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ASYMPTOTIC SHAPE AND THE SPEED OF PROPAGATION OF CONTINUOUS-TIME CONTINUOUS-SPACE BIRTH PROCESSES

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Abstract

We formulate and prove a shape theorem for a continuous-time continuous-space stochastic growth model under certain general conditions. Similar to the classical lattice growth models, the proof makes use of the subadditive ergodic theorem. A precise expression for the speed of propagation is given in the case of a truncated free-branching birth rate.

Keywords: Shape theorem; spatial birth process; growth model

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1. Introduction

Shape theorems have a long history. Richardson [21] proved the shape theorem for the Eden model. Since then, shape theorems have been proven in various settings, most notably for first-passage percolation and permanent and nonpermanent growth models. Garet and Marchand [11] not only proved a shape theorem for the contact process in a random environment, but also provided a overview of existing results.

Most of the literature is devoted to discrete-space models. A continuous-space first-passage percolation model was analyzed by Howard and Newman [13]; see also the references therein. A shape theorem for a continuous-space growth model was proven by Deijfen [5]; see also Gouéré and Marchand [12]. Our model is naturally connected to that model; see the end of Section 2.

The questions addressed in this paper are motivated not only by probability theory but also by studies in natural sciences. In particular, we can mention a demand to incorporate spatial information in the description and analysis of ecology, bacteria populations, tumor growth, epidemiology, and phylogenetics among others; see, e.g. [24]–[27]. Authors often emphasize that it is preferable to use the continuous-space spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ as the basic or 'geographic'

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space; see, e.g. [26]. More on connections between theoretical studies and applications can be found in [20].

The paper is organized as follows. In Section 2 we describe the model and formulate our results, which are proven in Sections 3 and 4. Technical results, in particular on the construction of the process, are collected in Section 5. In Section 6 we present some further conjectures about the models treated in this paper and related models.

2. The model, assumptions, and results

We consider a growth model represented by a continuous-time continuous-space Markov birth process. Let Γ_0 be the collection of finite subsets of $\mathbb{R}^d$,

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of elements in η. Note that Γ_0 is also called the *configuration space* or the *space of finite configurations*.

The evolution of the spatial birth process on $\mathbb{R}^d$ admits the following description. Let $\mathcal{B}(X)$ be the Borel σ-algebra on the Polish space X. If the system is in state $\eta \in \Gamma_0$ at time t then the probability that a new particle appears (a 'birth') in a bounded set $B \in \mathcal{B}(\mathbb{R}^d)$ over time interval $[t;\ t + \Delta t]$ is

$$\Delta t \int_B b(x, \eta)\, \mathrm{d}x + o(\Delta t),$$

and with probability 1 no two births happen simultaneously. Here $b \colon \mathbb{R}^d \times \Gamma_0 \to \mathbb{R}_+$ is some function which is called the *birth rate*. Using a slightly different terminology, we can say that the rate at which a birth occurs in B is $\int_B b(x, \eta)\, \mathrm{d}x$. We note that it is conventional to call the function b the 'birth rate', even though it is not a rate in the usual sense (as in, e.g. 'the Poisson process (N_t) has unit jumps at rate 1 meaning that $\mathbb{P}\{N_{t+\Delta t} - N_t = 1\}/\Delta t = 1$ as $\Delta t \to 0$') but rather a version of the Radon–Nikodym derivative of the rate with respect to the Lebesgue measure.

Remark 2.1. We characterize the birth mechanism by the birth rate $b(x, \eta)$ at each spatial position. Often the birth mechanism is given in terms of contributions of individual particles: a particle at y, $y \in \eta$, gives a birth at x at rate $c(x, y, \eta)$ (often $c(x, y, \eta) = \gamma(y, \eta)k(y, x)$, where $\gamma(y, \eta)$ is the proliferation rate of the particle at y, whereas the dispersion kernel $k(y, x)$ describes the distribution of the offspring); see, e.g. Fournier and Méléard [10]. As long as we are not interested in the induced genealogical structure, the two ways of describing the process are equivalent under our assumptions. Indeed, given c, we may set

$$b(x, \eta) = \sum_{y \in \eta} c(x, y, \eta),$$

or, conversely, given b, we may set

$$c(y, x, \eta) = \frac{g(x - y)}{\sum_{y \in \eta} g(x - y)} b(x, \eta),$$

where $g \colon \mathbb{R}^d \to (0, \infty)$ is a continuous function. Note that b is uniquely determined by c but not vice versa.

We equip Γ_0 with the σ-algebra $\mathcal{B}(\Gamma_0)$ induced by the sets

$$\text{ball}(\eta, r) = \{\zeta \in \Gamma_0 \mid |\eta| = |\zeta|, \text{ dist}(\eta, \zeta) < r\}, \qquad \eta \in \Gamma_0, \ r > 0, \tag{2.1}$$

where $\text{dist}(\eta, \zeta) = \min\{\sum_{i=1}^{|\eta|} |x_i - y_i| \mid \eta = \{x_1, \dots, x_{|\eta|}\}, \zeta = \{y_1, \dots, y_{|\eta|}\}\}$. For more detail on configuration spaces; see, e.g. [16] or [22]. In particular, the distance above coincides with the restriction to the space of finite configurations of the metric ρ used in [22], and the σ-algebra $\mathcal{B}(\Gamma_0)$ introduced above coincides with the σ-algebra from [16].

We say that a function $f : \mathbb{R}^d \to \mathbb{R}_+$ has an exponential moment if there exists $\theta > 0$ such that

$$\int_{\mathbb{R}^d} e^{\theta |x|} f(x) \, dx < \infty.$$

Of course, if f has an exponential moment then automatically $f \in L^1(\mathbb{R}^d)$.

Assumptions on b. We will need several assumptions on the birth rate b.

Condition 2.1. (Sublinear growth.) *The birth rate b is measurable and there exists a function* $a : \mathbb{R}^d \to \mathbb{R}_+$ *with an exponential moment such that*

$$b(x, \eta) \le \sum_{y \in \eta} a(x - y). \tag{2.2}$$

Condition 2.2. (Monotonicity.) *For all* $\eta \subset \zeta$,

$$b(x, \eta) \le b(x, \zeta), \qquad x \in \mathbb{R}^d.$$

The previous condition ensures attractiveness; see below.

Condition 2.3. (Rotation and translation invariance.) *The birth rate b is translation and rotation invariant: for every* $x, y \in \mathbb{R}^d$, $\eta \in \Gamma_0$, *and* $M \in \text{SO}(d)$,

$$b(x + y, \eta + y) = b(x, \eta), \qquad b(Mx, M\eta) = b(x, \eta).$$

Here $\text{SO}(d)$ *is the orthogonal group of linear isometries on* $\mathbb{R}^d$, *and for a Borel set* $B \in \mathcal{B}(\mathbb{R}^d)$ *and* $y \in \mathbb{R}^d$,

$$B + y = \{z \mid z = x + y, \ x \in B\}, \qquad MB = \{z \mid z = Mx, \ x \in B\}.$$

Condition 2.4. (Nondegeneracy.) *Let* $c_0, r > 0$ *such that*

$$b(x, \eta) \ge c_0 \quad \textit{wherever} \ \min_{y \in \eta} |x - y| \le r. \tag{2.3}$$

Remark 2.2. Condition 2.4 is used to ensure that the system grows at least linearly. The condition could be weakened, e.g. as follows.

- For some $r_2 > r_1 \ge 0$ and all $x, y \in \mathbb{R}^d$, $b(y, \{x\}) \ge c_0 \mathbf{1}_{\{r_1 \le |x-y| \le r_2\}}$.

Accordingly, the proof would become more intricate.

Remark 2.3. If b is as in (2.4) and f has polynomial tails, then the result of Durrett [6] suggests that we would expect a superlinear growth. This is in contrast with Deijfen's model, for which Gouéré and Marchand [11] gave a sharp condition on the distribution of the outbursts for linear or superlinear growth.

Examples of a birth rate are

$$b(x, \eta) = \lambda \sum_{y \in \eta} f(|x - y|) \tag{2.4}$$

and

$$b(x, \eta) = k \wedge \left(\lambda \sum_{y \in \eta} f(|x - y|) \right), \tag{2.5}$$

where λ, k are positive constants and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nonnegative, nonincreasing function with compact support.

We denote the underlying probability space by $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$. A random element A in Γ_0 is $\mathcal{A}$-measurable if

$$\Omega \ni \omega \to A = A(\omega) \in \Gamma_0$$

is a measurable map from the measure space $(\Omega, \mathcal{A})$ to $(\Gamma_0, \mathcal{B}(\Gamma_0))$. Such an A will also be called an $\mathcal{A}$-*measurable finite random* set.

The birth process will be obtained as a unique solution to a certain stochastic equation. The construction and the proofs of key properties, such as the rotation invariance and the strong Markov property, are given in Section 6. We place the construction toward the end because it is rather technical and the methods used there do not shed much light on the ideas of the proofs of our main results. Denote by $(\eta_t^{s,A})_{t \geq s} = (\eta_t^{s,A}, \, t \geq s)$ the process started at time $s \geq 0$ from an $\mathcal{S}_s$-measurable finite random set A. Here $(\mathcal{S}_s)_{s \geq 0}$ is a filtration of σ-algebras to which $(\eta_t^{s,A})_{t \geq s}$ is adapted; it is introduced after (5.3). Furthermore, $(\eta_t^{s,A})_{t \geq s}$ is a strong Markov process with respect to $(\mathcal{S}_s)_{s \geq 0}$; see Proposition 5.2.

The construction method we use has the advantage that the stochastic equation approach resembles a graphical representation (see, e.g. [7] or [18]) in the fact that it preserves monotonicity: if $s \geq 0$ and almost surely (a.s.) $A \subset B$, A and B being $\mathcal{S}_s$-measurable finite random sets, then a.s.

$$\eta_t^{s,A} \subset \eta_t^{s,B}, \qquad t \geq s. \tag{2.6}$$

This property is proven in Lemma 5.1 and is often referred to as *attractiveness*.

The process started from a single particle at $\mathbf{0}$ at time zero will be denoted by $(\eta_t)_{t \geq 0}$; thus, $\eta_t = \eta_t^{0,\{\mathbf{0}\}}$. Let

$$\xi_t := \bigcup_{x \in \eta_t} B(x, r) \tag{2.7}$$

and, similarly,

$$\xi_t^{s,A} := \bigcup_{x \in \eta_t^{s,A}} B(x, r),$$

where $B(x, r)$ is the closed ball of radius r centered at x (recall that r appears in (2.3)).

The following theorem represents the main result of the paper.

Theorem 2.1. *There exists* $\mu > 0$ *such that, for all* $\varepsilon \in (0, 1)$, *a.s.*

$$(1 - \varepsilon) B(\mathbf{0}, \mu^{-1}) \subset \frac{\xi_t}{t} \subset (1 + \varepsilon) B(\mathbf{0}, \mu^{-1}) \quad \textit{for sufficiently large } t. \tag{2.8}$$

Proof. See Section 3. □

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Remark 2.4. We note that the statement of Theorem 2.1 does not depend on our choice for the radius in (2.7) to be r; we could just as well take any positive constant, e.g.

$$\bigcup_{x\in\eta_t} B(x, 1).$$

In particular, μ in (2.8) does not depend on r.

It is common to write the ball radius as the reciprocate μ^{-1}, probably because μ comes up in the proof as the limiting value of a certain sequence of random variables after applying the subadditive ergodic theorem; see, e.g. [5] or [7]. We decided to retain the tradition not only for historic reasons, but also because μ comes up as a certain limit in our proof too, even though we do not obtain μ directly from the subadditive ergodic theorem. The value μ^{-1} is called the *speed of propagation*. The subadditive ergodic theorem is a cornerstone in the majority of shape theorem proofs, and our proof relies on it.

Formal connection to Deijfen's model. The model introduced in [5] with deterministic outburst radius, that is, when in the notation of [5], the distribution F of the radii of the outburst balls is the Dirac measure: $F = \delta_R$ for some $R \geq 0$ can be identified with

$$\zeta_t^R = \bigcup_{x\in\eta_t} B(x, R)$$

for the birth process (η_t) with birth rate

$$b(x, \eta) = \mathbf{1}_{\{\text{there exists } y\in\eta\colon\, |x-y|\leq R\}}.$$

Explicit growth speed for a particular model. The precise evaluation of speed appears to be a difficult problem. For a general one-dimensional branching random walk, the speed of propagation was determined by Biggins [3]. An overview of related results for different classes of models can be found in [1].

We now provide the speed for a model with interaction.

