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ON MORDELL–WEIL GROUPS OF JACOBIANS OVER FUNCTION FIELDS

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Abstract We study the arithmetic of abelian varieties over $K = k(t)$ where k is an arbitrary field. The main result relates Mordell–Weil groups of certain Jacobians over K to homomorphisms of other Jacobians over k. Our methods also yield completely explicit points on elliptic curves with unbounded rank over $\overline{\mathbb{F}}_p(t)$ and a new construction of elliptic curves with moderately high rank over $\mathbb{C}(t)$.

Keywords: abelian variety; Jacobian; Mordell–Weil group; function field; rank

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1. Introduction

In [24] we showed that large analytic ranks are ubiquitous in towers of function fields. For example, if k is a finite field and E is an elliptic curve over $K = k(t)$ whose j-invariant is not in k, then after replacing k by a finite extension, there are extensions of K isomorphic to $k(u)$ such that the order of vanishing at $s = 1$ of the L-functions $L(E/k(u), s)$ is unbounded. In many cases, the finite extension of k is not needed and we may take extensions of the form $k(t^{1/d})$ for d varying through integers relatively prime to the characteristic of k.

We were also able to show that the Birch and Swinnerton-Dyer conjecture holds for certain elliptic curves and higher genus Jacobians and thus exhibit abelian varieties with Mordell–Weil groups of arbitrarily large rank. However, the class of abelian varieties for which we can prove the BSD conjecture is much more limited than the class for which results on large analytic rank apply.

In her thesis (see [3]), Lisa Berger gave a construction of a large class of curves for which the BSD conjecture holds *a priori*. (Here by *a priori* BSD, we mean that the equality Rank = ord is shown to hold without necessarily calculating Rank or ord.) Combined with the L-function results of [24], this yields *families* of abelian varieties over function fields with large analytic and algebraic rank. The key point in Berger's work is a construction of towers of surfaces each of which is dominated by a product of curves in the sense of [17]. Roughly speaking, Berger shows that certain covers of products of curves are themselves dominated by products of curves by using a fundamental group argument.

The aim of this article is to elucidate the geometry of Berger's construction and to use it to study the arithmetic of abelian varieties over $K = k(t)$ where k is an arbitrary field. The main result relates Mordell–Weil groups of certain Jacobians over K to homomorphisms of other Jacobians over k.

More precisely, let $\mathcal{C}$ and $\mathcal{D}$ be two smooth, irreducible curves over k and choose non-constant, separable rational functions f on $\mathcal{C}$ and g on $\mathcal{D}$. Consider the closed subset Y of $\mathcal{C} \times_k \mathcal{D} \times_k \operatorname{Spec} K$ defined by $f - tg = 0$. Under mild hypotheses on f and g, Y is an irreducible curve and has a smooth proper model $X \to \operatorname{Spec} K$. Let J be the Jacobian of X. The main result (Theorem 6.4) is a formula relating the rank of the Mordell–Weil group of J over the fields $K_d = \overline{k}(t^{1/d})$ to homomorphisms between the Jacobians of covers of $\mathcal{C}$ and $\mathcal{D}$ over $\overline{k}$. (Here $\overline{k}$ is the algebraic closure of k.) Thus hard questions about certain abelian varieties over K_d are reduced to questions over the simpler field $\overline{k}$.

The idea of using domination by a product of curves to study Néron–Severi and Mordell–Weil groups appeared already in [18], and Shioda used these ideas in [19] to produce explicit points on an elliptic curve of large rank over $\overline{\mathbb{F}}_q(t)$. The dominating curves in Shioda's work are quotients of Fermat curves and the elliptic curves that he studies are isotrivial. The principal innovation in this paper is that we systematically use families of curves as in [3] and this leads to results about non-isotrivial abelian varieties and also to interesting specialization phenomena. It also demonstrates that high ranks are not fundamentally a byproduct of supersingularity.

Here is an outline of the paper. In §§ 2 and 3 we discuss some background material. In §§ 4–6 we explain the construction in more detail and prove the rank formula. The last three sections of the paper give two additional applications of these ideas. One yields completely explicit points on elliptic curves with unbounded rank over $\overline{\mathbb{F}}_p(t)$, and the other is a new construction of elliptic curves over $\mathbb{C}(t)$ with moderately high rank.

2. Curves and surfaces

2.1. Definitions

If F is a field and X is a scheme over $\operatorname{Spec} F$, the phrase 'X is smooth over F' means that the morphism $X \to \operatorname{Spec} F$ is smooth. This implies that X is a regular scheme. When F is not perfect, the converse is false: X being a regular scheme does not imply that X is smooth over F.

If F is perfect and X is a noetherian scheme over F, then X is smooth over F if and only if X is regular and in this case, $X \times_F \operatorname{Spec} F' \to \operatorname{Spec} F'$ is smooth for every field extension F' of F. (See for example [13, Section 4.3] for this and the previous paragraph. Also, a particularly clear treatment appears in § V.4 of a forthcoming book of Mumford and Oda.)

Throughout the paper, k is a field, $\bar{k}$ is an algebraic closure of k, and K is the rational function field $k(t)$. Note that if k has positive characteristic, then K is not perfect.

A *variety* is by definition a separated, reduced scheme of finite type over a field. A *curve* (resp. *surface*) is by definition a purely one-dimensional (resp. two-dimensional) variety.

2.2. Curves and surfaces

Let $\mathcal{X}$ be a smooth and proper surface over k, equipped with a non-constant k-morphism $\pi : \mathcal{X} \to \mathbb{P}^1_k$ to the projective line over k. Let $X \to \operatorname{Spec} K$ be the generic fiber of π, so X is a curve over K. We assume that π is generically smooth in the sense that $X \to \operatorname{Spec} K$ is smooth. (This is equivalent to the existence of a non-empty open subset $U \subset \mathbb{P}^1_k$ over which π is smooth.) If $\mathcal{X}$ is proper over k, then X is proper over K.

Conversely, if X is a smooth curve over K, then there exists a regular surface $\mathcal{X}$ over k equipped with a k-morphism $\pi : \mathcal{X} \to \mathbb{P}^1_k$ whose generic fiber is the given X. If X is proper over K then we may choose $\mathcal{X}$ to be proper over k, and if X has genus >0 and we further insist that π be relatively minimal (i.e., have no (-1)-curves in its fibers), then the pair $(\mathcal{X}, \pi)$ is unique up to isomorphism. (A convenient, modern reference for this and related facts is [13, Chapter 9].)

In the situation of the previous paragraph, when k is not perfect it could happen that $\mathcal{X} \to k$ is not smooth. To rule this out, we explicitly assume that $\mathcal{X} \to k$ is smooth. In the situations considered in this paper, the starting point will always be a surface $\mathcal{X}$ smooth over k, so this will not be an issue.

2.3. Points and sections

With notation as above, there is a bijection between sections of π and K-rational points of X. More generally, the map

$$D \mapsto D \times_{\mathbb{P}^1_k} \operatorname{Spec} K$$

induces a homomorphism $\operatorname{Div}(\mathcal{X}) \to \operatorname{Div}(X)$ from divisors on $\mathcal{X}$ to divisors on X. Under this homomorphism, multi-sections (reduced and irreducible subschemes of codimension 1 in $\mathcal{X}$ flat over $\mathbb{P}^1_k$) map to closed points of X; their residue fields are finite extensions of K of degree equal to the degree of the multi-section over $\mathbb{P}^1_k$. Divisors on $\mathcal{X}$ supported in fibers of π map to the zero (empty) divisor on X. The homomorphism $\operatorname{Div}(\mathcal{X}) \to \operatorname{Div}(X)$ is surjective since a closed point of X is the generic fiber of its scheme-theoretic closure in $\mathcal{X}$.

2.4. The Shioda–Tate formula

In this subsection, we review the Shioda–Tate formula [20], which relates the Néron–Severi group of $\mathcal{X}$ to the Mordell–Weil group of the Jacobian of X. Throughout, we assume that k is algebraically closed.

Assume that $\mathcal{X}$ is a geometrically irreducible, smooth, proper surface over k with a k-morphism $\pi : \mathcal{X} \to \mathbb{P}^1_k$ whose generic fiber $X \to \operatorname{Spec} K$ is smooth.

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Let $\mathrm{Div}\,(\mathcal{X})$ be the group of divisors on $\mathcal{X}$, let $\mathrm{Pic}\,(\mathcal{X})$ be the Picard group of $\mathcal{X}$, and let $NS(\mathcal{X})$ be the Néron–Severi group of $\mathcal{X}$. Let $\mathrm{Pic}^0(\mathcal{X})$ be the kernel of the natural surjection $\mathrm{Pic}\,(\mathcal{X}) \to NS(\mathcal{X})$.

Let $\mathrm{Div}\,(X)$ be the group of divisors on X, let $\mathrm{Pic}\,(X)$ be the Picard group of X, $\mathrm{Pic}^0(X)$ its degree zero subgroup, and $J = J_X$ the Jacobian variety of X, so $\mathrm{Pic}^0(X) = J(K)$.

Recall the natural homomorphism $\mathrm{Div}\,(\mathcal{X}) \to \mathrm{Div}\,(X)$ induced by taking the fiber product with K. For a divisor $D \in \mathrm{Div}\,(\mathcal{X})$, write $D \cdot X$ for the degree of the image of D in $\mathrm{Div}\,(X)$. Let $L^1\mathrm{Div}\,(\mathcal{X})$ be the subgroup of divisors D such that $D \cdot X = 0$ and let $L^1\mathrm{Pic}\,(\mathcal{X}) \subset \mathrm{Pic}\,(\mathcal{X})$ and $L^1NS(\mathcal{X}) \subset NS(\mathcal{X})$ be the subgroups generated by $L^1\mathrm{Div}\,(\mathcal{X})$. It is clear that $NS(\mathcal{X})/L^1NS(\mathcal{X})$ is an infinite cyclic group. The homomorphism $L^1\mathrm{Div}\,(\mathcal{X}) \to \mathrm{Div}^0(X)$ (divisors of degree 0) descends to the Picard group and yields an exact sequence

$$0 \to L^2\mathrm{Pic}\,(\mathcal{X}) \to L^1\mathrm{Pic}\,(\mathcal{X}) \to \mathrm{Pic}^0(X) \to 0 \tag{2.4.1}$$

where $L^2\mathrm{Pic}\,(\mathcal{X})$ is defined to make the sequence exact. A simple geometric argument (see p. 363 of [20]) shows that $L^2\mathrm{Pic}\,(\mathcal{X})$ is the subgroup of $\mathrm{Pic}\,(\mathcal{X})$ generated by the classes of divisors supported in the fibers of π.

Let $L^2NS(\mathcal{X})$ be the image of $L^2\mathrm{Pic}\,(\mathcal{X})$ in $NS(\mathcal{X})$. It is well known that $L^2NS(\mathcal{X})$ is free abelian of rank $1 + \sum_v (f_v - 1)$ where the sum is over closed points of $\mathbb{P}^1$ and f_v is the number of irreducible components of the fiber over v. (The main point being the fact that the intersection form restricted to the components of a fiber of π is negative semi-definite, with kernel equal to rational multiples of the fiber. See, e.g., [2, Corollary VIII.4].)

Let $\mathrm{PicVar}\,(\mathcal{X})$ be the Picard variety of $\mathcal{X}$, so $\mathrm{Pic}^0(\mathcal{X}) = \mathrm{PicVar}\,(\mathcal{X})(k)$. The homomorphism $L^1\mathrm{Div}\,(\mathcal{X}) \to \mathrm{Div}^0(X)$ induces a morphism of abelian varieties $\mathrm{PicVar}\,(\mathcal{X}) \to J$. Let (B, τ) be the K/k-trace of J. (See [5] for a modern account.) By the universal property of the K/k trace, the homomorphism above factors through $\mathrm{PicVar}\,(\mathcal{X}) \to B$.

