Probability in the Engineering and Informational Sciences, **31**, 2017, 420–435.
doi:10.1017/S0269964817000092

PROCESSOR SHARING G-QUEUES WITH INERT CUSTOMERS AND CATASTROPHES: A MODEL FOR SERVER AGING AND REJUVENATION

J.-M. Fourneau
DAVID, UVSQ, Université Paris-Saclay,
45, Av. des États-Unis,
78035 Versailles,
France
E-mail: *Jean-Michel.Fourneau@uvsq.fr*

Y. Ait El Majhoub
DAVID, UVSQ, Université Paris-Saclay,
45, Av. des États-Unis,
78035 Versailles,
France

We consider open networks of queues with Processor-Sharing discipline and signals. The signals deletes all the customers present in the queues and vanish instantaneously. The customers may be usual customers or inert customers. Inert customers do not receive service but the servers still try to share the service capacity between all the customers (inert or usual). Thus a part of the service capacity is wasted. We prove that such a model has a product-form steady-state distribution when the signal arrival rates are positive.

Keywords: aging, catastrophes, G-networks with customers and signals, inert customers, product – form steady-state distribution, rejuvenation

1. INTRODUCTION

The theory of queues with signals (or G-networks) had received a considerable attention since the seminal paper on positive and negative customers [10] published by Gelenbe more than 20 years ago. Traditional queueing networks models are used to represent contention among customers for a set of resources. Customers move from server to server where they wait for service according to a scheduling discipline. Apart this competition, they do not interact among themselves.

In a network of queues with signals, signals interact at their arrival into a queue with the queue or with customers already backlogged. Signals are never queued. At their arrival, they try to interact immediately. After their trial of interaction, they disappear irrespective of their failure or success or they can migrate to another queue. Furthermore, customers are allowed to change into signals at the completion of their service. This mechanism allows us to model complex interaction between customers in several queues.

 0269-9648/17 $25.00

Despite this deep modification of the classical queueing network model, G-networks still preserve the product form property for the steady-state distribution of some Markovian queueing networks under some technical conditions on the processes involved (Poisson arrivals for signals and customers from the outside, exponential service times for customers, Markovian routing of customers and signals, open topology, independence, etc.).

The first type of signal was introduced as negative customers in [10]. A negative customer deletes a positive customer at its arrival at a non-empty queue. Positive customers are usual customers, which are queued and receive service or are deleted by negative customers. It must be clear that the results are more complex than Jackson's networks. The G-networks flow equations exhibit some uncommon properties: they are neither linear as in closed queueing networks nor contracting as in Jackson queueing networks. Therefore, the existence of a solution had to be proved [21] by new techniques from the theory of fixed point equation and numerical algorithms had to be proved to solve the flow equations [5,8]. Many types of signal have been studied and they all lead to product form solution: triggers, which redirect other customers among the queues [11]; catastrophes, which flush all the customers out of a queue and batches of deletion [12]; resets [17]. Extensions with multiple classes of customers have also been derived [18].

G-networks had also motivated many new important results in the theory of queues. As negative customers lead to customer deletions, the original description of quasi-reversibility by arrivals and departures does not hold anymore and a new version had been proposed by Chao et al. in [2]. A different approach, based on Stochastic Process Algebra, was proposed by Harrison [22,23]. The main results (CAT and RCAT theorems and their extensions [1,22,23]) give some sufficient conditions for product form stationary distributions. This technique clearly has a different range of applications as it allows to represent component based models, which are much more general and more detailed than networks of queues.

Network of positive and negative customers were introduced to model neural networks where neurones exchange inhibitory and exciting signals [13,16]. G-networks and Random Neural Networks were also used in the design of the learning process for Cognitive Packet Networks [19] or application of the Random Neural Networks to quality of service [25] or to model the interaction between energy and the Data plane in telecom networks [14,15]. Currently there are several hundred references devoted to the subject and two books [2,20] provide insight into some of the research issues, developments and applications in the area of networks of queues with customers and signals.

Here we consider Processor-Sharing (PS) queues with inert customers and signals. Inert customers were introduced by Dao Thi et al. in [3]. Inert customers are customers which do not use the service capacity, but they stay in the queue until they interact with the signal. More precisely, in a PS queue, the service is shared among all the customers whatever they are usual customers or inert customers. But the inert customers do not use the server and this part of the service capacity is wasted. The signal is a catastrophe or a disaster: it removes all the customers (both inert or usual). Note that the arrival rate of signal must be positive to obtain a stationary system. Indeed, the signal is the only possibility to let the inert customers leave the queue. We depict in Fig. 1 a typical sample-path for a queue with both usual and inert customers obtained by the simulation tool in XBorne [6]. We will see that the queue is less and less efficient with aging until it is refreshed by the signal.

The technical part of the paper is as follows. In Section 2, we introduce networks with inert customers and catastrophes signals. We state that the steady-state distribution has a product form if the chain is ergodic (the proof based on the analysis of the Kolmogorov equation at steady state in postpone in an appendix for the sake of readability.