Theorem 2.2. *Let $d = 1$ and*

$$b(x, \eta) = 2 \wedge \left(\sum_{y\in\eta} \mathbf{1}_{\{|x-y|\leq 1\}}\right).$$

Then the speed of propagation is given by

$$\mu^{-1} = \frac{144\ln(3) - 144\ln(2) - 40}{25} \approx 0.735\,48\ldots.$$

Proof. The proof can be found in Section 4. □

3. Proof of Theorem 2.1

Outline of the proof. The proof can roughly be divided into three parts. First, we show that the system grows no faster than linearly, which is the content of Proposition 3.1. The proof of Proposition 3.1 relies on Lemma 5.1, which allows a comparison of birth processes with different rates, and on the results on the spread of the supercritical branching random walk by Biggins [3].

Second, we show that the system grows at least linearly. Strictly speaking, in this part we give only exponential estimates on the probability of certain linearly growing balls not to be filled with the particles of our system (Lemma 3.3) as opposed to an a.s. statement about the

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entire trajectory as in Proposition 3.1. This is, however, sufficient for our purposes. The main ingredients here are exponential estimates for the Eden model (or first-passage percolation model), comparison of the Eden model with our process, and once again Lemma 5.1.

Finally, the most technical in our opinion, we actually prove the theorem using the previous two parts. We define a specially designed collection of stopping times $\{T_\lambda(x),\ x \in \mathbb{R}^d\}$ and $\{T_\lambda(x, y),\ x, y \in \mathbb{R}^d\}$, depending on an additional parameter $\lambda > 0$ (see (3.5) and (3.6)). The strong Markov property of (η_t) (Proposition 5.2 and Corollary 5.1) allows us to apply Liggett's subadditive ergodic theorem to show that, for any $x \in \mathbb{R}^d$, $(T_\lambda(tx))_{t \geq 0}$ grows linearly with t ((3.10) and Lemma 3.6). We then move on to prove that the limit $\lim_{t\to\infty} T_\lambda(tx)/t$ does not depend on x (Lemma 3.7) and is strictly positive (Lemma 3.8). The bulk of the final part of the proof of Theorem 2.1 is contained in Lemmas 3.10 and 3.11, where we show the necessary a.s. inclusions removing the dependence on λ along the way.

Proposition 3.1. *There exists* $C_{\text{upb}} > 0$ *such that, a.s. for large* t*,*

$$\eta_t \subset B(\mathbf{0}, C_{\text{upb}} t). \tag{3.1}$$

Remark 3.1. The index 'upb' hints on 'upper bound'.

Proof of Proposition 3.1. It is sufficient to show that, for $\boldsymbol{e} = (1, 0, \ldots, 0) \in \mathbb{R}^d$, there exists $C > 0$ such that, a.s. for large t,

$$\max\{\langle x, \boldsymbol{e}\rangle : x \in \eta_t\} \subset Ct. \tag{3.2}$$

Indeed, if (3.2) holds then, by Proposition 5.1, replacing $\boldsymbol{e}$ with any other unit vector along any of the $2d$ directions in $\mathbb{R}^d$, (3.1) also holds.

For $z \in \mathbb{R}$, $y = (y_1, \ldots, y_{\text{d}-1}) \in \mathbb{R}^{\text{d}-1}$ we define $z \circ y$ to be the concatenation $(z, y_1, \ldots, y_{\text{d}-1}) \in \mathbb{R}^d$. In this proof we denote by $(\bar{\eta}_t)$ the birth process with $\bar{\eta}_0 = \eta_0$ and the birth rate given by the right-hand side of (2.2), namely

$$\bar{b}(x, \eta) = \sum_{y \in \eta} a(x - y). \tag{3.3}$$

Since $b(x, \eta) \leq \bar{b}(x, \eta)$, $x \in \mathbb{R}^d$, $\eta \in \Gamma_0$, we have, by Lemma 5.1, a.s. $\eta_t \subset \bar{\eta}_t$ for all $t \geq 0$. Thus, it is sufficient to prove the proposition for $(\bar{\eta}_t)$. The process $(\bar{\eta}_t)$ with rate (3.3) is, in fact, a continuous-time continuous-space branching random walk (for an overview of branching random walks and related topics; see, e.g. [23]). Denote by $\bar{\eta}_t^{\boldsymbol{e}}$ the element-wise projection of $\bar{\eta}_t$ onto the line determined by $\boldsymbol{e}$; that is, $\bar{\eta}_t^{\boldsymbol{e}} = \{x \in \mathbb{R}^1 \mid x = \langle y, \boldsymbol{e}\rangle \text{ for some } y \in \eta_t\}$. The process $(\bar{\eta}_t^{\boldsymbol{e}})$ is itself a branching random walk, and, by Corollary 2 of [3], the position of the rightmost particle $X_t^{\boldsymbol{e}}$ of $(\bar{\eta}_t^{\boldsymbol{e}})$ at time t satisfies

$$\lim_{t\to\infty} \frac{X_t^{\boldsymbol{e}}}{t} \to \gamma \quad \text{for a certain } \gamma \in (0, \infty). \tag{3.4}$$

The conditions from Corollary 2 of [3] are satisfied due to Condition 2.1. Indeed, $(\bar{\eta}_t^{\boldsymbol{e}})$ is the branching random walk with the birth kernel

$$\bar{a}^{\boldsymbol{e}}(z) = \int_{y \in \mathbb{R}^{\text{d}-1}} a(z \circ y)\, \mathrm{d}y,$$

that is, $(\bar{\eta}_t^{\boldsymbol{e}})$ is a birth process on $\mathbb{R}^1$ with birth rate

$$\bar{b}(x, \eta) = \sum_{y \in \eta} \bar{a}^{\boldsymbol{e}}(x - y), \qquad x \in \mathbb{R},\ \eta \in \Gamma_0(\mathbb{R}).$$

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Note that $a^{\boldsymbol{e}}(z) = a(z)$ if $d = 1$. Hence, in the notation of [3], for $\theta < 0$,

$$\begin{aligned} m(\theta, \phi) &= \int_{\mathbb{R}\times\mathbb{R}_+} \mathrm{e}^{-\theta z}\mathrm{e}^{-\phi\tau}\bar{a}^{\boldsymbol{e}}(z)\,\mathrm{d}z\,\mathrm{d}\tau \\ &= \frac{1}{\phi}\int_{\mathbb{R}} \mathrm{e}^{-\theta|z|}\bar{a}^{\boldsymbol{e}}(z)\,\mathrm{d}z \\ &= \frac{1}{\phi}\int_{\mathbb{R}} \mathrm{e}^{-\theta|z|}\,\mathrm{d}z\int_{y\in\mathbb{R}^{d-1}} a(z\circ y)\,\mathrm{d}y \\ &= \frac{1}{\phi}\int_{\mathbb{R}^d} \mathrm{e}^{-\theta|\langle x,\boldsymbol{e}\rangle|}a(x)\,\mathrm{d}x \\ &\le \frac{1}{\phi}\int_{\mathbb{R}^d} \mathrm{e}^{-\theta|x|}a(x)\,\mathrm{d}x \end{aligned}$$

and, thus, $\alpha(\theta) < \infty$ for a negative θ satisfying $\int_{\mathbb{R}^d} \mathrm{e}^{-\theta|x|}a(x)\,\mathrm{d}x < \infty$ (the functions $m(\theta, \phi)$ and $\alpha(\theta)$ are defined in [3] at the beginning of Section 3).

Since (3.2) follows from (3.4), the proof of the proposition is now complete. □

Next, using a comparison with the Eden model (see [8]), we will show that the system grows no slower than linearly (in the sense of Lemma 3.3 below). The Eden model is a model of tumor growth on the lattice $\mathbb{Z}^d$. The evolution starts from a single particle at the origin. A site once occupied stays occupied forever. A vacant site becomes occupied at rate $\lambda > 0$ if at least one of its neighbors is occupied. We mention that this model is closely related to the first-passage percolation model; see, e.g. [1] and [15]. In fact, the two models coincide if the passage times have exponential distribution; see [15].

For $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, let $|z|_1 = \sum_{i=1}^d |z_i|$.

Lemma 3.1. *Consider the Eden model starting from a single particle at the origin. Then there exists a constant* $\widetilde{C} > 0$ *such that, for every* $z \in \mathbb{Z}^d$ *and time* $t \ge 4\mathrm{e}^2/\lambda^2(\mathrm{e}-1)^2 \vee \widetilde{C}|z|_1$,

$$\mathbb{P}\{z \text{ is vacant at } t\} \le \mathrm{e}^{-\sqrt{t}}.$$

Proof. Let σ_z be the time when z becomes occupied. Let v be a path on the integer lattice of length $m = \mathrm{length}(v)$ starting from $\mathbf{0}$ and ending in z, so that $v_0 = \mathbf{0}$, $v_m = z$, $v_i \in \mathbb{Z}^d$, and $|v_i - v_{i-1}| = 1$, $i = 1, \dots, m$. Define $\sigma(v)$ as the time it takes for the Eden model to move along the path v; that is, if $v_0, \dots, v_j$ are occupied then a birth can occur only at v_{j+1}. By construction, $\sigma(v)$ is distributed as the sum of $\mathrm{length}(v)$ independent unit exponentials (the so-called passage times; see, e.g. [1] or [15]). We have

$$\sigma_z = \inf\{\sigma(v) \colon v \text{ is a path from } \mathbf{0} \text{ to } z\}.$$

Hence, σ_z is dominated by the sum of $|z|_1$ independent unit exponentials, say $\sigma_z \le Z_1 + \dots + Z_{|z|_1}$.

We have the equality of the events

$$\{z \text{ is vacant at } t\} = \{\sigma_z > t\}.$$

Note that

$$\mathbb{E}\exp\biggl\{\lambda\biggl(1 - \frac{1}{\mathrm{e}}\biggr)Z_1\biggr\} = \mathrm{e}.$$

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Using Chebyshev's inequality $\mathbb{P}\{Z > t\} \le \mathbb{E}\exp\{\lambda(1-1/\mathrm{e})(Z-t)\}$, we obtain

$$\begin{aligned}\mathbb{P}\{\sigma_z > t\} &\le \mathbb{P}\{Z_1 + \cdots + Z_{|z|_1} > t\}\\ &\le \mathbb{E}\exp\left\{\lambda\left(1-\frac{1}{\mathrm{e}}\right)(Z_1+\cdots+Z_{|z|_1}-t)\right\}\\ &= \left[\mathbb{E}\exp\left\{\lambda\left(1-\frac{1}{\mathrm{e}}\right)Z_1\right\}\right]^{|z|_1}\exp\left\{-\lambda\left(1-\frac{1}{\mathrm{e}}\right)t\right\}\\ &= \exp\{|z|_1\}\exp\left\{-\lambda\left(1-\frac{1}{\mathrm{e}}\right)t\right\}.\end{aligned}$$

Since

$$\frac{1}{2}\lambda\left(1-\frac{1}{\mathrm{e}}\right)t \ge \sqrt{t},$$

for $t \ge 4\mathrm{e}^2/\lambda^2(\mathrm{e}-1)^2$, we may take $\widetilde{C} = 2\mathrm{e}/\lambda(\mathrm{e}-1)$. □

We now continue to work with the Eden model.

Lemma 3.2. *For the Eden model starting from a single particle at the origin, there are constants* $c_1, t_0 > 0$ *such that*

$$\mathbb{P}\{\text{there is a vacant site in } B(0, c_1 t) \cap \mathbb{Z}^d \text{ at } t\} \le \exp\{-\sqrt[4]{t}\}, \qquad t \ge t_0.$$

Proof. By the previous lemma, for $c_1 < 1/\widetilde{C}$,

$$\begin{aligned}\mathbb{P}\{\text{there is a vacant site in } B(0,c_1t)\cap\mathbb{Z}^d \text{ at } t\} &\le \sum_{z\in B(0,c_1t)\cap\mathbb{Z}^d}\mathbb{P}\{z \text{ is vacant at } t\}\\ &\le |B(0,c_1t)|\exp\{-\sqrt{t}\},\end{aligned}$$

where $|B(0, c_1t)|$ is the number of integer points (that is, points whose coordinates are integers) inside $B(0, c_1t)$. It remains to note that $|B(0, c_1t)|$ grows only polynomially fast in t. □

Definition 3.1. Let the growth process $(\alpha_t)_{t\ge0}$ be a $\mathbb{Z}_+^{\mathbb{Z}^d}$-valued process with

$$\alpha(z) \to \alpha(z)+1 \quad \text{at rate } \lambda\mathbf{1}_{\{\sum_{y\in\mathbb{Z}^d:\,|z-y|\le1}\alpha(y)>0\}},\ z\in\mathbb{Z}^d,\ \alpha\in\mathbb{Z}_+^{\mathbb{Z}^d}, \qquad \sum_{y\in\mathbb{Z}^d}\alpha(y)<\infty,$$

where $\lambda > 0$.