Proposition 2.5 (Shioda–Tate). *With the notation and hypotheses as above, we have:*

(1) *The canonical morphism of k-abelian varieties $\mathrm{PicVar}\,(\mathcal{X}) \to B = \mathrm{Tr}_{K/k}J$ is an isomorphism.*

(2) *Defining the Mordell–Weil group of J to be $MW(J) = J(K)/\tau B(k)$, we have*

$$MW(J) \cong \frac{L^1NS(\mathcal{X})}{L^2NS(\mathcal{X})}.$$

(3) *In particular,*

$$\mathrm{Rank}\, MW(J) = \mathrm{Rank}\, NS(\mathcal{X}) - 2 - \sum_v (f_v - 1)$$

where the sum is over the closed points of $\mathbb{P}^1_k$ and f_v is the number of irreducible components in the fiber of π over v.

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It is well known, but apparently not written down in detail, that this result holds under considerably more general hypotheses, for example when k is finite. We will discuss this and related questions in a future publication.

2.6. Heights

In [20], Shioda also defines a height pairing on $MW(J)$. We briefly review the construction.

Choose a class $c \in MW(J)$. Shioda proves that there is an element $[\tilde{D}] \in L^1NS(\mathcal{X}) \otimes \mathbb{Q}$ which is congruent to c in $MW(J) \otimes \mathbb{Q} \cong (L^1NS(\mathcal{X})/L^2NS(\mathcal{X})) \otimes \mathbb{Q}$ and which is orthogonal to $L^2(\mathcal{X})$ under the intersection pairing. Roughly speaking, one 'lifts' c to a divisor D on $\mathcal{X}$ whose class is in $L^1(\mathcal{X})$ and then 'corrects' D by an element of $L^2NS(\mathcal{X}) \otimes \mathbb{Q}$ to ensure orthogonality with $L^2NS(\mathcal{X})$. The resulting $[\tilde{D}]$ is unique up to the addition of a multiple of F, the class of a fiber of π in $NS(\mathcal{X})$. If c and c' are elements of $MW(J)$ with corresponding divisors $\tilde{D}$ and $\tilde{D}'$, then Shioda defines their inner product by

$$\langle c, c' \rangle = -\tilde{D} \cdot \tilde{D}'$$

where the dot on the right signifies the intersection product on $\mathcal{X}$. Shioda proves that $\langle \cdot, \cdot \rangle$ yields a positive definite symmetric inner product on the real vector space $MW(J) \otimes \mathbb{R}$.

When $k = \mathbb{F}_q$ is finite, it is well known (and not difficult to show) that $\langle \cdot, \cdot \rangle \log q$ enjoys the properties characterizing the Néron–Tate canonical height.

3. Domination by a product of curves

3.1. DPC

A variety $\mathcal{X}$ over k is said to be 'dominated by a product of curves' (or 'DPC') if there is a dominant rational map $\prod_i \mathcal{C}_i \dashrightarrow \mathcal{X}$ where each $\mathcal{C}_i$ is a curve over k. This notion was introduced and studied by Schoen in [17], following Deligne [10]. If $\mathcal{X}$ has dimension n and is DPC then it admits a dominant rational map from a product of n curves [17, Lemma 6.1]. We will be concerned only with the case $n = 2$, and therefore with rational maps from products of two curves $\mathcal{C} \times_k \mathcal{D}$ to a surface $\mathcal{X}$.

Schoen develops several invariants related to the notion of DPC and obstructions to a variety being DPC. In particular, he shows that for any field k, there are surfaces over k which are not DPC.

3.2. DPC and NS

Part of the interest of the notion of DPC is that the Néron–Severi group of a product of curves is well understood. Indeed (under the assumption that $\mathcal{C}$ and $\mathcal{D}$ have k-rational points), we have a canonical isomorphism

$$NS(\mathcal{C} \times \mathcal{D}) \cong \mathbb{Z}^2 \times \mathrm{Hom}_{k-av}(J_{\mathcal{C}}, J_{\mathcal{D}})$$

where $J_{\mathcal{C}}$ and $J_{\mathcal{D}}$ are the Jacobians of $\mathcal{C}$ and $\mathcal{D}$ and the homomorphisms are of abelian varieties over k. The factor $\mathbb{Z}^2$ corresponds to classes of divisors which are 'vertical' or 'horizontal', i.e., contained in the fibers of the projections to $\mathcal{C}$ or $\mathcal{D}$. The map from

NS to Hom sends a divisor class to the action of the induced correspondence on the Jacobians. (This is classical. See [14, Corollary 6.3] for a modern account.)

If $\mathcal{C} \times \mathcal{D} \dashrightarrow \mathcal{X}$ is a dominant rational map, then after finitely many blow-ups, we have a morphism $\widetilde{\mathcal{C} \times \mathcal{D}} \to \mathcal{X}$. The effect of a blow-up on NS is well known (each blow-up adds a factor of $\mathbb{Z}$ to NS) and so we have good control of $NS(\widetilde{\mathcal{C} \times \mathcal{D}})$. In favorable cases, this can be extended to good control of $NS(\mathcal{X})$.

3.3. Finitely generated k

In the case where k is finitely generated over its prime field, there is a further favorable consequence of DPC, namely the Tate conjecture on divisors and cohomology classes. Indeed, this conjecture has been proven for products of curves (by Tate when k is finite [21], by Zarhin when k is finitely generated, of characteristic $p > 0$ [25], and by Faltings when k is finitely generated, of characteristic 0 [12]). It follows immediately for products of curves blown up at finitely many closed points and it then descends to images of these varieties. The details of the descent are explained in [23]. The upshot is that for $\mathcal{X}$ a smooth and proper variety of any dimension over a finitely generated field k which is DPC, and for any ℓ not equal to the characteristic of k, we have

$$\operatorname{Rank} NS(\mathcal{X}) = \dim_{\mathbb{Q}_\ell} H^2(\mathcal{X} \times \overline{k}, \mathbb{Q}_\ell)\,(1)^{\mathrm{Gal}\,(\overline{k}/k)}.$$

The cohomology on the right is ℓ-adic étale cohomology and the superscript denotes the space of invariants under the natural action of Galois.

3.4. Finite k

Now assume that k is finite, $\mathcal{X}$ is a smooth and proper surface over k, and $\mathcal{X}$ is equipped with a morphism $\pi : \mathcal{X} \to \mathbb{P}^1$ as in § 2.2. Let X/K be the generic fiber of π and $J = J_X$ the Jacobian of X. If $\mathcal{X}$ is DPC, then the Tate conjecture holds for $\mathcal{X}$. It is well known that the Tate conjecture for $\mathcal{X}$ is equivalent to the Birch and Swinnerton-Dyer (BSD) conjecture for J [22]. Thus DPC of $\mathcal{X}$ gives *a priori* BSD for J. Explicitly, we have

$$\operatorname{Rank} MW(J) = \operatorname{ord}_{s=1} L(J, s) = \operatorname{ord}_{s=1} L(X, s).$$

(We also have the finiteness of $Ш(J/K)$ and the refined conjecture of Birch and Swinnerton-Dyer, but we will not discuss these facts in detail here.)

3.5. DPCT

As noted in the introduction, when k is finite [24] gives a large supply of curves X over K such that $\operatorname{ord}_{s=1} L(X/K_d, s)$ is unbounded as d varies through integers prime to the characteristic of k and $K_d = k(t^{1/d})$. If one has the BSD conjecture for $J = \operatorname{Jac}(X)$ in each layer of the tower, then one obtains many examples of Mordell–Weil groups of unbounded rank.

As we have seen, domination by a product of curves provides *a priori* BSD. However, we arrive at the following problem: Suppose that $\mathcal{X}$ is DPC and $\pi : \mathcal{X} \to \mathbb{P}^1$ is as in § 2.2, so that we have BSD for $J = J_X$ over K. Let $\mathcal{X}_d$ be a smooth proper model over k of the

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fiber product

$$\begin{array}{ccc} \mathbb{P}^1_k \times_{\mathbb{P}^1_k} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi_1 \\ \mathbb{P}^1_k & \xrightarrow[r_d]{} & \mathbb{P}^1_k \end{array}$$

where $r_d^*(t) = u^d$. Then it is not true in general that $\mathcal{X}_d$ is DPC and we cannot conclude that BSD holds for J_X/K_d. The motivation for Berger's thesis was to find a large supply of surfaces $\mathcal{X}$ which are fibered over $\mathbb{P}^1$ such that the base changed surfaces as above remain DPC. She calls this 'domination by a product of curves in a tower' or 'DPCT'. If $\mathcal{X} \to \mathbb{P}^1$ is DPCT, then the generic fibers of $\pi_d : \mathcal{X}_d \to \mathbb{P}^1$ are curves over $K_d = k(u) = k(t^{1/d})$ which will often have unbounded analytic and algebraic rank. In fact, Berger's construction leads to families of elliptic curves over K with unbounded rank.

The aim of the rest of this paper is to elucidate the geometry of Berger's construction and to use it to study ranks over arbitrary fields k.

4. Berger's construction

4.1. Special pencils

Fix two smooth, proper, irreducible curves $\mathcal{C}$ and $\mathcal{D}$ over k, and choose non-constant, separable rational functions $f : \mathcal{C} \to \mathbb{P}^1$ and $g : \mathcal{D} \to \mathbb{P}^1$. For simplicity, we assume that the zeros and poles of f and g are k-rational. (This is not essential for any of the results below, but it simplifies the formulas.) Write the divisors of f and g as

$$\mathrm{div}\,(f) = \sum_{i=1}^{k} a_i P_i - \sum_{i'=1}^{k'} a'_{i'} P'_{i'} \quad \text{and} \quad \mathrm{div}\,(g) = \sum_{j=1}^{\ell} b_j Q_j - \sum_{j'=1}^{\ell'} b'_{j'} Q'_{j'}$$

with $a_i, a'_{i'}, b_i, b'_{j'}$ positive integers and P_i, $P'_{i'}$, Q_j, and $Q'_{j'}$ distinct k-rational points. (We hope the use of k for the ground field and for the number of zeros of f will not cause confusion.) Let

$$m = \sum_{i=1}^{k} a_i = \sum_{i'=1}^{k'} a'_{i'} \quad \text{and} \quad n = \sum_{j=1}^{\ell} b_j = \sum_{j'=1}^{\ell'} b'_{j'}.$$

We make two standing assumptions on the multiplicities a_i, $a'_{i'}$, b_j, and $b'_{j'}$:

$$\gcd(a_1, \ldots, a_k, a'_1, \ldots, a'_{k'}, b_1, \ldots, b_\ell, b'_1, \ldots, b'_{\ell'}) = 1$$
$$\text{for all } i, j, \mathrm{char}(k) \nmid \gcd(a_i, b_j) \text{ and for all } i', j', \mathrm{char}(k) \nmid \gcd(a'_{i'}, b'_{j'}). \quad (4.1.1)$$

Here $\mathrm{char}(k)$ is the characteristic of the ground field k.

Define a rational map $\pi_0 : \mathcal{C} \times_k \mathcal{D} \dashrightarrow \mathbb{P}^1_k$ by the formula

$$\pi_0(x, y) = [f(x) : g(y)]$$

or, in terms of the standard coordinate t on P^1_k, $t = f(x)/g(y)$. It is clear that π_0 is defined away from the 'base points' (P_i, Q_j) $(1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant \ell)$ and $(P'_{i'}, Q'_{j'})$ $(1 \leqslant i' \leqslant k'$,

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$1 \leqslant j' \leqslant \ell'$). Let U be $\mathcal{C} \times_k \mathcal{D}$ with the base points removed, so that π_0 defines a morphism $\pi_0|_U : U \to \mathbb{P}^1_k$.