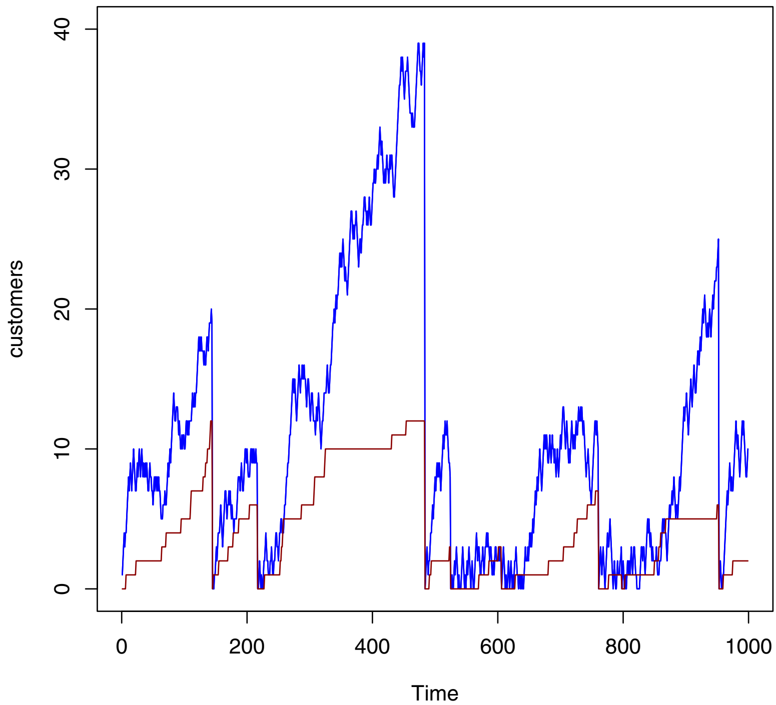

FIGURE 1. Sample-paths for a queue with catastrophes and customers (usual in blue, inert in red). Parameters: $\lambda^{\mathrm{I}} = 0.1$, $\lambda^{S} = 0.01$, $\lambda^{\mathrm{U}} = 1.0$, $\mu = 1.0$.

In Section 3, we prove that under some technical assumptions, the flow equations have a solution. We present in Section 4, a more complex catastrophe signal, which only deletes the inert customers.

2. MODEL AND PRODUCT-FORM STEADY-STATE DISTRIBUTION

We consider an open network with N PS queues and two types of customers: usual customer and inert customers. Furthermore, the queues can receive signals from the outside or sent by another queue at the completion of service for an usual customer (see Fig. 2).

Both types of customers arrive from the outside at queue i according to Poisson processes with rate λ_i^{U} for usual customers and λ_i^{I} for inert ones. Usual customers receive service at queue i with a exponentially distributed duration with rate μ_i. Inert customers have a service rate equal to 0: they do not receive service. However, a part of the service capacity of the server is given to inert customers according to the PS discipline. Therefore, it is wasted.

At the completion of its service at queue i, an usual customer may join queue j either as an usual customer (routing matrix $P(i,j)$), or a signal (routing matrix $C(i,j)$) or an inert customer (routing matrix $N(i,j)$), or it can leave the network with probability d_i. Of course, we have for all i,

$$d_i + \sum_j P(i,j) + \sum_j N(i,j) + \sum_j C(i,j) = 1.$$

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

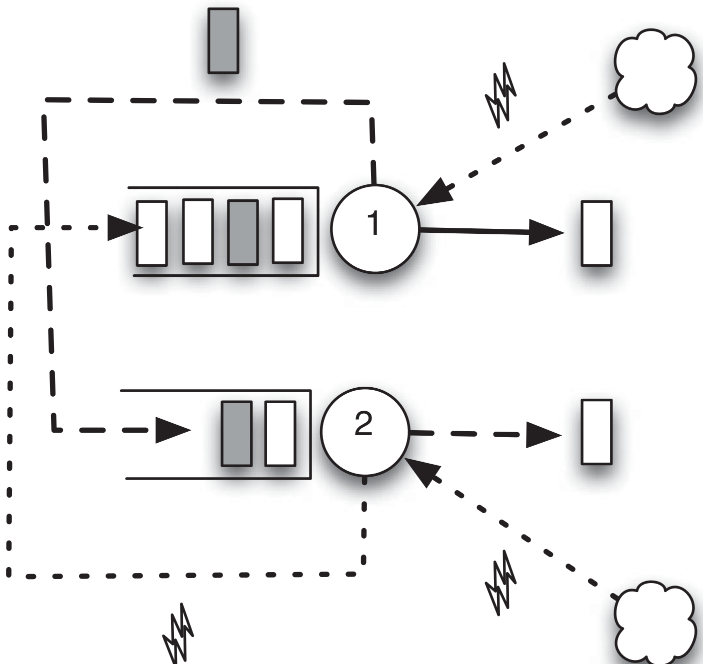

FIGURE **2.** Two PS queue with usual customers (white boxes), inert customers (gray boxes) and catastrophe signals.

We also assume that there is no loop in the routing matrices: for all i

$$P(i,i) = 0, \quad N(i,i) = 0, \quad C(i,i) = 0.$$

Signals may also arrive from the outside following Poisson processes of rate λ_i^S for queue i. A signal entering queue i deletes all the customers present in the queue irrespective of their types. A signal is never queued. It disappear immediately after its arrival. Such a signal has been previously studied in the literature [4,7,24]. It is a particular case of the batch deletion of customers proposed in [12].

Note that the open topology is mandatory because in a closed queuing network with catastrophes, the customers disappear and all the queues are empty at steady state with probability one.

We consider an open network with N queues. The state of queue i is $x_i = (x_i^{\mathrm{U}}, x_i^{\mathrm{I}})$. Under the classical assumptions we have presented, $(x)_t = (x_1, \ldots, x_i, \ldots x_N)_t$ is a Markov chain.

Let us first introduce some notation. Let:

- $\|x_i\| = x_i^{\mathrm{U}} + x_i^{\mathrm{I}}$.
- e_i^{U} is a vector whose all entries are equal to 0 except entry (i, U), which is equal to 1.
- Similarly, e_i^{I} is a vector whose all entries are equal to 0 except entry (i, I), which is equal to 1.

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

THEOREM 2.1: *Assume that the Markov chain $(x)_t = (x_1, \ldots, x_i, \ldots x_N)_t$ is ergodic. If the following flow equations have a solution such that $\rho_i^U + \rho_i^I < 1$ for all i,*

$$\rho_i^U = \frac{\lambda_i^U + \sum_j \mu_j \rho_j^U P(j,i)}{\mu_i + (\lambda_i^S + \sum_j \mu_j \rho_j^U C(j,i))R_i}, \tag{1}$$

and

$$\rho_i^I = \frac{\lambda_i^I + \sum_j \mu_j \rho_j^U N(j,i)}{(\lambda_i^S + \sum_j \mu_j \rho_j^U C(j,i))R_i}, \tag{2}$$

where

$$R_i = \sum_{k=0}^{\infty} (\rho_i^U + \rho_i^I)^k = (1 - \rho_i^U + \rho_i^I)^{-1}, \tag{3}$$

then the steady-state distribution has a product form solution:

$$\pi(x) = \prod_{i=1}^{N} (1 - \rho_i^U - \rho_i^I) \frac{\|x_i\|!}{x_i^U!\; x_i^I!} (\rho_i^U)^{x_i^U} (\rho_i^I)^{x_i^I}. \tag{4}$$

