Proceedings of the Royal Society of Edinburgh, **141A**, 717–737, *2011*

Boundary blow-up solutions for elliptic equations with gradient terms and singular weights: existence, asymptotic behaviour and uniqueness

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(MS received 28 February 2010; accepted 20 September 2010)

This paper deals with the non-negative boundary blow-up solutions of the equation $\Delta u = b(x)u^p + c(x)u^\sigma|\nabla u|^q$ in $\Omega \subset \mathbb{R}^N$, where $b(x), c(x) \in C^\gamma(\Omega, \mathbb{R}^+)$ for some $0 < \gamma < 1$ and can be vanishing or singular on the boundary, and p, σ and q are non-negative constants. The existence and asymptotic behaviour of such a solution near the boundary are investigated, and we show how the nonlinear gradient term affects the results. As a consequence of the asymptotic behaviour, we also show the uniqueness result.

1. Introduction and main results

In this work we analyse the existence, asymptotic behaviour near the boundary and uniqueness of non-negative solutions to the singular boundary-value problem

$$\left.\begin{aligned} \Delta u &= b(x)u^p + c(x)u^\sigma|\nabla u|^q, \quad && x \in \Omega, \\ u &= \infty, && x \in \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

As usual, $u = \infty$ on $\partial\Omega$ means that $u(x) \to \infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$, and such non-negative solutions will be called 'boundary blow-up' or, for brevity, 'large' solutions. Here Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geqslant 1$, exponents p, σ and q are non-negative, $b(x), c(x) \in C^\gamma(\Omega, \mathbb{R}^+)$, $\mathbb{R}^+ := [0, \infty)$ for some $0 < \gamma < 1$, and henceforth, $C^\gamma(\Omega)$ means that $C^\gamma_{\mathrm{loc}}(\Omega)$. Moreover, sometimes we need the following structural assumption.

(Hb) The weighted function $b(x)$ is non-trivial in Ω, and if $x_0 \in \Omega$ such that $b(x_0) = 0$, then there exists a sub-domain $\Omega_0 \subset \Omega$ such that $x_0 \in \Omega_0$ and $b(x) > 0$ on $\partial\Omega_0$.

For the large solution of the particular problem

$$\left.\begin{aligned} \Delta u &= b(x)u^p, \quad && x \in \Omega, \\ u &= \infty, && x \in \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

https://doi.org/10.1017/S0308210510000363 Published online by Cambridge University Press

with $p > 1$, almost everything is known: existence and non-existence, uniqueness, boundary behaviour of the solution and its normal derivatives, second-order estimates near the boundary, etc. The assumptions on the weighted function $b(x)$ have varied over the years: for instance,

- in [2, 18], when b is smooth and positive up to the boundary,
- in [8], when b is smooth but is zero on $\partial\Omega$ with a prescribed behaviour,
- in [20, 21], where b satisfies (Hb) and $b \in C^\gamma(\Omega)$ and either
$$\sup_{x\in\Omega}[d(x)]^{-\mu} b(x) \leqslant C_0$$
for some $-2 < \mu \leqslant 0$ and some constant C_0, or $b \in L^r(\Omega)$ for some $r > N/2$.

Then [6] almost covered the three cases above about the weight and studied the more general equation $\Delta u = b(x)f(u)$; the upper-and-lower-solutions method was used there. Chuaqui [4] also studied the large solution of (1.2). It was assumed that $b \in C^\gamma(\Omega)$ and there exist constants $b_1, b_2 > 0$ and $\mu > -2$ such that

$$b_1 d^\mu(x) \leqslant b(x) \leqslant b_2 d^\mu(x) \quad \text{in } \Omega. \tag{1.3}$$

The following result was obtained in [4, theorem 1.1].

Proposition 1.1. *Assume b verifies hypotheses (1.3). Then problem (1.2) has no positive solution if $\mu \leqslant -2$, and it has a unique positive solution u when $-2 < \mu < 0$. Moreover,*

$$D_1 d^{-\alpha}(x) \leqslant u(x) \leqslant D_2 d^{-\alpha}(x) \quad \textit{in } \Omega,$$

where $\alpha = (2 + \mu)/(p - 1)$, and D_1 and D_2 are positive constants.

In fact, the existence of positive large solutions holds under the weaker assumption $0 < b(x) \leqslant b_2 d^\mu(x)$ in Ω for some $-2 < \mu < 0$ and $b_2 > 0$.

More recently, García-Melián [7] studied the corresponding results for p-Laplacian problem.

The presence of the gradient terms may have significant influence on the existence, uniqueness and asymptotic behaviour of the large solution. Problems of this type appear in stochastic control theory and were first studied by Lasry and Lions [15], who addressed existence, uniqueness, non-existence and the blow-up rate near the boundary of $C^2(\Omega)$-solutions of the problem

$$\Delta u = \alpha u + |\nabla u|^q + \Psi(x), \quad x \in \Omega; \qquad u = \infty, \quad x \in \partial\Omega.$$

where $\alpha > 0$, $1 < q < \infty$ are constants and Ψ is a smooth function. In recent years, there have been many works published on the problem

$$\Delta u \pm |\nabla u|^q = b(x)f(u), \quad x \in \Omega; \qquad u = \infty, \quad x \in \partial\Omega. \tag{1.4}$$

When $b(x) = 1$, $f(u) = \mathrm{e}^u$, Bandle and Giarrusso [1] gave the following results:

(i) if $q \geqslant 0$, then problem $(1.4)_+$ has one solution in $C^2(\Omega)$, and the same statement is true for problem $(1.4)_-$, provided $0 \leqslant q \leqslant 2$;

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(ii) if $0 \leqslant q \leqslant 2$, then every solution $u_\pm$ to problem $(1.4)_\pm$ satisfies

$$\lim_{d(x)\to 0} \frac{u_\pm(x)}{-\ln(d(x))} = 2;$$

(iii) if $q > 2$, then for any solution u_+ to problem $(1.4)_+$, it holds that

$$\lim_{d(x)\to 0} \frac{u_+(x)}{-\ln(d(x))} = q.$$

Whereas when $b(x) = 1, f(u) = u^p$, they gave the following results.

(i) If $p > 1$ and $0 < q < 2p/(p+1)$, then problem $(1.4)_\pm$ has at least one positive solution in $C^2(\Omega)$. Moreover, every solution satisfies

$$\lim_{d(x)\to 0} \frac{u_\pm(x)}{[2(p+1)/(p-1)^2]^{1/(p-1)}(d(x))^{-2/(p-1)}} = 1.$$

(ii) If $2p/(p+1) < q < p$, then any positive solution of problem $(1.4)_+$ satisfies

$$\lim_{d(x)\to 0} u_+(x)[(p-q)d(x)/q]^{q/(p-q)} = 1.$$

(iii) If $p > 0$ and $\max\{1, 2p/(p+1)\} < q < 2$, then problem $(1.4)_-$ possesses a positive solution. Moreover, every positive solution u of problem $(1.4)_-$ satisfies

$$\lim_{d(x)\to 0} u_-(x)(2-q)[(q-1)d(x)]^{(2-q)/(q-1)} = 1.$$

(iv) If $p > 0$ and $q = 2$, then problem $(1.4)_-$ has a positive solution which satisfies

$$\lim_{d(x)\to 0} \frac{u_-(x)}{\ln(d(x))} = 1.$$

Bandle and Giarrusso [1] extended the above results for more general function $f(u)$. It was also shown that if $f(t)/F^{q/2}(t) \to \infty$ as $t \to \infty$, then the solution $u_\pm$ of $(1.4)_\pm$ satisfies

$$\frac{1}{d(x)} \int_{u_\pm(x)}^{\infty} \frac{1}{\sqrt{2F(t)}}\,\mathrm{d}t \to 1 \quad \text{as } d(x) \to 0,$$

and if

$$\frac{f(t)}{F^{q/2}(t)} \to 0 \quad \text{as } t \to \infty,$$

then

$$u_\pm(x)[(q-1)d(x)]^{(2-q)/(q-1)} \to \frac{1}{2-q} \quad \text{as } d(x) \to 0,$$

where

$$F(t) = \int^t f(s)\,\mathrm{d}s.$$

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Giarrusso [10, 11] showed that if

$$\frac{f(t)}{F^{q/2}(t)} \to \kappa, \quad 0<\kappa<\infty, \quad \text{as } t\to\infty \quad \text{and} \quad 1<q<2,$$

then

$$u_\pm(x)[(q-1)d(x)]^{(2-q)/(q-1)} \to \left[\frac{2-q}{\sqrt{a_\pm}}\right]^{(2-q)/(q-1)} \quad \text{as } d(x)\to 0,$$

where $a_\pm$ is the solution of the equation

$$\frac{a}{2-q} \pm a^{q/2} = \kappa.$$

Later Ghergu *et al*. [9] and Zhang [22, 24] extended the above results for the more general function $f(u)$ and weighted function $b(x)$.