Clearly, Lemma 3.2 also applies to $(\alpha_t)_{t\ge0}$ since it dominates the Eden process. Recall that r appears in (2.3), and (ξ_t) is defined in (2.7).

Lemma 3.3. *There exist* $c, s_0 > 0$ *such that*

$$\mathbb{P}\{B(\mathbf{0}, cs) \not\subset \xi_s\} \le \exp\{-\sqrt[4]{s}\}, \qquad s \ge s_0.$$

Proof. For $x \in \mathbb{R}^d$, let $z_x \in (r/2d)\mathbb{Z}^d$ be uniquely determined by $x \in z_x + (-r/4d, r/4d]^d$. Recall that c_0 appears in Condition 2.4. Define

$$\bar{b}(x,\eta) = c_0\mathbf{1}_{\{z_x\sim z_y \text{ for some } y\in\eta\}},$$

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where '$z_x \sim z_y$' means that z_x and z_y are neighbors on $(r/2d)\mathbb{Z}^d$. Let $(\bar{\eta}_t)_{t\geq 0}$ be the birth process with birth rate $\bar{b}$. Note that, by (2.3), for every $\eta \in \Gamma_0$,

$$\bar{b}(x,\eta) \leq b(x,\eta), \qquad x \in \mathbb{R}^d,$$

hence, a.s. $\bar{\eta}_t \subset \eta_t$ by Lemma 5.1, $t \geq 0$. Then the 'projection' process defined by

$$\bar{\bar{\eta}}_t(z) = \sum_{x\in\bar{\eta}_t} \mathbf{1}_{\{x\in z+(-r/4d,r/4d]^d\}}, \qquad z \in \frac{r}{2d}\mathbb{Z}^d,$$

is the process $(\alpha_t)_{t\geq 0}$ from Definition 3.1 with $\lambda = c_0(r/2d)^d$ and the 'geographic' space $(r/2d)\mathbb{Z}^d$ instead of $\mathbb{Z}^d$, that is, taking values in $\mathbb{Z}_+^{(r/2d)\mathbb{Z}^d}$ instead of $\mathbb{Z}_+^{\mathbb{Z}^d}$. Since $\bar{\bar{\eta}}_t(z_x) > 0$ implies that $x \in \xi_t$, the desired result follows from Lemma 3.2 and the fact that Lemma 3.2 also applies to $(\alpha_t)_{t\geq 0}$. □

Notation and conventions. In what follows, for $x, y \in \mathbb{R}^d$, we define

$$[x, y] = \{z \in \mathbb{R}^d \mid z = tx + (1-t)y,\ t \in [0,1]\}.$$

We call $[x, y]$ an interval. Similarly, open or half-open intervals are defined, e.g.

$$(x, y] = \{z \in \mathbb{R}^d \mid z = tx + (1-t)y,\ t \in (0,1]\}.$$

We also adopt the convention $B(x,0) = \{x\}$.

For $x \in \mathbb{R}^d$ and $\lambda \in (0,1)$, we define a stopping time $T_\lambda(x)$ (here and below, all stopping times are considered with respect to the filtration $(\mathcal{S}_t)$ introduced after (5.3)) by

$$T_\lambda(x) = \inf\{t > 0 \colon |\eta_t \cap B(x, \lambda|x|)| > 0\}, \tag{3.5}$$

and, for $x, y \in \mathbb{R}^d$, we define

$$T_\lambda(x,y) = \inf\{t > T_\lambda(x) \colon |\eta_t^{T_\lambda(x),\{z_\lambda(x)\}} \cap B(y + z_\lambda(x) - x, \lambda|y-x|)| > 0\} - T_\lambda(x), \tag{3.6}$$

where $z_\lambda(x)$ is uniquely defined by $\{z_\lambda(x)\} = \eta_{T_\lambda(x)} \cap B(x,\lambda|x|)$. Note that $\{z_\lambda(x)\}$ is an $\mathcal{S}_{T_\lambda(x)}$-measurable finite random set. Also, $T_\lambda(\mathbf{0}) = 0$ and $T_\lambda(x,x) = 0$ for $x \in \mathbb{R}^d$. To reduce the number of double subscripts, we will sometimes write $z(x)$ instead of $z_\lambda(x)$.

Since, for $q \geq 1$,

$$\{x_1 + x_2 \colon x_1 \in B(x,\lambda|x|),\ x_2 \in B((q-1)x,\ \lambda(q-1)|x|)\} = B(qx, \lambda q|x|),$$

we have, by attractiveness (recall (2.6)),

$$\begin{aligned} T_\lambda(qx) &\leq T_\lambda(x) + (\inf\{t > 0 \colon |\eta_t^{T_\lambda(x),\,\eta_{T_\lambda(x)}} \cap B(qx,\lambda q|x|)| > 0\} - T_\lambda(x)) \\ &\leq T_\lambda(x) + (\inf\{t > 0 \colon |\eta_t^{T_\lambda(x),\{z_\lambda(x)\}} \cap B(z_\lambda(x) + (q-1)x, \lambda(q-1)|x|)| > 0\} \\ &\qquad - T_\lambda(x)), \end{aligned}$$

that is,

$$T_\lambda(qx) \leq T_\lambda(x) + T_\lambda(x, qx), \qquad x \in \mathbb{R}^d \setminus \{\mathbf{0}\}. \tag{3.7}$$

Note that, by the strong Markov property (Proposition 5.2 and Corollary 5.1), we have an equality in distribution:

$$T_\lambda(x, qx) \overset{\mathrm{D}}{=} T_\lambda((q-1)x). \tag{3.8}$$

The following elementary lemma is used in the proof of Lemma 3.5.

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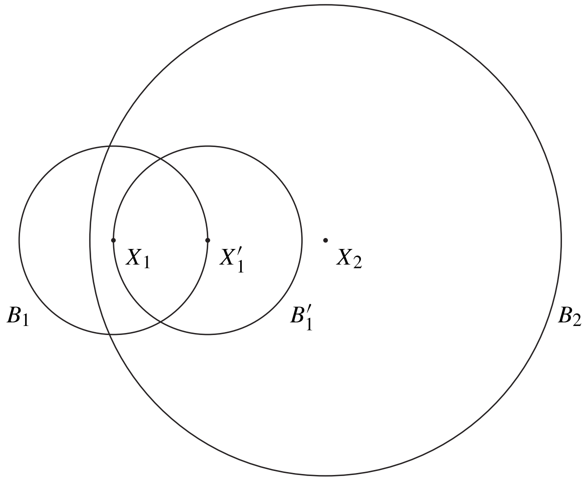

FIGURE 1: Representation of Lemma 3.4(i).

Lemma 3.4. *Let $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ be two d-dimensional balls.*

(i) *There exists a constant $c_{\text{ball}}(\mathrm{d}) > 0$ depending on* d *only such that if B_1 and B_2 are two balls in $\mathbb{R}^d$ and $x_1 \in B_2$, then*

$$\mathrm{vol}(B_1 \cap B_2) \geq c_{\text{ball}}(\mathrm{d})(\mathrm{vol}(B_1) \wedge \mathrm{vol}(B_2)),$$

where $\mathrm{vol}(B)$ *is the d-dimensional volume of B.*

(ii) *The intersection $B_1 \cap B_2$ contains a ball of radius r_3 provided that*

$$2r_3 \leq (r_1 + r_2 - |x_1 - x_2|) \wedge r_1 \wedge r_2.$$

Proof. (i) Without loss of generality we can assume that $r_1 \leq r_2$. Indeed, if $r_1 > r_2$ then $x_2 \in B_1$, so we can swap B_1 and B_2. Let $B'_1 = B(x'_1, r_1)$ be the shifted ball B_1 with $x'_1 = x_1 + r_1((x_2 - x_1)/|x_2 - x_1|)$ (see Figure 1). The intersection $B'_1 \cap B_1$ is a subset of B_2 and is a union of two identical d-dimensional hyperspherical caps with height $r_1/2$. Using the standard formula for the volume of a hyperspherical cap, we see that we can take

$$c_{\text{ball}}(\mathrm{d}) = \frac{V(B'_1 \cap B_1)}{V(B_1)} = 2\frac{\Gamma(d/2+1)}{\sqrt{\pi}\,\Gamma((d+1)/2)} \int_0^{\pi/3} \sin^d(s)\,\mathrm{d}s.$$

(ii) We have $B_3 \subset B_1 \cap B_2$, where $B_3 = B(x_3, r_3)$ and x_3 is the middle point of the interval $[x_1, x_2] \cap B_1 \cap B_2$. □

Lemma 3.5. *For every $x \in \mathbb{R}^d$ and $\lambda > 0$, there exist $A_{x,\lambda}, q_{x,\lambda} > 0$ such that*

$$\mathbb{P}\{T_\lambda(x) > s\} \leq A_{x,\lambda} \exp\{-q_{x,\lambda} \sqrt[4]{s}\}, \qquad s \geq 0.$$

Proof. Let

$$\tau_x = \inf\{s > 0 \colon x \in \xi_s\}$$

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(recall that (ξ_t) is defined in (2.7)), that is, τ_x is the moment when the first point in the ball $B(x, r)$ appears. By Lemma 3.3, for $s \geq s_0 \vee |x|/c$,

$$\mathbb{P}\{\tau_x > s\} \leq \mathbb{P}\{x \notin \xi_s\} \leq \mathbb{P}\{B(\mathbf{0}, |x|) \not\subseteq \xi_s\} \leq \mathbb{P}\{B(\mathbf{0}, cs) \not\subseteq \xi_s\} \leq \exp\{-\sqrt[4]{s}\}. \tag{3.9}$$

In the $r \leq \lambda|x|$ case, we have a.s. $T_\lambda(x) \leq \tau_x$, and the statement of the lemma follows from (3.9) since, for $s \geq s_0 \vee |x|/c$,

$$\mathbb{P}\{T_\lambda(x) > s\} \leq \mathbb{P}\{\tau_x > s\} \leq \exp\{-\sqrt[4]{s}\}.$$

We now consider the $r > \lambda|x|$ case. Denote by $\bar{x} \in B(x, r)$ the place where the particle is born at τ_x. For $t \geq 0$ on $\{t > \tau_x\}$, we have

$$\int_{y \in B(x, \lambda|x|)} b(y, \eta_t)\, \mathrm{d}y \geq \int_{y \in B(x, \lambda|x|)} b(y, \{\bar{x}\})\, \mathrm{d}y \geq \int_{y \in B(x, \lambda|x|)} c_0 \mathbf{1}_{\{y \in B(\bar{x}, r)\}}\, \mathrm{d}y,$$

so that, by Lemma 3.4 on $\{t > \tau_x\}$,

$$\begin{aligned} \int_{y \in B(x, \lambda|x|)} b(y, \eta_t)\, \mathrm{d}y &\geq \int_{y \in B(x, \lambda|x|)} c_0 \mathbf{1}_{\{y \in B(\bar{x}, r)\}}\, \mathrm{d}y \\ &= c_0 \operatorname{vol}(B(x, \lambda|x|) \cap B(\bar{x}, r)) \\ &\geq c_0 c_{\mathrm{ball}}(\mathrm{d}) \operatorname{vol}(B(x, \lambda|x|)) \\ &= c_0 c_{\mathrm{ball}}(\mathrm{d}) V_d \lambda^d |x|^d, \end{aligned}$$

where $V_d = \operatorname{vol}(B(\mathbf{0}, 1))$, hence,

$$\begin{aligned} \mathbb{P}\{T_\lambda(x) - \tau_x > s'\} &\leq \mathbb{P}\{\inf\{t > 0 \colon \eta_t^{\tau_x, \{\bar{x}\}} \cap B(x, r) \neq \varnothing\} - \tau_x > s'\} \\ &\leq \exp\{-c_0 c_{\mathrm{ball}}(\mathrm{d}) V_d \lambda^d |x|^d s'\}. \end{aligned}$$

Combining this with (3.9) yields the desired result. □

We fix an $x \in \mathbb{R}^d$, $x \neq \mathbf{0}$, and define, for $k, n \in \mathbb{N}$, $k < n$,

$$s_{k,n} = T_\lambda(kx, nx).$$

Note that the random variables $s_{k,n}$ are integrable by Lemma 3.5. The conditions of Liggett's subadditive ergodic theorem, see [17], are satisfied here. Indeed, Equation (1.7) of [17] is ensured by our (3.7), while Equations (1.8) and (1.9) of [17] follow from our (3.8) and the strong Markov property of (η_t) (Proposition 5.2 and Corollary 5.1). Thus, there exists $\mu_\lambda(x) \in [0, \infty)$ such that a.s. and in L^1,

$$\frac{s_{0,n}}{n} \to \mu_\lambda(x). \tag{3.10}$$

Lemma 3.6. *Let $\lambda > 0$. For every $x \neq \mathbf{0}$,*

$$\lim_{t \to \infty} \frac{T_\lambda(tx)}{t} = \mu_\lambda(x).$$

Proof. We know that, for every $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$,

$$\lim_{n \to \infty} \frac{T_\lambda(nx)}{n} = \mu_\lambda(x).$$

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Denote $\sigma_n = \inf_{y\in[nx,(n+1)x]} T_\lambda(y)$. Since there are only a finite number of particles born in a bounded time interval, this infinum is achieved. So, let $\widetilde{z}_n$ be such that $\eta_{\sigma_n} \setminus \eta_{\sigma_n -} = \{\widetilde{z}_n\}$. By the definition of σ_n, the set

$$\{y \in [nx, (n+1)x] \mid \widetilde{z}_n \in B(y, \lambda|y|)\}$$

is not empty. We see that $\{\widetilde{z}_n\}$ is an $\mathcal{S}_{\sigma_n}$-measurable finite random set, so we can apply Corollary 5.1 here.