The proof is based on the analysis of the global balance equation. The Kolmogorov equation at steady state is:

$$\begin{aligned}
&\pi(x) \left(\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_i \frac{x_i^{\mathrm{U}} \mu_i}{||x_i||} 1_{||x_i||>0} \right) \\
&= \sum_i \pi(x - e_i^{\mathrm{U}}) \lambda_i^{\mathrm{U}} 1_{x_i^{\mathrm{U}}>0} \\
&\quad + \sum_i \pi(x - e_i^{\mathrm{I}}) \lambda_i^{\mathrm{I}} 1_{x_i^{\mathrm{I}}>0} \\
&\quad + \sum_i \pi(x + e_i^{\mathrm{U}}) \frac{(x_i^{\mathrm{U}} + 1)\mu_i}{||x_i + e_i^{\mathrm{U}}||} d_i \\
&\quad + \sum_i \sum_j \pi(x + e_i^{\mathrm{U}} - e_j^{\mathrm{U}}) \frac{(x_i^{\mathrm{U}} + 1)\mu_i}{||x_i + e_i^{\mathrm{U}}||} P(i,j) 1_{x_j^{\mathrm{U}}>0} \\
&\quad + \sum_i \sum_j \pi(x + e_i^{\mathrm{U}} - e_j^{\mathrm{I}}) \frac{(x_i^{\mathrm{U}} + 1)\mu_i}{||x_i + e_i^{\mathrm{U}}||} N(i,j) 1_{x_j^{\mathrm{I}}>0} \\
&\quad + \sum_i \lambda_i^S \sum_{a\geq 0} \sum_{b\geq 0} \pi(x + e_i^{\mathrm{U}} a + e_i^{\mathrm{I}} b) 1_{||x_i||=0} \\
&\quad + \sum_i \mu_i \sum_j C(i,j) \sum_{a\geq 0} \sum_{b\geq 0} \pi(x + e_i^{\mathrm{U}} + e_j^{\mathrm{U}} a + e_j^{\mathrm{I}} b) 1_{||x_j||=0}.
\end{aligned}$$

Remark that this equation includes null transitions, on both sides of the equation, when the queue size is 0. For the sake of readability the proof is postponed to an appendix.

These queues exhibit a very interesting behavior, which is depicted in Fig. 3. As mentioned earlier, the part of the capacity given by the servers to the inert customers is lost, thus one can observe a waste of the server power. At state $(x^{\mathrm{U}}, x^{\mathrm{I}})$, the lost part is $x^{\mathrm{I}}/(x^{\mathrm{U}} + x^{\mathrm{I}})$. In Fig. 3, we have depicted a sample path of the remaining part of the service capacity.

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

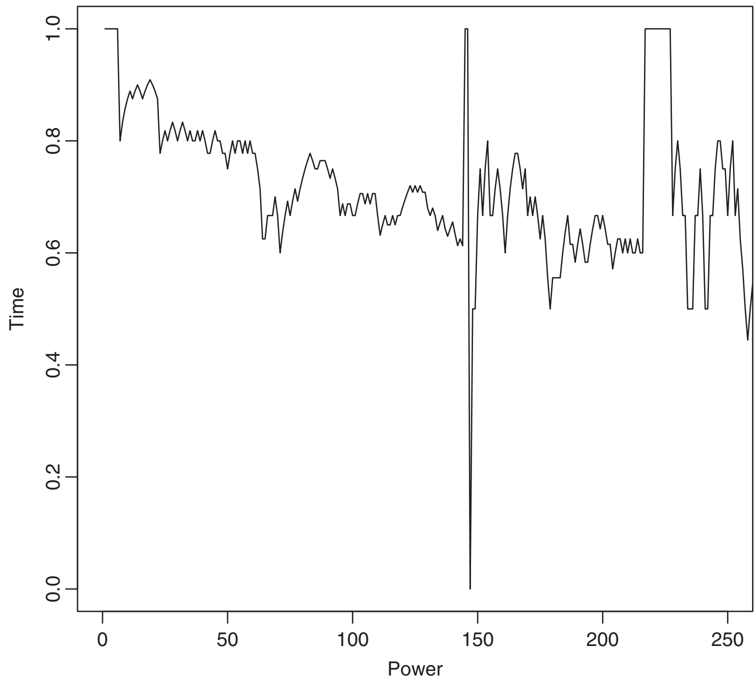

FIGURE 3. Sample paths for the effective capacity for a queue with catastrophes and both types of customers. Same parameters as in Fig. 1.

When $x^{\mathrm{U}} + x^{\mathrm{I}} = 0$, we have set this lost part to 0 (or equivalently, the remaining part is equal to 1). The service capacity evolves with time with a decreasing trend, which is due to the increasing number of inert customers. Clearly, the remaining service which is equal to $x^{\mathrm{U}}/(x^{\mathrm{U}} + x^{\mathrm{I}})$ decreases with the number of inert customers (i.e. x^{I}). As the number of inert customers increases with time until the next catastrophe, we obtain a queue where the service capacity decreases with age until a rejuvenation (i.e. the catastrophe) refreshes the server and its capacity. The small fluctuations are due to the number of usual customers which increases or decreases as a result of arrivals and departures. Finally at time 144, a signal occurs and all the customers are deleted. Another signal arrives at time 217 and also clears the queue. However, these two signals do not have exactly the same effect on the future of the sample-path.

The signal arriving at time 144 empties the queue. But the next event is an arrival of an inert customer and the remaining capacity of service jumps to 0 as the queue only contains one inert customer. When an usual customer arrives, the capacity jumps at 0.5 as the queue population is now one inert and one usual customer sharing the capacity.

At time 217, the signal is followed by the arrival of several usual customers (i.e. exactly 3). Thus, the capacity stays at 1 for a short period of time before decreasing when the first inert customer arrives.