For the general quasilinear boundary blow-up problem

$$\left.\begin{aligned} \Delta u - \psi(x,u,\nabla u) = 0, \quad & x\in\Omega, \\ u=\infty, \quad & x\in\partial\Omega, \end{aligned}\right\} \tag{1.5}$$

Goncalves *et al*. [14] showed the existence of non-negative solutions under the condition

$$a(x)g(t) \leqslant \psi(x,t,\xi) \leqslant h(t)(1+\Lambda|\xi|^2), \tag{1.6}$$

where a, g, h are continuous functions, $\Lambda>0$ is a constant, g and h are non-decreasing and satisfy $g(0)=0$, $g(t)>0$ for $t>0$ and $h(0)\geqslant 1$; in particular, g satisfies the so-called Keller–Osserman condition, namely

$$\int_1^\infty \frac{1}{\sqrt{G(t)}}\,\mathrm{d}t < \infty, \qquad G(t)=\int_0^t g(s)\,\mathrm{d}s.$$

Goncalves *et al*. applied the abstract result therein to prove the existence of large solutions of the problem

$$\Delta u = a(x)g(u) + \lambda u^\sigma|\nabla u|^q, \quad x\in\Omega; \qquad u=\infty, \quad x\in\partial\Omega,$$

where $a(x)\in C^\gamma(\Omega)\cap C(\bar\Omega)$, $0\leqslant q\leqslant 2$, and g satisfies the Keller–Osserman condition.

For other works on large solutions with nonlinear gradient terms, see [3, 9, 12, 16, 23] and the references therein.

Motivated by the above papers, in this work we study the existence, asymptotic behaviour near the boundary and uniqueness of large solutions for problem (1.1). To this end, we first study the radially symmetric case:

$$\left.\begin{aligned} v'' + \frac{N-1}{r}v' = a_1(r)(R-r)^\mu v^p + a_2(r)(R-r)^\nu v^\sigma|v'|^q, \quad & r\in(0,R), \\ v\geqslant 0, \quad r\in(0,R), & \\ v'(0)=0, \qquad \lim_{r\nearrow R} v(r)=\infty. & \end{aligned}\right\} \tag{1.7}$$

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We show the existence and asymptotic behaviour of radial large C^2-solutions of (1.7) by constructing suitable blow-up weak upper and lower solutions. Then we derive the existence of non-negative solutions of (1.1) in a general domain by a comparison argument. Finally, by using a perturbation method and constructing comparison functions, we obtain the exact asymptotic behaviour of any non-negative solution of (1.1) near the boundary. The uniqueness of the solution is shown by a standard argument.

Our main results are summarized in the following and, to the best of our knowledge, they are not covered by any of the references cited above.

THEOREM 1.2. *Assume that $R > 0$, $a_i \in C([0, R]; \mathbb{R}^+)$, $p, \sigma > 0$, $q \geqslant 0$, $\mu > -2$ and $\nu > -(2-q)$. If one of the following holds:*

(i)
$$p > \max\left\{1, \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1\right\}, \quad a_1(r) > 0 \text{ in } (0, R];$$

(ii)
$$p < \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1, \quad q+\sigma > 1 \quad \text{and} \quad a_2(r) > 0 \text{ in } (0, R];$$

(iii)
$$1 < p = \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1, \quad a_1(r) > 0 \text{ or } a_2(r) > 0 \text{ in } (0, R],$$

then, for each $\varepsilon > 0$, problem (1.7) has at least one non-negative solution v_ε and satisfies

$$1-\varepsilon \leqslant \liminf_{r \nearrow R} \frac{v_\varepsilon(r)}{\psi(R)(R-r)^{-\alpha}} \leqslant \limsup_{r \nearrow R} \frac{v_\varepsilon(r)}{\psi(R)(R-r)^{-\alpha}} \leqslant 1+\varepsilon, \tag{1.8}$$

where

$$\alpha = \frac{\mu+2}{p-1} \quad \text{and} \quad \psi(R) = \left(\frac{\alpha(\alpha+1)}{a_1(R)}\right)^{1/(p-1)}$$

when condition (i) holds,

$$\alpha = \frac{\nu+2-q}{q+\sigma-1} \quad \text{and} \quad \psi(R) = \left(\frac{\alpha^{1-q}(\alpha+1)}{a_2(R)}\right)^{1/(q+\sigma-1)}$$

when condition (ii) holds,

$$\alpha = \frac{\mu+2}{p-1}$$

and $\psi(R)$ is the unique positive solution of the equation

$$a_1(R)x^{p-1} + a_2(R)\alpha^q x^{q+\sigma-1} = \alpha(\alpha+1)$$

when condition (iii) holds.

Therefore, for each $x_0 \in \mathbb{R}^N$, the function

$$u_\varepsilon(x) := v_\varepsilon(r) \quad \text{with } r := |x - x_0|$$

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provides us with a radially symmetric non-negative solution of the problem

$$\begin{gathered}-\Delta u = a_1(r)d^{\mu}(x)u^p + a_2(r)d^{\nu}(x)|\nabla u|^q, \quad x \in B_R(x_0),\\ u = \infty, \quad x \in \partial B_R(x_0),\end{gathered}$$

and satisfies

$$1-\varepsilon \leqslant \liminf_{d(x)\searrow 0} \frac{u_\varepsilon(x)}{\psi(R)d^{-\alpha}(x)} \leqslant \limsup_{d(x)\searrow 0} \frac{u_\varepsilon(x)}{\psi(R)d^{-\alpha}(x)} \leqslant 1+\varepsilon,$$

where $d(x) := \operatorname{dist}(x, \partial B_R(x_0)) = R - |x - x_0| = R - r$, α *and* $\psi(R)$ *are defined above.*

THEOREM 1.3. *Suppose that there exist constants* $b_2, c_2 > 0$ *and* $\mu_2, \nu_2 > -1/N$, *such that*

$$b(x) \leqslant b_2 d^{\mu_2}(x), c(x) \leqslant c_2 d^{\nu_2}(x), \quad x \in \Omega.$$

Then problem (1.1) has at least one non-negative solution under one of the following conditions:

(i) $p > 1$, $b(x)$ *satisfies (Hb);*

(ii) $q + \sigma > 1$, $c(x) \geqslant c_1 d^{\nu_1}(x)$ *in* Ω *for some* $c_1 > 0$ *and* $\nu_1 \geqslant 0$.