It is useful to think of π_0 as giving a pencil of divisors on $\mathcal{C} \times_k \mathcal{D}$: for each closed point $t \in \mathbb{P}^1_k$, the Zariski closure in $\mathcal{C} \times_k \mathcal{D}$ of $(\pi_0|_U)^{-1}(t)$ is a divisor on $\mathcal{C} \times_k \mathcal{D}$ passing through the base points which we denote as $\overline{\pi_0^{-1}(t)}$. As t varies we get a pencil of divisors with base locus exactly $\{(P_i, Q_j), (P'_{i'}, Q'_{j'})\}$. A key feature of this pencil is that its fibers over 0 and ∞ are unions of 'horizontal' and 'vertical' divisors, i.e.,

$$\overline{\pi_0^{-1}(0)} = \left(\bigcup_{i=1}^{k} \{a_i P_i\} \times \mathcal{D} \right) \cup \left(\bigcup_{j'=1}^{\ell'} \mathcal{C} \times \{b'_{j'} Q'_{j'}\} \right)$$

and

$$\overline{\pi_0^{-1}(\infty)} = \left(\bigcup_{i'=1}^{k'} \{a'_{i'} P_{i'}\} \times \mathcal{D} \right) \cup \left(\bigcup_{j=1}^{\ell} \mathcal{C} \times \{b_j Q_j\} \right).$$

In the proof of the proposition below, we will construct a specific blow-up $\mathcal{X}_1$ of $\mathcal{C} \times_k \mathcal{D}$ such that the composed rational map $\mathcal{X}_1 \to \mathcal{C} \times_k \mathcal{D} \dashrightarrow \mathbb{P}^1$ is a generically smooth morphism $\pi_1 : \mathcal{X}_1 \to \mathbb{P}^1$. (There are many such $\mathcal{X}_1$, but the statement of the theorem below is independent of this choice.) Let $X_1 \to \operatorname{Spec} K$ be the generic fiber of π_1 so that X_1 is a smooth curve over K.

For each positive integer d prime to the characteristic of k, let $r_d : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the morphism with $r_d^*(t) = u^d$ (i.e., $u \mapsto t = u^d$ in the standard affine coordinates t and u) and form the base change

$$\begin{array}{ccc} \mathcal{S}_d := \mathbb{P}^1_k \times_{\mathbb{P}^1_k} \mathcal{X}_1 & \longrightarrow & \mathcal{X}_1 \\ \downarrow & & \downarrow \pi_1 \\ \mathbb{P}^1_k & \xrightarrow{\ r_d\ } & \mathbb{P}^1_k \end{array}$$

The fiber product $\mathcal{S}_d$ will usually not be smooth over k (or even normal) because both r_d and π_1 have critical points over 0 and ∞. Let $\mathcal{X}_d \to \mathcal{S}_d$ be a blow-up of the normalization of the fiber product such that $\mathcal{X}_d$ is smooth over k. (Again, there are many such, but the statement of the theorem is independent of the choice.) Let $\pi_d : \mathcal{X}_d \to \mathbb{P}^1_k$ be the composition of $\mathcal{X}_d \to \mathcal{S}_d$ with the projection onto $\mathbb{P}^1_k$. (The domain $\mathbb{P}^1_k$ for π_d is thus the $\mathbb{P}^1_k$ in the lower left of the displayed diagram.) Let K_d be the function field of this $\mathbb{P}^1_k$ so that $K_d = k(t^{1/d}) \cong k(u)$ is an extension of K of degree d. Finally, let $X_d \to \operatorname{Spec} K_d$ be the generic fiber of π_d. Then we have $X_d \cong X_1 \times_{\operatorname{Spec} K} \operatorname{Spec} K_d$.

With this notation, we can state the main results of Berger's thesis [3, Theorems 2.2 and 3.1].

Theorem 4.2 (Berger). *Choose data $k, \mathcal{C}, \mathcal{D}, f$, and g as above, subject to the hypotheses (4.1.1). For each d prime to the characteristic of k, construct the fibered surfaces $\pi_d : \mathcal{X}_d \to \mathbb{P}^1$ and the curves X_d/K_d as above.*

(1) *$X = X_1$ is a smooth, proper curve of genus*

$$g = mg_{\mathcal{D}} + ng_{\mathcal{C}} + (m-1)(n-1) - \sum_{i,j} \delta(a_i, b_j) - \sum_{i',j'} \delta(a'_{i'}, b'_{j'})$$

where $g_{\mathcal{C}}$ and $g_{\mathcal{D}}$ are the genera of $\mathcal{C}$ and $\mathcal{D}$ respectively, $m = \sum_{i=1}^{k} a_i$, $n = \sum_{j=1}^{\ell} b_j$, and $\delta(a, b) = (ab - a - b + \gcd(a, b))/2$.

(2) *X_d is irreducible and remains irreducible over $\bar{k}K_d$.*

(3) *$\mathcal{X}_d$ is dominated by a product of curves.*

Remarks 4.3. (1) Note that the generic fiber of π_0 is an open subset of the curve called Y in the introduction and so the curve X of the theorem is a smooth proper model of Y.

(2) We will sketch the proof of part (1) of the proposition, essentially along the lines of Berger's thesis, in the rest of this section.

(3) Berger proved that $\mathcal{X}_d$ is DPC by using a fundamental group argument, generalizing [17, Proposition 6.6]. In the next section, we will give a different and more explicit proof which will be the basis for a general rank formula.

(4) Berger proved a stronger irreducibility result, namely that X is absolutely irreducible. The result stated above is sufficient for our purposes and follows easily from the same ideas as were used to prove the domination by a product of curves.

4.4. Proof of Theorem 4.2, part (1)

What we have to prove is that there is a blow-up of $\mathcal{C} \times_k \mathcal{D}$ such that π_0 becomes a generically smooth morphism, and that its generic fiber has the properties announced in part (1).

Recall that $\pi_0|_U : U \to \mathbb{P}^1$ is the maximal open where π_0 is a morphism. An easy application of the Jacobian criterion shows that $\pi_0|_U$ has only finitely many critical values and so all but finitely many fibers of $\pi_0|_U$ are smooth (open) curves. But their closures $\overline{\pi_0^{-1}(t)}$ may be quite singular at the base points.

Let us focus attention near a base point $(P, Q) = (P_i, Q_j)$. (The argument near a base point $(P'_{i'}, Q'_{j'})$ is essentially identical and will be omitted.) Choose uniformizing parameters x and y on $\mathcal{C}$ and $\mathcal{D}$ at P and Q respectively. Then in a Zariski open neighborhood of (P, Q), the map π_0 is defined away from (P, Q) and given by

$$(\alpha, \beta) \mapsto [x(\alpha)^a : y(\beta)^b\, u(\alpha, \beta)]$$

where $(a, b) = (a_i, b_j)$ and where u is a unit in the local ring at (P, Q). We say more succinctly that π_0 is given by $[x^a : y^b u]$. Now blowing up (P, Q) and passing to suitable coordinates, $\tilde{\pi}_0$, the composition of the blow-up with π_0, is given by $[\tilde{x}^{a-b} : \tilde{y}^b \tilde{u}]$ if $a \geqslant b$ or by $[\tilde{x}^a : \tilde{y}^{b-a} \tilde{u}]$ if $a \leqslant b$. Thus if $a = b$, $\tilde{\pi}_0$ is a morphism in a neighborhood of the inverse image of (P, Q). If $a \neq b$, there is exactly one point of indeterminacy upstairs. We relabel $\tilde{\pi}_0$ as π_0 and iterate. After finitely many blow-ups, π_0 is given by $[x^c, u]$ where $c = \gcd(a, b)$, and is defined everywhere on an open neighborhood of the inverse image

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of (P, Q). We note for future use that the exceptional fibers of the blow-ups at stages where $a \neq b$ map to 0 or $\infty \in \mathbb{P}^1_k$ whereas the exceptional divisor at the last stage (when $a = b = c$) maps c-to-1 to $\mathbb{P}^1_k$.

For a pair of positive integers (a, b), let $\gamma(a, b)$ be the number of steps for proceeding from (a, b) to $(\gcd(a, b), 0)$ by subtracting the smaller of the pair from the larger at each step. (So, for example, $\gamma(1, 1) = 1$, $\gamma(a, a) = 1$, $\gamma(a, 1) = a$ and $\gamma(2, 3) = 3$.) The discussion above shows that if we blow up $\mathcal{C} \times_k \mathcal{D}$ at the base points (P_i, Q_j) $\gamma(a_i, b_j)$ times and at $(P'_{i'}, Q'_{j'})$ $\gamma(a'_{i'}, b'_{j'})$ times we arrive at a surface $\mathcal{X}_1 \to \mathcal{C} \times_k \mathcal{D}$ such that the composed rational map $\mathcal{X}_1 \to \mathcal{C} \times_k \mathcal{D} \dashrightarrow \mathbb{P}^1$ is a morphism $\pi_1 : \mathcal{X}_1 \to \mathbb{P}^1$. I claim that π_1 is generically smooth with fibers of genus as stated in the proposition.

We have already seen that most fibers of π_1 are smooth away from the base points. Near a base point, $\mathcal{F} = \overline{\pi_0^{-1}(t)}$ is given locally by $tx^a - y^b u = 0$. If $a \neq b$ then the tangent cone is a single line and the proper transform of $\mathcal{F}$ has a single point over the base point, at $x = y = 0$ in the coordinates used above. Continuing in this fashion, there is always a unique point over (P, Q) in the proper transform of $\mathcal{F}$, except at the penultimate step, where the proper transform of $\mathcal{F}$ is given locally by $tx^c - y^c u = 0$ and its tangent cone consists of c distinct lines. (Here we use the hypothesis (4.1.1) which implies that c is prime to the characteristic of k.) In the final blow-up, the proper transform has c points over the base point and π is smooth in a neighborhood of these points, as long as we avoid $t = 0$ and $t = \infty$.

This proves that all but finitely many fibers of π_1 are smooth proper curves. To find their genus, note that the fibers $\overline{\pi_0^{-1}(t)}$ are curves of bidegree (m, n) on $\mathcal{C} \times_k \mathcal{D}$ and thus have arithmetic genus $mg_{\mathcal{D}} + ng_{\mathcal{C}} + (m-1)(n-1)$. A blow-up when the local equation is $tx^a - y^b u = 0$ decreases the arithmetic genus by $e(e-1)/2$ where $e = \min(a, b)$. The value of the genus of the fibers of π_1 then follows by a simple calculation. (See [3, §§ 3.7 and 3.8] for more details.)

This completes the proof of part (1) of Theorem 4.2. □

It is worth noting that the morphism $\pi_1 : \mathcal{X}_1 \to \mathbb{P}^1_k$ constructed above may not be relatively minimal.

5. Explicit domination by a product of curves

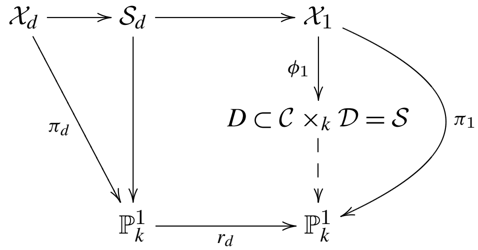
5.1. Kummer covers of $\mathcal{C}$ and $\mathcal{D}$

For each d, let $\mathcal{C}_d$ be the smooth, proper, (possibly reducible) covering of $\mathcal{C}$ of degree d defined by the equation $z^d = f(x)$ and let $\mathcal{D}_d$ be defined similarly by $w^d = g(y)$. Let $e = e_{d,f}$ be the largest divisor of d such that f is an eth power in $k(\mathcal{C})^\times$, so that $\mathcal{C}_d$ is irreducible over k if and only if $e_{d,f} = 1$. Note that e is a divisor of $\gcd(d, a_1, \ldots, a_k, a'_1, \ldots, a'_{k'})$, with equality when $k = \overline{k}$ and $\mathcal{C} = \mathbb{P}^1$. A similar discussion applies to g and $\mathcal{D}_d$. The hypothesis (4.1.1) implies that $\gcd(e_{d,f}, e_{d,g}) = 1$.