Note that even if these queues exhibit a very unusual behavior, they are still PS queues with a well-known steady-state distribution. The only difference is described by the flow equation, not by the distribution. Therefore, the usual formulas for PS queues are still valid and we obtain the average number of usual customers at queue i by:

$$E[N_i] = \frac{\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}}}{1 - \rho_i^{\mathrm{U}} - \rho_i^{\mathrm{I}}}.$$

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

3. STABILITY

Clearly Eqs. (1)–(3) define a nonlinear fixed point system. Due to the nonlinearity, existence of a fixed point solution is not a trivial question and must be addressed. Furthermore, one may expect that the arrival of catastrophes make the queueing process stationary.

THEOREM 3.1: *Assume that the chain is ergodic. If $\lambda_i^S > 0$ and $\lambda_i^U > 0$ for all queue i, then the solution of the fixed point exists and $\rho_i^U + \rho_i^I < 1$ for all i.*

PROOF: We first prove the existence with Brouwer's theorem. Let us define operator F on $(R^+)^{2N}$ by its components F_i^{U} and F_i^{I}:

- if $F_i^{\mathrm{U}} + F_i^{\mathrm{I}} < 1$

$$F_i^{\mathrm{U}}(F) = \frac{\lambda_i^{\mathrm{U}} + \sum_j \mu_j F_j^{\mathrm{U}} P(j,i)}{\mu_i + (\lambda_i^S + \sum_j \mu_j F_j^{\mathrm{U}} C(j,i)) R_i},$$

$$F_i^{\mathrm{I}}(F) = \frac{\lambda_i^{\mathrm{I}} + \sum_j \mu_j F_j^{\mathrm{U}} N(j,i)}{(\lambda_i^S + \sum_j \mu_j F_j^{\mathrm{U}} C(j,i)) R_i}$$

  and

$$R_i = (1 - F_i^{\mathrm{U}} - F_i^{\mathrm{I}})^{-1}$$

- and $F_i^{\mathrm{U}} = F_i^{\mathrm{I}} = 0$ otherwise

We investigate the fixed points of F. Remark that the system of flow equations and F define the same system when $\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}} < 1$. We now define a new operator, say G, on $(R^+)^{2N}$ by its components:

$$G_i^{\mathrm{U}}(F) = \frac{\lambda_i^{\mathrm{U}} + \sum_j \mu_j F_j^{\mathrm{U}} P(j,i)}{\mu_i},$$

and

$$G_i^{\mathrm{I}}(F) = \frac{\lambda_i^{\mathrm{I}} + \sum_j \mu_j F_j^{\mathrm{U}} N(j,i)}{\lambda_i^S}$$

Clearly, $F \preceq G$. It is sufficient to take into account that $\mu_j > 0$, $F_j^{\mathrm{U}} \geq 0$, $C(j,i) \geq 0$, and $R_i \geq 0$ or $R_i \geq 1$ in Eqs. (1) and (2).

Furthermore, operator G is non-negative, contracting, continuous and (G_i^{U}) is associated with a classical Jackson network. Thus, G has a fixed point $\hat{f}$.

Now, we define $\mathcal{S}$ as a subset of $(R^+)^{2N}$ as follows:

$$\mathcal{S} = \{q \in (R^+)^{2N} : 0 \preceq q \preceq \hat{f}\}.$$

Clearly, $\mathcal{S}$ is compact and convex. Since for all j, $\lambda_j^{\mathrm{U}} > 0$, we have $\hat{f} \succ 0$ and then interior of $\mathcal{S}$ is not empty.

As mentioned earlier, $F(q) \preceq G(q)$. Furthermore G is non-decreasing in $\mathcal{S}$, so for all q in $\mathcal{S}$ we have $G(q) \preceq G(\hat{f})$. Combining these inequalities and the fixed point, we get:

$$F(q) \preceq G(q) \preceq G(\hat{f}) = \hat{f}$$

and then $F(\mathcal{S}) \subseteq \mathcal{S}$.

$\mathcal{S}$ is compact convex and has a non-empty interior, F is continuous and $F(\mathcal{S}) \subseteq \mathcal{S}$. F satisfies assumptions of Brouwer's theorem [9]. Thus, F has a fixed point.

This sufficient condition of existence of fixed point of F is also a sufficient condition for system of Eqs. (1)–(3) to have a fixed point.

Finally, we prove that $\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}} < 1$. Clearly, if a fixed point exists with $\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}} > 1$ for some i, then $F_i^{\mathrm{U}} = 0$. But F_i^{U} cannot be equal to 0 for a fixed point. Indeed $F_i^{\mathrm{U}} > 0$ as $\lambda_i^{\mathrm{U}} > 0$ for all queue i. ■

Theorem 3.2: *If $\lambda_i^S > 0$ and $\lambda_i^U > 0$ for all queue i, then the chain is ergodic.*

Proof: As the rates are bounded, the chain is uniformizable and we consider the embedded Markov chain. Remember that N is the number of queues. At any time, there is a positive probability that the next N events are signals sent to the N queues and which empty all the queues. Therefore, the chain is ergodic. ■

4. PARTIAL REJUVENATION

We now assume that the effect of the signal is to delete some of the inert customers present in the queue. However, it does not delete all of them with probability 1. The probability of deletion is state dependent. It depends on the number of inert customers and the number of usual customers. Let the state be $(x^{\mathrm{U}}, x^{\mathrm{I}})$, the probability of a deletion of m customer is given by the following probability:

$$\Pr(m \text{ deletions given state}(x^{\mathrm{U}}, x^{\mathrm{I}})) = \beta(x^{\mathrm{U}}, x^{\mathrm{I}}, m) \tag{5}$$

Of course this is only defined for $m \leq x^{\mathrm{I}}$ and we have: $\sum_{m=0}^{x^{\mathrm{I}}} \beta(x^{\mathrm{U}}, x^{\mathrm{I}}, m) = 1$. We prove, in the following theorem, that for a well-defined distribution β, the steady-state distribution has a multiplicative form. For the sake of readability, we assume that matrix C is zero and the signals only arrive from the outside.