THEOREM 1.4. *Suppose that there exist constants* $\beta, \rho > 0$, $\mu > -2$, $\nu > -(2-q)$ *satisfying* $\mu\nu \geqslant 0$. *If one of the following holds:*

(i)

$$p > \max\left\{1, \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1\right\}, \quad \lim_{d(x)\searrow 0} \frac{b(x)}{d^{\mu}(x)} = \beta, \quad c(x) \leqslant c_2 d^{\nu}(x)$$

near the boundary;

(ii)

$$p < \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1, \quad q+\sigma > 1, \quad \lim_{d(x)\searrow 0} \frac{c(x)}{d^{\nu}(x)} = \rho, \quad b(x) \leqslant b_2 d^{\mu}(x)$$

near the boundary;

(iii)

$$1 < p = \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1, \qquad \lim_{d(x)\searrow 0} \frac{b(x)}{d^{\mu}(x)} = \beta, \qquad \lim_{d(x)\searrow 0} \frac{c(x)}{d^{\nu}(x)} = \rho$$

and

$$\mu \leqslant \frac{2}{2-q}\nu \quad \text{if } \nu < 0$$

($\nu < 0$ *implies* $q < 2$);

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then any non-negative solution $u(x)$ of (1.1) satisfies

$$\lim_{d(x)\searrow 0} \frac{u(x)}{d^{-\alpha}(x)} = l, \tag{1.9}$$

where

$$\alpha = \frac{\mu+2}{p-1} \quad \textit{and} \quad l = \left(\frac{\alpha(\alpha+1)}{\beta}\right)^{1/(p-1)}$$

when condition (i) holds,

$$\alpha = \frac{\nu+2-q}{q+\sigma-1} \quad \textit{and} \quad l = \left(\frac{\alpha^{1-q}(\alpha+1)}{\rho}\right)^{1/(q+\sigma-1)}$$

when condition (ii) holds,

$$\alpha = \frac{\mu+2}{p-1}$$

and l is the unique positive solution of the equation

$$\beta x^{p-1} + \rho\alpha^q x^{q+\sigma-1} = \alpha(\alpha+1) \tag{1.10}$$

when condition (iii) holds.

Furthermore, in any case, if $p > 1$ and $q + \sigma \geqslant 1$, or $p = 1, \sigma + q > 1$ and $q \geqslant 1$, then the solution of (1.1) is unique if it exists.

Remark 1.5. It is well known that case (i) also holds for the large solutions of the simple semilinear equation $\Delta u = b(x)u^p$. On the contrary, case (ii) shows that the asymptotic behaviour of u is really influenced by the term $c(x)u^\sigma|\nabla u|^q$ and does not depend on the particular $b(x)u^p$. On the other hand, case (iii) shows that the asymptotic behaviour of u is influenced not only by the term $c(x)u^\sigma|\nabla u|^q$ but also by the term $b(x)u^p$.

The rest of this paper is organized as follows: in § 2, we give a comparison principle for quasilinear equations and two related existence theorems. In § 3 we first study the large solution for the radially symmetric case, and then prove theorems 1.3 and 1.4.

2. Comparison principle and existence result

2.1. Comparison principle

First we consider a second-order quasilinear operator Q of the form

$$Qu := \sum_{i,j=1}^{N} a^{ij}(x,u,\nabla u)D_{ij}u + b(x,u,\nabla u),$$

where $x = (x_1,\dots,x_N)$ is contained in the domain Ω of $\mathbb{R}^N$. The coefficients of Q, namely the functions $a^{ij}(x,z,\xi)$, $i,j = 1,\dots,N$, $b(x,z,\xi)$ are assumed to be defined for all values of (x,z,ξ) in the set $\Omega\times\mathbb{R}\times\mathbb{R}^N$.

LEMMA 2.1 (Gilbarg and Trudinger [13, theorem 10.1]). *Let $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $Qu \geqslant Qv$ in Ω, and $u \leqslant v$ on $\partial\Omega$, where*

(i) *the operator Q is elliptic with respect to either u or v,*

(ii) *the coefficients a^{ij} are independent of z,*

(iii) *the coefficient b is non-increasing in z for each $(x, \xi) \in \Omega \times \mathbb{R}^N$,*

(iv) *the coefficients a^{ij}, b are continuously differentiable with respect to the ξ variables in $\Omega \times \mathbb{R} \times \mathbb{R}^N$.*

It then follows that $u \leqslant v$ in Ω. Furthermore, if $Qu > Qv$ in Ω, $u \leqslant v$ on $\partial\Omega$ and conditions (i)–(iii) (but not necessarily (iv)) hold, we have the strict inequality $u < v$ in Ω.

If $\psi : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ and $\psi(x, 0, 0) = 0$, we can extend the function $\psi(x, u, \nabla u)$ to the region $u \leqslant 0$ properly, and apply lemma 2.1 with the operator $Qu = \Delta u - \psi(x, u, \nabla u)$. The following comparison principle plays an important role in the proofs of theorems 1.3 and 1.4.

LEMMA 2.2. *Assume that $\psi : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$, $\psi(x, 0, 0) = 0$ and $\psi(x, t, \xi)$ is non-decreasing in t for each $(x, \xi) \in \Omega \times \mathbb{R}^N$. Let $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ be two non-negative functions.*

(i) *If u, v satisfy*

$$\Delta u - \psi(x, u, \nabla u) > \Delta v - \psi(x, v, \nabla v) \quad \text{in } \Omega, \quad u \leqslant v \text{ on } \partial\Omega,$$

it then follows that $u < v$ in Ω.

(ii) *If u, v satisfy*

$$\Delta u - \psi(x, u, \nabla u) \geqslant \Delta v - \psi(x, v, \nabla v) \quad \text{in } \Omega, \quad u \leqslant v \text{ on } \partial\Omega,$$

and in addition the coefficient ψ is continuously differentiable with respect to the ξ variable in $\Omega \times \mathbb{R}^+ \times \mathbb{R}^N$, it then follows that $u \leqslant v$ in Ω.

2.2. Existence results

Now we consider the general equation

$$\Delta u - \psi(x, u, \nabla u) = 0, \quad x \in \Omega. \tag{2.1}$$

DEFINITION 2.3. Let $1 < k \leqslant \infty$. A function $\bar{u} \in W^{1,k}(\Omega)$ is called a weak upper solution to (2.1) if

$$\psi(\cdot, \bar{u}(\cdot), \nabla\bar{u}(\cdot)) \in L^{k'}(\Omega) \quad \text{with } k' = \begin{cases} \dfrac{k}{k-1}, & k < \infty, \\ 1, & k = \infty, \end{cases}$$

and

$$\int_\Omega \nabla\bar{u} \cdot \nabla v \, dx \geqslant \int_\Omega \psi(x, \bar{u}, \nabla\bar{u}) v \, dx \quad \text{for all } v \in W_0^{1,k}(\Omega),\ v \geqslant 0 \text{ a.e. in } \Omega.$$

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A function $\underline{u} \in W^{1,k}(\Omega)$ is called a weak lower solution of (2.1) if the inequalities above are reversed. If $\underline{u} \leqslant \bar{u}$ a.e. in Ω, we say they are ordered. In addition, if $\bar{u} = \underline{u} = \infty$, we say that they are the ordered weak upper and lower solutions of (1.5).

We first determine the existence of the weak solution to the problem

$$\left.\begin{aligned} \Delta u - \psi(x, u, \nabla u) = 0, \qquad & x \in \Omega, \\ u = \phi(x), \quad & x \in \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

Assume that $\bar{u} \in W^{1,k}(\Omega)$ is a weak upper solution ($\underline{u} \in W^{1,k}(\Omega)$ is a weak lower solution) of (2.1). If $\bar{u} \geqslant \phi$ ($\underline{u} \leqslant \phi$) on $\partial\Omega$, we say that $\bar{u}$ ($\underline{u}$) is a weak upper solution (lower solution) of (2.2).

LEMMA 2.4 (Du [5, theorem 4.9]). *Let $\bar{u}, \underline{u} \in W^{1,k}(\Omega)$ be the ordered weak upper and lower solutions of (2.2); $\phi(x)$ can be extended to Ω such that $\phi \in W^{1,k}(\Omega)$ and $\underline{u} \leqslant \phi \leqslant \bar{u}$ a.e. in Ω. Assume that there exist a positive constant C_1 and a function $h_1 \in L^{k'}(\Omega)$ with $k' = k/(k-1)$, such that*

$$|\psi(x, t, \xi)| \leqslant h_1(x) + C_1|\xi|^{k-1} \quad \text{a.e. } x \in \Omega \text{ for all } \xi \in \mathbb{R}^N,\ t \in [\underline{u}, \bar{u}]. \tag{2.3}$$

Then there is a weak solution $u \in W^{1,k}(\Omega)$ of problem (2.2) such that $\underline{u} \leqslant u \leqslant \bar{u}$ a.e. in Ω.