Let $\mathcal{C}^o_d$ be $\mathcal{C}_d$ with the (inverse images of the) zeros and poles of f removed, and let $\mathcal{D}^o_d$ be $\mathcal{D}_d$ with the (inverse images of the) zeros and poles of g removed. Then $\mathcal{C}^o_d \to \mathcal{C}^o_1$ and $\mathcal{D}^o_d \to \mathcal{D}^o_1$ are étale torsors for μ_d. In particular, there is a free action of the étale group scheme μ_d over k on $\mathcal{C}^o_d$ and $\mathcal{D}^o_d$ and the quotients are $\mathcal{C}^o_1$ and $\mathcal{D}^o_1$.

5.2. Analysis of a fiber product

Let $\mathcal{S} = \mathcal{C} \times \mathcal{D}$ and let $D \subset \mathcal{S}$ be the reduced divisor supported on $\mathcal{C} \times \mathcal{D} \setminus \mathcal{C}^o \times \mathcal{D}^o$. Let $\phi_1 : \mathcal{X}_1 \to \mathcal{S}$ be the blow-up described in the proof of Theorem 4.2(1). For $d > 1$, let $\mathcal{S}_d$ be the fiber product $\mathbb{P}^1_k \times_{\mathbb{P}^1_k} \mathcal{X}_1$ defined in the previous section and let $\phi_d : \mathcal{X}_d \to \mathcal{S}$ be the composition $\mathcal{X}_d \to \mathcal{S}_d \to \mathcal{X}_1 \to \mathcal{S}$. The following diagram may be helpful in organizing the definitions:



(5.2.1)

For $d \geqslant 1$, we define $\mathcal{X}_d^o$ to be $\mathcal{X}_d$ minus the divisor $\phi_d^{-1}(D)$. It is clear that ϕ_1 induces an isomorphism $\mathcal{X}_1^o \cong \mathcal{C}^o \times \mathcal{D}^o$ and that $\mathcal{X}_d^o \cong \mathcal{S}_d^o$ is isomorphic to the fiber product of $\mathcal{X}_1^o \to \mathbb{P}^1 \setminus \{0, \infty\}$ with the étale morphism $\mathbb{P}^1 \setminus \{0, \infty\} \to \mathbb{P}^1 \setminus \{0, \infty\}$, $t \mapsto t^d$.

Proposition 5.3. *There is a canonical isomorphism*

$$\mathcal{X}_d^o \cong \left(\mathcal{C}_d^o \times \mathcal{D}_d^o\right) / \mu_d$$

where μ_d acts on $\mathcal{C}_d^o \times \mathcal{D}_d^o$ diagonally.

Proof. By definition, $\mathcal{X}_1^o$ is the open subset of $\mathcal{C} \times_k \mathcal{D}$ where $f(x)$ and $g(y)$ are $\neq 0, \infty$. By the discussion above, $\mathcal{X}_d^o$ is the closed subset of

$$\mathcal{X}_1^o \times_k \mathbb{A}^1_k = \mathcal{C}^o \times_k \mathcal{D}^o \times_k \mathbb{A}^1_k$$

(with coordinates (x, y, t)) where $f(x) = t^d g(y)$.

On the other hand, $\mathcal{C}_d^o \times_k \mathcal{D}_d^o$ is isomorphic to the closed subset of

$$\mathcal{C}^0 \times_k \mathbb{A}^1_k \times_k \mathcal{D}^o \times_k \mathbb{A}^1_k$$

(with coordinates (x, z, y, w)) where $f(x) = z^d$ and $g(y) = w^d$. It is then clear that the morphism $(x, z, y, w) \mapsto (x, y, z/w)$ presents $\mathcal{C}_d^o \times \mathcal{D}_d^o$ as a μ_d-torsor over $\mathcal{X}_d^o$. □

5.4. Proof of Theorem 4.2, parts (2) and (3)

Since $\mathcal{X}_d^o$ is birational to $\mathcal{X}_d$, Proposition 5.3 shows that $\mathcal{X}_d$ is dominated by a product of curves, i.e., we have part (3) of Theorem 4.2.

For part (2), we may assume that $k = \bar{k}$. In this case, $e_{d,f}$ and $e_{d,g}$ (defined in the first paragraph of § 5.1) are the numbers of irreducible components of $\mathcal{C}_d$ and $\mathcal{D}_d$ respectively. The components of $\mathcal{C}_d$ are a torsor for $\mu_{e_{d,f}}$; those of $\mathcal{D}_d$ are a torsor for $\mu_{e_{d,g}}$. The action of μ_d on the components is via the quotients $\mu_d \to \mu_{e_{d,f}}$ and $\mu_d \to \mu_{e_{d,g}}$. By hypothesis (4.1.1), $\gcd(e_{d,f}, e_{d,g}) = 1$. This implies that under the diagonal action, μ_d acts transitively on the set of components of $\mathcal{C}_d \times_k \mathcal{D}_d$ and so $(\mathcal{C}_d \times_k \mathcal{D}_d)/\mu_d$ is

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irreducible. It follows that $\mathcal{X}_d$ is geometrically irreducible and therefore that X_d is irreducible over $\bar{k}K_d$.

This completes the proof of Theorem 4.2. □

This proof suggests another way to construct $\mathcal{X}_d$, namely starting from $\mathcal{C}_d \times_k \mathcal{D}_d$. In the next section we will carry out this construction and use it to give a formula for the rank of $MW(J_{X_d})$.

5.5. The K/k trace of J_d

We end this section with an analysis of B_d, the K/k-trace of $J_d = \mathrm{Jac}(X_d)$ under the simplifying assumption that k is algebraically closed. Fix d and recall $e_{d,f}$ and $e_{d,g}$ from § 5.1. Fix also roots $f' = f^{1/e_{d,f}}$ and $g' = g^{1/e_{d,g}}$ and write $\mathcal{C}'_{d'}$ and $\mathcal{D}'_{d'}$ for the covers of $\mathcal{C}$ and $\mathcal{D}$ defined by the equations $z^{d'} = f'$ and $w^{d'} = g'$.

Proposition 5.6. *Assume that k is algebraically closed. With the notation as above, the K/k-trace B_d of $J_d = \mathrm{Jac}(X_d)$ is canonically isomorphic to $J_{\mathcal{C}'_{e_{d,g}}} \times_k J_{\mathcal{D}'_{e_{d,f}}}$.*

(The positions of $e_{d,f}$ and $e_{d,g}$ in this formula are not typos — see the proof below.)

Proof. By Proposition 2.5, B_d is isomorphic to the Picard variety of $\mathcal{X}_d$ and since the Picard variety is a birational invariant, we need to compute $\mathrm{PicVar}\,((\mathcal{C}_d \times_k \mathcal{D}_d)/\mu_d)$. Applying [20, Theorem 2] twice, we have that for irreducible curves $\mathcal{C}$ and $\mathcal{D}$ over k, $\mathrm{PicVar}\,(\mathcal{C} \times_k \mathcal{D}) \cong J_{\mathcal{C}} \times_k J_{\mathcal{D}}$.

Now the components of $\mathcal{C}_d$ (resp. $\mathcal{D}_d$) are indexed by $\mu_{e_{d,f}}$ (resp. $\mu_{e_{d,g}}$) and each one is isomorphic to $\mathcal{C}'_{d/e_{d,f}}$ (resp. $\mathcal{D}'_{d/e_{d,g}}$). The action of μ_d on the components $\mathcal{C}_d \times_k \mathcal{D}_d$ is transitive, with stabilizer $\mu_{d/(e_{d,f}e_{d,g})}$. Thus

$$\mathrm{PicVar}\,((\mathcal{C}_d \times_k \mathcal{D}_d)/\mu_d) \cong \mathrm{PicVar}\left((\mathcal{C}'_{d/e_{d,f}} \times_k \mathcal{D}'_{d/e_{d,g}})/\mu_{d/(e_{d,f}e_{d,g})}\right).$$

Applying the formula above for products of curves and passing to the $\mu_{d/(e_{d,f}e_{d,g})}$ invariants, we have

$$B_d \cong J_{\mathcal{C}'_{e_{d,g}}} \times_k J_{\mathcal{D}'_{e_{d,f}}}.$$

This completes the proof of the proposition. □

A nice example occurs in [3, Section 4.3(4)] where $\mathcal{C} = \mathcal{D} = \mathbb{P}^1_k$, $f(x) = x^2$ and $g(y)$ is a quadratic rational function with distinct zeros and poles so that $e_{d,f} = \gcd(2, d)$, $e_{d,g} = 1$. It follows that when d is odd, $B_d = 0$ and when d is even, B_d is the elliptic curve $w^2 = g(y)$. This is confirmed in [3] by explicit computation.

Remark 5.7. Let B'_d be the image of the K/k-trace $\tau : B_d \times_k K \to J_d$ and define an abelian variety A_d such that

$$0 \to B'_d \to J_d \to A_d \to 0$$

is exact. Since τ is purely inseparable [5, Theorem 6.12], we have $B'_d(K) = B_d(K)$ and since $K = k(\mathbb{P}^1_k)$, we have $B_d(K) = B_d(k)$. Thus the Mordell–Weil group $J_d(K)/\tau B_d(k)$ is a subgroup of the more traditional Mordell–Weil group $A_d(K)$.

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6. A rank formula

6.1. Numerical invariants

Throughout this section, we assume that k is algebraically closed. Fix data $\mathcal{C}$, $\mathcal{D}$, f, and g as in Berger's construction (§ 4.1), subject to the hypothesis (4.1.1). For each positive integer d not divisible by the characteristic of k, form covers $\mathcal{C}_d \to \mathcal{C}$ and $\mathcal{D}_d \to \mathcal{D}$ defined by the equations $z^d = f$ and $w^d = g$. Recall that (since $k = \bar{k}$) $e_{d,f}$ and $e_{d,g}$ are the numbers of irreducible components of $\mathcal{C}_d$ and $\mathcal{D}_d$ respectively.

