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LIFTING HOMEOMORPHISMS AND FINITE ABELIAN BRANCHED COVERS OF THE 2-SPHERE

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Abstract

We completely determine finite abelian regular branched covers of the 2-sphere S^2 with the property that each homeomorphism of S^2 preserving the branching set can be lifted.

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1. Result and method

As pointed out by Birman and Hilden [2] (see also [1, 4, 5]), an interesting problem is to find branched covers of a surface with the property that each homeomorphism of the base surface preserving the branching set can be lifted.

Fix a set $\mathfrak{B} = \{\mathfrak{x}_1, \ldots, \mathfrak{x}_n\} \subset S^2$, with $n \geq 2$. Let $\Sigma_{0,n} = S^2 \setminus \mathfrak{B}$. Let $x_i \in H_1(\Sigma_{0,n}; \mathbb{Z})$ denote the homology class of a small loop enclosing $\mathfrak{x}_i$. Given an abelian group A, a regular A-cover $\pi : \Sigma \to S^2$ with branching set $\mathfrak{B}$ is determined by an epimorphism $\phi_\pi : H_1(\Sigma_{0,n}; \mathbb{Z}) \twoheadrightarrow A$, which satisfies $\phi_\pi(x_1) + \cdots + \phi_\pi(x_n) = 0$. Call two such covers $\phi : \Sigma \to S^2$ and $\phi' : \Sigma' \to S^2$ *equivalent* if there exist homeomorphisms $\tilde{g} : \Sigma \to \Sigma'$, $g : S^2 \to S^2$ such that $g(\mathfrak{B}) = \mathfrak{B}$ and $g \circ \pi = \pi' \circ \tilde{g}$.

Denote by $\mathbf{e}_i$ the vector with 1 at the ith position and 0 elsewhere.

We extend the results of [1, 4] by proving the following result.

THEOREM 1.1. *Let A be a finite abelian p-group with exponent p^k and $\pi : \Sigma \to S^2$ be a regular A-cover with branching set $\mathfrak{B}$ such that each homeomorphism of S^2 preserving $\mathfrak{B}$ can be lifted. Then up to equivalence, one of the following occurs:*

(1) $A = \mathbb{Z}_{p^k}^{n-1}$ *and* $\phi_\pi(x_i) = \mathbf{e}_i$, $1 \leq i \leq n-1$;

(2) $A = \mathbb{Z}_{p^r}^{n-2} \times \mathbb{Z}_{p^k}$ *for some* $k > r > 0$ *with* $p^{k-r} \mid n$ *and*

$$\phi_\pi(x_i) = (\mathbf{e}_i, 1), \ 1 \leq i \leq n-2; \quad \phi_\pi(x_{n-1}) = (\mathbf{0}, 1);$$

(3) $A = \mathbb{Z}_{p^k}$ *with* $p^k \mid n$ *and* $\phi_\pi(x_i) = 1$ *for all* i.

According to [1, Section 3], this essentially solves the problem for abelian covers.

Let G denote the group of automorphisms of $H_1(\Sigma_{0,n};\mathbb{Z})$ induced by all homeomorphisms of $\Sigma_{0,n}$. Clearly, G is isomorphic to the permutation group on $\{x_1,\dots,x_n\}$. By [4, Lemma 2.1], a homeomorphism f of $\Sigma_{0,n}$ can be lifted if and only if there exists $\psi\in\mathrm{Aut}(A)$ such that $\psi\circ\phi_\pi=\phi_\pi\circ\alpha$, where $\alpha\in G$ is induced by f. This is equivalent to $\alpha(\ker\phi_\pi)=\ker\phi_\pi$, which in turn is equivalent to

$$\alpha(\ker\overline{\phi_\pi})=\ker\overline{\phi_\pi},$$

where $\overline{\phi_\pi}:H_1(\Sigma_{0,n};\mathbb{Z}_{p^k})\twoheadrightarrow A$ is the map induced by ϕ_π. Instead of dealing with $\mathrm{Aut}(A)$, we work directly on $\ker\overline{\phi_\pi}$, reducing the problem to finding all subgroups of $H_1(\Sigma_{0,n};\mathbb{Z}_{p^k})$ that are invariant under all $\alpha\in G$. From this viewpoint, π is equivalent to π' if and only if $\beta(\ker\overline{\phi_{\pi'}})=\ker\overline{\phi_\pi}$ for some $\beta\in G$.