Define now another stopping time

$$\widetilde{\sigma}_n = \inf\{t > 0 \colon \xi_t^{\sigma_n,\{\widetilde{z}_n\}} \supset B(\widetilde{z}_n, \lambda|x| + |x| + 2r)\}.$$

We show that

$$\sup_{y\in[nx,(n+1)x]} T_\lambda(y) \le \widetilde{\sigma}_n. \tag{3.11}$$

For any $y \in [nx, (n+1)x]$,

$$|y - \widetilde{z}_n| \le |\widetilde{z}_n - nx| \vee |\widetilde{z}_n - (n+1)x| \le \lambda(n+1)|x| + |x|.$$

Therefore, the intersection of the balls $B(\widetilde{z}_n, \lambda|x| + |x| + 2r)$ and $B(y, \lambda|y|)$ contains a ball $\widetilde{B}$ of radius r by Lemma 3.4(ii), since

$$\lambda|x| + |x| + 2r + \lambda|y| - \lambda(n+1)|x| - |x| \ge \lambda|x| + 2r + \lambda n|x| - \lambda(n+1)|x| = 2r.$$

Since the radius of $\widetilde{B}$ is r and $\xi_{\widetilde{\sigma}_n}^{\sigma_n,\{\widetilde{z}_n\}} \supset B(\widetilde{z}_n, \lambda|x| + |x| + 2r) \supset \widetilde{B}$,

$$\eta_{\widetilde{\sigma}_n}^{\sigma_n,\{\widetilde{z}_n\}} \cap \widetilde{B} \neq \varnothing,$$

and, hence,

$$\eta_{\widetilde{\sigma}_n} \cap \widetilde{B} \neq \varnothing. \tag{3.12}$$

Since $\widetilde{B} \subset B(y, \lambda|y|)$ for all $y \in [n|x|, (n+1)|x|]$, (3.12) implies (3.11).

For $q \ge (\lambda|x| + |x| + 2r) \vee cs_0$, by Lemma 3.3,

$$\begin{aligned}
\mathbb{P}\left\{\widetilde{\sigma}_n - \sigma_n \ge \frac{q}{c}\right\} &= \mathbb{P}\left\{B(\widetilde{z}_n, \lambda|x| + |x| + 2r) \not\subseteq \xi_{q/c+\sigma_n}^{\sigma_n,\{\widetilde{z}_n\}}\right\} \\
&\le \mathbb{P}\left\{B(\widetilde{z}_n, q) \not\subseteq \xi_{q/c+\sigma_n}^{\sigma_n,\{\widetilde{z}_n\}}\right\} \\
&\le \exp\left\{-\sqrt[4]{\frac{q}{c}}\right\},
\end{aligned}$$

hence,

$$\mathbb{P}\{\widetilde{\sigma}_n - \sigma_n \ge q'\} \le \exp\{-\sqrt[4]{q'}\}, \qquad q' \ge \left(\frac{\lambda|x| + |x| + 2r}{c}\right) \vee s_0.$$

By the Borel–Cantelli lemma,

$$\mathbb{P}\{\widetilde{\sigma}_n - \sigma_n > \sqrt{n} \text{ for infinitely many } n\} = 0,$$

and since $\sigma_n \le T_\lambda(nx) \le \widetilde{\sigma}_n$, a.s. for large n,

$$\widetilde{\sigma}_n < T_\lambda(nx) + \sqrt{n} \quad \text{and} \quad \sigma_n \ge T_\lambda(nx) - \sqrt{n}.$$

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By (3.11),

$$\limsup_{n\to\infty} \frac{\sup_{y\in[nx,\,(n+1)x]} T_\lambda(y)}{n} \le \limsup_{n\to\infty} \frac{\widetilde{\sigma}_n}{n} \le \limsup_{n\to\infty} \frac{T_\lambda(nx)+\sqrt{n}}{n} \le \mu_\lambda(x)$$

and

$$\liminf_{n\to\infty} \frac{\inf_{y\in[nx,\,(n+1)x]} T_\lambda(y)}{n} = \liminf_{n\to\infty} \frac{\sigma_n}{n} \ge \limsup_{n\to\infty} \frac{T_\lambda(nx)-\sqrt{n}}{n} \ge \mu_\lambda(x). \qquad \square$$

Lemma 3.7. *The ratio $\mu_\lambda(x)/|x|$ in (3.10) does not depend on x, $x \neq \mathbf{0}$.*

Proof. First, note that, for every $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and every $q > 0$,

$$\mu_\lambda(x) = \frac{\mu_\lambda(qx)}{q} \tag{3.13}$$

by Lemma 3.6.

On the other hand, if $|x| = |y|$ then, by Proposition 5.1,

$$\mu_\lambda(x) = \mu_\lambda(y), \tag{3.14}$$

since the distribution of (η_t) is invariant under rotation and we can consider $\mu_\lambda(x)$ as a functional acting on the trajectory $(\eta_t)_{t\ge 0}$. The proof follows from (3.13) and (3.14). $\square$

Set

$$\mu_\lambda := \frac{\mu_\lambda(x)}{|x|}, \qquad x \neq \mathbf{0}.$$

As λ decreases, $T_\lambda(x)$ increases and, therefore, μ_λ increases too. Denote

$$\mu = \lim_{\lambda\to 0+} \mu_\lambda. \tag{3.15}$$

Lemma 3.8. *The constants μ_λ and μ are strictly positive: $\mu_\lambda > 0$, $\mu > 0$.*

Proof. By Proposition 3.1, for x with large $|x|$,

$$\eta_{(1-\lambda)|x|/C_{\text{upb}}} \subset B(\mathbf{0}, (1-\lambda)|x|),$$

hence, for every $\lambda \in (0, 1)$ for x with large $|x|$,

$$T_\lambda(x) \ge \frac{(1-\lambda)|x|}{C_{\text{upb}}}.$$

Thus,

$$\mu_\lambda \ge \frac{1-\lambda}{C_{\text{upb}}} \quad \text{and} \quad \mu = \lim_{\lambda\to 0+} \mu_\lambda \ge \frac{1}{C_{\text{upb}}}. \qquad \square$$

Lemma 3.9. *Let q, $R > 0$. Suppose that, for all $\varepsilon \in (0, 1)$ a.s. for sufficiently large $n \in \mathbb{N}$,*

$$\frac{\eta_{qn}}{qn} \subset (1+\varepsilon)B(\mathbf{0}, R) \quad \left((1-\varepsilon)B(\mathbf{0}, R) \subset \frac{\xi_{qn}}{qn}, \text{ respectively}\right). \tag{3.16}$$

Then, for all $\varepsilon \in (0, 1)$ a.s. for sufficiently large $t \ge 0$,

$$\frac{\eta_t}{t} \subset (1+\varepsilon)B(\mathbf{0}, R) \quad \left((1-\varepsilon)B(\mathbf{0}, R) \subset \frac{\xi_t}{t}, \text{ respectively}\right).$$

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Proof. We consider the first case only – the proof of the other case is similar. Since $\varepsilon \in (0, 1)$ is arbitrary, (3.16) implies that, for all $\widetilde{\varepsilon} \in (0, 1)$ a.s. for large $n \in \mathbb{N}$,

$$\frac{\eta_{q(n+2)}}{qn} \subset (1 + \widetilde{\varepsilon})B(\mathbf{0}, R).$$

Since a.s. $(\eta_t)_{t \geq 0}$ is monotonically growing, it is sufficient to note that

$$\frac{\eta_t}{t} \subset (1 + \varepsilon)B(\mathbf{0}, R) \quad \text{if } \frac{\eta_{\lceil t/q \rceil q + q}}{\lfloor t/q \rfloor q} \subset (1 + \varepsilon)B(\mathbf{0}, R). \qquad \square$$

Recall that c is a constant from Lemma 3.3.

Lemma 3.10. *Let* $\varepsilon \in (0, 1)$. *Then a.s.*

$$(1 - \varepsilon)B(\mathbf{0}, \mu^{-1}) \subset \frac{\xi_m}{m} \tag{3.17}$$

for large m of the form $m = (1 + \lambda\mu_\lambda^{-1}/c)n,\ n \in \mathbb{N}$.

Proof. Let $\lambda = \lambda_\varepsilon > 0$ be chosen so small that

$$(1 - \varepsilon)\mu^{-1} \leq \frac{1 - \varepsilon/2}{1 + \lambda\mu_\lambda^{-1}/c}\mu_\lambda^{-1}.$$

Such a λ exists since

$$\lim_{\lambda \to 0+} \frac{\mu_\lambda^{-1}}{1 + \lambda\mu_\lambda^{-1}/c} = \mu^{-1}.$$

Choose a finite sequence of points $\{x_j,\ j = 1, \dots, N\}$ such that $x_j \in (1 - \varepsilon/2)B(\mathbf{0}, \mu_\lambda^{-1})$ and

$$\bigcup_j B\Big(x_j, \frac{\varepsilon}{4}c\Big) \supset \Big(1 - \frac{\varepsilon}{2}\Big)B(\mathbf{0}, \mu_\lambda^{-1}).$$

Let $\delta > 0$ be so small that $(1 + \delta)(1 - \varepsilon/2) \leq (1 - \varepsilon/4)$. Since a.s.

$$\frac{T_\lambda(nx_j)}{n|x_j|} \to \mu_\lambda,$$

for large n for every $j \in \{1, \dots, N\}$,

$$T_\lambda(nx_j) \leq n|x_j|(1 + \delta)\mu_\lambda \leq n\big(1 - \tfrac{1}{2}\varepsilon\big)(1 + \delta) \leq n\big(1 - \tfrac{1}{4}\varepsilon\big), \tag{3.18}$$

so that the system reaches the ball $B(nx_j, \lambda n|x_j|)$ before the time $n(1 - \varepsilon/4)$. Let Q_n be the random event

$$\big\{T_\lambda(nx_j) \leq n\big(1 - \tfrac{1}{4}\varepsilon\big) \text{ for } j = 1, \dots, N\big\}$$
$$= \{\eta_{n(1-\varepsilon/4)} \cap B(nx_j, \lambda n|x_j|) \neq \varnothing \text{ for } j = 1, \dots, N\}.$$

Note that $\mathbb{P}(Q_n) \to 1$ by (3.18), and even

$$\mathbb{P}\bigg\{\bigcup_{m \in \mathbb{N}} \bigcap_{i=m}^{\infty} Q_i\bigg\} = 1. \tag{3.19}$$

In other words, a.s. for large i, all Q_i occur.

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Let $\bar{z}(nx_j)$ be defined as $z(nx_j)$ on Q_n and as nx_j on the complement $\Omega \setminus Q_n$ (recall that $z(x) = z_\lambda(x)$, $x \in \mathbb{R}^d$, was defined after (3.6)). The set $\{\bar{z}(nx_j)\}$ is a finite random $\mathcal{S}_{n(1-\varepsilon/4)}$-measurable set.