For each i (resp. i', j, j'), let $r_{d,i}$ (resp. $r'_{d,i'}$, $s_{d,j}$, $s'_{d,j'}$) be the number of closed points of $\mathcal{C}_d$ over P_i (resp. of $\mathcal{C}_d$ over $P'_{i'}$, of $\mathcal{D}_d$ over Q_j, of $\mathcal{D}_d$ over $Q'_{j'}$). Also, for each pair (i, j) (resp. (i', j')) let $t_{d,i,j}$ (resp. $t'_{d,i',j'}$) be the number of closed points of $(\mathcal{C}_d \times_k \mathcal{D}_d)/\mu_d$ (the quotient by the diagonal action) over (P_i, Q_j) (resp. over $(P'_{i'}, Q'_{j'})$). Since k is algebraically closed, we have the equalities

$$\begin{aligned} r_{d,i} &= \gcd(a_i, d) & r'_{d,i'} &= \gcd(a'_{i'}, d) \\ s_{d,j} &= \gcd(b_j, d) & s'_{d,j'} &= \gcd(b'_{j'}, d) \\ t_{d,i,j} &= \gcd(a_i, b_j, d) & t'_{d,i',j'} &= \gcd(a'_{i'}, b'_{j'}, d). \end{aligned}$$

On $\mathcal{C}_d$, z has a zero of order $a_i/r_{d,i}$ at each point over P_i and a pole of order $a'_{i'}/r'_{d,i'}$ at each point over $P'_{i'}$, and similarly for w and $\mathcal{D}_d \to \mathcal{D}$. Let

$$k_d = \sum_{i=1}^{k} r_{d,i} \quad k'_d = \sum_{i'=1}^{k'} r'_{d,i'}$$
$$\ell_d = \sum_{j=1}^{\ell} s_{d,j} \quad \ell'_d = \sum_{j'=1}^{\ell'} s'_{d,j'}.$$

6.2. A nice model for X_d

Now consider the rational map $\mathcal{C}_d \times \mathcal{D}_d \dashrightarrow \mathbb{P}^1_k$ defined by z/w. This is defined away from $k_d\ell_d + k'_d\ell'_d$ base points. Blowing up each one as in § 4.4 several times (more precisely, $\gamma(a_i/r_{d,i}, b_j/s_{d,j})$ times at points over (P_i, Q_j) and $\gamma(a'_{i'}/r'_{d,i'}, b'_{j'}/s'_{d,j'})$ times at points over $(P'_{i'}, Q'_{j'})$), we arrive at a surface $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}$ equipped with a generically smooth morphism to $\mathbb{P}^1_k$ which we again denote as z/w. Over each base point, all but one component of the exceptional locus maps to 0 or ∞ in $\mathbb{P}^1_k$ and the remaining component (the exceptional divisor of the last blow-up at that base point) maps finitely to $\mathbb{P}^1_k$. Write

$$\gamma_{d,i,j} = \gamma\left(\frac{a_i}{r_{d,i}}, \frac{b_j}{s_{d,j}}\right)$$

$(i = 1, \ldots, k, j = 1, \ldots, \ell)$ and

$$\gamma'_{d,i',j'} = \gamma\left(\frac{a'_{i'}}{r'_{d,i'}}, \frac{b'_{j'}}{s'_{d,j'}}\right)$$

$(i' = 1, \ldots, k', j' = 1, \ldots, \ell')$ for the number of blow-ups needed at each base point.

The diagonal action of μ_d on $\mathcal{C}_d \times_k \mathcal{D}_d$ lifts canonically to the blow-up $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}$. The locus where the action has non-trivial stabilizers has divisorial components and sometimes has isolated fixed points (depending on the order of zeros or poles of z and w at the base points). The isolated fixed points all map via z/w to 0 or ∞ in $\mathbb{P}^1_k$. Let $\mathcal{Y}_d$ be the quotient $(\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d})/\mu_d$. The morphism $z/w : \widetilde{\mathcal{C}_d \times_k \mathcal{D}_d} \to \mathbb{P}^1_k$ descends to a morphism which is generically smooth and which we again denote as $z/w : \mathcal{Y}_d \to \mathbb{P}^1$. If the action of μ_d on $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}$ has any isolated fixed points, then $\mathcal{Y}_d$ has cyclic quotient singularities as studied by Hirzebruch and Jung. The resolution of these singularities is well known: the exceptional locus is a chain of $\mathbb{P}^1$s whose length and self-intersections are given by a Hirzebruch–Jung continued fraction (see [1, Section III.5]). Let $\mathcal{X}_d \to \mathcal{Y}_d$ be the canonical resolution of the quotient singularities and let HJ_d be the number of irreducible components of the exceptional locus. The only point about the singularities that will be relevant for us is that every component of the exceptional locus lies over 0 or $\infty \in \mathbb{P}^1_k$. Write π_d for the composition of $\mathcal{X}_d \to \mathcal{Y}_d$ with z/w. Then $\mathcal{X}_d$ is smooth and proper over k, $\pi_d : \mathcal{X}_d \to \mathbb{P}^1_k$ is generically smooth, and $\mathcal{X}_d$ is birational to the fiber product $\mathcal{S}_d$ of § 4.1. So $\mathcal{X}_d$ is a nice model of X_d and we can use it to compute the rank of $MW(J_d)$.

6.3. More numerical invariants

To state our main result, we define two more numerical invariants. For each closed point v of $\mathbb{P}^1_k$, let $f_{d,v}$ be the number of irreducible components in the fiber of π_d over v. Define

$$c_1(d) = \sum_{v \neq 0,\infty} (f_{d,v} - 1) = d \sum_{v \neq 0,\infty} (f_{1,v} - 1).$$

Here the first sum is over places of the $\mathbb{P}^1$ in the lower left corner of diagram (5.2.1) and the second is over places of the $\mathbb{P}^1$ in the lower right of the same diagram. The second equality uses that k is algebraically closed.

Define a second function $c_2(d)$ by

$$\begin{aligned} c_2(d) = & \sum_{i=1,\dots,k, j=1,\dots,\ell} t_{d,i,j} - \sum_{i=1,\dots,k} \gcd(a_i, e_{d,g}) - \sum_{j=1,\dots,\ell} \gcd(b_j, e_{d,f}) + 1 \\ & + \sum_{i'=1,\dots,k', j'=1,\dots,\ell'} t'_{d,i',j'} - \sum_{i'=1,\dots,k'} \gcd(a'_{i'}, e_{d,g}) \\ & - \sum_{j'=1,\dots,\ell'} \gcd(b'_{j'}, e_{d,f}) + 1. \end{aligned} \tag{6.3.1}$$

For d prime to the multiplicities a_i, $a'_{i'}$, b_j, $b'_{j'}$, we have

$$c_2(d) = c_2(1) = (k-1)(\ell-1) + (k'-1)(\ell'-1).$$

For all d, we have $c_2(d) \geqslant c_2(1)$ and $c_2(d)$ is periodic and so bounded.

We note that c_2 is of a combinatorial nature – it depends only on the multiplicities a_i, $a'_{i'}$, b_j, and $b'_{j'}$ – whereas c_1 in general depends also on the positions of the points P_i, $P'_{i'}$, Q_j, and $Q'_{j'}$.

Before stating the theorem, we recall that $e_{d,f}$ is the largest divisor of d such that f is an $e_{d,f}$th power in $k(\mathcal{C})^\times$ and similarly for $e_{d,g}$. Choose roots $f' = f^{1/e_{d,f}}$ and $g' = g^{1/e_{d,g}}$ and let $\mathcal{C}'_{d/e_{d,f}}$ and $\mathcal{D}'_{d/e_{d,g}}$ be the covers of $\mathcal{C}$ and $\mathcal{D}$ defined by $z^{d/e_{d,f}} = f'$ and $w^{d/e_{d,g}} = g'$ respectively. Their Jacobians will be denoted $J_{\mathcal{C}'_{d/e_{d,f}}}$ and $J_{\mathcal{D}'_{d/e_{d,g}}}$ respectively.

Theorem 6.4. *Assume that k is algebraically closed. Choose data $\mathcal{C}$, $\mathcal{D}$, f, and g as above, subject to the hypotheses (4.1.1). Let X be the smooth proper model of*

$$\{f - tg = 0\} \subset \mathcal{C} \times_k \mathcal{D} \times_k \operatorname{Spec} K$$

over $K = k(t)$ constructed in § 4. For each d prime to the characteristic of k let $K_d = k(t^{1/d})$, and let J_d be the Jacobian of X over K_d. Then with notation as above,

$$\operatorname{Rank} MW(J_d) = \operatorname{Rank} \operatorname{Hom}_{k-av}\left(J_{\mathcal{C}'_{d/e_{d,f}}}, J_{\mathcal{D}'_{d/e_{d,g}}}\right)^{\mu_{d/e_{d,f}e_{d,g}}} - c_1(d) + c_2(d).$$

Here $\operatorname{Hom}_{k-av}$ denotes homomorphisms of abelian varieties over k and the exponent $\mu_{d/e_{d,f}e_{d,g}}$ signifies those homomorphisms which commute with the action of $\mu_{d/e_{d,f}e_{d,g}}$.

In the case where $e_{d,f} = e_{d,g} = 1$, and so $\mathcal{C}_d$ and $\mathcal{D}_d$ are irreducible, the rank formula simplifies to

$$\operatorname{Rank} MW(J_d) = \operatorname{Rank} \operatorname{Hom}_{k-av}(J_{\mathcal{C}_d}, J_{\mathcal{D}_d})^{\mu_d} - c_1(d) + c_2(d).$$

Proof. The proof consists of computing the rank of the Néron–Severi group of $\mathcal{X}_d$ using the construction of $\mathcal{X}_d$ via $\mathcal{C}_d \times_k \mathcal{D}_d$ and then applying the Shioda–Tate formula.

First note that $\mathcal{C}_d \times_k \mathcal{D}_d$ is isomorphic to the disjoint union of $e_{d,f}e_{d,g}$ copies of $\mathcal{C}'_{d/e_{d,f}} \times_k \mathcal{D}'_{d/e_{d,g}}$ so the Néron–Severi group of $\mathcal{C}_d \times_k \mathcal{D}_d$ is isomorphic to

$$\left(\operatorname{Hom}_{k-av}(J_{\mathcal{C}'_{d/e_{d,f}}}, J_{\mathcal{D}'_{d/e_{d,g}}}) \oplus \mathbb{Z}^2\right)^{e_{d,f}e_{d,g}}.$$

Each blow-up in passing from $\mathcal{C}_d \times_k \mathcal{D}_d$ to $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}$ contributes a factor of $\mathbb{Z}$ to the Néron–Severi group. Grouping these according to the base point (P_i, Q_j) or $(P'_{i'}, Q'_{j'})$ over which they lie, we have

$$\begin{aligned} NS(\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}) \cong & \left(\operatorname{Hom}_{k-av}(J_{\mathcal{C}'_{d/e_{d,f}}}, J_{\mathcal{D}'_{d/e_{d,g}}}) \oplus \mathbb{Z}^2\right)^{e_{d,f}e_{d,g}} \\ & \bigoplus_{i,j} \mathbb{Z}^{r_{d,i}s_{d,j}\gamma_{d,i,j}} \\ & \bigoplus_{i',j'} \mathbb{Z}^{r'_{d,i'}s'_{d,j'}\gamma'_{d,i',j'}}. \end{aligned}$$

Now consider the action of μ_d on $NS(\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d})$. The group μ_d acts transitively on the components of $\mathcal{C}_d \times_k \mathcal{D}_d$ and the stabilizer of a component is $\mu_{d/e_{d,f}e_{d,g}}$. Thus the group of invariants under μ_d of the first summand above is

$$\operatorname{Hom}_{k-av}\left(J_{\mathcal{C}'_{d/e_{d,f}}}, J_{\mathcal{D}'_{d/e_{d,g}}}\right)^{\mu_{d/e_{d,f}e_{d,g}}} \oplus \mathbb{Z}^2$$

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where we use an exponent to denote invariants. For a fixed (i, j), the action of μ_d on the base points of z/w on $\mathcal{C}_d \times_k \mathcal{D}_d$ over $(P_i, Q_j) \in \mathcal{C} \times_k \mathcal{D}$ has $t_{d,i,j}$ orbits. The stabilizer of one of these points acts trivially on the part of the exceptional locus of $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d} \to \mathcal{C}_d \times_k \mathcal{D}_d$ over that point. Thus the contribution to the rank of $NS(\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}/\mu_d)$ corresponding to (P_i, Q_j) is $t_{d,i,j}\gamma_{d,i,j}$. A similar discussion applies to the base points $(P'_{i'}, Q'_{j'})$. Thus the rank of the Néron–Severi group of $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}/\mu_d$ is

$$\operatorname{Rank} \operatorname{Hom}_{k-av}\left(J_{\mathcal{C}'_{d/e_{d,f}}}, J_{\mathcal{D}'_{d/e_{d,g}}}\right)^{\mu_{d/e_{d,f}e_{d,g}}} + 2 + \sum_{i,j} t_{d,i,j}\gamma_{d,i,j} + \sum_{i',j'} t'_{d,i',j'}\gamma'_{d,i',j'}.$$