Theorem 4.1: *Assume that for all queue, the effect of the signal on inert customers is given by probability:*

$$\beta(x^U, x^I + m, m) = \frac{x^U}{x^U + x^I}\left(\prod_{k=1}^{m} \frac{x^I + k}{x^U + x^I + k}\right) = \frac{x^U}{x^U + x^I}\frac{(x^I + m)!(x^U + x^I)!}{(x^U + x^I + m)!x^I!})1_{||x||>0}, \tag{6}$$

and $\beta(0,0,0) = 1$. Assume that the Markov chain $(x)_t = (x_1, \ldots, x_i, \ldots x_N)_t$ is ergodic. If the following flow equations have a solution such that $\rho_i^U + \rho_i^I < 1$ for all i,

$$\rho_i^U = \frac{\lambda_i^U + \sum_j \mu_j \rho_j^U P(j,i)}{\mu_i + \lambda_i^S}, \tag{7}$$

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

and

$$\rho_i^I = \frac{\lambda_i^I + \sum_j \mu_j \rho_j^U N(j,i)}{\frac{\lambda_i^S}{1-\rho_i^I}}, \tag{8}$$

then the steady-state distribution has a product form solution:

$$\pi(x) = \prod_{i=1}^{N} (1 - \rho_i^U - \rho_i^I) \frac{||x_i||!}{x_i^U!\ x_i^I!} (\rho_i^U)^{x_i^U} (\rho_i^I)^{x_i^I}. \tag{9}$$

PROOF: once again, the proof is based on the analysis of the Kolmogorov equation at steady state:

$$\begin{aligned}
\pi(x)\left(\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_i \frac{x_i^{\mathrm{U}}\mu_i}{||x_i||} 1_{||x_i||>0}\right) &= \sum_i \pi(x - e_i^{\mathrm{U}})\lambda_i^{\mathrm{U}} 1_{x_i^{\mathrm{U}}>0} \\
&+ \sum_i \pi(x - e_i^{\mathrm{I}})\lambda_i^{\mathrm{I}} 1_{x_i^{\mathrm{I}}>0} \\
&+ \sum_i \pi(x + e_i^{\mathrm{U}}) \frac{(x_i^{\mathrm{U}}+1)\mu_i}{||x_i + e_i^{\mathrm{U}}||} d_i \\
&+ \sum_i \sum_j \pi(x + e_i^{\mathrm{U}} - e_j^{\mathrm{U}}) \frac{(x_i^{\mathrm{U}}+1)\mu_i}{||x_i + e_i^{\mathrm{U}}||} P(i,j) 1_{x_j^{\mathrm{U}}>0} \\
&+ \sum_i \sum_j \pi(x + e_i^{\mathrm{U}} - e_j^{\mathrm{I}}) \frac{(x_i^{\mathrm{U}}+1)\mu_i}{||x_i + e_i^{\mathrm{U}}||} N(i,j) 1_{x_j^{\mathrm{I}}>0} \\
&+ \sum_i \lambda_i^S \sum_{m\geq 0} \pi(x + e_i^{\mathrm{I}} m)\beta(x_i^{\mathrm{U}}, x_i^{\mathrm{I}} + m, m).
\end{aligned}$$

Once again this equation includes null transitions, on both sides of the equation, when the queue size is 0. We use the same arguments as in the previous proof of product form.

$$\begin{aligned}
\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_i \frac{x_i^{\mathrm{U}}\mu_i}{||x_i||} 1_{||x_i||>0} &= \sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}}||x_i||} 1_{||x_i||>0} \\
&+ \sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}}||x_i||} 1_{||x_i||>0} \\
&+ \sum_i \rho_i^{\mathrm{U}} \mu_i d_i \\
&+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}}||x_j||} P(i,j) 1_{||x_j||>0} \\
&+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}}||x_j||} N(i,j) 1_{||x_j||>0} \\
&+ \sum_i \lambda_i^S \sum_{m\geq 0} (\rho_i^{\mathrm{I}})^m \frac{(||x_i||+m)!x_i^{\mathrm{I}}!}{||x_i||!\ (x_i^{\mathrm{I}}+m)!} \beta(x_i^{\mathrm{U}}, x_i^{\mathrm{I}} + m, m).
\end{aligned}$$

We use the definition of the distribution β. After substitution, we obtain:

$$\begin{aligned}\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_i \frac{x_i^{\mathrm{U}}\mu_i}{||x_i||} 1_{||x_i||>0} = &\sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}}||x_i||} 1_{||x_i||>0} \\ &+ \sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}}||x_i||} 1_{||x_i||>0} \\ &+ \sum_i \rho_i^{\mathrm{U}} \mu_i d_i \\ &+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}}||x_j||} P(i,j) 1_{||x_j||>0} \\ &+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}}||x_j||} N(i,j) 1_{||x_j||>0} \\ &+ \sum_i \lambda_i^S \frac{x_i^{\mathrm{U}}}{||x_i||} \sum_{m\geq 0} (\rho_i^{\mathrm{I}})^m 1_{||x_i||>0} \\ &+ \sum_i \lambda_i^S \sum_{m\geq 0} (\rho_i^{\mathrm{I}})^m 1_{||x_i||=0}.\end{aligned}$$

As $\rho_i^{\mathrm{I}} < 1$, $\sum_{m\geq 0} (\rho_i^{\mathrm{I}})^m = (1-\rho_i^{\mathrm{I}})^{-1}$. We make the same decomposition of λ_i^S into

$$\lambda_i^S \frac{x_i^{\mathrm{U}}}{||x_i||} + \lambda_i^S \frac{x_i^{\mathrm{I}}}{||x_i||}.$$

Furthermore,

$$\frac{x_i^{\mathrm{U}}}{||x_i||} = 1 - \frac{x_i^{\mathrm{I}}}{||x_i||}$$

and we move the negative part to the left-hand side (l.h.s.).

$$\begin{aligned}&\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_i \frac{\lambda_i^S}{1-\rho_i^{\mathrm{I}}} \frac{x_i^{\mathrm{I}}}{||x_i||} 1_{||x_i||>0} + \sum_i \frac{x_i^{\mathrm{U}}(\mu_i + \lambda_i^S)}{||x_i||} 1_{||x_i||>0} \\ &= \sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}}||x_i||} 1_{||x_i||>0} \\ &\quad + \sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}}||x_i||} 1_{||x_i||>0} \\ &\quad + \sum_i \rho_i^{\mathrm{U}} \mu_i d_i \\ &\quad + \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}}||x_j||} P(i,j) 1_{||x_j||>0}\end{aligned}$$