Using Lemma 3.3, we will show that after an additional time interval of length $(\varepsilon/4 + \lambda\mu_\lambda^{-1}/c)n$, the entire ball $(1-\varepsilon/2)nB(\mathbf{0}, \mu_\lambda^{-1})$ is covered by (ξ_t), that is, a.s. for large n,

$$\big(1 - \tfrac{1}{2}\varepsilon\big)nB(\mathbf{0}, \mu_\lambda^{-1}) \subset \xi_{n(1-\varepsilon/4)+(\varepsilon/4+\lambda\mu_\lambda^{-1}/c)n} = \xi_{n+\lambda n\mu_\lambda^{-1}/c}. \tag{3.20}$$

Indeed, since

$$B\Big(nx_j, c\frac{\varepsilon}{4}n\Big) \subset B\Big(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda|x_j|n\Big) \subset B\Big(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda\mu_\lambda^{-1}n\Big),$$

the series

$$\sum_{n\in\mathbb{N}} \mathbb{P}\Big\{B\Big(nx_j, c\frac{\varepsilon}{4}n\Big) \not\subset \xi_{n+\lambda\mu_\lambda^{-1}n/c}^{(n(1-\varepsilon/4),\,\{\bar{z}(nx_j)\})} \text{ for some } j\Big\}$$
$$\leq \sum_{n\in\mathbb{N}} \mathbb{P}\Big\{B\Big(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda\mu_\lambda^{-1}n\Big) \not\subset \xi_{n+\lambda\mu_\lambda^{-1}n/c}^{(n(1-\varepsilon/4),\,\{\bar{z}(nx_j)\})} \text{ for some } j\Big\}$$

converges by Lemma 3.3, thus, a.s. for large n,

$$B\Big(nx_j, c\frac{\varepsilon}{4}n\Big) \subset \xi_{n+\lambda\mu_\lambda^{-1}n/c}^{(n(1-\varepsilon/4),\,\{\bar{z}(nx_j)\})}, \qquad j = 1, \dots, N.$$

By (3.19), a.s. for large n,

$$B\Big(nx_j, c\frac{\varepsilon}{4}n\Big) \subset \xi_{n+\lambda\mu_\lambda^{-1}n/c}^{(n(1-\varepsilon/4),\,\{z(nx_j)\})}, \qquad j = 1, \dots, N. \tag{3.21}$$

Hence, the choice of $\{x_j,\ j = 1, \dots, N\}$ and (3.21) yield (3.20). Due to our choice of λ,

$$(1-\varepsilon)nB(\mathbf{0}, \mu^{-1}) \subset \frac{1-\varepsilon/2}{1+\lambda\mu_\lambda^{-1}/c}nB(\mathbf{0}, \mu_\lambda^{-1}),$$

which, in conjunction with (3.20), implies that (3.17) holds a.s. for large m of the form $(1 + \lambda\mu_\lambda^{-1}/c)n$, where $n \in \mathbb{N}$. □

Lemma 3.11. *Let $\varepsilon \in (0, 1)$. Then, a.s. for large $n \in \mathbb{N}$,*

$$\frac{\eta_n}{n} \subset (1+\varepsilon)B(\mathbf{0}, \mu^{-1}). \tag{3.22}$$

Proof. Let $\lambda = \lambda_\varepsilon > 0$ be so small that

$$\big(1 + \tfrac{1}{2}\varepsilon\big)B(\mathbf{0}, \mu_\lambda^{-1}) \subset (1+\varepsilon)B(\mathbf{0}, \mu^{-1}). \tag{3.23}$$

Let $q \in (\varepsilon, \infty)$ and A be the annulus

$$A := (1+q)B(\mathbf{0}, \mu_\lambda^{-1}) \setminus \big(1 + \tfrac{1}{2}\varepsilon\big)B(\mathbf{0}, \mu_\lambda^{-1}),$$

and $\{x_j,\ j = 1, \dots, N\}$ be a finite sequence such that $x_j \in A$ and

$$\bigcup_j B(x_j, \lambda |x_j|) \supset A.$$

Define $F := \{\eta_n \cap nA \neq \varnothing \text{ infinitely often}\}$. On F there exists a (random) $i \in \{1, \dots, N\}$ such that the intersection $\eta_n \cap nB(x_i, \lambda |x_i|)$ is nonempty infinitely often. Define also

$$F_i := \{\eta_n \cap nB(x_i, \lambda |x_i|) \neq \varnothing \text{ infinitely often}\}.$$

Note that $F \subset \bigcup_{i=1}^{N} F_i$.

On F_i, we have $T_\lambda(nx_i) \le n$ infinitely often, hence, our choice of A implies that

$$\liminf_{n\to\infty} \frac{T_\lambda(nx_i)}{n|x_i|} \le \liminf_{n\to\infty} \frac{n}{(1+\varepsilon/2)\mu_\lambda^{-1} n} = \mu_\lambda \frac{1}{1+\varepsilon/2}.$$

The last inequality and Lemma 3.6 imply that $\mathbb{P}(F_i) = 0$ for every $i \in \{1, \dots, N\}$. Hence, $\mathbb{P}(F) = 0$ too. Setting $q = 2\mu_\lambda C_{\text{upb}} + 1$, so that the radius of the ball on the left-hand side of (3.23) is

$$q\mu_\lambda^{-1} > 2C_{\text{upb}},$$

by Proposition 3.1 and the definition of F, we obtain, a.s. for large n,

$$\frac{\eta_n}{n} \subset \big(1 + \tfrac{1}{2}\varepsilon\big) B(\mathbf{0}, \mu_\lambda^{-1}) \tag{3.24}$$

and the statement of the lemma follows from (3.23) and (3.24). □

Proof of Theorem 2.1. The theorem follows from Lemmas 3.9–3.11. Note that

$$\frac{\xi_n}{n} \subset (1+\varepsilon) B(\mathbf{0}, \mu^{-1})$$

is obtained from Lemma 3.11 by replacing ε in (3.22) with $\varepsilon/2$. □

4. Proof of Theorem 2.2

We precede the proof of Theorem 2.2 with an auxiliary lemma concerning Markovian functionals of a general Markov chain.

Let $(S, \mathcal{B}(S))$ be a Polish (state) space. Consider a (time-homogeneous) Markov chain on $(S, \mathcal{B}(S))$ as a family of probability measures on S^∞. Namely, on the measurable space $(\bar{\Omega}, \mathcal{F}) = (S^\infty, \mathcal{B}(S^\infty))$ consider a family of probability measures $\{\mathbb{P}_s\}_{s\in S}$ such that, for the coordinate mappings,

$$X_n \colon \bar{\Omega} \to S, \qquad X_n(s_1, s_2, \dots) = s_n,$$

the process $X := \{X_n\}_{n\in\mathbb{Z}_+}$ is a Markov chain such that, for all $s \in S$,

$$\mathbb{P}_s\{X_0 = s\} = 1, \qquad \mathbb{P}_s\{X_{n+m_j} \in A_j,\ j = 1, \dots, l \mid \mathcal{F}_n\} = \mathbb{P}_{X_n}\{X_{m_j} \in A_j,\ j = 1, \dots, l\}.$$

Here $A_j \in \mathcal{B}(S)$, $m_j \in \mathbb{N}$, $l \in \mathbb{N}$, and $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. The space S is separable, hence, there exists a transition probability kernel $Q \colon S \times \mathcal{B}(S) \to [0, 1]$ such that

$$Q(s, A) = \mathbb{P}_s\{X_1 \in A\}, \qquad s \in S,\ A \in \mathcal{B}(S).$$

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Consider a transformation of the chain X, $Y_n = f(X_n)$, where $f\colon S \to \mathbb{R}$ is a Borel-measurable function. We now provide sufficient conditions for $Y = \{Y_n\}_{n\in\mathbb{Z}_+}$ to be a Markov chain. A very similar question was discussed by Burke and Rosenblatt [4] for discrete-space Markov chains.

Lemma 4.1. *Assume that, for any bounded Borel function $h\colon S \to S$,*

$$\mathbb{E}_s h(X_1) = \mathbb{E}_q h(X_1) \quad \textit{whenever } f(s) = f(q). \tag{4.1}$$

Then Y is a Markov chain.

Remark 4.1. Condition (4.1) is the equality of distributions of X_1 under two different measures $\mathbb{P}_s$ and $\mathbb{P}_q$.

Proof of Lemma 4.1. For the natural filtrations of the processes X and Y, we have an inclusion

$$\mathcal{F}_n^X \supset \mathcal{F}_n^Y, \qquad n \in \mathbb{N}, \tag{4.2}$$

since Y is a function of X. For $k \in \mathbb{N}$ and bounded Borel functions $h_j\colon \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \dots, k$,

$$\begin{aligned}
&\mathbb{E}_s\biggl[\prod_{j=1}^k h_j(Y_{n+j}) \biggm| \mathcal{F}_n^X\biggr] \\
&\quad = \mathbb{E}_{X_n} \prod_{j=1}^k h_j(f(X_j)) \\
&\quad = \int_S Q(x_0, \mathrm{d}x_1) h_1(f(x_1)) \int_S Q(x_1, \mathrm{d}x_2) h_2(f(x_2)) \cdots \int_S Q(x_{n-1}, \mathrm{d}x_n) h_n(f(x_n)) \bigg|_{x_0 = X_n}.
\end{aligned}$$

To transform the last integral, we introduce a new kernel: for $y \in f(S)$, choose $x \in S$ with $f(x) = y$, and then, for $B \in \mathcal{B}(\mathbb{R})$, define

$$\bar{Q}(y, B) = Q(x, f^{-1}(B)).$$

The expression on the right-hand side does not depend on the choice of x due to (4.1). To obtain the kernel $\bar{Q}$ defined on $\mathbb{R} \times \mathcal{B}(\mathbb{R})$, we set

$$\bar{Q}(y, B) = \mathbf{1}_{\{0 \in B\}}, \qquad y \notin f(S).$$

Then, setting $z_n = f(x_n)$, we obtain, from the change of variables formula for the Lebesgue integral,

$$\int_S Q(x_{n-1}, \mathrm{d}x_n) h_n(f(x_n)) = \int_{\mathbb{R}} \bar{Q}(f(x_{n-1}), \mathrm{d}z_n) h_n(z_n).$$

Likewise, setting $z_{n-1} = f(x_{n-1})$, we obtain

$$\begin{aligned}
&\int_S Q(x_{n-2}, \mathrm{d}x_{n-1}) h_n(f(x_{n-1})) \int_S Q(x_{n-1}, \mathrm{d}x_n) h_n(f(x_n)) \\
&\quad = \int_S Q(x_{n-2}, \mathrm{d}x_{n-1}) h_n(f(x_{n-1})) \int_{\mathbb{R}} \bar{Q}(f(x_{n-1}), \mathrm{d}z_n) h_n(z_n) \\
&\quad = \int_{\mathbb{R}} \bar{Q}(f(x_{n-2}), \mathrm{d}z_{n-1}) h_n(z_{n-1}) \int_{\mathbb{R}} \bar{Q}(z_{n-1}, \mathrm{d}z_n) h_n(z_n).
\end{aligned}$$

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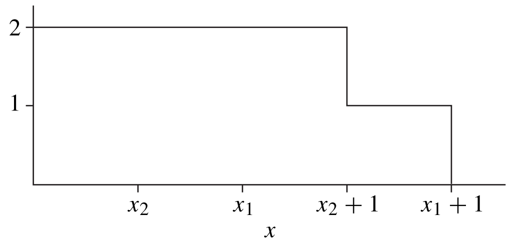

FIGURE 2: The plot of $b(\cdot, \eta_t)$.

Proceeding further, we obtain

$$\int_S Q(x_0, \mathrm{d}x_1)h_1(f(x_1)) \int_S Q(x_1, \mathrm{d}x_2)h_2(f(x_2)) \cdots \int_S Q(x_{n-1}, \mathrm{d}x_n)h_n(f(x_n))$$
$$= \int_{\mathbb{R}} \bar{Q}(z_0, \mathrm{d}z_1)h_1(z_1) \int_{\mathbb{R}} \bar{Q}(z_1, \mathrm{d}z_2)h_2(z_2) \cdots \int_{\mathbb{R}} \bar{Q}(z_{n-1}, \mathrm{d}z_n)h_n(z_n),$$

where $z_0 = f(x_0)$.

Thus,

$$\mathbb{E}_s\biggl[\prod_{j=1}^{k} h_j(Y_{n+j}) \biggm| \mathcal{F}_n^X\biggr]$$
$$= \int_{\mathbb{R}} \bar{Q}(f(X_0), \mathrm{d}z_1)h_1(z_1) \int_{\mathbb{R}} \bar{Q}(z_1, \mathrm{d}z_2)h_2(z_2) \cdots \int_{\mathbb{R}} \bar{Q}(z_{n-1}, \mathrm{d}z_n)h_n(z_n).$$

This equality and (4.2) imply that Y is a Markov chain. □

Remark 4.2. From the proof, it follows that $\bar{Q}$ is the transition probability kernel for the chain $\{f(X_n)\}_{n\in\mathbb{Z}_+}$.

Remark 4.3. Clearly, this result holds for a Markov chain which is not necessarily defined on a canonical state space since for the property of a process to be a Markov chain depends on its distribution only.

