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On regularity criteria of a weak solution to the three-dimensional magnetohydrodynamics equations in Lorentz space

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We give a weak-L^p Serrin-type regularity criterion for a weak solution to the three-dimensional magnetohydrodynamics equations in a bounded domain $\Omega \subset \mathbb{R}^3$.

Keywords: magnetohydrodynamics equations; weak solutions; regularity condition; bounded domains

2010 *Mathematics subject classification:* Primary 35Q30
Secondary 35K15

1. Introduction

We study the three-dimensional magnetohydrodynamics (3D MHD) equations

$$\left.\begin{aligned} u_t - \Delta u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla\pi = 0,\\ b_t - \Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0,\\ \operatorname{div} u = 0, \qquad \operatorname{div} b = 0, \end{aligned}\right\} \quad \text{in } Q_T := \Omega\times[0,T), \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^3$. Here, $u\colon Q_T \to \mathbb{R}^3$ is the flow velocity vector, $b\colon Q_T \to \mathbb{R}^3$ is the magnetic vector and $\pi = p+\frac{1}{2}|b|^2\colon Q_T \to \mathbb{R}$ is the total pressure. We consider the initial–boundary-value problem of (1.1), which requires initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad b(x,0) = b_0(x), \quad x\in\Omega, \tag{1.2}$$

together with the boundary conditions defined as follows: either

$$u = 0 \quad \text{and} \quad b\cdot n = 0,\ (\nabla\times b)\times n = 0 \tag{1.3}$$

or

$$u\cdot n = 0,\ (\nabla\times u)\times n = 0 \quad \text{and} \quad b\cdot n = 0,\ (\nabla\times b)\times n = 0. \tag{1.4}$$

Here, n is the outward unit normal vector along the boundary $\partial\Omega$. The initial conditions satisfy the compatibility condition, i.e. $\nabla\cdot u_0(x) = 0$ and $\nabla\cdot b_0(x) = 0$. The notion of weak solutions will be introduced in definition 2.2.

The MHD equations describe the dynamics of the interaction of electrically conducting fluids and electromagnetic forces, for example, plasma and liquid metals (see, for example, [7]).

Definition 1.1. A weak solution pair (u, b) of the 3D MHD equations (1.1), (1.2) with boundary conditions (1.3) or (1.4) is regular in Q_T provided that $\|u\|_{L^\infty(Q_T)} + \|b\|_{L^\infty(Q_T)} < \infty$.

In this paper, we list only some results relevant to our concerns. It has been shown that global weak solutions for the MHD equations exist in finite energy space (see [8]) and classical solutions can exist locally in time in a 3D space. Namely, the weak solutions exist globally in time (see [8]); however, as shown in [16], if weak solutions (u, b) are additionally in $L^\infty(0, T; H^1(\mathbb{R}^3))$, they become regular in the 3D case. In view of the regularity conditions in Lorentz space, He and Wang proved in [11] that a weak solution pair (u, b) becomes regular in the presence of a certain type of integral condition, typically referred to as Serrin's condition, namely, $u \in L^{q,\infty}(0, T; L_x^{p,\infty}(\mathbb{R}^3))$ with $3/p + 2/q \leqslant 1$ and $p > 3$, or $\nabla u \in L^{q,\infty}(0, T; L_x^{p,\infty}(\mathbb{R}^3))$ with $3/p + 2/q \leqslant 2$ and $p > 3/2$. These results are restricted to the problem for the whole space.

Our study is motivated by the work of He and Wang [11], that is, we obtain the regularity conditions for a weak solution to the 3D MHD equations (1.1)–(1.4) in a 3D bounded domain. In particular, for bounded domains, the difficulty lies in treating the pressure. For this, we consider the vorticity equations for the 3D MHD equations to avoid the estimate of terms containing the pressure term. Our proof is based on *a priori* estimates for the vorticities $w := \nabla \times u$ and $j := \nabla \times b$. On the other hand, to deal with the pressure, we use the Stokes estimate for the Stokes systems (see lemma A.1).

Our main results reads as follows.

Theorem 1.2. *Suppose that (u, b) is a weak solution of (1.1), (1.2) with initial condition $u_0, b_0 \in H^1(\Omega)$ and boundary condition (1.3) or (1.4). If the velocity u satisfies*

$$u \in L^q((0, T); L^{p,\infty}(\Omega)), \quad \frac{3}{p} + \frac{2}{q} = 1, \ 3 < p \leqslant \infty, \tag{1.5}$$

then (u, b) is regular in $\overline{Q_T}$.

The proof of this part is almost same as that in [12, theorem 1]. Furthermore, we modify it by replacing the Sobolev norm for a velocity u by the Lorentz norm via the interpolation theorem. For the convenience of the reader, we include the proof in the appendix.