For problems of this kind, a method was developed in [3].

Some notations and conventions. For a ring R, let $R^{\ell,m}$ denote the set of $\ell\times m$ matrices over R. For $X\in R^{\ell,m}$, let $X_{i,j}$ denote its (i,j)-entry and let $\langle X\rangle$ denote the subgroup of R^m generated by the row vectors of X. For $Y\in\mathbb{Z}^{\ell,m}$, abusing the notation, we denote its image under the map $\mathbb{Z}^{\ell,m}\to\mathbb{Z}_{p^k}^{\ell,m}$ induced by the quotient map $\mathbb{Z}\twoheadrightarrow\mathbb{Z}_{p^k}$ also by Y. Let S_m denote the permutation group on m elements. Embed S_m as a subgroup of $\mathrm{GL}(m,\mathbb{Z})$ by identifying $\sigma\in S_m$ with the matrix (denoted by the same notation) whose (i,j)-entry is $\delta_{i,\sigma(j)}$, where δ is the Kronecker symbol.

LEMMA 1.2 [3, Theorem 3.9]. *Each subgroup $C\leqslant\mathbb{Z}_{p^k}^m$ is of the form $\langle PQ\omega\rangle$ for some $P\in\mathbb{Z}_{p^k}^{\ell,m}$, $Q\in\mathrm{GL}(m,\mathbb{Z})$, $\omega\in S_m$ such that*

- *$0\le\ell\le m$, $P_{i,j}=\delta_{i,j}p^{r_i}$ with $0\le r_1\le\cdots\le r_\ell<k$;*
- *$Q_{i,i}=1$ for all i, and $Q_{j,i}=0\le Q_{i,j}<p^{r_j-r_i}$ for all $i<j$, where $r_i=k$ for $\ell<i\le m$.*

Clearly, $\mathbb{Z}_{p^k}^m/C\cong\mathbb{Z}_{p^{r_\iota}}\times\cdots\times\mathbb{Z}_{p^{r_m}}$, with $\iota=\min\{i\colon r_i>0\}$.

Let $b=n-1$. Take $x_1,\dots,x_b$ as generators for $H_1(\Sigma_{0,n};\mathbb{Z})\cong\mathbb{Z}^b$. For each $\alpha\in G\cong S_{b+1}$, let $T^\alpha\in\mathrm{GL}(b,\mathbb{Z})$ denote the matrix determined by

$$\alpha(x_i)=\sum_{j=1}^{b}(T^\alpha)_{i,j}\cdot x_j,\quad i=1,\dots,b.$$

Obviously, if $\alpha\in S_b$, by which we mean $\alpha(x_{b+1})=x_{b+1}$, then $T^\alpha=\alpha$.

If $\ker\overline{\phi_\pi}=\langle PQ\omega\rangle$, then $\alpha(\ker\overline{\phi_\pi})=\langle PQ\omega T^\alpha\rangle$. Taking $\beta=\omega^{-1}\in S_b$, we have $\beta(\ker\overline{\phi_\pi})=\langle PQ\rangle$. Hence, up to equivalence, we can assume $\ker\overline{\phi_\pi}=\langle PQ\rangle$. Then $\alpha(\ker\overline{\phi_\pi})=\langle PQT^\alpha\rangle$ for each $\alpha\in G$. By the criterion given by [3, Lemma 3.11], $\alpha(\ker\overline{\phi_\pi})=\ker\overline{\phi_\pi}$ is equivalent to

$$p^{r_j-r_i}\mid(QT^\alpha Q^{-1})_{i,j}\quad\text{for all } i<j.$$

Thus, it suffices to find tuples $(r_1,\dots,r_b;Q)$ consisting of integers $0\le r_1\le\cdots\le r_b>0$ and a matrix $Q\in\mathrm{GL}(b,\mathbb{Z})$, such that

$$Q_{i,i}=1,1\le i\le b;\quad Q_{j,i}=0\le Q_{i,j}<p^{r_j-r_i}\quad\text{for all } i<j;\tag{1.1}$$

$$p^{r_j-r_i} \mid (QT^{\alpha}Q^{-1})_{i,j} \quad \text{for all } i<j \text{ and } \alpha \in S_{b+1}. \tag{1.2}$$