Proof of Theorem 2.2. Without any loss of generality, we will consider the speed of propagation in one direction only, say toward $+\infty$. Let $x_1(t)$ and $x_2(t)$ denote the positions of the rightmost particle and the second rightmost particle, respectively ($x_2(t) = 0$ until the first two births occur inside $(0, +\infty)$). We observe that $b(x, \eta_t) \equiv 2$ on $(0, x_2(t) + 1]$ (see Figure 2), and $X = (x_1(t), x_2(t))$ is a continuous-time pure jump Markov process on $\{(x_1, x_2) \mid x_1 \geq x_2 \geq 0,\ x_1 - x_2 \leq 1\}$ with transition densities

$$\begin{aligned} (x_1, x_2) &\to (v, x_1) \quad \text{at rate } 1,\ v \in (x_2+1, x_1+1],\\ (x_1, x_2) &\to (v, x_1) \quad \text{at rate } 2,\ v \in (x_1, x_2+1],\\ (x_1, x_2) &\to (x_1, v) \quad \text{at rate } 2,\ v \in (x_2, x_1] \end{aligned}$$

(to be precise, the above holds from the moment the first birth inside $\mathbb{R}_+$ occurs).

Furthermore, $z(t) := x_1(t) - x_2(t)$ satisfies

$$\mathbb{E}\{f(z(t+\delta)) \mid x_1(t) = x_1,\ x_2(t) = x_2\} = \mathbb{E}\{f(z(t+\delta)) \mid x_1(t) = x_1 + h,\ x_2(t) = x_2 + h\}$$

for every $h > 0$ and every Borel bounded function f. In other words, the transition rates of $(z(t))_{t\geq 0}$ are entirely determined by the current state of $(z(t))_{t\geq 0}$. Therefore, by Lemma 4.1,

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$(z(t))_{t\geq 0}$ is itself a pure jump Markov process on [0, 1] (Lemma 4.1 ensures that the embedded Markov chain of $(z(t))_{t\geq 0}$ is indeed a discrete-time Markov process). The transition densities of $(z(t))_{t\geq 0}$ are

$$q(x,y)=\begin{cases}4\mathbf{1}_{\{y\leq x\}}+2\mathbf{1}_{\{x\leq y\leq 1-x\}}+\mathbf{1}_{\{y\geq 1-x\}}, & x\leq \frac{1}{2},\ y\in[0,1],\\ 4\mathbf{1}_{\{y\leq 1-x\}}+3\mathbf{1}_{\{1-x\leq y\leq x\}}+\mathbf{1}_{\{y\geq x\}}, & x\geq \frac{1}{2},\ y\in[0,1].\end{cases}$$

Note that the total jump rate out of x is $q(x) := \int_0^1 q(x,y)\,\mathrm{d}y = 2+x$. The process $(z(t))_{t\geq 0}$ is a regular Harris recurrent Feller process with the Lebesgue measure on [0, 1] being a supporting measure (see, e.g. [14, Chapter 20]). Hence, a unique invariant measure exists and has a density g with respect to the Lebesgue measure. The equation for g is

$$\int_0^1 q(x,y)g(x)\,\mathrm{d}x = q(y)g(y). \tag{4.3}$$

Set

$$f(x)=g(x)q(x)\bigg(\int_0^1 g(y)q(y)\,\mathrm{d}y\bigg)^{-1}, \qquad x\in[0,1].$$

It is clear that f is again a density (as an aside we point out that f is the density of an invariant distribution of the embedded Markov chain of $(z(t))_{t\geq 0}$). Equation (4.3) becomes

$$f(y)=\int_0^1 \frac{q(x,y)}{q(x)}f(x)\,\mathrm{d}s,$$

which after some calculation transforms into

$$f(y)=\begin{cases} 2\displaystyle\int_0^{1/2}\frac{f(x)\,\mathrm{d}x}{2+x}+2\int_y^{1/2}\frac{f(x)\,\mathrm{d}x}{2+x}+3\int_{1/2}^{1}\frac{f(x)\,\mathrm{d}x}{2+x} \\ \quad +\displaystyle\int_{1/2}^{1-y}\frac{f(x)\,\mathrm{d}x}{2+x}, & y\leq\frac{1}{2}, \quad (4.4)\\ \displaystyle\int_0^{1/2}\frac{f(x)\,\mathrm{d}x}{2+x}+\int_0^{1-y}\frac{f(x)\,\mathrm{d}x}{2+x}+\int_{1/2}^{1}\frac{f(x)\,\mathrm{d}x}{2+x} \\ \quad +2\displaystyle\int_y^{1}\frac{f(x)\,\mathrm{d}x}{2+x}, & y\geq\frac{1}{2}. \quad (4.5)\end{cases}$$

Differentiating (4.4) and (4.5) with respect to y, we find that f solves the equation

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x)=-2\frac{f(x)}{2+x}-\frac{f(1-x)}{3-x}, \qquad x\in[0,1]. \tag{4.6}$$

Let

$$\varphi(x) := [(2+x)^2(3-x)^2]f(x), \qquad x\in[0,1].$$

Then (4.6) becomes

$$(3-x)\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)+2\varphi(x)+\varphi(1-x)=0, \qquad x\in[0,1]. \tag{4.7}$$

Looking for solutions to (4.7) among polynomials, we find that $\varphi(x)=c(4-3x)$ is a solution. By direct substitution, we can check that

$$f(x)=\frac{c(4-3x)}{(2+x)^2(3-x)^2}, \qquad x\in[0,1],$$

solves (4.4) and (4.5). The constant $c>0$ can be computed but is irrelevant for our purposes.

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Hence, after some more computation

$$g(x) = \frac{36(4-3x)}{(2+x)^3(3-x)^2}, \qquad x \in [0, 1]. \tag{4.8}$$

Note that we do not prove analytically that (4.4) and (4.5) have a unique solution. However, uniqueness for nonnegative integrable solutions follows from the uniqueness of the invariant distribution for $(z(t))_{t\geq 0}$. Let l be the Lebesgue measure on $\mathbb{R}$. By an ergodic theorem for Markov processes, see, e.g. Theorem 20.21(i) of [14], for any $0 \leq p < p' \leq 1$,

$$\lim_{t\to\infty} \frac{l\{s : z(s) \in [p, p'],\ 0 \leq s \leq t\}}{t} \to \int_p^{p'} g(x)\,\mathrm{d}x.$$

Conditioned on $z(t) = z$, the transition densities of $x_1(t)$ are

$$x_1 \to \begin{cases} x_1 + v \text{ at rate } 2, & v \in (0, 1-z], \\ x_1 + v \text{ at rate } 1, & v \in (1-z, 1]. \end{cases}$$

Hence, by (4.8), the speed of propagation is

$$\begin{aligned}\int_0^1 g(z)\,\mathrm{d}z\bigg[\int_0^{1-z} 2y\,\mathrm{d}y + \int_{1-z}^1 y\,\mathrm{d}y\bigg] &= \int_0^1 g(z)\big(1 - z + \tfrac{1}{2}z^2\big)\,\mathrm{d}z \\ &= \frac{144\ln(3) - 144\ln(2) - 40}{25}.\end{aligned}$$

□

Remark 4.4. We see from the proof that the speed can be computed in a similar way for the birth rates of the form

$$b_k(x, \eta) = k \wedge \bigg(\sum_{y\in\eta} \mathbf{1}_{\{|x-y|\leq 1\}}\bigg), \tag{4.9}$$

where $k \in (1, 2)$. However, the computations quickly become unwieldy.

5. The construction and properties of the process

We now proceed to construct the process as a unique solution to a stochastic integral equation. Such a scheme was first carried out by Massoulié [19]. This method can be deemed an analog of the construction from a graphical representation. Here we follow the approach of [2].

Remark 5.1. Of course, the process starting from a fixed initial condition we consider here can be constructed as the minimal jump process (pure jump type Markov process in the terminology of [14]) as was carried out in, e.g. [9]. Note, however, that we use coupling of infinitely many processes starting at different time points from different initial conditions, so here we employ another method.

Recall that

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

and the σ-algebra on Γ_0 that was introduced in (2.1). To construct the family of processes $(\eta_t^{q,A})_{t\geq q}$, we consider the stochastic equation with Poisson noise:

$$|\eta_t \cap B| = \int_{(q,t]\times B\times[0,\infty)} \mathbf{1}_{[0,b(x,\eta_{s-})]}(u) N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u) + |\eta_q \cap B|,$$

$$t \geq q,\ B \in \mathscr{B}(\mathbb{R}^d), \tag{5.1}$$

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where $(\eta_t)_{t\geq q}$ is a càdlàg Γ_0-valued solution process, N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$, and the mean measure of N is $\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}u$. We require the processes N and η_0 to be independent of each other. Equation (5.1) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq q$. In the integral on the right-hand side of (5.1), x is the location and s is the time of birth of a new particle. Thus, the integral over B from q to t represents the number of births inside B which have occurred before t.

We assume for convenience that $q = 0$. We will make the following assumption on the initial condition:

$$\mathbb{E}|\eta_0| < \infty. \tag{5.2}$$

We say that the process N is *compatible* with an increasing, right-continuous, and complete filtration of σ-algebras $(\mathcal{F}_t,\ t \geq 0)$ if N is adapted, that is, all random variables of the type $N(\bar{T}_1, U)$, $\bar{T}_1 \in \mathcal{B}([0; t])$, $U \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$, are $\mathcal{F}_t$-measurable, and all random variables of the type $N(t + h, U) - N(t, U)$, $h \geq 0$, $U \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$, are independent of $\mathcal{F}_t$, $N(t, U) = N([0; t], U)$.

Definition 5.1. A *(weak) solution* of (5.1) is a triple $((\eta_t)_{t\geq 0}, N)$, $(\Omega, \mathcal{F}, \mathbb{P})$, $(\{\mathcal{F}_t\}_{t\geq 0})$, where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing, right-continuous, and complete filtration of sub-σ-algebras of $\mathcal{F}$;

(ii) N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity $\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}u$;

(iii) η_0 is a random $\mathcal{F}_0$-measurable element in Γ_0 satisfying (5.2);

(iv) the processes N and η_0 are independent, N is compatible with $\{\mathcal{F}_t\}_{t\geq 0}$;

(v) $(\eta_t)_{t\geq 0}$ is a càdlàg Γ_0-valued process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, $\eta_t|_{t=0} = \eta_0$;

(vi) all integrals in (5.1) are well defined,

$$\mathbb{E}\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} b(x, \eta_{s-})\,\mathrm{d}x < \infty, \qquad t > 0;$$

(vii) (5.1) holds a.s. for all $t \in [0, \infty]$ and all Borel sets B.

Let

$$\mathcal{S}_t^0 = \sigma\{\eta_0, N([0, q] \times B \times C), q \in [0, t], B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}(\mathbb{R}_+)\}, \tag{5.3}$$

and let $\mathcal{S}_t$ be the completion of $\mathcal{S}_t^0$ under $\mathbb{P}$. Note that $\{\mathcal{S}_t\}_{t\geq 0}$ is a right-continuous filtration (see Remark A.1).

Definition 5.2. A solution of (5.1) is called *strong* if $(\eta_t)_{t\geq 0}$ is adapted to $(\mathcal{S}_t,\ t \geq 0)$.

Remark 5.2. In the definition above, we considered solutions as processes indexed by $t \in [0, \infty)$. The reformulations for the $t \in [0, T]$, $0 < T < \infty$, case is straightforward. This remark also applies to many of the results below.

