

$$F_{C_k}(x) \geq F_{D_k}(x) \quad \textit{for all } x \in \mathcal{R} \tag{3}$$

or (ii)

$$F_{C_k}(x) \leq F_{D_k}(x) \quad \textit{for all } x \in \mathcal{R}. \tag{4}$$

https://doi.org/10.1017/S0269964801153039 Published online by Cambridge University Press

We set $A_k = C_k$ and $B_k = D_k$ for case (i) and $A_k = D_k$ and $B_k = C_k$ for case (ii). We also define

$$\tilde{A} = \bigcup_{k=1}^{m} A_k \quad \textit{and} \quad \tilde{B} = \Omega \backslash \tilde{A} = \left(\bigcup_{k=1}^{m} B_k \right) \cup J'.$$

Then, we have

$$E[C] \le C_U(\tilde{A}, \tilde{B}) = \min_{(A,B)\in\Phi} C_U(A,B).$$

Proof: Any feasible division $(Z, Z^c) \in \Phi$ must satisfy that either ($A_k \subset Z$ and $B_k \subset Z^c$) or ($B_k \subset Z$ and $A_k \subset Z^c$) for each $k \in \{1, 2, \ldots, n\}$ because adjacent stations with positive successive times must belong to different groups. For $k = 1, \ldots, m$, let

$$Y_{2k-1} = \max_{i\in A_k} I_i \quad \text{and} \quad Y_{2k} = \max_{i \in B_k} I_i.$$

We also define

$$\Omega' = \{Y_1, \ldots, Y_{2m}, I_{i_1^c}, \ldots, I_{i_l^c}\} \quad \text{and} \quad Q' = \{(Y_1, Y_2), (Y_3, Y_4), \ldots, (Y_{2m-1}, Y_{2m})\}.$$

We associate each feasible division (Z, Z^c) of Ω with one division (U, U^c) of Ω' under constraint Q' as follows: $Y_{2k-1} \in U$ if and only if $A_k \subset Z$, and $X_{i_k^c} \in U$ if and only if $i_k^c \in Z$. We denote a division of Ω' associated with $(\tilde{A}, \tilde{B})$ by $(\tilde{U}, \tilde{U}^c)$. Then,

$$(\tilde{U}, \tilde{U}^c) = (\{Y_1, Y_3, \ldots, Y_{2m-1}\}, \{Y_2, Y_4, \ldots, Y_{2m}, I_{i_1^c}, \ldots, I_{i_l^c}\}).$$

Since $Y_{2k-1} \le_{\text{st}} Y_{2k}$ for $k = 1, \ldots, m$, by Proposition 1 we have $(\tilde{U}, \tilde{U}^c) \prec_{j,c} (U, U^c)$ for any division (U, U^c) of Ω' under constraint Q'. Therefore, by applying Lemma 2 we have

$$\begin{aligned} E\left[\max_{i\in\tilde{A}} I_i\right] + E\left[\max_{i\in B} I_i\right] &= E\left[\max_{X\in\tilde{U}} X\right] + E\left[\max_{X\in\tilde{U}^c} X\right] \\ &\le E\left[\max_{X\in U} X\right] + E\left[\max_{X\in U^c} X\right] \end{aligned}$$

for any division (U, U^c), where X is a random variable representing Y_k or $I_{i_k^c}$. Since each $(A, B) \in \Phi$ is associated with a division of Ω' under Q', we obtain the theorem. ■

Note that the assumption of the theorem, (3) or (4), is satisfied in the following cases:

1. If $I_{i_k^a} =_{\text{st}} I_{i_k^a+1} =_{\text{st}} \cdots =_{\text{st}} I_{i_k^b}$, then the sequence $(i_k^a, i_k^a + 1, \ldots, i_k^b)$ satisfies (4). Note that if the number of stations in the sequence, $i_k^b - i_k^a + 1$, is odd, then the number of stations in D_k is smaller than that in C_k and we have $F_{D_k}(x) \ge F_{C_k}(x)$ for all $x \in \mathcal{R}$.