Theorem 1.3. *Suppose that (u, b) is a weak solution of (1.1), (1.2) with the initial condition $u_0, b_0 \in H^1(\Omega)$ and boundary condition (1.4). If the vorticity w satisfies*

$$w \in L^q((0, T); L^{p,\infty}(\Omega)), \quad \frac{3}{p} + \frac{2}{q} = 2, \ \frac{3}{2} < p \leqslant \infty,$$

then (u, b) is regular in $\overline{Q_T}$.

REMARK 1.4. In theorems 1.2 and 1.3 we do not obtain results like

$$u \in L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{q} \leqslant 1,\ p>3,$$

or

$$w \in L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{q} \leqslant 2,\ p>\frac{3}{2}.$$

Unfortunately, the argument in [6, 14] has not worked in our proof. However, in case of the whole space, we can get the result above. Its proof is contained in a forthcoming paper because of technical reasons to do with the results.

This paper is organized as follows. In § 2 we recall the notion of weak solutions and review some known results. In § 3 we present the proof of theorem 1.3. Lastly, in the appendix we give a proof of the regularity condition for the vorticity in the 3D MHD equations in Sobolev space via a different method to that in [4, theorem 1.4].

2. Preliminaries

In this section we collect notation and definitions used throughout this paper. We also recall some lemmas that are useful for our analysis. Let Ω be a bounded open domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $n \geqslant 3$, and let $I=(0,T)$ be a finite time-interval. For $1 \leqslant q \leqslant \infty$, $W^{k,q}(\Omega)$ indicates the usual Sobolev space with standard norm $\|\cdot\|_{k,q}$, i.e. $W^{k,q}(\Omega)=\{u \in L^q(\Omega)\colon D^\alpha u \in L^q(\Omega),\ 0 \leqslant |\alpha| \leqslant k\}$. In the case in which $q=2$, we write $W^{k,q}(\Omega)$ as $H^k(\Omega)$. Also, we denote $\{f \in L^2(\Omega)\colon \nabla\cdot f=0\}$ by $L^2_\sigma(\Omega)$. All generic constants will be denoted by C, which may vary from line to line.

2.1. Lorentz space

Let $m(\varphi,t)$ be the Lebesgue measure of the set $\{x \in \Omega\colon |\varphi(x)|>t\}$, i.e.

$$m(\varphi,t) := m\{x \in \Omega\colon |\varphi(x)|>t\}.$$

We denote the Lorentz space by $L^{p,q}(\Omega)$ with $1 \leqslant p$, $q \leqslant \infty$ and with the norm [18]

$$\|\varphi\|_{L^{p,q}(\Omega)} = \begin{cases} \left(\displaystyle\int_0^\infty t^q (m(\varphi,t))^{q/p} \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty & \text{for } 1 \leqslant q < \infty, \\ \sup\limits_{t \geqslant 0}\{t(m(\varphi,t))^{1/p}\} < \infty & \text{for } q=\infty. \end{cases} \tag{2.1}$$

The Lorentz space $L^{p,\infty}(\Omega)$ is also called the weak $L^p(\Omega)$ space, with norm equivalent to

$$\|f\|_{L^{q,\infty}(\Omega)} = \sup_{0<|\Omega|<\infty} |\Omega|^{1/q-1} \int_\Omega |f(x)|\,\mathrm{d}x. \tag{2.2}$$

Following [18], the Lorentz space $L^{p,q}(\Omega)$ may be defined by real interpolation methods as

$$L^{p,q}(\Omega) = (L^{p_1}(\Omega), L^{p_2}(\Omega))_{\alpha,q}, \tag{2.3}$$

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with

$$\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \quad 1 \leqslant p_1 < p < p_2 \leqslant \infty.$$

From the interpolation method above, we note that

$$L^{2p/(p-1),2}(\Omega) = (L^2(\Omega), L^6(\Omega))_{3/2p,2}. \tag{2.4}$$

We also need the Hölder inequality in Lorentz spaces (see [15]).

LEMMA 2.1. *Assume that* $1 \leqslant p_1,\ p_2 \leqslant \infty,\ 1 \leqslant q_1,\ q_2 \leqslant \infty$ *and* $u \in L^{p_1,q_1}(\Omega)$, $v \in L^{p_2,q_2}(\Omega)$. *Then* $uv \in L^{p_3,q_3}(\Omega)$ *with*

$$\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q_3} \leqslant \frac{1}{q_1} + \frac{1}{q_2},$$

and the inequality

$$\|uv\|_{L^{p_3,q_3}(\Omega)} \leqslant C\|u\|_{L^{p_1,q_1}(\Omega)}\|v\|_{L^{p_2,q_2}(\Omega)} \tag{2.5}$$

is valid.