Definition 5.3. We say that *joint uniqueness in law* holds for (5.1) with an initial distribution ν if any two (weak) solutions $((\eta_t), N)$ and $((\eta_t'), N')$ of (5.1), $\mathrm{law}(\eta_0) = \mathrm{law}(\eta_0') = \nu$, have the same joint distribution:

$$\mathrm{law}((\eta_t), N) = \mathrm{law}((\eta_t'), N').$$

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Theorem 5.1. *Pathwise uniqueness, strong existence, and joint uniqueness in law hold for (5.1). The unique solution is a Markov process.*

Proof. Without loss of generality, assume that $\mathbb{P}\{\eta_0 \neq \varnothing\} = 1$. Define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where

$$\sigma_{n+1} = \inf\left\{t > 0 \colon \int_{(\sigma_n, \sigma_n + t] \times B \times [0,\infty)} \mathbf{1}_{[0, b(x, \zeta_{\sigma_n})]}(u) N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u) > 0\right\} + \sigma_n, \qquad \sigma_0 = 0,$$

and

$$\zeta_0 = \eta_0, \qquad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^d \colon N(\{\sigma_{n+1}\} \times \{x\} \times [0, b(x, \zeta_{\sigma_n})]) > 0\}$. The points z_n are uniquely determined a.s. Furthermore, $\sigma_{n+1} > \sigma_n$ a.s. and σ_n are finite a.s by (2.3). We define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n, \sigma_{n+1})$. Then, by induction on n, it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is $\mathcal{F}_{\sigma_n}$-measurable. By direct substitution, we see that $(\zeta_t)_{t \geq 0}$ is a strong solution to (5.1) on the time interval $t \in [0, \lim_{n\to\infty} \sigma_n)$. Although we have not defined what is a solution, or a strong solution, on a random time interval, we do not discuss it here. Instead, we are going to show that

$$\lim_{n\to\infty} \sigma_n = \infty \quad \text{a.s.}$$

The process $(\zeta_t)_{t \in [0, \lim_{n\to\infty} \sigma_n)}$ has the Markov property, since the process N has the strong Markov property and independent increments. Indeed, conditioning on $\mathcal{I}_{\sigma_n}$,

$$\mathbb{E}[\mathbf{1}_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x \in B\}} \mid \mathcal{I}_{\sigma_n}] = \frac{\int_B b(x, \zeta_{\sigma_n})\, \mathrm{d}x}{\int_{\mathbb{R}^d} b(x, \zeta_{\sigma_n})\, \mathrm{d}x},$$

thus, the chain $\{\zeta_{\sigma_n}\}_{n \in Z_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n \in Z_+}$, $\sigma_{n+1} - \sigma_n$ are distributed exponentially,

$$\mathbb{E}\{\mathbf{1}_{\{\sigma_{n+1} - \sigma_n > a\}} \mid \{\zeta_{\sigma_n}\}_{n \in Z_+}\} = \exp\left\{-a \int_{\mathbb{R}^d} b(x, \zeta_{\sigma_n})\, \mathrm{d}x\right\}.$$

Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1}) \int_{\mathbb{R}^d} b(x, \zeta_{\sigma_n})\, \mathrm{d}x$ constitute, independent of $\{\zeta_{\sigma_n}\}_{n \in Z_+}$, a sequence of independent unit exponentials. Theorem 12.18 of [14] implies that $(\zeta_t)_{t \in [0, \lim_{n\to\infty} \sigma_n)}$ is a pure jump type Markov process.

The jump rate of $(\zeta_t)_{t \in [0, \lim_{n\to\infty} \sigma_n)}$ is given by

$$c(\alpha) = \int_{\mathbb{R}^d} b(x, \alpha)\, \mathrm{d}x.$$

Condition 2.1 implies that $c(\alpha) \leq \|a\|_1 \cdot |\alpha|$, where $\|a\|_1 = \|a\|_{L^1(\mathbb{R}^d)}$. Consequently,

$$c(\zeta_{\sigma_n}) \leq \|a\|_1 \cdot |\zeta_{\sigma_n}| = \|a\|_1 \cdot |\eta_0| + n\|a\|_1.$$

We see that $\sum_n 1/c(\zeta_{\sigma_n}) = \infty$ a.s., hence, Proposition 12.19 of [14] implies that $\sigma_n \to \infty$.

We have proved the existence of a strong solution. The uniqueness follows by induction on jumps of the process. Namely, let $(\widetilde{\zeta}_t)_{t \geq 0}$ be a solution to (5.1). From Definition 5.1(vii) and

$$\int_{(0, \sigma_1) \times \mathbb{R}^d \times [0,\infty]} \mathbf{1}_{[0, b(x, \eta_0)]}(u) N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u) = 0,$$

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it follows that $\mathbb{P}\{\widetilde{\zeta}$ has a birth before $\sigma_1\} = 0$. At the same time,

$$\int_{\{\sigma_1\}\times\mathbb{R}^d\times[0,\infty]} \mathbf{1}_{[0,b(x,\eta_0)]}(u)N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u) = 1,$$

which holds a.s., yields that $\widetilde{\zeta}$ too has a birth at the moment σ_1, and in the same point of space. Therefore, $\widetilde{\zeta}$ coincides with ζ up to σ_1 a.s. Similar reasoning shows that they coincide up to σ_n a.s. and, since $\sigma_n \to \infty$ a.s.,

$$\mathbb{P}\{\widetilde{\zeta}_t = \zeta_t \text{ for all } t \geq 0\} = 1.$$

Thus, pathwise uniqueness holds. Joint uniqueness in law follows from the functional dependence between the solution to the equation, and the 'input' η_0 and N. □

Proposition 5.1. *If b is rotation invariant then so is* (η_t).

Proof. It is sufficient to note that $(M_d\eta_t)$, where $M_d \in \mathrm{SO}(d)$ is the unique solution to (5.1), with N replaced by $M_d^{-1}N$, defined by

$$M_d^{-1}N([0,q]\times B\times C) = N([0,q]\times M_d^{-1}B\times C), \qquad q\geq 0,\ B\in\mathcal{B}(\mathbb{R}^d),\ C\in\mathcal{B}(\mathbb{R}_+).$$

Then $M_d^{-1}N$ is a Poisson point process with the same intensity, therefore, by uniqueness in law $(M_d\eta_t) \stackrel{\mathrm{D}}{=} (\eta_t)$. □

Proposition 5.2. (The strong Markov property.) *Let* τ *be an* $(\mathcal{S}_t,\ t\geq 0)$*-stopping time and let* $\widetilde{\eta}_0 \stackrel{\mathrm{D}}{=} \eta_\tau$. *Then*

$$(\eta_{\tau+t},\ t\geq 0) \stackrel{\mathrm{D}}{=} (\widetilde{\eta}_t,\ t\geq 0). \tag{5.4}$$

Furthermore, for any $\mathcal{D}\in\mathcal{B}(D_{\Gamma_0}[0,\infty))$,

$$\mathbb{P}\{(\eta_{\tau+t},\ t\geq 0)\in\mathcal{D} \mid \mathcal{S}_\tau\} = \mathbb{P}\{(\eta_{\tau+t},\ t\geq 0)\in\mathcal{D} \mid \eta_\tau\};$$

that is, given η_τ, $(\eta_{\tau+t},\ t\geq 0)$ *is conditionally independent of* $(\mathcal{S}_t,\ t\geq 0)$.

Proof. Note that

$$|\eta_{\tau+t}\cap B| = \int_{(\tau,\tau+t]\times B\times[0,\infty)} \mathbf{1}_{[0,b(x,\eta_{s-})]}(u)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}u) + |\eta_\tau\cap B|,$$

$$t\geq 0,\ B\in\mathcal{B}(\mathbb{R}^d),$$

Since the unique solution is adapted to the filtration generated by the noise and initial condition, the conditional independence follows, and (5.4) follows from the uniqueness in law. We rely here on the strong Markov property of the Poisson point process; see Proposition A.1 below. □

Corollary 5.1. *Let* τ *be an* $(\mathcal{S}_t,\ t\geq 0)$*-stopping time and* $\{y\}$ *be an* $\mathcal{S}_\tau$*-measurable finite random singleton. Then*

$$\left(\eta_{\tau+t}^{\tau,\{y\}} - y\right)_{t\geq 0} \stackrel{\mathrm{D}}{=} (\eta_t)_{t\geq 0}.$$

Proof. This is a consequence of Theorem 5.1 and Proposition 5.2. □

Consider two growth processes $(\zeta^{(1)})_t$ and $(\zeta^{(2)})_t$ defined on the common probability space and satisfying equations of the form of (5.1),

$$|\zeta_t^{(k)}\cap B| = \int_{(q,t]\times B\times[0,\infty)} \lambda\mathbf{1}_{[0,b_k(x,\zeta_{s-}^{(k)})]}(u)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}u) + |\zeta_q^{(k)}\cap B|, \qquad k=1,2. \tag{5.5}$$

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Assume that (5.5) and the rates b_1 and b_2 satisfy the conditions imposed on b in Section 2. Let $(\zeta_t^{(k)})_{t\in[0,\infty)}$ be the unique strong solution.

Lemma 5.1. *Assume that a.s.* $\zeta_0^{(1)} \subset \zeta_0^{(2)}$*, and, for any two finite configurations* $\eta^1 \subset \eta^2$*,*

$$b_1(x, \eta^1) \le b_2(x, \eta^2), \qquad x \in \mathbb{R}^d. \tag{5.6}$$

Then a.s.

$$\zeta_t^{(1)} \subset \zeta_t^{(2)}, \qquad t \in [0, \infty).$$

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}$ be the ordered sequence of the moments of birth for $(\zeta_t^{(1)})$, that is, $t \in (\sigma_n)_{n\in\mathbb{N}}$ if and only if $|\zeta_t^{(1)} \setminus \zeta_{t-}^{(1)}| = 1$. It suffices to show that, for each $n \in \mathbb{N}$, σ_n is a moment of birth for $(\zeta_t^{(2)})_{t\in[0,\infty)}$ too, and the birth occurs at the same place. We use induction on n.

Here we deal only with the base case, the induction step is carried out in the same way. Assume that

$$\zeta_{\sigma_1}^{(1)} \setminus \zeta_{\sigma_1-}^{(1)} = \{x_1\}.$$

The process $(\zeta^{(1)})_{t\in[0,\infty)}$ satisfies (5.5), therefore, $N(\{x\} \times [0, b_k(x_1, \zeta_{\sigma_1-}^{(1)})]) = 1$. Since

$$\zeta_{\sigma_1-}^{(1)} = \zeta_0^{(1)} \subset \zeta_0^{(2)} \subset \zeta_{\sigma_1-}^{(2)},$$

by (5.6),

$$N_1(\{x\} \times \{\sigma_1\} \times [0, b_k(x_1, \zeta_{\sigma_1-}^{(2)})]) = 1,$$

hence, $\zeta_{\sigma_1}^{(2)} \setminus \zeta_{\sigma_1-}^{(2)} = \{x_1\}$. □

6. Conjectures

In this section we collect some conjectures concerning the models treated in this paper and related models.

W now set d = 1. Denote by $s(k)$ the speed of propagation of the system with the birth rate (4.9). Thus, $s(k)$ is μ^{-1} in the notation of Theorem 2.1, if the birth rate is as in (4.9). In Figure 3 we present plots of $s(k)$ for the truncated birth rate (4.9).

Conjecture 6.1. *We conjecture that*

$$s(k) \to s_*, \qquad k \to \infty, \tag{6.1}$$

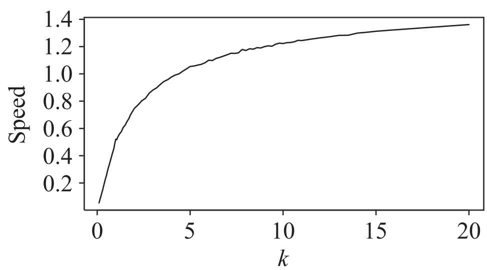

FIGURE 3: The distance to the furthest particle divided by time against k for the birth rate (3.15) at time $t = 100$.

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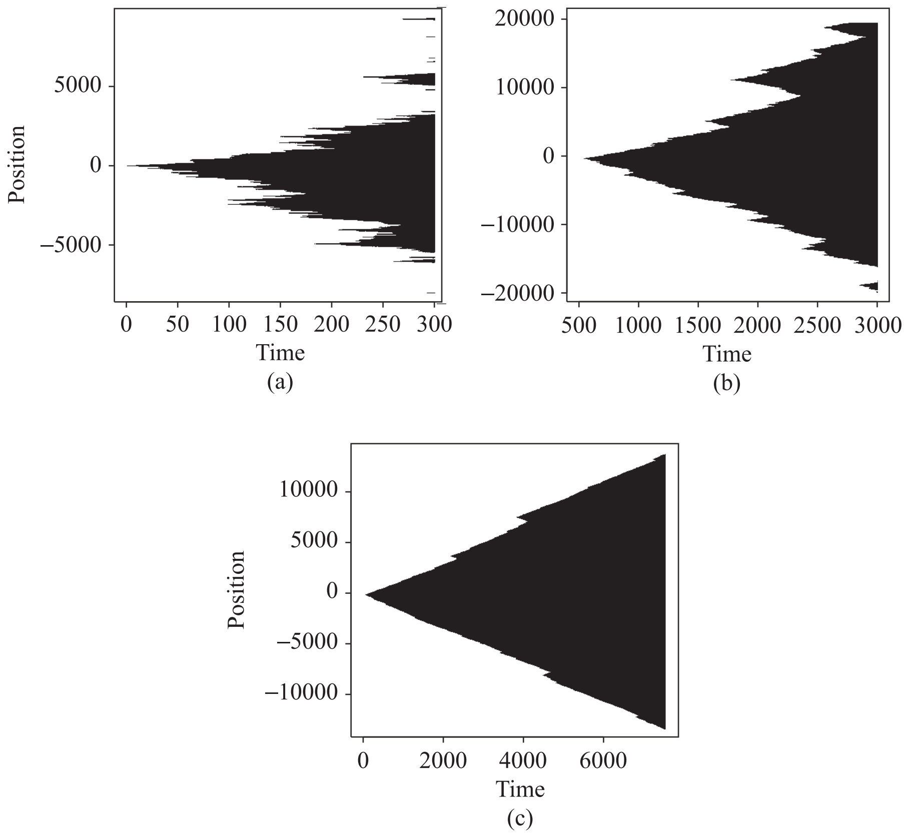

FIGURE 4: Positions of the occupied sites varying with time for the discrete-space model with the birth rate (3.21) and (a) $\alpha = 2.8$, (b) $\alpha = 3.5$, and (c) $\alpha = 4.2$.

where s_* is the speed of propagation of the process with the birth rate b_∞ given by

$$b_\infty(x, \eta) = \sum_{y \in \eta} \mathbf{1}_{\{|x-y| \le 1\}}.$$

Using the exact formula for the speed of propagation of a general branching random walk, see Proposition 1 of [3], we obtain

$$s_* = \inf_{a>0} \left\{ \inf_{\theta>0} (\mathrm{e}^{\theta} - \mathrm{e}^{-\theta} - a\theta^2) < 0 \right\} \approx 1.81\ldots.$$

The question concerning the speed of convergence in (6.1) is more subtle.