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

$$+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}} ||x_j||} N(i,j) 1_{||x_j||>0}$$
$$+ \sum_i \lambda_i^S \frac{1}{1-\rho_i^{\mathrm{I}}} 1_{||x_i||>0}$$
$$+ \sum_i \lambda_i^S \frac{1}{1-\rho_i^{\mathrm{I}}} 1_{||x_i||=0}.$$

The last two terms of the right-hand side (r.h.s.) are gathered and we cancel the term λ_i^S, which is present on both sides on the equation.

$$\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}}) + \sum_i \frac{\lambda_i^S}{1-\rho_i^{\mathrm{I}}} \frac{x_i^{\mathrm{I}}}{||x_i||} 1_{||x_i||>0} + \sum_i \frac{x_i^{\mathrm{U}}(\mu_i + \lambda_i^S)}{||x_i||} 1_{||x_i||>0}$$
$$= \sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}} ||x_i||} 1_{||x_i||>0}$$
$$+ \sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}} ||x_i||} 1_{||x_i||>0}$$
$$+ \sum_i \rho_i^{\mathrm{U}} \mu_i d_i$$
$$+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}} ||x_j||} P(i,j) 1_{||x_j||>0}$$
$$+ \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}} ||x_j||} N(i,j) 1_{||x_j||>0}$$
$$+ \sum_i \lambda_i^S \frac{\rho_i^{\mathrm{I}}}{1-\rho_i^{\mathrm{I}}}.$$

We decompose into three equations:

$$\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}}) = \sum_i \rho_i^{\mathrm{U}} \mu_i d_i + \sum_i \lambda_i^S \frac{\rho_i^{\mathrm{I}}}{1-\rho_i^{\mathrm{I}}}, \tag{10}$$

$$\sum_i \frac{\lambda_i^S}{1-\rho_i^{\mathrm{I}}} \frac{x_i^{\mathrm{I}}}{||x_i||} 1_{||x_i||>0} = \sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}} ||x_i||} 1_{||x_i||>0} + \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}} ||x_j||} N(i,j) 1_{||x_j||>0}, \tag{11}$$

$$\sum_i \frac{x_i^{\mathrm{U}}(\mu_i + \lambda_i^S)}{||x_i||} 1_{||x_i||>0} = \sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}} ||x_i||} 1_{||x_i||>0} + \sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}} ||x_j||} P(i,j) 1_{||x_j||>0}. \tag{12}$$

The second and third equations are satisfied due to the definition of ρ_i^{I} (Eq. (8)) and ρ_i^{U} (Eq. (7)). It remains to prove that the first equation is a global flow equation between

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

the network and the outside. We consider Eq. (8) and we multiply by the denominator:

$$\frac{\lambda_i^S}{1-\rho_i^{\mathrm{I}}}\rho_i^{\mathrm{I}} = \lambda_i^{\mathrm{I}} + \sum_j \mu_j \rho_j^{\mathrm{U}} N(j,i).$$

We proceed the same way for Eq. (7):

$$\mu_i \rho_i^{\mathrm{U}} = \lambda_i^{\mathrm{U}} + \sum_j \mu_j \rho_j^{\mathrm{U}} P(j,i).$$

We sum for all queue index i and we add the two equalities:

$$\sum_i \mu_i \rho_i^{\mathrm{U}} + \sum_i \frac{\lambda_i^S}{1-\rho_i^{\mathrm{I}}}\rho_i^{\mathrm{I}} = \sum_i \lambda_i^{\mathrm{U}} + \sum_i \sum_j \mu_j \rho_j^{\mathrm{U}} P(j,i) + \sum_i \lambda_i^{\mathrm{I}} + \sum_i \sum_j \mu_j \rho_j^{\mathrm{U}} N(j,i).$$

Taking into account that for all i, $\sum_j P(i,j) + \sum_j N(i,j) = 1 - d_i$, as matrix C is zero, we get:

$$\sum_i \mu_i \rho_i^{\mathrm{U}} d_i + \sum_i \frac{\lambda_i^S}{1-\rho_i^{\mathrm{I}}}\rho_i^{\mathrm{I}} = \sum_i \lambda_i^{\mathrm{U}} + \sum_i \lambda_i^{\mathrm{I}}.$$

And we find the first flow equation and that concludes the proof. ■

5. CONCLUSION

We have presented here a new types of G-networks with a new type of customers. Typically, we represent an age-dependent server, with a service capacity, which decreases with time until a rejuvenation takes place. We hope that such a theoretical result will help to develop new models for G-networks in the performability domain. It is possible to extend this result for a more general partial rejuvenation with a more complex state-dependent distribution of destruction.

References

1. Balsamo, S., Harrison P.G., & Marin A. (2010). An unifying approach to product-forms in networks with finite capacity constraints. In *SIGMETRICS 2010, Proceedings of the 2010 ACM SIGMETRICS International Conference on Measurement and Modeling of Computer Systems*, USA. In V. Misra, Barford, P., & Squillante M.S. (eds.), ACM, pp. 25–36.
2. Chao, X., Miyazawa, M., & Pinedo, M. (1999). *Queueing networks customers, signals and product form solutions*. Chichester, UK, John Wiley & Sons.
3. Dao-Thi, T.-H., Fourneau, J.-M., & Tran, M.-A. (2013). Network of queues with inert customers and signals. In *7th International Conference on Performance Evaluation Methodologies and Tools, ValueTools '13*, Italy, ICST/ACM, pp. 155–164.
4. Dudin, A.N. & Karolik, A.V. (2001). BMAP/SM/1 queue with Markovian input of disasters and non-instantaneous recovery. *Performance Evaluation* 45(1): 19–32.
5. Fourneau, J.-M. (1991). Computing the steady-state distribution of networks with positive and negative customers. In *13th IMACS World Congress on Computation and Applied Mathematics*, Dublin.
6. Fourneau, J.-M., Ait El Mahjoub, Y., Quessette, F., & Vekris, D. (2016). Xborne 2016: A brief introduction. In *Computer and Information Sciences – 31st International Symposium, ISCIS 2016, Krakow*, Poland, vol. 659, Communications in Computer and Information Science. In T. Czachórski, E. Gelenbe, K. Grochla, & R. Lent (eds.), Springer.
7. Fourneau, J.-M., Kloul, L., & Quessette, F. (1995). Multiple class G-Networks with jumps back to zero. In *MASCOTS '95: Proceedings of the 3rd International Workshop on Modeling, Analysis, and*