Such a tuple determines a regular branched A-cover π, with $A = \mathbb{Z}_{p^{r_\iota}} \times \cdots \times \mathbb{Z}_{p^{r_b}}$, $\iota = \min\{i\colon r_i > 0\}$, and

$$\phi_\pi(x_i) = ((Q^{-1})_{i,\iota}, \ldots, (Q^{-1})_{i,b}), \quad 1 \le i \le b, \tag{1.3}$$

with the understanding that $\phi_\pi(x_{b+1}) = -\phi_\pi(x_1) - \cdots - \phi_\pi(x_b)$.

2. Proof of Theorem 1.1

As a simple observation, (1.2) holds if and only if

$$p^{r_j-r_i} \mid (QXQ^{-1})_{i,j} \quad \text{for all } i<j, \tag{2.1}$$

for all X in the subgroup of $\mathbb{Z}^{b,b}$ generated by T^α, $\alpha \in S_{b+1}$.

Let I denote the identity matrix. Let E_u^v denote the matrix whose (u,v)-entry is 1 and the other entries are all 0.

For the permutation $\eta_i \in S_{b+1}$ switching i and $b+1$,

$$T^{\eta_i} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ -1 & \cdots & -1 & \cdots & -1 \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = I - E_i^i - \sum_{j=1}^{b} E_i^j.$$

Clearly, for each $\alpha \in S_{b+1} \setminus S_b$, there uniquely exists $u \in \{1, \ldots, b\}$, $\sigma \in S_b$, such that $\alpha = \eta_u\sigma$. Then

$$T^\sigma - T^\alpha = (I - T^{\eta_u})T^\sigma = E_u^{\sigma^{-1}(u)} + \sum_{j=1}^{b} E_u^j. \tag{2.2}$$

The difference of two such matrices can give rise to $E_u^v - E_u^w$ for any $v \neq w$.

Taking $i = 1$ and $X = E_1^v - E_1^w$ in (2.1), we obtain

$$(Q^{-1})_{v,j} \equiv (Q^{-1})_{w,j} \pmod{p^{r_j-r_1}}. \tag{2.3}$$

In particular, setting $w = j = b-1$ and $v = b$ leads to $r_{b-1} = r_1$ (so that $r_i = r_1$ for all $i < b$). By (1.1), $Q_{i,j} = 0$ for all i, j with $1 \le i < j < b$.

If $k = r_b = r_1$, then $Q_{i,b} = 0$ for $1 \le i < b$, so that $Q = I$. In this case, $A = \mathbb{Z}_{p^k}^b$, and by (1.3), $\phi(x_i) = \mathbf{e}_i$.

Now suppose $k > r_1$. Setting $w = j = b$ in (2.3) leads to

$$(Q^{-1})_{v,b} \equiv 1 \pmod{p^{r_b-r_1}}.$$

Hence $Q_{v,b} \equiv -1 \pmod{p^{r_b-r_1}}$ for all $v < b$. From (2.2), we see that for $i < b$,

$$(Q(T^\sigma - T^\alpha)Q^{-1})_{i,b} \equiv (\delta_{u,i} - \delta_{u,b})(b+1) \pmod{p^{r_b-r_1}},$$

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so $p^{r_b - r_1} \mid (Q(T^\sigma - T^\alpha)Q^{-1})_{i,b}$ is equivalent to

$$p^{r_b - r_1} \mid b + 1.$$

Once this holds, for each $\sigma \in S_b$ and all $i < b$,

$$(QT^\sigma Q^{-1})_{i,b} \equiv \sum_{v=1}^{b} (Q\sigma)_{i,v} \equiv 0 \pmod{p^{r_b - r_1}}.$$

Therefore, (1.2) is fulfilled.

There are two possible cases.

- If $r_1 > 0$, then $A = \mathbb{Z}_{p^{r_1}}^{b-1} \times \mathbb{Z}_{p^k}$ and, by (1.3),

$$\phi(x_i) = (\mathbf{e}_i, 1), 1 \le i \le b - 1; \quad \phi(x_b) = (\mathbf{0}, 1).$$

- If $r_1 = 0$, then $A = \mathbb{Z}_{p^k}$ and $\phi(x_i) = 1$ for each i.

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