The resolution of the Hirzebruch–Jung singularities contributes an additional HJ_d (by definition) to the rank of $NS(\mathcal{X}_d)$, so we have

$$\begin{aligned} \operatorname{Rank} NS(\mathcal{X}_d) = {} & \operatorname{Rank} \operatorname{Hom}_{k-av}\left(J_{\mathcal{C}'_{d/e_{d,f}}}, J_{\mathcal{D}'_{d/e_{d,g}}}\right)^{\mu_{d/e_{d,f}e_{d,g}}} + 2 \\ & + \sum_{i,j} t_{i,j}\gamma_{d,i,j} + \sum_{i',j'} t'_{d,i',j'}\gamma'_{d,i',j'} + HJ_d. \end{aligned} \tag{6.4.1}$$

Now consider the morphism $\pi_d : \mathcal{X}_d \to \mathbb{P}^1_k$ and the Shioda–Tate formula. There are three sources of components of $\mathcal{X}_d$ over 0 or ∞ in $\mathbb{P}^1_k$: (i) the images in $\mathcal{X}_d$ of components of curves on $\mathcal{C}_d \times_k \mathcal{D}_d$ over the curves $\{P_i\} \times_k \mathcal{D}$, $\{P_{i'}\} \times_k \mathcal{D}$, $\mathcal{C} \times_k \{Q_j\}$, and $\mathcal{C} \times_k \{Q_{j'}\}$ in $\mathcal{C} \times_k \mathcal{D}$; (ii) the images in $\mathcal{X}_d$ of the exceptional divisors of the blow-ups needed to pass from $\mathcal{C}_d \times_k \mathcal{D}_d$ to $\widetilde{\mathcal{C}_d \times_k \mathcal{D}_d}$ (more precisely the exceptional divisors for all but the last blow-up at each base point – see the third paragraph of § 5.4); and (iii) the exceptional divisors of the blow-ups needed to resolve any Hirzebruch–Jung singularities. Taking into account the action of μ_d as in the previous paragraph we find that

$$\begin{aligned} f_{d,\infty} + f_{d,0} = {} & \sum_{i=1,\dots,k} \gcd(a_i, e_{d,g}) + \sum_{i'=1,\dots,k'} \gcd(a'_{i'}, e_{d,g}) \\ & + \sum_{j=1,\dots,\ell} \gcd(b_j, e_{d,f}) + \sum_{j'=1,\dots,\ell'} \gcd(b'_{j'}, e_{d,f}) \\ & + \sum_{ij} t_{d,i,j}(\gamma_{d,i,j} - 1) + \sum_{i'j'} t'_{d,i',j'}(\gamma'_{d,i',j'} - 1) + HJ_d. \end{aligned}$$

Combining this and (6.4.1) with the Shioda–Tate formula Proposition 2.5 yields the stated formula for $\operatorname{Rank} MW(J_d)$.

This completes the proof of Theorem 6.4. □

Remarks 6.5. (1) We used the hypothesis that k is algebraically closed in two ways. First, it is used in the Shioda–Tate formula. In fact, this formula is known to hold in much greater generality; this will be discussed in detail in a forthcoming publication. Second, without this hypothesis, rationality issues would come into the definitions of the numerical invariants. This is more of an inconvenience than a fundamental issue. Thus, with some adjustments, we have a rank formula without the assumption that k is algebraically closed.

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(2) The reader will note that there is a great deal of flexibility in choosing the input data for Berger's construction. It is useful to think of first choosing the topological data, namely the genera of $\mathcal{C}$ and $\mathcal{D}$ and the multiplicities in the divisors of f and g, and then consider the (connected) family of choices of the isomorphism classes of the curves and the locations of the zeros and poles.

(3) There is an interesting difference in nature of the three terms in the rank formula. The term c_2 depends only on the topological data. The term c_1 depends on the continuous data in an algebro-geometric way – the sets on which it is constant are constructible. The $\operatorname{Rank}\operatorname{Hom}(\cdots)$ term is more arithmetic in nature – it may jump up on a countable collection of subschemes. Thus we may expect that the rank may go up countably often. The examples in the remainder of the paper illustrate two facets of this remark. One involves ranks going up in all positive characteristics and the other involves ranks going up for countably many values of a complex parameter.

In the rest of the paper we discuss two examples which illustrate certain aspects of Berger's construction and the rank formula. Further applications of these ideas appear in [15] and papers based on it.

7. A first example

The first interesting examples occur already when $\mathcal{C}$ and $\mathcal{D}$ are rational and f and g are quadratic.

7.1. Data

Let $\mathcal{C} = \mathcal{D} = \mathbb{P}^1_k$, and let

$$f(x) = x(x-1) \quad \text{and} \quad g(y) = 1/f(1/y) = y^2/(1-y).$$

The hypotheses (4.1.1) on f and g are satisfied no matter what the characteristic of k is, and for all d, $e_{d,f} = e_{d,g} = 1$.

7.2. The surface $\mathcal{X}_1$

The surface $\mathcal{X}_1$ is obtained by blowing up $\mathcal{C} \times \mathcal{D}$ twice $(\gamma(2,1) = 2)$ at each of the four base points $(x, y) = (0, 0)\ (1, 0), (\infty, 1)$, and (∞, ∞).

Straightforward calculation shows that the morphism $\pi_1 : \mathcal{X}_1 \to \mathbb{P}^1_k$ is smooth away from $t = 0, \infty$, and, if $\operatorname{char}(k) \neq 2$, $t = 1/16$. The smooth fibers have genus 1. The fiber over 0 is a configuration of Kodaira type I_4, the fiber over ∞ is a configuration of Kodaira type I_1^*, and the fiber over $t = 1/16$ is an irreducible rational curve with one node, i.e., a configuration of Kodaira type I_1.

It follows immediately that $c_1(d) = c_2(d) = 0$ for all d.

7.3. The curve X

The curve

$$Y = \{f - tg = 0\} \subset \mathbb{P}^1_k \times_k \mathbb{P}^1_k \times \operatorname{Spec} K \cong \mathbb{P}^1_K \times_K \mathbb{P}^1_K$$

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is smooth of bidegree (2, 2) and so has genus 1. The change of coordinates $x = -y/(x+t)$, $y = -x/t$ brings it into the Weierstrass form

$$X : y^2 + xy + ty = x^3 + tx^2.$$

The discriminant of this model is $\Delta = t^4(1 - 16t)$ and the j-invariant is $j = (16t^2 - 16t + 1)^3/\Delta$. (Dick Gross points out that over $\mathbb{Z}[1/2]$, X is the universal elliptic curve over the modular curve $X_1(4) \cong \mathbb{P}^1$.)

Since X is elliptic, it is its own Jacobian and by Proposition 5.6 (or because $j(X) \notin k$), the K/k trace of X_d is 0 for all d. Thus the rank formula in Theorem 6.4 says that if k is algebraically closed

$$\operatorname{Rank} X_d(K_d) = \operatorname{Rank} \operatorname{Hom}_{k-av}(J_{\mathcal{C}_d}, J_{\mathcal{D}_d})^{\mu_d}.$$

7.4. Homomorphisms and endomorphisms

Note that $\mathcal{C}_d \cong \mathcal{D}_d$ via $(x, z) \mapsto (y = 1/x, w = 1/z)$. This isomorphism, call it σ, anti-commutes with the μ_d action in the sense that $\sigma \circ [\zeta_d] = [\zeta_d^{-1}] \circ \sigma$. Using it to identify $\mathcal{C}_d$ with $\mathcal{D}_d$, we have

$$\operatorname{Hom}_{k-av}(J_{\mathcal{C}_d}, J_{\mathcal{D}_d})^{\mu_d} \cong \operatorname{End}_{k-av}(J_{\mathcal{C}_d})^{anti-\mu_d}$$

where the superscript denotes those endomorphisms anti-commuting with the μ_d action. Thus if k is algebraically closed we have

$$\operatorname{Rank} X_d(K_d) = \operatorname{Rank} \operatorname{End}_{k-av}(J_{\mathcal{C}_d})^{anti-\mu_d}.$$

We can make this much more explicit. Note that since $\mathcal{C}_d$ is defined over the prime field, every endomorphism of $J_{\mathcal{C}_d}$ is defined over an algebraic extension of the prime field. Thus there is no loss in assuming that k is the algebraic closure of the prime field.

Let $\phi(e)$ be the cardinality of $(\mathbb{Z}/e\mathbb{Z})^\times$ and let $o_e(q)$ be the order of q in $(\mathbb{Z}/e\mathbb{Z})^\times$.

Theorem 7.5. *If* $\operatorname{char}(k) = 0$, *then* $\operatorname{Rank} X_d(K_d) = 0$ *for all* d. *If* $\operatorname{char}(k) = p > 0$, *then* $\operatorname{Rank} X_d(K_d)$ *is unbounded as* d *varies. More precisely, suppose that* $k = \mathbb{F}_q$ *and* $d = p^n + 1$. *Then*

$$\operatorname{Rank} X_d(K_d) \geqslant \sum_{e|d, e>2} \phi(e)/o_e(q) \geqslant (p^n - 1)/2n.$$

If k *contains the* d *th roots of unity,* $\operatorname{Rank} X_d(K_d) = p^n - 1$ *is* p *is odd and* p^n *if* $p = 2$.

The proof of the theorem will occupy the rest of this section. In the following section, we will see that the points asserted to exist here can be made quite explicit.

7.6. The case char$(k) = 0$

We begin with the case $\operatorname{char}(k) = 0$ where we may assume that k is algebraically closed. Note that $\mathcal{C}_d$ has genus $(d-1)/2$ when d is odd and $(d-2)/2$ when d is even. Let $\zeta_d \in k$ be a primitive dth root of unity and $[\zeta_d]$ the corresponding endomorphism of $J_{\mathcal{C}_d}$. We consider the faithful action of $\operatorname{End}(J_{\mathcal{C}_d})$ on $H^0(J_{\mathcal{C}_d}, \Omega^1) = H^0(\mathcal{C}_d, \Omega^1)$. It is easy to write down an explicit basis of the 1-forms and to see that the eigenvalues of $[\zeta_d]$ on this space

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are ζ_d^i $i=1,\ldots,g_{\mathcal{C}_d}$. In particular, the eigenvalues of $[\zeta_d]$ and those of $[\zeta_d^{-1}]$ are disjoint. Therefore, no non-zero endomorphism ϕ of $J_{\mathcal{C}_d}$ can satisfy $\phi\circ[\zeta_d]=[\zeta_d^{-1}]\circ\phi$. Thus

$$\operatorname{Rank} X_d(K_d)=\operatorname{Rank}\operatorname{End}_{k-av}(J_{\mathcal{C}_d})^{anti-\mu_d}=0$$

and this proves the first part of the theorem.

7.7. Old and new

Between here and § 7.10, we assume that k is algebraically closed, of characteristic $p>0$. The modifications needed to treat non-algebraically closed k will be given in that subsection.

The discussion in this subsection is not specific to the example under consideration and makes sense in the general context of Berger's construction (§ 4).