Simulation of Computer and Telecommunication Systems, Washington, DC, USA, IEEE Computer Society, pp. 28–32
8. Fourneau, J.-M. & Quessette, F. (2006). Computing the steady-state distribution of G-networks with synchronized partial flushing. In *Computer and Information Sciences – ISCIS 2006, 21th International Symposium*, Istanbul, Turkey, volume 4263 of *Lecture Notes in Computer Science*. In A. Levi, E. Savas, H. Yenigün, S. Balcisoy, & Y Saygin (eds.), Springer, pp. 887–896.
9. Garcia, C.D. & Zangwill, W.I. (1981). *Pathways to solutions, fixed points, and equilibria*. Englewood Cliffs, NJ: Prentice–Hall.
10. Gelenbe, E. (1991). Product-form queuing networks with negative and positive customers. *Journal of Applied Probability* 28: 656–663.
11. Gelenbe, E. (1993). G-networks with instantaneous customer movement. *Journal of Applied Probability* 30(3): 742–748.
12. Gelenbe, E. (1993). G-networks with signals and batch removal. *Probability in the Engineering and Informational Sciences* 7: 335–342.
13. Gelenbe, E. (1994). G-networks: An unifying model for queuing networks and neural networks. *Annals of Operations Research* 48(1–4): 433–461.
14. Gelenbe, E. (2014). Adaptive management of energy packets. In *IEEE 38th Annual Computer Software and Applications Conference, COMPSAC Workshops*, IEEE Computer Society, pp. 1–6.
15. Gelenbe, E. & Ceran, E.T. (2016). Energy packet networks with energy harvesting. *IEEE Access* 4: 1321–1331.
16. Gelenbe, E. & Fourneau, J.-M. (1999). Random neural networks with multiple classes of signals. *Neural Computation* 11(4): 953–963.
17. Gelenbe, E. & Fourneau, J.-M. (2002). G-networks with resets. *Performance Evaluation* 49(1–4): 179–191.
18. Gelenbe, E. & Labed, A. (1998). G-networks with multiple classes of signals and positive customers. *European Journal of Operations Research* 108:293–305.
19. Gelenbe, E., Lent, R. & Xu, Z. (2001). Design and performance of cognitive packet networks. *Performance Evaluation* 46(2–3): 155–176.
20. Gelenbe, E. & Mitrani, I. (2010). *Analysis and synthesis of computer systems*. London: Imperial College Press.
21. Gelenbe, E. & Schassberger, R. (1992). Stability of g-networks. *Probability in the Engineering and Informational Sciences* 6: 271–276.
22. Harrison, P.G. (2003). Turning back time in Markovian process algebra. *Theoretical Computer Science* 290(3): 1947–1986.
23. Harrison, P.G. (2004). Compositional reversed Markov processes, with applications to G-networks. *Performance Evaluation* 57(3): 379–408.
24. Krishna Kumar, B. & Arivudainambi, D. (2000). Transient solution of an m/m/1 queue with catastrophes. *Computers & Mathematics With Applications* 40(10–11): 1233–1240.
25. Mohamed, S., Rubino, G., & Varela, M. (2004). Performance evaluation of real-time speech through a packet network: a random neural networks-based approach. *Performance Evaluation* 57(2): 141–161.

APPENDIX

We assume that the solution as a product form solution and each queue has distribution at steady state given by Eq. (4). After simplification, and exchanging indices i and j in the last term of the r.h.s., we get:

$$\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^{S}) + \sum_i \frac{x_i^{\mathrm{U}}\mu_i}{||x_i||} 1_{||x_i||>0} = \sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}}||x_i||} 1_{||x_i||>0} + \sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}}||x_i||} 1_{||x_i||>0} + \sum_i \rho_i^{\mathrm{U}} \mu_i d_i$$

$$+\sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}}||x_j||} P(i,j) 1_{||x_j||>0}$$

$$+\sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}}||x_j||} N(i,j) 1_{||x_j||>0}$$

$$+\sum_i \lambda_i^S \sum_{a\geq 0} \sum_{b\geq 0} (\rho_i^{\mathrm{U}})^a (\rho_i^{\mathrm{I}})^b 1_{||x_i||=0}$$

$$+\sum_j \mu_j \rho_j^{\mathrm{U}} \sum_i C(j,i) \sum_{a\geq 0} \sum_{b\geq 0} \frac{(a+b)!(\rho_i^{\mathrm{U}})^a (\rho_i^{\mathrm{I}})^b}{a!b!} 1_{||x_i||=0}.$$

First, let us consider the double summation in the last two terms of the r.h.s.:

$$\frac{(a+b)!(\rho_i^{\mathrm{U}})^a (\rho_i^{\mathrm{I}})^b}{a!b!}.$$

It is well known that, if $\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}} < 1$, then:

$$\sum_{a\geq 0} \sum_{b\geq 0} \frac{(a+b)!(\rho_i^{\mathrm{U}})^a (\rho_i^{\mathrm{I}})^b}{a!b!} = (1-\rho_i^{\mathrm{U}} - \rho_i^{\mathrm{I}})^{-1}.$$

Thus, this double summation is equal to R_i. On the r.h.s., we now write $1_{||x_i||=0} = 1 - 1_{||x_i||>0}$ and the move the negative terms on the l.h.s. to factorize.