As for the existence of classical solutions, we have the following theorem.

THEOREM 2.5. *Let $\psi \in C^{\gamma}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ for some $0 < \gamma < 1$, and $\bar{u}, \underline{u} \in W^{1,\infty}(\Omega)$ be the ordered weak upper and lower solutions of (2.2); $\phi(x)$ can be extended to Ω such that $\phi \in C^{2+\gamma}(\Omega)$ and $\underline{u} \leqslant \phi \leqslant \bar{u}$ a.e. in Ω. Assume that there exist constants $k > 1$, $C_1 > 0$, $m > N$ and a function $h_1 \in C^{\mu}(\Omega) \cap L^m(\Omega)$ for some $0 < \mu < 1$, such that (2.3) holds. Then there is a $C^{2,\beta}(\Omega)$-solution u of (2.2) for some $0 < \beta < 1$ and satisfying $\underline{u} \leqslant u \leqslant \bar{u}$ in Ω.*

Proof. Choose $l > k$ so large that $l/(k-1) > m$. For such fixed l, by the inequality (2.3) and Young's inequality, we have that there exist two positive constants C_2 and C_3 such that

$$|\psi(x, t, \xi)| \leqslant h_1(x) + C_2 + C_3|\xi|^{l-1} \quad \text{a.e. } x \in \Omega \text{ for all } \xi \in \mathbb{R}^N,\ t \in [\underline{u}, \bar{u}].$$

It is obvious that $\bar{u}, \underline{u} \in W^{1,l}(\Omega)$ are the ordered weak upper and lower solutions of (2.2). In view of lemma 2.4, problem (2.2) has a weak solution $u \in W^{1,l}(\Omega)$ and satisfies $\underline{u} \leqslant u \leqslant \bar{u}$ in Ω, which implies $\nabla u \in L^l(\Omega)$. So, $|\nabla u|^{k-1} \in L^{l/(k-1)}(\Omega) \hookrightarrow L^m(\Omega)$. By the inequality (2.3), we have

$$|\psi(x, u, \nabla u)| \leqslant h_1(x) + C_1|\nabla u|^{k-1} \in L^m(\Omega).$$

Thus, by the L^p theory for elliptic equations, we have $u \in W^{2,m}(\Omega)$. Since $m > N$, we have that $\nabla u \in C^{\alpha}(\bar{\Omega})$ with $0 < \alpha = 1 - N/m < 1$ by the Sobolev imbedding theorem. Applying the Schauder theory, we obtain that problem (2.2) has a solution $u \in C^{2,\beta}(\Omega)$ for some $0 < \beta < 1$. This completes the proof. □

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The following theorem is needed in the next section.

THEOREM 2.6. *Let $\psi \in C^{\gamma}(\Omega \times \mathbb{R}^+ \times \mathbb{R}^N)$, and non-negative functions $\bar{u}, \underline{u} \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be the ordered weak upper and weak lower solutions of (1.5). Assume that there exist constants $k > 1$, $C_1 > 0$ and two functions $h_1 \in C^{\mu}(\Omega)$ and $g \in L^{\infty}_{\mathrm{loc}}([0,\infty))$ for some $0 < \mu < 1$, such that*

$$|\psi(x,t,\xi)| \leqslant h_1(x) + g(t) + C_1|\xi|^{k-1} \quad \text{a.e. } x \in \Omega \text{ for all } \xi \in \mathbb{R}^N,\ t \in [\underline{u}, \bar{u}]. \tag{2.4}$$

Then there is a $C^{2,\beta}(\Omega)$-solution u of (1.5) for some $0 < \beta < 1$ such that $\underline{u} \leqslant u \leqslant \bar{u}$ in Ω.

Proof. Let $\delta > 0$ and $\Omega_\delta := \{x \in \Omega :\ d(x) > \delta\}$. Consider the problem

$$\left.\begin{aligned} \Delta u &= \psi(x,u,\nabla u), \quad && x \in \Omega_\delta, \\ u &= \bar{u}, && x \in \partial\Omega_\delta. \end{aligned}\right\} \tag{2.5}$$

Since $\bar{u} \in W^{1,\infty}(\Omega_\delta)$ and $\underline{u} \in W^{1,\infty}(\Omega_\delta)$, by (2.4) we see that there is constant $C_2 = C_2(\delta) > 0$ such that

$$|\psi(x,t,\xi)| \leqslant h_1(x) + C_2 + C_1|\xi|^{k-1} \quad \text{a.e. } x \in \Omega_\delta \text{ for all } \xi \in \mathbb{R}^N,\ t \in [\underline{u}, \bar{u}].$$

It is obvious that $\bar{u}|_{\Omega_\delta}$ and $\underline{u}|_{\Omega_\delta}$ are the ordered upper and lower solutions of (2.5), and $h_1 \in C^{\mu}(\bar{\Omega}_\delta)$. By theorem 2.5, there exists a solution $u_\delta \in C^{2,\beta}(\Omega_\delta)$ of (2.5) such that $\underline{u} \leqslant u_\delta \leqslant \bar{u}$ in Ω_δ. In view of the standard inner Schauder estimate and compact imbedding theorem, we conclude the existence of a sequence $\delta_n \to 0$ such that $u_{\delta_n} \to u$ in $C^{2,\beta}(\Omega)$. It follows that u is a solution to (1.5) and $\underline{u} \leqslant u \leqslant \bar{u}$ in Ω. □

3. Radially symmetric case

Proof of theorem 1.2.
(i) First we show that, for each $\varepsilon > 0$ sufficiently small, a constant $A_\varepsilon > 0$ exists for which the function

$$\bar{v}_\varepsilon(r) := A + B_+\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha} \tag{3.1}$$

provides us with a positive upper solution of (1.7) for each $A > A_\varepsilon$ if

$$B_+ = (1+\varepsilon)\psi(R), \tag{3.2}$$

where

$$\alpha = \frac{\mu+2}{p-1} \quad \text{and} \quad \psi(R) = \left(\frac{\alpha(\alpha+1)}{a_1(R)}\right)^{1/(p-1)}.$$

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Indeed, $\bar{v}_\varepsilon'(0) = 0$ and $\lim_{r \nearrow R} \bar{v}_\varepsilon(r) = \infty$, since $\alpha > 0$. Thus, $\bar{v}_\varepsilon$ is an upper solution of (1.7) if and only if

$$\begin{aligned} \frac{B_+}{R^2}[\alpha(\alpha+1)r^2 + 4\alpha r(R-r) + 2(R-r)^2] + (N-1)\frac{B_+}{R^2}(R-r)(2(R-r)+\alpha r) \\ \leqslant a_1(r)(R-r)^{\mu-\alpha p+\alpha+2}\left(A(R-r)^\alpha + \frac{B_+ r^2}{R^2}\right)^p \\ + a_2(r)(R-r)^{\nu-\alpha\sigma+\alpha+2-(\alpha+1)q}\left(A(R-r)^\alpha + \frac{B_+ r^2}{R^2}\right)^\sigma \\ \times \left(\left|\frac{B_+ r}{R^2}(2(R-r)+\alpha r)\right|^q\right). \end{aligned} \tag{3.3}$$

Note that $\nu - \alpha\sigma + \alpha + 2 - (\alpha+1)q > \mu - \alpha p + \alpha + 2 = 0$ at $r = R$; thus, (3.3) becomes

$$\alpha(\alpha+1)B_+ \leqslant B_+^p a_1(R),$$

which is satisfied if and only if

$$B_+ \geqslant \psi(R),$$

since $p > 1$. Therefore, by choosing B_+ as in (3.2), inequality (3.3) is satisfied in a left neighbourhood of $r = R$, say $(R-\delta, R]$ for some $\delta = \delta(\varepsilon) > 0$. Finally, by choosing A sufficiently large, it is clear that the inequality is satisfied in the whole interval $[0, R]$, since $p > 0$ and $a_1(r)$ is non-zero. This concludes the proof of the claim above.