We recall first the definition of weak solutions.

DEFINITION 2.2 (weak solutions). Let $u_0, b_0 \in L^2_\sigma(\Omega)$. We say that (u, b) is a weak solution of (1.1) if u and b satisfy the following.

(i) We have

$$u \in L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^1(\Omega)),$$
$$b \in L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^1(\Omega)).$$

(ii) (u, b) satisfies (1.1) in the sense of distributions; that is,

$$\int_0^T \int_\Omega \left(\frac{\partial \phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\right) u \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0\phi(x,0)\,\mathrm{d}x = \int_0^T \int_\Omega (b\cdot\nabla)\phi b\,\mathrm{d}x\,\mathrm{d}t,$$
$$\int_0^T \int_\Omega \left(\frac{\partial \phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\right) b \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega b_0\phi(x,0)\,\mathrm{d}x = \int_0^T \int_\Omega (b\cdot\nabla)\phi u\,\mathrm{d}x\,\mathrm{d}t$$

for all $\phi \in C_0^\infty(\Omega \times [0,T))$ with $\operatorname{div}\phi = 0$, and

$$\int_\Omega u\cdot\nabla\psi\,\mathrm{d}x = 0, \qquad \int_\Omega b\cdot\nabla\psi\,\mathrm{d}x = 0$$

for every $\psi \in C_0^\infty(\Omega)$.

2.2. Useful inequalities

Next, we recall a Gagliardo–Nirenberg inequality (see, for example, [13, theorem 2.2]).

LEMMA 2.3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, $n \geqslant 1$, *and let* $\partial\Omega$ *be locally Lipschitz. Assume that* $u \in W^{1,p}(\Omega)$ *and* $\int_\Omega u\,\mathrm{d}x = 0$. *For every fixed number* $p, q \geqslant 1$ *and* $r \geqslant 1$, *there exists a constant* $C = C(n,p,r,\Omega)$ *such that*

$$\|u\|_{L^q(\Omega)} \leqslant C\|\nabla u\|^\theta_{L^p(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}, \tag{2.6}$$

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where $p, q \geqslant 1$, and

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1}.$$

Next, we recall an estimate regarding to the gradient vector (see [19]).

LEMMA 2.4. *Let Ω be a bounded domain in $\mathbb{R}^3$. Suppose that $u \in W^{1,p}(\Omega)$ for some $1 < p < \infty$ with $u \cdot n = 0$ on $\partial\Omega$. Then the following estimate is satisfied:*

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C(\|\nabla \cdot u\|_{L^p(\Omega)} + \|\nabla \times u\|_{L^p(\Omega)}).$$

Next, we recall estimates regarding smooth vector fields under the slip boundary condition (see [2, lemma 2.2], [3, theorem 2.1] and [5, lemmas 2.1 and 2.2]).

LEMMA 2.5. *Let Ω be a smooth domain in $\mathbb{R}^3$. Then, for each $q > 1$ and regular smooth vector fields f, the following hold.*

(a) *We have*

$$-\int_\Omega \Delta f \cdot f|f|^{q-2}\,\mathrm{d}x = \frac{1}{2}\int_\Omega |f|^{q-2}|\nabla f|^2\,\mathrm{d}x + \frac{4(q-2)}{q^2}\int_\Omega |\nabla|f|^{q/2}|^2\,\mathrm{d}x - \int_{\partial\Omega} |f|^{q-2}(n\cdot\nabla f)f\cdot f\,\mathrm{d}S.$$

(b) *Moreover, using the vector identity,*

$$(n\cdot\nabla)f\cdot f = (f\cdot\nabla)f\cdot n + ((\nabla\times f)\times n)\cdot f,$$

we can also deduce that

$$\begin{aligned}-\int_\Omega \Delta f \cdot f|f|^{q-2}\,\mathrm{d}x = \frac{1}{2}\int_\Omega |f|^{q-2}|\nabla f|^2\,\mathrm{d}x &+ \frac{4(q-2)}{p^2}\int_\Omega |\nabla|f|^{q/2}|^2\,\mathrm{d}x \\ &- \int_{\partial\Omega} |f|^{p-2}(f\cdot\nabla)f\cdot n\,\mathrm{d}S \\ &- \int_{\partial\Omega} |f|^{p-2}((\nabla\times f)\times n)f\,\mathrm{d}S.\end{aligned}$$