If e is a divisor of d then there is a natural map $\mathcal{C}_d\to\mathcal{C}_e$ $(z\mapsto z^{d/e})$ which induces an injective homomorphism $J_{\mathcal{C}_e}\to J_{\mathcal{C}_d}$. We write $A_d^{\mathrm{old}}\subset J_{\mathcal{C}_d}$ for the sum over all $e|d$, $e\neq d$, of the images of these maps. Choose a complement to A_d^{old} in $J_{\mathcal{C}_d}$ (defined only up to isogeny), denoted as A_d^{new}, so that we have an isogeny $J_{\mathcal{C}_d}\to A_d^{\mathrm{old}}\oplus A_d^{\mathrm{new}}$. It is clear that the action of μ_d on $J_{\mathcal{C}_d}$ preserves A_d^{old} and therefore induces a homomorphism $\mu_d\to\operatorname{Aut}(A_d^{\mathrm{new}})\subset\operatorname{End}(A_d^{\mathrm{new}})$.

Lemma 7.7.1. *Let G be a cyclic group of order d and let R be the group ring $\mathbb{Q}[G]$. Let $R\to\operatorname{End}(J_{\mathcal{C}_d})\otimes\mathbb{Q}$ be a homomorphism which sends a generator of G to the action of a primitive d th root of unity on $J_{\mathcal{C}_d}$. Suppose that A_d^{new} is non-zero. Then the image of the composed map*

$$R\to\operatorname{End}(J_{\mathcal{C}_d})\otimes\mathbb{Q}\to\operatorname{End}(A_d^{new})\otimes\mathbb{Q}$$

is isomorphic to the field $\mathbb{Q}(\mu_d)$.

Proof. When k has characteristic zero, the lemma follows easily from a consideration of the action of ζ_d on 1-forms, but we need a proof that works over any k. For this, étale cohomology serves as a good replacement for the 1-forms.

Choose a prime ℓ not equal to the characteristic of k. Consider the (faithful) action of $\operatorname{End}^0(J_{\mathcal{C}_d})=\operatorname{End}(J_{\mathcal{C}_d})\otimes\mathbb{Q}$ on

$$H^1(J_{\mathcal{C}_d},\mathbb{Q}_\ell)=\bigoplus_{e|d}H^1(A_e^{\mathrm{new}},\mathbb{Q}_\ell).$$

It will suffice to determine the image of R in $\operatorname{End}(H^1(A_d^{\mathrm{new}},\mathbb{Q}_\ell))$.

If r is an element of R, we will write $H^1(\cdots)^{r=0}$ for the kernel of r on $H^1(\cdots)$.

Let $\Phi_e(x)$ be the cyclotomic polynomial of order e, so $x^d-1=\prod_{e|d}\Phi_e(x)$. Since $R\cong\mathbb{Q}[x]/(x^d-1)$, the Chinese remainder theorem implies that

$$R\cong\prod_{e|d}\mathbb{Q}(\mu_e).$$

Now the quotient of $\mathcal{C}_d$ by the group generated by a primitive e th root of unity $\zeta_d^{d/e}$ is $\mathcal{C}_e$. This implies that

$$H^1(J_{\mathcal{C}_d},\mathbb{Q}_\ell)^{[\zeta_d^{d/e}]-1=0}=H^1(J_{\mathcal{C}_e},\mathbb{Q}_\ell)\subset H^1(A_d^{\mathrm{old}},\mathbb{Q}_\ell).$$

This in turn implies that in the map $R \to \mathrm{End}\,(H^1(A_d^{\mathrm{new}}, \mathbb{Q}_\ell))$, all the factors $\mathbb{Q}(\mu_e)$ with $e < d$ go to zero. But if $A_d^{\mathrm{new}} \neq 0$, then $R \to \mathrm{End}\,(H^1(A_d^{\mathrm{new}}, \mathbb{Q}_\ell))$ is not the zero homomorphism and this forces its restriction to the factor $\mathbb{Q}(\mu_d)$ to be non-zero. Since $\mathbb{Q}(\mu_d)$ is a field, it maps isomorphically onto its image and this proves our claim. □

7.8. Dimensions

In the context of the data chosen at the beginning of this section, the Riemann–Hurwitz formula implies that $A_e^{\mathrm{new}} = 0$ for $e = 1$ and $e = 2$, and that it has dimension $\phi(e)/2$ when $e > 2$. The lemma implies that the image of R in $\mathrm{End}^0(J_{\mathcal{C}_d})$ has dimension $d - 1$ when d is odd and $d - 2$ when d is even.

7.9. Special endomorphisms

Now suppose that d has the special form $d = p^n + 1$. Let $\mathrm{Fr}_q : \mathcal{C}_d \to \mathcal{C}_d$ be the q-power Frobenius where $q = p^n$. Then it is immediate that $[\zeta_d] \circ \mathrm{Fr}_q = \mathrm{Fr}_q \circ [\zeta_d^{-1}]$, i.e., Fr_q gives an element of $\mathrm{End}\ (J_{\mathcal{C}_d})^{anti-\mu_d}$. The same calculation shows that $\mathrm{Fr}_q \circ [\zeta_d^i] \in \mathrm{End}\ (J_{\mathcal{C}_d})^{anti-\mu_d}$ for $i = 0, \dots, d-1$. Since Fr_q is not a zero divisor in $\mathrm{End}\,(J_{\mathcal{C}_d})$, the relations among the $\mathrm{Fr}_q \circ [\zeta_d^i]$ are the same as the relations among the $[\zeta_d^i]$. Lemma 7.7.1 shows that the endomorphisms $[\zeta_d^i]$, $(i = 0, \dots, d-1)$ generate a subalgebra of $\mathrm{End}\,(J_{\mathcal{C}_d})$ of rank $p^n - 1$ when p is odd and of rank p^n when $p = 2$. Thus the endomorphisms $\mathrm{Fr}_q \circ [\zeta_d^i]$, $(i = 0, \dots, d-1)$ span a sublattice of $\mathrm{End}\ (J_{\mathcal{C}_d})^{anti-\mu_d}$ of the same rank. This shows that when k has characteristic p and $d = p^n + 1$, then the rank of $X_d(K_d)$ is bounded below by $p^n - 1$ when p is odd and by p^n when $p = 2$.

There are several ways to see that these lower bounds are equalities. One is to examine more closely the structure of $\mathrm{End}^0(J_{\mathcal{C}_d}^{\mathrm{new}})$. Another is to reduce to the case where k is the algebraic closure of a finite field and then note that by the Grothendieck–Ogg–Shafarevitch formula, the degree of the L-function of X_d over K_d is $p^n - 1$ (p odd) or p^n ($p = 2$) and so we have an *a priori* upper bound on the rank. We omit the details.

7.10. General k

For k not necessarily algebraically closed, we can compute the rank of $X_d(K_d)$ by passing to $\bar{k}$ and then taking invariants under $\mathrm{Gal}\,(\bar{k}/k)$. The calculations are straightforward and so again we omit the details.

Remarks 7.11. (1) The example above, at least as regards analytic ranks, can also be treated by the methods of [24, Theorem 4.7].

(2) It is evident that if k has characteristic p, then starting with any $\mathcal{C} = \mathcal{D}$ and f, g with $g(y) = 1/f(y)$ we will arrive at an X with unbounded Mordell–Weil rank in the tower of fields K_d as d varies through the 'special' values $p^n + 1$. It seems that these examples are not all covered by the methods of [24].

(3) Still assuming that k has positive characteristic, a different construction leads to examples of ranks which grow linearly in the tower K_d. In other words, we have non-trivial lower bounds on ranks for all d, not just those dividing $p^n + 1$ for some n. This is one of the main results of [15].

8. Explicit points

The interesting term in the rank formula in Theorem 6.4 is $\mathrm{Hom}_{k-av}(J_{\mathcal{C}_d}, J_{\mathcal{C}_d})$ which is a quotient of $NS(\mathcal{C}_d \times_k \mathcal{D}_d)$. In the example of the previous section, large ranks came from endomorphisms $\mathrm{Fr}_q \circ [\zeta_d^i]$ of $J_{\mathcal{C}_d}$ which in turn are induced by certain explicit maps from $\mathcal{C}_d$ to itself. As we will see in this section, tracing through the geometry of Berger's construction leads to remarkable explicit expressions for points on the elliptic curve in the last section.

Theorem 8.1. *Fix a prime number p, let $k = \overline{\mathbb{F}}_p$, and let $K = k(t)$. Let X be the elliptic curve*

$$y^2 + xy + ty = x^3 + tx^2.$$

For each d prime to p, let $K_d = k(t^{1/d}) \cong k(u)$. We write ζ_d for a fixed primitive d th root of unity in k.

(1) *For all d, the torsion subgroup $X(K_d)_{tor}$ of $X(K_d)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and is generated by $Q = (0,0)$. We have $2Q = (-t, 0)$ and $3Q = (0, -t)$.*

(2) *If $p = 2$, and $d = p^n + 1$, let $q = p^n$ and*

$$P(u) = \left(u^q(u^q - u),\, u^{2q} \left(\sum_{j=1}^{n} u^{2^j} \right) \right).$$

Then the points $P_i = P(\zeta_d^i u)$ for $i = 0, \ldots, d-1$ lie in $X(K_d)$ and they generate a finite index subgroup of $X(K_d)$, which has rank $d-1$. The relation among them is that $\sum_{i=0}^{d-1} P_i$ is torsion.

(3) *If $p > 2$, and $d = p^n + 1$, let $q = p^n$ and*

$$P(u) = \left(\frac{u^q(u^q - u)}{(1+4u)^q}, \frac{u^{2q}(1 + 2u + 2u^q)}{2\,(1+4u)^{(3q-1)/2}} - \frac{u^{2q}}{2\,(1+4u)^{q-1}} \right).$$

Then the points $P_i = P(\zeta_d^i u)$ for $i = 0, \ldots, d-1$ lie in $X(K_d)$ and they generate a finite index subgroup of $X(K_d)$, which has rank $d-2$. The relations among them are that $\sum_{i=0}^{d-1} P_i$ and $\sum_{i=0}^{d-1} (-1)^i P_i$ are torsion.

Remarks 8.2. (1) The expression for $P(u)$ when p is odd is remarkable: it is in closed form (no ellipses or summations) and depends on p only through the exponents.

(2) The lack of dependence of P on p fits very well with some recent speculations of de Jong (cf. [9]).

(3) We saw in Theorem 7.5 that X has large rank already over $\mathbb{F}_p(t^{1/d})$. We can find generators for a subgroup of finite index by summing the explicit points over orbits under $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. But it is not clear whether it is possible to find expressions for these points which are as explicit and compact as those above.

(4) We will explain how to get the points in the theorem from Berger's construction, but once they are in hand, that they are on the curve can be verified by direct calculation and that they generate a large rank subgroup can be checked by simple height computations which we will sketch in § 8.8 below.

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(5) As an aid to the reader who wants to check that $P(u) \in X(K_d)$, we note that for p odd, the right-hand and left-hand sides of the Weierstrass equation evaluated at $P(u)$ are both equal to $u^{4q}(u^{2q} - 2u^{q+1} + u^2)/\,(1+4u)^{3q-1}$.

(6) Ricardo Conceição has recently produced isotrivial elliptic curves over $\mathbb{F}_q(t)$ of large rank with many explicit, independent polynomial points. See [4].

8.3. The plan

We will sketch the proof of the theorem in the rest of this section. The calculation of the torsion subgroup is straightforward using the Shioda–Tate isomorphism $MW(J_X) \cong L^1NS(\mathcal{X})/L^2NS(\mathcal{X})$ explained in § 2.4 and the explicit construction of $\mathcal{X}_d$ given in § 6. We omit the details.

8.4. Graphs of endomorphisms

For the rest of the section, we assume that d has the form $d = p^n + 1$ and we let $q = p^n$.