Now we will construct an adequate weak lower solution of (1.7). We claim that, for each sufficiently small $\varepsilon > 0$, there exists $C < 0$ such that

$$\underline{v}_\varepsilon(r) := \max\{0, C + B_-(r/R)^2(R-r)^{-\alpha}\} \tag{3.4}$$

provides us with a weak lower solution of (1.7), here

$$B_- = (1-\varepsilon)\psi(R). \tag{3.5}$$

Indeed, $\underline{v}_\varepsilon$ is a weak lower solution of (1.7) if, in the region where

$$C(R-r)^\alpha + \frac{B_- r^2}{R^2} \geqslant 0, \tag{3.6}$$

the following inequality is satisfied:

$$\begin{aligned} \frac{B_-}{R^2}[\alpha(\alpha+1)r^2 + 4\alpha r(R-r) + 2(R-r)^2] + (N-1)\frac{B_-}{R^2}(R-r)(2(R-r)+\alpha r) \\ \geqslant a_1(r)(R-r)^{\mu-\alpha p+\alpha+2}\left(C(R-r)^\alpha + \frac{B_- r^2}{R^2}\right)^p \\ + a_2(r)(R-r)^{\nu-\alpha\sigma+\alpha+2-(\alpha+1)q}\left(C(R-r)^\alpha + \frac{B_- r^2}{R^2}\right)^\sigma \\ \times \left(\left|\frac{B_- r}{R^2}(2(R-r)+\alpha r)\right|^q\right). \end{aligned} \tag{3.7}$$

Now, note that for each $C < 0$, a constant

$$Z = Z(C) \in (0, R)$$

exists for which

$$C(R-r)^\alpha + \frac{B_- r^2}{R^2} \leqslant 0 \quad \text{if } r \in [0, Z(C)),$$

while

$$C(R-r)^\alpha + \frac{B_- r^2}{R^2} \geqslant 0 \quad \text{if } r \in [Z(C), R).$$

Moreover, $Z(C)$ is decreasing and

$$\lim_{C \searrow -\infty} Z(C) = R, \qquad \lim_{C \nearrow 0} Z(C) = 0. \tag{3.8}$$

At $r = R$, (3.7) becomes

$$\alpha(\alpha+1)B_- \geqslant B_-^p a_1(R),$$

which is satisfied if and only if

$$B_- \leqslant \psi(R).$$

Therefore, by choosing B_- as in (3.5), inequality (3.7) is satisfied in a left neighbourhood of $r = R$, say $(R - \delta, R]$ for some $\delta = \delta(\varepsilon) > 0$. Moreover, due to (3.8), there exists $C < 0$ such that

$$Z(C) = R - \delta(\varepsilon).$$

For this choice of C, it readily follows that $\underline{v}_\varepsilon$ provides us with a weak lower solution of (1.7).

It is easy to see that (2.4) holds, owing to $p, \sigma > 0$, $q \geqslant 0$. Since $\underline{v}_\varepsilon(r), \bar{v}_\varepsilon(r) \in W^{1,\infty}_{\mathrm{loc}}(0, R)$ are the ordered non-negative weak lower and upper solutions of (1.7), the existence of a classical solution u of (1.7) is followed by theorem 2.6, and $\underline{v}_\varepsilon(r) \leqslant v \leqslant \bar{v}_\varepsilon(r)$ in Ω.

Finally, since

$$\lim_{r \nearrow R} \frac{\bar{v}_\varepsilon(r)}{B_+(R-r)^{-\alpha}} = \lim_{r \nearrow R} \frac{\underline{v}_\varepsilon(r)}{B_-(R-r)^{-\alpha}} = 1, \tag{3.9}$$

where B_+ and B_- are the constants defined through (3.2) and (3.5), we conclude the asymptotic behaviour for this case.

The proofs of cases (ii) and (iii) are similar to that of case (i). Here we omit the details. The remaining assertions of the theorem are easy consequences from these features. □

REMARK 3.1. From the proof we can see that, for case (i), if $a_2(r) \equiv 0$ in $(0, R)$, it suffices to require that $p > 1$, $a_1(r) > 0$ in $(0, R]$ to ensure the existence and asymptotic behaviour of the solution near the boundary. Similarly, for case (ii), if $a_1(r) \equiv 0$ in $(0, R)$, it suffices to require that $\sigma + q > 1$, $a_2(r) > 0$ in $(0, R]$ to ensure the corresponding results.

REMARK 3.2. If $\sigma = 0$, $q > 1$, $\mu > -2$, $\nu = -(2-q)$ and $a_1(r), a_2(r) > 0$, then (1.7) has a solution v_ε satisfying

$$\begin{aligned} 1-\varepsilon &\leqslant \liminf_{r \nearrow R} \frac{-v_\varepsilon(r)}{(a_2(R))^{-1/(q-1)} \ln(R-r)} \\ &\leqslant \limsup_{r \nearrow R} \frac{-v_\varepsilon(r)}{(a_2(R))^{-1/(q-1)} \ln(R-r)} \\ &\leqslant 1+\varepsilon. \end{aligned} \tag{3.10}$$

Indeed, we just set

$$\bar{v}_\varepsilon(r) = A - (1+\varepsilon)(a_2(R))^{-1/(q-1)} \left(\frac{r}{R}\right)^2 \ln(R-r),$$

$$\underline{v}_\varepsilon(r) = C - (1-\varepsilon)(a_2(R))^{-1/(q-1)} \left(\frac{r}{R}\right)^2 \ln(R-r).$$

With a similar argument to that in the proof of theorem 1.2, we obtain (3.10).

4. Proofs of theorems 1.3 and 1.4

Proof of theorem 1.3. For $n \geqslant 1$, consider the following problem:

$$\left.\begin{aligned} \Delta u &= b(x)u^p + c(x)u^\sigma |\nabla u|^q, \quad && x \in \Omega, \\ u &= n, && x \in \partial\Omega. \end{aligned}\right\} \tag{4.1}$$

Our aim is to pass to the limit in (4.1) as $n \to \infty$. This requires a few steps.

STEP 1 (existence of solutions for (4.1)). The (constant) function $\underline{u} = 0$ and $\bar{u} = n$ are the ordered lower and upper solutions of (4.1). Owing to $0 \leqslant b(x) \leqslant b_2 d^{\mu_2}(x)$, $0 \leqslant c(x) \leqslant c_2 d^{\nu_2}(x)$, by Young's inequality, we have that, for any $\theta > q$,

$$\begin{aligned} 0 &\leqslant b(x)u^p + c(x)u^\sigma |\nabla u|^q \\ &\leqslant b_2 n^p d^{\mu_2}(x) + c_2 n^\sigma \left(\frac{\theta-q}{\theta} d^{\nu_2\theta/(\theta-q)}(x) + \frac{q}{\theta}|\nabla u|^\theta\right) \quad \text{for all } x \in \Omega,\ 0 \leqslant u \leqslant n. \end{aligned}$$

Since $\mu_2, \nu_2 > -1/N$, we can choose $\theta > 1$ so large that $N\nu_2\theta/(\theta-q) > -1$, which implies

$$b_2 n^p d^{\mu_2}(x) + c_2 n^\sigma \frac{\theta-q}{\theta} d^{\nu_2\theta/(\theta-q)}(x) \in L^m(\Omega)$$

for some $m > N$. By theorem 2.5, we see that problem (4.1) has at least one solution $u_n(x) \in C^2(\Omega)$ such that $0 \leqslant u_n(x) \leqslant n$.

STEP 2 ($\{u_n\}$ is non-decreasing in Ω as n increases). Indeed, u_n and $n+1$ are the ordered lower and upper solutions of (4.1) when the boundary condition is replaced with $u = n+1$, $x \in \partial\Omega$. So we have $u_n \leqslant u_{n+1}$, $n \geqslant 1$.