LEMMA 2.6. *Assume that u is regular enough and satisfies the boundary condition (1.4) on $\partial\Omega$. Then the following identity for $w = \nabla\times u$ holds true:*

$$-\frac{\partial w}{\partial n}\cdot w = (\varepsilon_{1jk}\varepsilon_{1\beta\gamma} + \varepsilon_{2jk}\varepsilon_{2\beta\gamma} + \varepsilon_{3jk}\varepsilon_{3\beta\gamma})w_j w_\beta \partial_k n_\gamma \quad \text{on } \partial\Omega,$$

where ε_{ijk} denotes the totally antisymmetric tensor such that $(a\times b) = \varepsilon_{1jk}a_j b_k$. In particular,

$$\int_\Omega \Delta w\cdot w\,\mathrm{d}x \leqslant -\int_\Omega |\nabla w|^2\,\mathrm{d}x + C\int_{\partial\Omega} |w|^2\,\mathrm{d}x.$$

3. Proof of theorem 1.3

As mentioned in the introduction, to eliminate the pressure term we consider the vorticity equation

$$\left.\begin{aligned} w_t - \Delta w + (u\cdot\nabla)w - (u\cdot\nabla)w - (b\cdot\nabla)j + (j\cdot\nabla)b = 0,\\ j_t - \Delta j + (u\cdot\nabla)j - (j\cdot\nabla)u - (b\cdot\nabla)w + (w\cdot\nabla)b = 2F(\nabla b,\nabla u),\end{aligned}\right\} \tag{3.1}$$

where

$$F(\nabla b,\nabla u) = \begin{pmatrix} \partial_2 b\cdot\partial_3 u - \partial_3 b\cdot\partial_2 u\\ \partial_3 b\cdot\partial_1 u - \partial_1 b\cdot\partial_3 u\\ \partial_1 b\cdot\partial_2 u - \partial_2 b\cdot\partial_1 u\end{pmatrix}, \qquad w = \nabla\times u, \qquad j = \nabla\times b.$$

Proof of theorem 1.3. First, we prove theorem 1.3 under the condition (1.5). Multiplying the first equation of (3.1) by w and the second equation of (3.1) by j, integrating over Ω and adding them, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|w|^2+|j|^2) + \int_\Omega(|\nabla w|^2+|\nabla j|^2) &\\ \leqslant \int_\Omega |w|\,|\nabla u|\,|w| + \int_{\partial\Omega}\left|\frac{\partial w}{\partial n}\cdot w\right| + \int_\Omega |j|\,|\nabla u|\,|j| + \int_{\partial\Omega}\left|\frac{\partial j}{\partial n}\cdot j\right| &\\ + \int_\Omega |(j\cdot\nabla)b|\,|w| + \int_\Omega |(w\cdot\nabla b)|\,|j| + 2\int_\Omega |F(\nabla u,\nabla b)|\,|j| &\\ := \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4 + \mathrm{II}_5 + \mathrm{II}_6 + \mathrm{II}_7, & \end{aligned}$$

where we use lemmas 2.5 and 2.6. Using Hölder's inequality, lemmas 2.1, 2.3 and 2.4, the interpolation inequality for Lorentz space, and Young's inequality, the terms II_1 and II_7 are estimated as follows:

$$\begin{aligned} \mathrm{II}_1 &\leqslant \|\nabla u\|_{L^{p,\infty}(\Omega)}\|w\|^2_{L^{2p/(p-1),2}(\Omega)}\\ &\leqslant C\|\nabla u\|_{L^{p,\infty}(\Omega)}\|w\|^{2-3/p}_{L^2(\Omega)}\|\nabla w\|^{3/p}_{L^2(\Omega)}\\ &\leqslant C\|\nabla u\|^{2p/(2p-3)}_{L^{p,\infty}(\Omega)}\|w\|^2_{L^2(\Omega)} + \tfrac{1}{512}\|\nabla w\|^2_{L^2(\Omega)}\\ &\leqslant C\|w\|^{2p/(2p-3)}_{L^{p,\infty}(\Omega)}\|w\|^2_{L^2(\Omega)} + \tfrac{1}{512}\|\nabla w\|^2_{L^2(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \mathrm{II}_7 &\leqslant \|\nabla u\|_{L^{p,\infty}(\Omega)}\|j\|_{L^{2p/(p-1),2}(\Omega)}\|\nabla b\|_{L^{2p/(p-1),2}(\Omega)}\\ &\leqslant \|\nabla u\|^{2p/(2p-3)}_{L^{p,\infty}(\Omega)}\|j\|_{L^{2p/(p-1),2}(\Omega)}\|j\|_{L^{2p/(p-1),2}(\Omega)}\\ &\leqslant C\|\nabla u\|_{L^{p,\infty}(\Omega)}\|j\|^{2-3/p}_{L^2(\Omega)}\|\nabla j\|^{3/p}_{L^2(\Omega)}\\ &\leqslant C\|\nabla u\|^{2p/(2p-3)}_{L^{p,\infty}(\Omega)}\|j\|^2_{L^2(\Omega)} + \tfrac{1}{512}\|\nabla j\|^2_{L^2(\Omega)}\\ &\leqslant C\|w\|^{2p/(2p-3)}_{L^{p,\infty}(\Omega)}\|j\|^2_{L^2(\Omega)} + \tfrac{1}{512}\|\nabla j\|^2_{L^2(\Omega)}. \end{aligned}$$