$$\sum_i (\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_i \frac{x_i^{\mathrm{U}} \mu_i}{||x_i||} 1_{||x_i||>0} + \sum_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\mathrm{U}} C(j,i)\right) R_i 1_{||x_i||>0}$$

$$= \sum_i \lambda_i^{\mathrm{U}} \frac{x_i^{\mathrm{U}}}{\rho_i^{\mathrm{U}}||x_i||} 1_{||x_i||>0}$$

$$+\sum_i \lambda_i^{\mathrm{I}} \frac{x_i^{\mathrm{I}}}{\rho_i^{\mathrm{I}}||x_i||} 1_{||x_i||>0}$$

$$+\sum_i \rho_i^{\mathrm{U}} \mu_i d_i$$

$$+\sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{U}}}{\rho_j^{\mathrm{U}}||x_j||} P(i,j) 1_{||x_j||>0}$$

$$+\sum_i \sum_j \mu_i \rho_i^{\mathrm{U}} \frac{x_j^{\mathrm{I}}}{\rho_j^{\mathrm{I}}||x_j||} N(i,j) 1_{||x_j||>0}$$

$$+\sum_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\mathrm{U}} C(j,i)\right) R_i.$$

Now we decompose R_i into $R_i(x_i^{\mathrm{U}}/||x_i||) + R_i(x_i^{\mathrm{I}}/||x_i||)$ and we substitute in the l.h.s. After substitution, factorization and exchanging indices i and j in the fifth and sixth terms of the r.h.s., we get:

$$\sum_i (\lambda_i^{\rm U} + \lambda_i^{\rm I} + \lambda_i^S) + \sum_i \left(\mu_i + (\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i)) R_i \right) \frac{x_i^{\rm U}}{||x_i||} 1_{||x_i||>0}$$

$$+ \sum_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) R_i \frac{x_i^{\rm I}}{||x_i||} 1_{||x_i||>0}$$

$$= \sum_i \lambda_i^{\rm U} \frac{x_i^{\rm U}}{\rho_i^{\rm U} ||x_i||} 1_{||x_i||>0}$$

$$+ \sum_i \lambda_i^{\rm I} \frac{x_i^{\rm I}}{\rho_i^{\rm I} ||x_i||} 1_{||x_i||>0}$$

$$+ \sum_i \rho_i^{\rm U} \mu_i d_i$$

$$+ \sum_i \sum_j \mu_j \rho_j^{\rm U} \frac{x_i^{\rm U}}{\rho_i^{\rm U} ||x_i||} P(j,i) 1_{||x_i||>0}$$

$$+ \sum_i \sum_j \mu_j \rho_j^{\rm U} \frac{x_i^{\rm I}}{\rho_i^{\rm I} ||x_i||} N(j,i) 1_{||x_i||>0}$$

$$+ \sum_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) R_i,$$

which can be decomposed into three parts:

$$\sum_i (\lambda_i^{\rm U} + \lambda_i^{\rm I} + \lambda_i^S) = \sum_i \rho_i^{\rm U} \mu_i d_i + \sum_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) R_i$$

$$\times \sum_i \left(\mu_i + \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) R_i \right) \frac{x_i^{\rm U}}{||x_i||} 1_{||x_i||>0}$$

$$= \sum_i \left(\lambda_i^{\rm U} + \sum_j \mu_j \rho_j^{\rm U} P(j,i) \right) \frac{x_i^{\rm U}}{\rho_i^{\rm U} ||x_i||} 1_{||x_i||>0},$$

and

$$\sum_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) R_i \frac{x_i^{\rm I}}{||x_i||} 1_{||x_i||>0} = \sum_i \left(\lambda_i^{\rm I} + \sum_j \mu_j \rho_j^{\rm U} N(j,i) \right) \frac{x_i^{\rm I}}{\rho_i^{\rm I} ||x_i||} 1_{||x_i||>0}.$$

The last two equations hold because of the flow equations (i.e. Eqs. (1) and (2)). It remains to prove that the first equation is consistent with Eqs. (1) and (2) and describes the flow between the network and the outside. From these equations, we obtain:

$$\rho_i^{\rm U} \mu_i + \rho_i^{\rm U} R_i \left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) = \lambda_i^{\rm U} + \sum_j \mu_j \rho_j^{\rm U} P(j,i)$$

and

$$\left(\lambda_i^S + \sum_j \mu_j \rho_j^{\rm U} C(j,i) \right) R_i \rho_i^{\rm I} = \lambda_i^{\rm I} + \sum_j \mu_j \rho_j^{\rm U} N(j,i).$$

https://doi.org/10.1017/S0269964817000092 Published online by Cambridge University Press

Thus, adding the two equalities:

$$\rho_i^{\mathrm{U}}\mu_i + (\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}})R_i\left(\lambda_i^S + \sum_j \mu_j\rho_j^{\mathrm{U}}C(j,i)\right) = \lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \sum_j \mu_j\rho_j^{\mathrm{U}}(P(j,i) + N(j,i)).$$

But $R_i(\rho_i^{\mathrm{U}} + \rho_i^{\mathrm{I}}) = R_i - 1$. After substitution and summation on i, we get:

$$\sum_i \rho_i^{\mathrm{U}}\mu_i + \sum_i\left(\lambda_i^S + \sum_j \mu_j\rho_j^{\mathrm{U}}C(j,i)\right)(R_i - 1) = \sum_i(\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}}) + \sum_j \mu_j\rho_j^{\mathrm{U}}\sum_i(P(j,i) + N(j,i)).$$

Moving the negative terms on the r.h.s., we get:

$$\sum_i \rho_i^{\mathrm{U}}\mu_i + \sum_i\left(\lambda_i^S + \sum_j \mu_j\rho_j^{\mathrm{U}}C(j,i)\right)R_i = \sum_i(\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) + \sum_j \mu_j\rho_j^{\mathrm{U}}\sum_i(P(j,i) + N(j,i) + C(j,i)).$$

Remember that due to the normalization, we have for all i

$$d_i + \sum_j P(i,j) + \sum_j N(i,j) + \sum_j C(i,j) = 1.$$

Therefore, after cancellation of terms, we get:

$$\sum_i(\lambda_i^{\mathrm{U}} + \lambda_i^{\mathrm{I}} + \lambda_i^S) = \sum_i \rho_i^{\mathrm{U}}\mu_i d_i + \sum_i\left(\lambda_i^S + \sum_j \mu_j\rho_j^{\mathrm{U}}C(j,i)\right)R_i.$$

This concludes the proof.