STEP 3 ($\{u_n\}$ is uniformly bounded in Ω for each n). First we solve (i), i.e. $p > 1$, $b(x)$ satisfies (Hb) and $b(x) \leqslant b_2 d^{\mu_2}(x)$. By [20, theorem 1], there exists a positive solution $v(x)$ to the problem

$$\begin{aligned} \Delta v &= b(x)v^p, \quad x \in \Omega, \\ v &= \infty, \qquad\quad x \in \partial\Omega. \end{aligned}$$

Since

$$\Delta u_n - b(x)u_n^p - c(x)u_n^\sigma|\nabla u_n|^q \leqslant \Delta u_n - b(x)u_n^p, \quad x \in \Omega,$$

and $u_n(x) \leqslant v(x) = \infty$ on $\partial\Omega$, by lemma 2.2 we have $u_n(x) \leqslant v(x)$ for all n.

We now solve (ii), i.e. $q + \sigma > 1$, $c(x) \geqslant c_1 d^{\nu_1}(x)$. Fix a point $x_0 \in \Omega$, and consider a small ball B centred at x_0 and contained properly in Ω. By remark 3.1, there exists a positive solution $v(x)$ to the problem

$$\begin{aligned} \Delta v &= c_1(\operatorname{dist}(x, \partial B))^{\nu_1} v^\sigma |\nabla v|^q, \quad x \in B, \\ v &= \infty, \qquad\qquad\qquad\qquad\qquad\quad x \in \partial B. \end{aligned}$$

Hence, we have

$$\begin{aligned} \Delta u_n - b(x)u_n^p - c(x)u_n^\sigma|\nabla u_n|^q &\leqslant \Delta u_n - c_1 d^{\nu_1}(x)u_n^\sigma|\nabla u_n|^q \\ &< \Delta u_n - c_1(d(x, \partial B))^{\nu_1}u_n^\sigma|\nabla u_n|^q, \quad x \in B. \end{aligned}$$

By lemma 2.2 we also have $u_n(x_0) \leqslant v(x_0)$ for all n.

STEP 4 (the limit process). Standard elliptic regularity arguments show that the limit $\lim_{n\to\infty} u_n(x) = u^*(x)$ exists and $u^*(x)$ satisfies the differential equation in (1.1). To prove that $u^*(x)$ is a non-negative solution of (1.1), we merely verify $u^*(x)|_{\partial\Omega} = \infty$. If this is not true, then there exist a positive constant M, a sequence $\{x_k\} \subset \Omega$ and $x_0 \in \partial\Omega$, such that $x_k \to x_0$ and $u^*(x_k) \leqslant M$. For any fixed k, note that $u_n(x_k) \to u^*(x_k)$ as $n \to \infty$; it follows that there exists $N_k > 0$, such that $u_n(x_k) \leqslant 1 + M$ for all $n \geqslant N_k$. Note that u_n is increasing in n. We have

$$u_n(x_k) \leqslant 1 + M$$

for every $n > 0$. Now fix a $n > 1 + M$ and let $k \to \infty$ in the above inequality. It follows that $u_n(x_0) \leqslant 1 + M < n$, since $x_k \to x_0$, which is a contradiction of $u_n(x_0) = n$. The theorem is proved. □

Proof of theorem 1.4. If

$$\lim_{d(x)\searrow 0} \frac{b(x)}{d^\mu(x)} = \beta > 0 \quad \text{and} \quad \lim_{d(x)\searrow 0} \frac{c(x)}{d^\nu(x)} = \rho > 0,$$

then, given $0 < \varepsilon < \min\{\beta/2, \rho/2\}$, there exists $\delta = \delta(\varepsilon) > 0$, so that, for all $x \in \Omega$ with $d(x) < 2\delta$,

$$(\beta - \varepsilon)d^\mu(x) \leqslant b(x) \leqslant (\beta + \varepsilon)d^\mu(x), \qquad (\rho - \varepsilon)d^\nu(x) \leqslant c(x) \leqslant (\rho + \varepsilon)d^\nu(x). \tag{4.2}$$

Now we define $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$ with $\partial\Omega_\delta = \{x \in \Omega : d(x) = \delta\}$. It is easy to prove that, by diminishing $\delta > 0$ if necessary,

$$d(x) \in C^2(\bar{\Omega}_{2\delta}), \quad |\nabla d(x)| \equiv 1 \quad \text{on } \Omega_{2\delta}.$$

To prove the limit (1.9), we consider two cases.

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Case 1 ($\mu, \nu \geqslant 0$). Since the proof of each case is similar, here we just prove (iii). Let

$$\left.\begin{aligned} u^+(x) &= B_+(d(x)-\varrho)^{-\alpha}, \quad x \in D_\varrho^+ := \Omega_{2\delta}/\bar{\Omega}_\varrho, \\ u^-(x) &= B_-(d(x)+\varrho)^{-\alpha}, \quad x \in D_\varrho^- := \Omega_{2\delta-\varrho}, \end{aligned}\right\} \tag{4.3}$$

where $\varrho \in \Gamma := (0,\delta)$, α is defined in theorem 1.4 for case (iii), and

$$B_+ = \max\left\{l\left(1+\frac{2\varepsilon}{\beta-\varepsilon}\right)^{1/(p-1)}, l\left(1+\frac{2\varepsilon}{\rho-\varepsilon}\right)^{1/(\sigma+q-1)}\right\},$$
$$B_- = \min\left\{l\left(1-\frac{2\varepsilon}{\beta+\varepsilon}\right)^{1/(p-1)}, l\left(1-\frac{2\varepsilon}{\rho+\varepsilon}\right)^{1/(\sigma+q-1)}\right\},$$

where l is the positive solution of (1.10).

Since $\mu, \nu \geqslant 0$, we have $b(x) \geqslant (\beta-\varepsilon)(d(x)-\varrho)^\mu$ and $c(x) \geqslant (\rho-\varepsilon)(d(x)-\varrho)^\nu$ for any $x \in D_\varrho^+$. Consequently, it follows by $\mu-\alpha p+\alpha+2 = \nu-\alpha\sigma-(\alpha+1)q+\alpha+2 = 0$ and $p>1$, $\sigma+q>1$ that

$$\begin{aligned} \Delta u^+ - b(x)(u^+)^p - c(x)(u^+)^\sigma|\nabla u^+|^q &\leqslant B_+(d(x)-\varrho)^{-\alpha-2}[\alpha(\alpha+1) - \alpha(d(x)-\varrho)\Delta d(x) \\ &\qquad - (\beta-\varepsilon)B_+^{p-1}(d(x)-\varrho)^{\mu-\alpha p+\alpha+2} \\ &\qquad - \alpha^q(\rho-\varepsilon)B_+^{\sigma+q-1}(d(x)-\varrho)^{\nu-\alpha\sigma-(\alpha+1)q+\alpha+2}] \\ &= B_+(d(x)-\varrho)^{-\alpha-2}[\alpha(\alpha+1) - (\beta-\varepsilon)B_+^{p-1} \\ &\qquad - \alpha^q(\rho-\varepsilon)B_+^{\sigma+q-1} - \alpha(d(x)-\varrho)\Delta d(x)] \\ &< B_+(d(x)-\varrho)^{-\alpha-2}[\alpha(\alpha+1) - \beta l^{p-1} - \alpha^q\rho l^{\sigma+q-1} - \alpha(d(x)-\varrho)\Delta d(x)]. \end{aligned}$$

Since

$$\lim_{\substack{(x,\varrho)\in D_\varrho^+\times\Gamma \\ (d(x),\varrho)\searrow(0,0)}} \alpha(d(x)-\varrho)\Delta d(x) = 0,$$

we have

$$\Delta u^+ - b(x)(u^+)^p - c(x)(u^+)^\sigma|\nabla u^+|^q < 0, \quad x \in D_\varrho^+.$$

Similarly, we have

$$\Delta u^- - b(x)(u^-)^p - c(x)(u^-)^\sigma|\nabla u^-|^q > 0, \quad x \in D_\varrho^-.$$

Case 2 ($\mu, \nu \leqslant 0$). This situation is more complicated. The following method is motivated by [19].