Through a similar method to that used for II_1 and II_7, we obtain

$$\begin{aligned}&\mathrm{II}_1 + \mathrm{II}_3 + \mathrm{II}_5 + \mathrm{II}_6 + \mathrm{II}_7 \\ &\qquad \leqslant \|\nabla u\|_{L^{p,\infty}(\Omega)}^{2p/(2p-3)} (\|w\|_{L^2(\Omega)}^2 + \|j\|_{L^2(\Omega)}^2) + \tfrac{1}{2}(\|\nabla w\|_{L^2(\Omega)}^2 + \|\nabla j\|_{L^2(\Omega)}^2).\end{aligned}$$

Next, we can easily estimate II_2. Indeed, we use the trace theorem (see, for example, [9, pp. 257–258]) and smoothness of the boundary to obtain

$$\mathrm{II}_2 \leqslant \int_{\partial\Omega} \left| \frac{\partial w}{\partial n} \cdot w \right| \leqslant C \int_{\Omega} |w|^2. \tag{3.2}$$

Similarly, we estimate II_4 as follows:

$$\mathrm{II}_4 \leqslant \int_{\partial\Omega} \left| \frac{\partial j}{\partial n} \cdot j \right| \leqslant C \int_{\Omega} |j|^2. \tag{3.3}$$

Summing up the estimates $\mathrm{II}_1, \mathrm{II}_2, \ldots, \mathrm{II}_7$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} (|w|^2 + |j|^2) + \frac{1}{2} \int_{\Omega} (|\nabla w|^2 + |\nabla j|^2) \leqslant C(1 + \|w\|_{L^{p,\infty}(\Omega)}^{2p/(2p-3)}) \int_{\Omega} (|w|^2 + |j|^2). \tag{3.4}$$

Applying Gronwall's inequality to (3.4), we have the desired result. □

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Appendix A.

A.1. Stokes system

We consider the following Stokes system, which is the linearized Navier–Stokes equations,

$$v_t - \Delta v + \nabla p = f, \quad \operatorname{div} v = 0, \quad \text{in } Q_T := \Omega \times (0, T) \tag{A 1}$$

with initial data $v(x, 0) = v_0(x)$. As in (1.3) and (1.4), the boundary data of v are again assumed to be either no-slip or slip conditions, namely,

$$v(x, t) = 0, \quad x \in \partial\Omega, \tag{A 2}$$

or

$$v \cdot n = 0, \quad (\nabla \times v) \times n = 0, \quad x \in \partial\Omega. \tag{A 3}$$

Next, we recall maximal estimates of the Stokes system in terms of mixed norms (see [10, theorem 5.1] and [17, theorem 1.2] for no-slip and slip boundary cases, respectively).

Lemma A.1. *Let $1 < l, m < \infty$. Suppose that $f \in L^{l,m}_{x,t}(Q_T)$ and $v_0 \in D^{1-1/m,m}_l$, where $D^{1-1/m,m}_l$ is a Banach space with the norm (see, for example, [10])*

$$D^{1-1/m,m}_l(\Omega) := \left\{ w \in L^l_\sigma(\Omega); \quad \|w\|_{D^{1-1/m,m}_l} = \|w\|_{L^l} + \left(\int_0^\infty \|t^{1/m} A_l \mathrm{e}^{-tA_l} w\|^m_{L^l} \frac{\mathrm{d}t}{t} \right)^{1/m} < \infty \right\},$$

where A_l is the Stokes operator (see [10, 17] for the details). If (v, p) is the solution of the Stokes system (A 1) satisfying one of the boundary conditions (A 2) or (A 3), then the following estimate is satisfied:

$$\|v_t\|_{L^{l,m}_{x,t}(Q_T)} + \|\nabla^2 v\|_{L^{l,m}_{x,t}(Q_T)} + \|\nabla p\|_{L^{l,m}_{x,t}(Q_T)} \leqslant C\|f\|_{L^{l,m}_{x,t}(Q_T)} + \|v_0\|_{D^{1-1/m,m}_l(\Omega)}. \tag{A 4}$$

Since

$$D^{1-1/m,m}_l(\Omega) := [L_l(\Omega), W^{1,l}((\Omega))]_{1-1/m,m},$$

we note that $\|v_0\|_{D^{1-1/m,m}_l(\Omega)} \leqslant \|v_0\|_{W^{1,l}(\Omega)}$ (see, for example, [1, ch. 7]) and, therefore, $\|v_0\|_{D^{1-1/m,m}_l(\Omega)}$ in (A 4) can be replaced by $\|v_0\|_{W^{1,l}(\Omega)}$.