In Figures 4(a)–(c), we present representations of the evolution of the discrete version of the truncated model (2.5): the process evolves in $\mathbb{Z}^{\mathbb{Z}_+}$ and the birth rate is

$$b(x, \eta) = k \wedge \left(\sum_{y \in \eta} a_{\text{pow}}(x - y) \right)$$

with

$$a_{\text{pow}}(x) = c_{\text{pow}} \frac{1}{(|x| + 1)^{\alpha}}, \qquad x \in \mathbb{Z} \setminus \{0\}, \qquad a_{\text{pow}}(0) = 2c_{\text{pow}},$$

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where $\alpha > 2$ and $c_{\text{pow}} = c_{\text{pow}}(\alpha)$ is the normalizing constant. We have $\alpha = 2.8$, $\alpha = 3.5$, and $\alpha = 4.2$ in Figures 4(a)–(c), respectively. These representations allow us to observe the development of the set of occupied sites. We see that even for a large time, the set of occupied sites is not a connected interval for $\alpha = 2.8$, whereas the representation appears to be rather smooth for $\alpha = 4.2$. We conjecture that the speed of propagation is superlinear for $\alpha = 2.8$, but is linear for $\alpha = 4.2$. The proof of this is the subject of a forthcoming paper.

We also think that the speed of propagation has superadditive structure. For a birth rate b satisfying our assumptions, let $s(b)$ be the speed of propagation.

Conjecture 6.2. *For any birth rates b_1, b_2 satisfying our assumptions, we have*

$$s(b_1) + s(b_2) \le s(b_1 + b_2).$$

Appendix A. The strong Markov property of a Poisson point process

We need the strong Markov property of a Poisson point process. Denote $X := \mathbb{R}^d \times \mathbb{R}_+$ (compare the proof of Proposition 5.2) and let l be the Lebesgue measure on X. Consider a Poisson point process N on $\mathbb{R}_+ \times X$ with intensity measure $\mathrm{d}t \times l$. Let N be compatible with a right-continuous complete filtration $\{\mathcal{F}_t\}_{t\ge 0}$ and τ be a finite a.s. $\{\mathcal{F}_t\}_{t\ge 0}$-stopping time. Introduce another Point process $\bar{N}$ on $\mathbb{R}_+ \times X$,

$$\bar{N}([0; s] \times U) = N((\tau; \tau + s] \times U), \qquad U \in \mathcal{B}(X).$$

Proposition A.1. *The process $\bar{N}$ is a Poisson point process on $\mathbb{R}_+ \times X$ with intensity* $\mathrm{d}t \times l$*, independent of $\mathcal{F}_\tau$.*

Proof. To prove the proposition, it suffices to show the following:

(i) for any $b > a > 0$ and open bounded $U \subset X$, $\bar{N}((a; b), U)$ is a Poisson random variable with mean $(b - a)l(U)$, and

(ii) for any $b_k > a_k > 0$, $k = 1, \dots, m$, and any open bounded $U_k \subset X$, such that $((a_i; b_i) \times U_i) \cap ((a_j; b_j) \times U_j) = \varnothing$, $i \ne j$, the collection $\{\bar{N}((a_k; b_k) \times U_k)\}_{k=1,m}$ is a sequence of independent random variables, independent of $\mathcal{F}_\tau$.

Indeed, $\bar{N}$ is determined completely by values on sets of type $(b - a)\beta(U)$, a, b, U as in (i), therefore it must be an, independent of $\mathcal{F}_\tau$, Poisson point process if (i) and (ii) hold.

Let τ_n be the sequence of $\{\mathcal{F}_t\}_{t\ge 0}$-stopping times, $\tau_n = k/2^n$ on $\{\tau \in ((k-1)/2^n; k/2^n]\}$, $k \in \mathbb{N}$. Then $\tau_n \downarrow \tau$ and $\tau_n - \tau \le 1/2^n$. Note that the stopping times τ_n take countably many values only. The process N satisfies the strong Markov property for τ_n: the processes $\bar{N}_n$, defined by

$$\bar{N}_n([0; s] \times U) := N((\tau_n; \tau_n + s] \times U),$$

are Poisson point processes, independent of $\mathcal{F}_{\tau_n}$. To prove this, take k with $\mathbb{P}\{\tau_n = k/2^n\} > 0$ and note that on $\{\tau_n = k/2^n\}$, $\bar{N}_n$ coincides with the Poisson point process $\widetilde{N}_{k/2^n}$ given by

$$\widetilde{N}_{k/2^n}([0; s] \times U) := N\biggl(\biggl(\frac{k}{2^n}; \frac{k}{2^n} + s\biggr] \times U\biggr), \qquad U \in \mathcal{B}(\mathbb{R}^d).$$

Conditionally on $\{\tau_n = k/2^n\}$, $\widetilde{N}_{k/2^n}$ is again a Poisson point process, with the same intensity. Furthermore, conditionally on $\{\tau_n = k/2^n\}$, $\widetilde{N}_{k/2^n}$ is independent of $\mathcal{F}_{k/2^n}$, hence it is independent of $\mathcal{F}_\tau \subset \mathcal{F}_{k/2^n}$.

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To prove (i), note that $\bar{N}_n((a;b)\times U)\to \bar{N}((a;b)\times U)$ a.s. and all random variables $\bar{N}_n((a;b)\times U)$ have the same distribution, therefore $\bar{N}((a;b)\times U)$ is a Poisson random variable with mean $(b-a)\lambda(U)$. The random variables $\bar{N}_n((a;b)\times U)$ are independent of $\mathcal{F}_\tau$, hence $\bar{N}((a;b)\times U)$ is independent of $\mathcal{F}_\tau$, too. Similarly, (ii) follows. □

Remark A.1. We assumed in Proposition A.1 that there exists an increasing, right-continuous and complete filtration $\{\mathcal{S}_t\}_{t\ge 0}$ compatible with N. We show that such filtrations exist.

Introduce the natural filtration of N,

$$\bar{\mathcal{S}}_t^0 = \sigma\{N_k(C,B),\, B\in\mathcal{B}(\mathbb{R}^d),\, C\in\mathcal{B}([0;t])\},$$

and let $\bar{\mathcal{S}}_t$ be the completion of $\bar{\mathcal{S}}_t^0$ under $\mathbb{P}$. Then N is compatible with $\{\bar{\mathcal{S}}_t\}$. We claim that $\{\bar{\mathcal{S}}_t\}_{t\ge 0}$, defined in such a way, is right-continuous (this may be regarded as an analog of Blumenthal's 0–1 law). Indeed, as in the proof of Proposition A.1, we can check that $\widetilde{N}_a$ is independent of $\bar{\mathcal{S}}_{a+}$. Since $\bar{\mathcal{S}}_\infty = \sigma(\widetilde{N}_a)\vee\bar{\mathcal{S}}_a$, $\sigma(\widetilde{N}_a)$ and $\bar{\mathcal{S}}_a$ are independent and $\bar{\mathcal{S}}_{a+}\subset\bar{\mathcal{S}}_\infty$, we see that $\bar{\mathcal{S}}_{a+}\subset\bar{\mathcal{S}}_a$. Thus, $\bar{\mathcal{S}}_{a+}=\bar{\mathcal{S}}_a$.

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References

[1] Auffinger, A., Damron, M. and Hanson, J. (2017). 50 Years of First-Passage Percolation (University Lecture Series **68**). American Mathematical Society.

[2] Bezborodov, V. (2015). Spatial birth-and-death Markov dynamics of finite particle systems. Preprint. Available at https://arxiv.org/abs/1507.05804.

[3] Biggins, J. D. (1995). The growth and spread of the general branching random walk. *Ann. Appl. Prob.* **5,** 1008–1024.

[4] Burke, C. J. and Rosenblatt, M. (1958). A Markovian function of a Markov chain. *Ann. Math. Statist.* **29,** 1112–1122.

[5] Deijfen, M. (2003). Asymptotic shape in a continuum growth model. *Adv. Appl. Prob.* **35,** 303–318.

[6] Durrett, R. (1983). Maxima of branching random walks. *Z. Wahrscheinlichkeitsth.* **62,** 165–170.

[7] Durrett, R. (1988). *Lecture Notes on Particle Systems and Percolation*. Wadsworth & Brooks/Cole, Pacific Grove, CA.

[8] Eden, M. (1961). A two-dimensional growth process. In *Proc. 4th Berkeley Symp. Math. Statist. Prob.*, Vol. IV, University of California Press, Berkeley, CA, pp. 223–239.

[9] Eibeck, A. and Wagner, W. (2003). Stochastic interacting particle systems and nonlinear kinetic equations. *Ann. Appl. Prob.* **13,** 845–889.

[10] Fournier, N. and Méléard, S. (2004). A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Prob.* **14,** 1880–1919.
[11] Garet, O. and Marchand, R. (2012). Asymptotic shape for the contact process in random environment. *Ann. Appl. Prob.* **22,** 1362–1410.
[12] Gouéré, J.-B. and Marchand, R. (2008). Continuous first-passage percolation and continuous greedy paths model: linear growth. *Ann. Appl. Prob.* **18,** 2300–2319.
[13] Howard, C. D. and Newman, C. M. (1997). Euclidean models of first-passage percolation. *Prob. Theory Relat. Fields* **108,** 153–170.
[14] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd edn. Springer, New York.
[15] Kesten, H. (1987). Percolation theory and first-passage percolation. *Ann. Prob.* **15,** 1231–1271.
[16] Kondratiev, Y. G. and Kutoviy, O. V. (2006). On the metrical properties of the configuration space. *Math. Nachr.* **279,** 774–783.
[17] Liggett, T. M. (1985). An improved subadditive ergodic theorem. *Ann. Prob.* **13,** 1279–1285.
[18] Liggett, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, Berlin.
[19] Massoulié, L. (1998). Stability results for a general class of interacting point processes dynamics, and applications. *Stoch. Process. Appl.* **75,** 1–30.
[20] Møller, J. and Waagepetersen, R. (2004). *Statistical Inference and Simulation for Spatial Point Processes*. Chapman & Hall/CRC, Boca Raton, FL.
[21] Richardson, D. (1973). Random growth in a tessellation. *Proc. Camb. Philos. Soc.* **74,** 515–528.
[22] Röckner, M. and Schied, A. (1999). Rademacher's theorem on configuration spaces and applications. *J. Funct. Anal.* **169,** 325–356.
[23] Shi, Z. (2015). *Branching Random Walks* (Lecture Notes Math. **2151**). Springer, Cham.
[24] Tartarini, D. and Mele, E. (2016). Adult stem cell therapies for wound healing: biomaterials and computational models. *Frontiers Bioeng. Biotech.* **3,** 10.3389/fbioe.2015.00206.
[25] Treloar, K. *et al.* (2013). Multiple types of data are required to identify the mechanisms influencing the spatial expansion of melanoma cell colonies. *BMC Systems Biol.* **7,** 137.
[26] Vo, B. N., Drovandi, C. C., Pettitt, A. N. and Pettet, G. J. (2015). Melanoma cell colony expansion parameters revealed by approximate Bayesian computation. *PLOS Comput. Biol.* **11,** e1004635.
[27] Waclaw, B. *et al.* (2015). A spatial model predicts that dispersal and cell turnover limit intratumour heterogeneity. *Nature* **525,** 261–264.