For (i), we define

$$\left.\begin{aligned} u^+(x) &= B_+(d^{(\mu+2)/2}(x) - \varrho^{(\mu+2)/2})^{-2/(p-1)}, \quad x \in D_\varrho^+, \\ u^-(x) &= B_-(d^{(\mu+2)/2}(x) + \varrho^{(\mu+2)/2})^{-2/(p-1)}, \quad x \in D_\varrho^-, \end{aligned}\right\} \tag{4.4}$$

where ϱ and $D_\varrho^\pm$ are defined as in (4.3), $B_\pm = (1\pm\varepsilon)l$ and l is a constant defined in theorem 1.4 for case (i). Set $D_\pm(x) = d^{(\mu+2)/2}(x) \mp \varrho^{(\mu+2)/2}$ for simplicity. A direct calculation implies

$$
\begin{aligned}
\Delta u^{\pm} - b(x)(u^{\pm})^p - c(x)(u^{\pm})^{\sigma}|\nabla u^{\pm}|^q
&= B_{\pm}\biggl[\frac{(p+1)(\mu+2)^2}{2(p-1)^2}D_{\pm}^{-2p/(p-1)}(x)d^{\mu}(x) \\
&\qquad - \frac{(\mu+2)\mu}{2(p-1)}D_{\pm}^{-(p+1)/(p-1)}(x)d^{(\mu-2)/2}(x) \\
&\qquad - \frac{\mu+2}{p-1}D_{\pm}^{-(p+1)/(p-1)}(x)d^{\mu/2}(x)\Delta d(x)\biggr] \\
&\quad - b(x)B_{\pm}^{p}D_{\pm}^{-2p/(p-1)}(x) \\
&\quad - c(x)B_{\pm}^{\sigma+q}\left(\frac{\mu+2}{p-1}\right)^{q}D_{\pm}^{-(2\sigma+(p+1)q)/(p-1)}(x)d^{\mu q/2}(x)
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
&= B_{\pm}D_{\pm}^{-2p/(p-1)}(x)d^{\mu}(x) \\
&\quad \times\biggl[\frac{(p+1)(\mu+2)^2}{2(p-1)^2} - \frac{(\mu+2)\mu}{2(p-1)}D_{\pm}(x)d^{-(\mu+2)/2}(x) \\
&\qquad - \frac{\mu+2}{p-1}D_{\pm}(x)d^{-\mu/2}(x)\Delta d(x) - B_{\pm}^{p-1}\frac{b(x)}{d^{\mu}(x)} \\
&\qquad - B_{\pm}^{\sigma+q-1}\frac{c(x)}{d^{\nu}(x)}\left(\frac{\mu+2}{p-1}\right)^{q} \\
&\qquad\qquad \times D_{\pm}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x)d^{(2\nu+\mu q-2\mu)/2}(x)\biggr].
\end{aligned} \tag{4.6}
$$

Since

$$
\begin{aligned}
D_{+}(x) &\leqslant d^{(\mu+2)/2}(x) \quad \text{for } x \in D_{\varrho}^{+}, \\
d^{(\mu+2)/2}(x) \leqslant D_{-}(x) &\leqslant 2d^{(\mu+2)/2}(x) \quad \text{for } x \in D_{\varrho}^{-},
\end{aligned}
$$

and $-2 < \mu \leqslant 0$, we have

$$0 \leqslant -\frac{(\mu+2)\mu}{2(p-1)}D_{+}(x)d^{-(\mu+2)/2}(x) \leqslant -\frac{(\mu+2)\mu}{2(p-1)}, \tag{4.7}$$

$$-\frac{(\mu+2)\mu}{2(p-1)} \leqslant -\frac{(\mu+2)\mu}{2(p-1)}D_{-}(x)d^{-(\mu+2)/2}(x) \leqslant -\frac{(\mu+2)\mu}{(p-1)}. \tag{4.8}$$

Consequently,

$$\lim_{\substack{(x,\varrho)\in D_{\varrho}^{\pm}\times\Gamma \\ (d(x),\varrho)\searrow(0,0)}} -\frac{\mu+2}{p-1}D_{\pm}(x)d^{-\mu/2}(x)\Delta d(x) = 0. \tag{4.9}$$

Now we estimate the last term in the bracket of (4.6). If

$$-\frac{2\sigma+(p+1)q-2p}{p-1} \geqslant 0,$$

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it follows that

$$D_{\pm}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x)d^{(2\nu+\mu q-2\mu)/2}(x) \leqslant Cd^{\xi}(x),$$

where

$$C = 2^{-(2\sigma+(p+1)q-2p)/(p-1)}, \qquad \xi = \frac{\nu+2-q}{p-1}\left[p - \left(\frac{(\mu+2)(\sigma+q-1)}{\nu+2-q}+1\right)\right] > 0.$$

On the other hand, if

$$-\frac{2\sigma+(p+1)q-2p}{p-1} \leqslant 0,$$

we also find that

$$D_{-}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x)d^{(2\nu+\mu q-2\mu)/2}(x) \leqslant d^{\xi}(x),$$

Hence, by $c(x) \leqslant c_2 d^{\nu}(x)$, we have

$$\lim_{\substack{(x,\varrho)\in D_{\varrho}^{-}\times\Gamma \\ (d(x),\varrho)\searrow(0,0)}} B_{-}^{\sigma+q-1}\frac{c(x)}{d^{\nu}(x)}\left(\frac{\mu+2}{p-1}\right)^{q} \times D_{-}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x)d^{(2\nu+\mu q-2\mu)/2}(x) = 0. \tag{4.10}$$

Note that

$$\lim_{d(x)\to 0}\frac{b(x)}{d^{\mu}(x)} = \beta.$$

Combining (4.6) with (4.8)–(4.10), for $\varrho \in \Gamma$ with δ sufficiently small, we conclude that

$$\Delta u^{-} - b(x)(u^{-})^{p} - c(x)(u^{-})^{\sigma}|\nabla u^{-}|^{q} > 0, \quad x \in D_{\varrho}^{-}. \tag{4.11}$$

Furthermore, by

$$B_{+}^{\sigma+q-1}\frac{c(x)}{d^{\nu}(x)}\left(\frac{\mu+2}{p-1}\right)^{q}D_{+}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x)d^{(2\nu+\mu q-2\mu)/2}(x) \geqslant 0,$$

and combining (4.6) with (4.7) and (4.9), we also conclude that

$$\Delta u^{+} - b(x)(u^{+})^{p} - c(x)(u^{+})^{\sigma}|\nabla u^{+}|^{q} < 0, \quad x \in D_{\varrho}^{+}. \tag{4.12}$$

For case (ii), we set

$$u^{\pm}(x) = B^{\pm}(d^{(\nu+2-q)/2} \mp \varrho^{(\nu+2-q)/2})^{-2/(q+\sigma-1)}, \quad x \in D_{\varrho}^{\pm},$$

where $B_{\pm} = (1 \pm \varepsilon)l$, and l is a constant defined in theorem 1.4 for case (ii). Set

$$D_{\pm}(x) = d^{(\nu+2-q)/2}(x) \mp \varrho^{(\nu+2-q)/2}.$$

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A direct calculation implies

$$\begin{aligned}\Delta u^{\pm} - b(x)(u^{\pm})^p - c(x)(u^{\pm})^{\sigma}|\nabla u^{\pm}|^q &= B_{\pm} D_{\pm}^{-(q(q+\sigma+1)+2\sigma)/(q+\sigma-1)}(x) d^{(2\nu+(\nu-q)q)/2}(x) \\ &\quad\times \bigg[\frac{(q+\sigma+1)(\nu+2-q)^2}{2(q+\sigma-1)^2} D_{\pm}^{q}(x) d^{-(\nu+2-q)q/2} \\ &\qquad + \frac{(\nu+2-q)(-\nu+q)}{2(q+\sigma-1)} D_{\pm}^{q+1}(x) d^{-(\nu+2-q)(q+1)/2}(x) \\ &\qquad - \frac{\nu+2-q}{q+\sigma-1} D_{\pm}^{q+1}(x) d^{-(\nu+2-q)(q+1)/2+1}(x)\Delta d(x) \\ &\qquad - B_{\pm}^{p-1}\frac{b(x)}{d^{\mu}(x)} D_{\pm}^{(2(\sigma-p)+q(q+\sigma+1))/(q+\sigma-1)}(x) \\ &\qquad \times d^{(2(\mu-\nu)-(\nu-q)q\mu)/2}(x) - B_{\pm}^{\sigma+q-1}\frac{c(x)}{d^{\nu}(x)}\left(\frac{\nu+2-q}{q+\sigma-1}\right)^q\bigg].\end{aligned}$$