A.2. Proof of theorem 1.2

In [12, proposition 1], we are shown the following result.

Let $1 \leqslant q < \infty$ and introduce a function space X^q_t defined as follows:

$$X^q_t = \left\{ f\colon \Omega \times [0, t) \to \mathbb{R}^3 \;\middle|\; \|f\|_{X^q_t} := \limsup_{\tau < t} \|f(\tau)\|_{W^{1,q}(\Omega)} + \|f\|_{L^q((0,t);W^{2,q}(\Omega))} < \infty \right\}.$$

Proposition A.2 (local existence). *Let $3 < q < \infty$ and let Ω be either a bounded domain in $\mathbb{R}^3$ or a half-space $\mathbb{R}^3_+$. There exists $T_{\max} \in (0, \infty]$, the maximal time of existence, such that if $u_0, b_0 \in H^1(\Omega) \cap W^{1,q}(\Omega)$, then there is a unique solution pair (u, b) in (1.1) with boundary conditions (1.3) or (1.4) satisfying $u, b \in X^q_t$ for any $t < T_{\max}$.*

Proof of theorem 1.2. We argue by contradiction. Suppose that T^* is the first time of singularity with $T^* \leqslant T$. Then u and b must satisfy, for any $\delta > 0$,

$$\limsup_{t \nearrow T^*} (\|u(\cdot, t)\|^4_{L^4_x} + \|b(\cdot, t)\|^4_{L^4_x}) + \lim_{t \nearrow T^*} \left(\int_{T^*-\delta}^t \||\nabla u(\cdot, \tau)||u(\cdot, \tau)|\|^2_{L^2_x} + \||\nabla b(\cdot, \tau)||b(\cdot, \tau)|\|^2_{L^2_x} \right) = \infty. \tag{A 5}$$

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Multiplying the first equation of (1.1) by $|u|^2u$, the equation of the magnetic field by $|b|^2b$, integrating over Ω and summing the above estimates, we have

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u|^4+|b|^4)+\int_\Omega(|\nabla u|^2|u|^2+|\nabla b|^2|b|^2)+\frac{1}{2}\int_\Omega(|\nabla|u|^2|^2+|\nabla|b|^2|^2)\\ =-\int_\Omega\nabla\pi|u|^2u-\int_\Omega b\nabla(|u|^2u)b-\int_\Omega b\nabla(|b|^2b)u\\ +\sum_{i,j=1}^3\int_{\partial\Omega}u_{j,x_i}u_j|u|^2n_i+\sum_{i,j=1}^3\int_{\partial\Omega}b_{j,x_i}b_j|b|^2n_i.\end{aligned}\tag{A 6}$$

Let ε be a sufficiently small positive number, which will be specified later. Integrating (A 6) in time over $(T^*-\varepsilon,\tau)$ for any τ with $T^*-\varepsilon<\tau<T^*$, we observe that

$$\begin{aligned}&\frac{1}{4}\int_\Omega(|u(\cdot,\tau)|^4\,\mathrm{d}x+|b(\cdot,\tau)|^4)\,\mathrm{d}x-\frac{1}{4}\int_\Omega(|u(\cdot,T^*-\varepsilon)|^4+|b(\cdot,T^*-\varepsilon)|^4)\,\mathrm{d}x\\ &\quad+\int_{T^*-\varepsilon}^\tau\int_\Omega|\nabla u|^2|u|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{T^*-\varepsilon}^\tau\int_\Omega|\nabla b|^2|b|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac{1}{2}\int_{T^*-\varepsilon}^\tau\int_\Omega|\nabla|u|^2|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_{T^*-\varepsilon}^\tau\int_\Omega|\nabla|b|^2|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant\int_{T^*-\varepsilon}^\tau\int_\Omega|\nabla\pi|\,|u|^2|u|\,\mathrm{d}x\,\mathrm{d}t+\int_{T^*-\varepsilon}^\tau\int_\Omega|b|^2|u|\,|u|\,|\nabla u|\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\quad+\int_{T^*-\varepsilon}^\tau\int_\Omega|b|^2|u|\,|b|\,|\nabla b|\,\mathrm{d}x\,\mathrm{d}t+\int_{T^*-\varepsilon}^\tau\int_\Omega|u|^3|\nabla u|\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\quad+\int_{T^*-\varepsilon}^\tau\int_\Omega|b|^3|\nabla b|\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad:=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}$$