Table 1. Upper Bounds for Expected Cycle Times in Tandem Queuing Systems

Example	$(E[I_1],\ldots,E[I_{11}])$	$E[C]$	$C_U(\tilde{A},\tilde{B})$	$\tilde{A}$	$\tilde{B}$
a	(1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1)	2.21338 ± 0.00124	2.52380	{4, 8}	{1, 3, 5, 7, 9, 11}
b	(1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1)	2.18043 ± 0.00092	2.58333	{4, 7, 10}	{1, 3, 6, 8, 11}
c	(1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1)	2.25691 ± 0.00114	2.58333	{4, 6, 9}	{1, 3, 5, 8, 11}
d	(1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1)	2.36696 ± 0.00116	2.60714	{4, 6, 8}	{1, 3, 5, 7, 9, 11}
e	(1, 0, 2, 1, 2, 0, 2, 1, 2, 0, 1)	3.41866 ± 0.00178	3.76574	{4, 8}	{1, 3, 5, 7, 9, 11}
f	(1, 0, 2, 1, 2, 2, 0, 1, 2, 0, 1)	4.03445 ± 0.00330	4.53912	{4, 6, 8}	{1, 3, 5, 9, 11}
g	(1, 0, 2, 1, 2, 1, 2, 1, 2, 0, 1)	3.58095 ± 0.00161	3.84907	{4, 6, 8}	{1, 3, 5, 7, 9, 11}

2. If $I_{i_k^a} \leq_{\text{st}} I_{i_k^a+1} \geq_{\text{st}} I_{i_k^a+2} \leq_{\text{st}} \cdots \leq_{\text{st}} I_{i_k^b}$, $I_i \neq_{\text{st}} I_j$ for some $i, j \in \{i_k^a, \ldots, i_k^b\}$ and $i_k^b - i_k^a + 1$ is even, then the sequence satisfies (3).
3. If $I_{i_k^a} \geq_{\text{st}} I_{i_k^a+1} \leq_{\text{st}} I_{i_k^a+2} \geq_{\text{st}} \cdots \geq_{\text{st}} I_{i_k^b}$ and $I_i \neq_{\text{st}} I_j$ for some $i, j \in \{i_k^a, \ldots, i_k^b\}$, then the sequence satisfies (4).

In Table 1, we show numerical examples. We assume that the processing times at station k have a uniform distribution with its interval $[\frac{1}{2}E[I_k], \frac{3}{2}E[I_k]]$. The expected cycle times $E[C]$ are computed by simulation, which is run 20 times and each includes 20,000 cycles. The mean value and their 95% confidence interval for each simulation are shown in Table 1.

As shown in Table 1, the upper bounds have 7.5–18.5% relative errors. In particular, if it holds that either the number of elements in $\tilde{A}$ are 2 or the expected processing times at stations in $\tilde{A}$ are smaller than those at the other stations with positive processing times, then the bounds show good performance. Table 1 also shows that the upper bound is good if there is a long sequence of successive stations with positive processing times. The reason seems that jobs in such systems behave as those in the synchronous systems. Note that when $A = \{4,7,9\}$, $B = \{1,3,5,8,11\}$ in Example a of Table 1, $C_U(A, B)$ is 2.58333, which is greater than $C_U(\tilde{A}, \tilde{B})$ in Example a. This illustrates the result of Theorem 1.

5. CONCLUDING REMARKS

In this article, we show effective upper bounds for cycle times in tandem queues with communication blocking by considering the synchronous systems and introducing the order on pairs of random variable sets. This method for deriving upper bounds seems to be able to be applied for queuing networks with communication blocking.

Acknowledgment

The author expresses his appreciation to an anonymous referee for his/her valuable comments.

References

1. Altiok, T. (1988). Approximate analysis of queues in series with phase-type service times and blocking. *Operations Research* 37: 601–610.
2. Baccelli, F. & Makowski, A.M. (1989). Multidimensional stochastic ordering and associated random variables. *Operations Research* 37: 478–487.
3. Baccelli, F., Makowski, A.M., & Shwartz, A. (1989). The fork–join queue and related systems with synchronization constraints: Stochastic ordering and computable bounds. *Advances in Applied Probability* 21: 629–660.
4. Buzacott, J.A. & Shanthikumar, J.G. (1993). *Stochastic models of manufacturing systems*. Englewood Cliffs, NJ: Prentice-Hall, Chap. 5.
5. Nakade, K. (2000). New bounds for expected cycle times in tandem queues with blocking. *European Journal of Operational Research* 125: 85–93.
6. Shaked, M. & Shanthikumar, J.G. (1994). *Stochastic orders and their applications*. Boston: Academic Press.
7. van Dijk, N.M. & Miyazawa, M. (1997). Error bounds on a practical approximation for finite tandem queues. *Operations Research Letters* 21: 201–208.