Analogously to case (i), we have

$$\begin{aligned} D_{-}^{q}(x) d^{-(\nu+2-q)q/2}(x) \leqslant 1, \qquad D_{-}^{q+1}(x) d^{-(\nu+2-q)(q+1)/2}(x) \leqslant 1, \\ D_{+}^{q}(x) d^{-(\nu+2-q)q/2}(x) \geqslant 1, \qquad D_{+}^{q+1}(x) d^{-(\nu+2-q)(q+1)/2}(x) \geqslant 1, \end{aligned}$$

and

$$\lim_{\substack{(x,\varrho)\in D_{\varrho}^{\pm}\times\Gamma \\ (d(x),\varrho)\searrow(0,0)}} D_{\pm}^{q+1}(x) d^{-(\nu+2-q)(q+1)/2+1}(x)\Delta d(x) = 0,$$

$$\lim_{\substack{(x,\varrho)\in D_{\varrho}^{-}\times\Gamma \\ (d(x),\varrho)\searrow(0,0)}} B_{-}^{p-1}\frac{b(x)}{d^{\mu}(x)} D_{-}^{(2(\sigma-p)+q(q+\sigma+1))/(q+\sigma-1)}(x) d^{(2(\mu-\nu)-(\nu-q)q\mu)/2}(x) = 0.$$

So, arguing as in case (i), (4.11) and (4.12) also hold.

For (iii), as in case (i), we define $u^{\pm}$ by (4.3), except that $B_{\pm} = (1 \pm l)\varepsilon$, where l is defined in theorem 1.4 for case (iii). Since $\nu < 0$ (which implies $q < 2$),

$$\mu \leqslant \frac{2\nu}{2-q} \quad \text{and} \quad p = \frac{(\mu+2)(\sigma+q-1)}{\nu+2-q} + 1,$$

we conclude that

$$\frac{2\sigma + (p+1)q - 2p}{p-1} \geqslant 0.$$

It then follows that

$$\left.\begin{aligned} D_{-}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x) d^{(2\nu+\mu q-2\mu)/2}(x) \leqslant 1, \quad x \in D_{\varrho}^{-}, \\ D_{+}^{-(2\sigma+(p+1)q-2p)/(p-1)}(x) d^{(2\nu+\mu q-2\mu)/2}(x) \geqslant 1, \quad x \in D_{\varrho}^{+}. \end{aligned}\right\} \tag{4.13}$$

By

$$\lim_{d(x)\to 0}\frac{b(x)}{d^{\mu}(x)} = \beta, \qquad \lim_{d(x)\to 0}\frac{c(x)}{d^{\mu}(x)} = \rho,$$

https://doi.org/10.1017/S0308210510000363 Published online by Cambridge University Press

and combining (4.6) with (4.7)–(4.9) and (4.13), for $\varrho \in \Gamma$ with δ sufficiently small, we conclude that (4.11) and (4.12) hold.

Let u be any solution of (1.1) and let

$$M_1(\delta) = \max_{d(x) \geqslant 2\delta} u(x), \qquad M_2(\delta) = B_-(2\delta)^{-\alpha}.$$

We see that

$$\left.\begin{aligned} u(x) &\leqslant u^+(x) + M_1(\delta), \quad x \in \partial D_\varrho^+, \\ u_-(x) &\leqslant u(x) + M_2(\delta), \quad x \in \partial D_\varrho^-. \end{aligned}\right\} \tag{4.14}$$

On the other hand, as $p, \sigma \geqslant 0$, we have

$$\left.\begin{aligned} &\Delta[u^+ + M_1(\delta)] - b(x)[u^+ + M_1(\delta)]^p \\ &\qquad -c(x)[u^+ + M_1(\delta)]^\sigma |\nabla[u^+ + M_1(\delta)]|^q < 0, \quad x \in D_\varrho^+, \\ &\Delta[u + M_2(\delta)] - b(x)[u + M_2(\delta)]^p \\ &\qquad -c(x)[u + M_2(\delta)]^\sigma |\nabla[u + M_2(\delta)]|^q < 0, \quad x \in D_\varrho^-. \end{aligned}\right\} \tag{4.15}$$

By (4.14), (4.15), it follows by lemma 2.2 that

$$\begin{aligned} u(x) &\leqslant u^+(x) + M_1(\delta), \quad x \in D_\varrho^+, \\ u_-(x) &\leqslant u(x) + M_2(\delta), \quad x \in D_\varrho^-. \end{aligned}$$

Hence, for $x \in D_\varrho^+ \cap D_\varrho^-$, letting $\varrho \to 0$, we see that

$$B_- d^{-\alpha}(x) \leqslant u + M_2(\delta) \leqslant B_+ d^{-\alpha}(x) + M_1(\delta) + M_2(\delta),$$

which implies

$$B_- \leqslant \liminf_{d(x) \searrow 0} \frac{u(x)}{d^{-\alpha}(x)} \leqslant \limsup_{d(x) \searrow 0} \frac{u(x)}{d^{-\alpha}(x)} \leqslant B_+.$$

Letting $\varepsilon \to 0$, we obtain

$$\lim_{d(x) \searrow 0} \frac{u(x)}{d^{-\alpha}(x)} = l,$$

which is in agreement with (1.10).

The final step is to prove the uniqueness. Let u_1 and u_2 be two positive solutions of (1.1). By (1.10), we thus have

$$\lim_{d(x) \searrow 0} \frac{u_1(x)}{u_2(x)} = 1.$$

The uniqueness follows from this fact and lemma 2.2 by a standard argument [17].

Indeed, for $\theta > 0$ arbitrary, set $(1 + \theta)u_i = w_i$, for $i = 1, 2$. It follows that

$$\lim_{d(x) \searrow 0} (u_1 - w_2)(x) = \lim_{d(x) \searrow 0} (u_2 - w_1)(x) = -\infty.$$

When $p > 1$, $\sigma + q \geqslant 1$, we have that

$$\Delta w_i - b(x) w_i^p - c(x) w_i^\sigma |\nabla w_i|^q < (1+\theta)[\Delta u_i - b(x) u_i^p - c(x) u_i^\sigma |\nabla u_i|^q] = 0, \quad x \in \Omega.$$

Therefore, by lemma 2.2(i), we may infer that

$$u_1 \leqslant (1+\theta)u_2, \quad u_2 \leqslant (1+\theta)u_1, \quad x \in \Omega. \tag{4.16}$$

When $p = 1$, $\sigma + q > 1$ and $q \geqslant 1$, we have that

$$\Delta w_i - b(x)w_i^p - c(x)w_i^\sigma |\nabla w_i|^q \leqslant (1+\theta)[\Delta u_i - b(x)u_i^p - c(x)u_i^\sigma |\nabla u_i|^q], \quad x \in \Omega.$$

By lemma 2.2(ii), we also obtain (4.16).

Passing to the limit $\theta \to 0^+$ in (4.16), we get $u_1 = u_2$ in Ω. This completes the proof of theorem 1.4. □

Acknowledgements

The authors are grateful to the referees and editors for their helpful comments and suggestions. This work was supported by PRC Grant no. NSFC 11071049.

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(*Issued 5 August 2011*)