For convenience, we denote $\Omega\times(T^*-\varepsilon,\tau)$ by Q_τ. Using Hölder's inequality, the first term I can be estimated as

$$\begin{aligned}\mathrm{I}&\leqslant\int_{T^*-\varepsilon}^\tau\|\nabla\pi\|_{L^2_x}\|u^2\|_{L_x^{2p/(p-2),2}}\|u\|_{L_x^{p,\infty}}\\ &\leqslant C\int_{T^*-\varepsilon}^\tau\|\nabla\pi\|_{L^2_x}\|u^2\|_{L^2_x}^{\theta}\|\nabla|u^2|\|_{L^2_x}^{1-\theta}\|u\|_{L_x^{p,\infty}}\\ &\leqslant C\|\nabla\pi\|_{L^2(Q_\tau)}\|\nabla|u|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q},\end{aligned}$$

where $\theta=2/q$. For convenience, we denote

$$\|u(\cdot,T^*-\varepsilon)\|_{W^{1,2}(\Omega)}$$

by $\mathcal{C}_\varepsilon$. Using the estimate (A 4), we continue to estimate I as

$$\begin{aligned}\mathrm{I}&\leqslant C(\|(u\cdot\nabla)u\|_{L^2(Q_\tau)}+\|(b\cdot\nabla)b\|_{L^2(Q_\tau)}+\mathcal{C}_\varepsilon)\\ &\quad\times\|\nabla|u|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}\end{aligned}$$

$$\leqslant C\||u||\nabla u|\|_{L^2(Q_\tau)}^{2(q-1)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C\||b||\nabla b|\|_{L^2(Q_\tau)}\|\nabla|u|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C\mathcal{C}_\varepsilon\sum_{k=1}^{3}\|\nabla|u|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}. \qquad \text{(A 7)}$$

Next we estimate II. Following similar computations to those for I, we get

$$\mathrm{II} \leqslant \int_{T^*-\varepsilon}^{\tau}\int_\Omega |b|^2|u||u||\nabla u|$$
$$\leqslant C\sum_{k=1}^{3}||u||\nabla u||_{L^2(Q_\tau)}\|\nabla|b|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|b(\cdot,t)\|_{L_x^4}^{4/q}. \qquad \text{(A 8)}$$

In the same manner, we estimate III:

$$\mathrm{III} \leqslant C\sum_{k=1}^{3}||b||\nabla b||_{L^2(Q_\tau)}\|\nabla|b|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|b(\cdot,t)\|_{L_x^4}^{4/q}. \qquad \text{(A 9)}$$

For IV and V, using Hölder's inequality, we have

$$\mathrm{IV}+\mathrm{V} \leqslant C\varepsilon^{1/2}(||u||\nabla u||_{L^2(Q_\tau)}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^2$$
$$+ ||b|\nabla b|||_{L^2(Q_\tau)}\sup_{T^*-\varepsilon<t<\tau}\|b(\cdot,t)\|_{L_x^4}^2).$$

Summing up (A 7)–(A 9) and using Young's inequality, we obtain

$$\frac{1}{4}\int_\Omega(|u(\cdot,\tau)|^4\,\mathrm{d}x+|b(\cdot,\tau)|^4)\,\mathrm{d}x-\frac{1}{4}\int_\Omega(|u(\cdot,T^*-\varepsilon)|^4+|b(\cdot,T^*-\varepsilon)|^4)\,\mathrm{d}x$$
$$+\int_{T^*-\varepsilon}^{\tau}\int_\Omega|\nabla u|^2|u|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{T^*-\varepsilon}^{\tau}\int_\Omega|\nabla b|^2|b|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{1}{2}\int_{T^*-\varepsilon}^{\tau}\int_\Omega|\nabla|u|^2|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_{T^*-\varepsilon}^{\tau}\int_\Omega|\nabla|b|^2|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\||u||\nabla u|\|_{L^2(Q_\tau)}^{2(q-1)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C\||b||\nabla b|\|_{L^2(Q_\tau)}\|\nabla|u|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C\mathcal{C}_\varepsilon\|\nabla|u|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C||u||\nabla u||_{L^2(Q_\tau)}\|\nabla|b|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|b(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C||b||\nabla b||_{L^2(Q_\tau)}\|\nabla|b|^2\|_{L^2(Q_\tau)}^{(q-2)/q}\|\|u\|_{L_x^{p,\infty}(\Omega)}\|_{L_t^q}\sup_{T^*-\varepsilon<t<\tau}\|b(\cdot,t)\|_{L_x^4}^{4/q}$$
$$+ C\varepsilon^{1/2}(||u||\nabla u||_{L^2(Q_\tau)}\sup_{T^*-\varepsilon<t<\tau}\|u(\cdot,t)\|_{L_x^4}^2$$

$$
\begin{aligned}
&+ \||b|\nabla b|\|_{L^2(Q_\tau)} \sup_{T^*-\varepsilon<t<\tau} \|b(\cdot,t)\|^2_{L^4_x}) \\
&\leqslant \tfrac{1}{2}\||u||\nabla u|\|^2_{L^2(Q_\tau)} + \tfrac{1}{2}\||b||\nabla b|\|^2_{L^2(Q_\tau)} + CC^2_\varepsilon \\
&\quad + C(\|\|u\|_{L^{p,\infty}_x(\Omega)}\|_{L^q_t} + \varepsilon)\Big(\sup_{T^*-\varepsilon<t<\tau} \|u(\cdot,t)\|^4_{L^4_x} + \sup_{T^*-\varepsilon<t<\tau} \|b(\cdot,t)\|^4_{L^4_x}\Big).
\end{aligned}
$$

Since the above estimate holds for all t with $T^* - \varepsilon < t < \tau$, we obtain

$$
\begin{aligned}
&\sup_{T^*-\varepsilon<t<\tau} (\|u(\cdot,t)\|^4_{L^4_x} + \|b(\cdot,t)\|^4_{L^4_x}) + \int_{T^*-\varepsilon}^{\tau} \int_\Omega (|\nabla u|^2|u|^2| + |\nabla b|^2|b|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2}\int_{T^*-\varepsilon}^{\tau} \int_\Omega (|\nabla|u|^2|^2 + |\nabla|b|^2|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \int_\Omega |u(\cdot,T^*-\varepsilon)|^4\,\mathrm{d}x + \int_\Omega |b(\cdot,T^*-\varepsilon)|^4\,\mathrm{d}x + CC^2_\varepsilon \\
&\quad\quad + C(\|\|u\|_{L^{p,\infty}_x(\Omega)}\|_{L^q_t} + \varepsilon)\Big(\sup_{T^*-\varepsilon<t<\tau} \|u(\cdot,t)\|^4_{L^4_x} + \sup_{T^*-\varepsilon<t<\tau} \|b(\cdot,t)\|^4_{L^4_x}\Big).
\end{aligned}
$$

With sufficiently small ε so that $(\|\|u\|_{L^{p,\infty}_x(\Omega)}\|_{L^q_t} + \varepsilon) \leqslant 1/2C$ with a constant C in the above estimate, we have

$$
\begin{aligned}
&\|u(\cdot,t)\|^4_{L^{4,\infty}_{x,t}(Q_\tau)} + \|b(\cdot,t)\|^4_{L^{4,\infty}_{x,t}(Q_\tau)} + \tfrac{1}{2}\||\nabla u||u|\|^2_{L^2(Q_\tau)} \\
&\quad + \tfrac{1}{2}\||\nabla b||b|\|^2_{L^2(Q_\tau)} + \tfrac{1}{2}\|\nabla|u|^2\|^2_{L^2(Q_\tau)} + \tfrac{1}{2}\|\nabla|b|^2\|^2_{L^2(Q_\tau)} \\
&\quad\quad \leqslant 2(\|u(\cdot,T-\varepsilon)\|^4_{L^4_x(\Omega)} + \|b(\cdot,T-\varepsilon)\|^4_{L^4_x(\Omega)}) + CC^2_\varepsilon.
\end{aligned}
$$

For simplicity, we denote $\Omega \times (T^* - \varepsilon, T^*)$ by Q_ε. Since τ is arbitrary with $\tau < T^*$, we obtain

$$
\begin{aligned}
&\|u(\cdot,t)\|^4_{L^{4,\infty}_{x,t}(Q_\varepsilon)} + \|b(\cdot,t)\|^4_{L^{4,\infty}_{x,t}(Q_\varepsilon)} + \tfrac{1}{2}\||\nabla u||u|\|^2_{L^2(Q_\varepsilon)} \\
&\quad + \tfrac{1}{2}\||\nabla b||b|\|^2_{L^2(Q_\varepsilon)} + \tfrac{1}{2}\|\nabla|u|^2\|^2_{L^2(Q_\varepsilon)} + \tfrac{1}{2}\|\nabla|b|^2\|^2_{L^2(Q_\varepsilon)} \leqslant C,
\end{aligned}
$$

where C is a constant depending on $\|u(\cdot,T^*-\varepsilon)\|_{W^{1,2}(\Omega)}$. This is contrary to the hypothesis of (A 5). Therefore, T^* cannot be a maximal time of existence less than or equal to T. This completes the proof. □

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