Recall that $\mathrm{End}_{k-av}(J_{\mathcal{C}_d})$ and $\mathrm{Hom}_{k-av}(J_{\mathcal{C}_d}, J_{\mathcal{D}_d})$ are quotients of $NS(\mathcal{C}_d \times_k \mathcal{C}_d)$ and $NS(\mathcal{C}_d \times_k \mathcal{D}_d)$ respectively, via the homomorphism which sends a divisor to the action of the induced correspondence on the Jacobians. The endomorphism $\mathrm{Fr}_q \circ [\zeta_d^i]$ of $J_{\mathcal{C}_d}$ is induced by the graph of the morphism $\mathrm{Fr}_q \circ [\zeta_d^i] : \mathcal{C}_d \to \mathcal{C}_d$. The statements about the relations among the P_i follow from Lemma 7.7.1. To find explicit coordinates, it suffices to treat the case $i = 0$ since the other points P_i are the images of P_0 under $\mathrm{Gal}\,(K_d/K)$. So from now on we take $i = 0$.

Using the isomorphism $\mathcal{C}_d \to \mathcal{D}_d$ which sends (x, z) to $(y, w) = (1/x, 1/z)$, the graph of Fr_q maps to the curve

$$\{y = x^{-q}, w = z^{-q}\} \subset \mathcal{C}_d \times_k \mathcal{D}_d$$

where x, z, y, and w are the coordinates on $\mathcal{C}_d$ and $\mathcal{D}_d$ introduced in § 5.

Taking the image in $(\mathcal{C}_d \times_k \mathcal{D}_d)/\mu_d$ and using the isomorphism $(\mathcal{C}_d^o \times_k \mathcal{D}_d^o)/\mu_d \cong \mathcal{X}_d^o$, the graph maps to the curve

$$\{(x, y, u) = (x, x^{-q}, f(x))\} \subset \mathcal{X}_d^o.$$

(Here u is the coordinate on the $\mathbb{P}^1_k$ in the lower left of the diagram (5.2.1) and $f(x) = x(x-1)$.)

Since f has degree 2, the Zariski closure of this curve is a bi-section of $\pi_d : \mathcal{X}_d \to \mathbb{P}^1$. Passing to the generic fiber, we get a point defined over a quadratic extension. More precisely, let $k(a)$ be the quadratic extension of $K_d = k(u)$ with $a(a-1) = u$. The point in question has coordinates $(x, y) = (a, a^{-q})$ in the model $Y = V(f - tg) \subset \mathcal{C}_d \times_k \mathcal{D}_d$. Passing to the Weierstrass model X, our point becomes

$$(x, y) = (-a\,(1-a)^{q+1}, a^2\,(1-a)^{2q+1}).$$

The next step is to take the trace of this point down to $X(K_d)$.

8.5. Taking a trace

We use a prime to denote the action of $\mathrm{Gal}\,(k(a)/K_d)$, so $a' = (1-a) = b$ and we have $a + b = 1$ and $ab = -u$.

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We have to compute the sum of

$$(x, y) = (-ab^{q+1}, a^2b^{2q+1})$$

and

$$(x', y') = (-a^{q+1}b, a^{2q+1}b^2)$$

in $X(k(a))$, the result being $P = P(u) \in X(K_d)$. Actually, it will be more convenient to let P be $-((x, y) + (x', y'))$ (the third point of intersection of X and the line joining (x, y) and (x', y')).

To that end, let

$$\lambda = \frac{y' - y}{x' - x} = \frac{a^{2q}b - ab^{2q}}{(b-a)^q}$$

and

$$\nu = \frac{x'y - xy'}{x' - x} = \frac{(ab)^{q+1}(a^qb - ab^q)}{(b-a)^q}.$$

(Both of these expressions lie in K_d but it is convenient to leave them in this form for the moment.)

The coordinates of P are then $x_P = \lambda^2 + \lambda - u^d - x - x'$ and $y_P = \lambda x_P + \nu$.

8.6. The case $p = 2$

Assume that $p = 2$. Then $b - a = 1$, and since $a^2 = u + a$, $b^2 = u + b$,

$$\begin{aligned}\lambda &= a^{2q}b - ab^{2q}\\ &= u^q + a^q - a\\ &= u^q + (a^q - a^{q/2}) + \cdots + (a^2 - a)\\ &= u^q + u^{q/2} + \cdots + u.\end{aligned}$$

Similarly $\nu = u^{q+1}(u^{q/2} + \cdots + u)$. Straightforward calculation then leads to the coordinates of P.

8.7. The case $p > 2$

Now assume that $p > 2$. Introduce $c = b - a = 1 - 2a$ so that $c' = -c$ and $c^2 = 1 + 4u$. Then again using $a^2 = u + a$ and $b^2 = u + b$,

$$\begin{aligned}\lambda &= \frac{u^q(b-a) + a^q - a}{(b-a)^q}\\ &= \frac{2cu^q + c - c^q}{2c^q}\end{aligned}$$

and

$$\begin{aligned}\nu &= \frac{u^{q+1}(a^q - a)}{(b-a)^q}\\ &= \frac{u^{q+1}(c - c^q)}{2c^q}.\end{aligned}$$

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Straightforward calculation then leads to expressions for x_P and y_P which are visibly in $k(c^2) = k(u) = K_d$.

This completes our sketch of the proof of the theorem. □

8.8. Heights

Recall that the Mordell–Weil group $X(K_d)$ has a canonical height pairing $\langle\cdot,\cdot\rangle$ discussed in § 2.6 above. Because the points in Theorem 8.1 are so explicit, we can easily compute their heights.

Proposition 8.9. *Let the notation and hypotheses be as in Theorem 8.1. We view the points P_i as being indexed by $i \in \mathbb{Z}/d\mathbb{Z}$.*

(1) *If $p = 2$, then*

$$\langle P_i, P_j\rangle = \begin{cases} \dfrac{(d-1)^2}{d} & \text{if } i = j \\ \dfrac{(1-d)}{d} & \text{if } i \neq j. \end{cases}$$

(2) *If $p > 2$, then*

$$\langle P_i, P_j\rangle = \begin{cases} \dfrac{(d-1)(d-2)}{d} & \text{if } i = j \\ \dfrac{2(1-d)}{d} & \text{if } i-j \text{ is even and } \neq 0 \\ 0 & \text{if } i-j \text{ is odd.} \end{cases}$$

Remarks 8.10. (1) The proposition gives another proof that the sums $\sum_i P_i$ and $\sum_i (-1)^i P_i$ are torsion when p is odd.

(2) In the notation of [6, Section 4.6.6], the Mordell–Weil lattice $X(K_d)$ modulo torsion is a scaling of A^*_{d-1} when $p = 2$ and a scaling of $A^*_{d/2-1} \oplus A^*_{d/2-1}$ when $p > 2$.

(3) Let V_d be the subgroup of $E(K_d)$ generated by Q and the P_i, $i = 0, \ldots, d-1$ (V for 'visible'). The BSD formula together with the height calculation above yields a simple relationship between the order of the Tate–Shafarevitch group of X over K_d and the index of V_d in $X(K_d)$:

$$|Ш(X/K_d)| = [X(K_d) : V_d]^2.$$

8.11. Proof of Proposition 8.9

First note that Gal (K_d/K) acts on $X(K_d)$ permuting the P_i cyclically and so it will suffice to compute $\langle P_0, P_j\rangle$. We first treat the case where p is odd.

We write P_j both for the point in $X(K_d)$ and for the corresponding section of $\pi_d : \mathcal{X}_d \to \mathbb{P}^1_k$; and similarly for O, the origin of the group law on X. In order to compute $\langle P_0, P_j\rangle$, we have to find a divisor D supported on components of fibers such that $P_0 - O + D$ is orthogonal to all components of fibers; then $\langle P_0, P_j\rangle = -(P_0 - O + D)\cdot(P_j - O)$. The key input to finding D and computing the intersection is to find the reduction of the P_j at each place of $\mathbb{P}^1_k$.

Recall that P_j is the fiberwise sum of a bisection of π_d which we denote as B_j. The divisors $B_j - 2O$ and $P_j - O$ are linearly equivalent on $\mathcal{X}_d$ up to the addition of a sum of components of fibers; thus in the height calculation we may use $B_j - 2O$ in place of $P_j - O$. This is a considerable simplification because B_j has a very compact expression: it is the closure in $\mathcal{X}_d$ of the point of X defined over the quadratic extension $k(a)/k(u)$ where $\zeta_d^j u = a(a-1)$ with coordinates

$$(-a\,(1-a)^{q+1}, a^2\,(1-a)^{2q+1}).$$

(With respect to Remark 8.2(4), we note that using B_j is not essential – the heights can be computed knowing only the P_j, not the B_j.)

The canonical divisor of $\mathcal{X}_d$ is $(d/2-2)$ times a fiber. Since B_j and O are smooth rational curves, the adjunction formula implies that $O^2 = -d/2$ and $B_j^2 = 2-d$. It turns out that $O \cdot B_j = 0$ for all j, $B_0 \cdot B_j = 1$ if j is even and $\neq 0$, and $B_0 \cdot B_j = 0$ if j is odd. So the geometric part of the height pairing is

$$-(B_0 - 2O)\cdot(B_j - 2O) = \begin{cases} 3d-2 & \text{if } j = 0 \\ 2d-1 & \text{if } j \text{ is even and } \neq 0 \\ 2d & \text{if } j \text{ is odd.} \end{cases}$$

To find the ‘correction factors’ $D\cdot(B_j - 2O)$ we calculate the reduction of B_j at the bad places. Over $u = 0$, $\mathcal{X}$ has a fiber of type I_{4d} and if we label the components as usual (with elements of $\mathbb{Z}/4d\mathbb{Z}$ such that the identity component is 0), the B_j reduce to distinct points on the components labelled 1 and $2d+1$ (or -1 and $2d-1$ depending on the orientation of the labelling). It follows that the correction factor at $u = 0$ is $-(d+2-1/d)$ for all j. Over $u = \infty$, $\mathcal{X}$ has a fiber of type I_d. Using the standard labelling of components, if j is even, B_j reduces to the point of intersection of the components labelled $d/2-1$ and $d/2$ and if j is odd, B_j reduces to the intersection of the components labelled $d/2$ and $d/2+1$ (again up to a suitable choice of orientation). It follows that the correction factor at $u = \infty$ is $-(d-1-1/d)$ if j is even and $-(d-2+1/d)$ if j is odd. Adding the geometric and correction factors gives the result.

The calculations for $p = 2$ are quite similar. One interesting twist is that at ∞, the reduction is of type I_d^* and B_j lands on one of the components of multiplicity 2. Thus the linear algebra required to find the correction term is not carried out in the standard references.

This completes the proof of the proposition. □

9. A second example

9.1. Data

Throughout this section, $k = \mathbb{C}$, the field of complex numbers. Let $\mathcal{C} = \mathcal{D} = \mathbb{P}^1_k$. Let

$$f(x) = x(x-1)(x-a) \quad \text{and} \quad g(y) = y(y-1)(y-a)$$

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where a is a parameter in $U = k \setminus \{0, 1, -1, 1/2, 2, \zeta_6, \zeta_6^{-1}\}$. (The reason that we exclude -1, $1/2$, 2, and the primitive sixth roots is that f and g have extra symmetry in those cases and they turn out to be less interesting.) For all d, we have $e_{d,f} = e_{d,g} = 1$.

9.2. The surface $\mathcal{X}_1$

The surface $\mathcal{X}_1$ is obtained by blowing up $\mathcal{C} \times_k \mathcal{D}$ once at each of ten base points. Straightforward calculation shows that $\mathcal{X}_1 \to \mathbb{P}^1_k$ is smooth away from $t = 0$, 1, and ∞ and two other points specified below. The fibers over 0 and ∞ consist of a rational curve of multiplicity 3 and three rational curves of multiplicity 1 each meeting the multiplicity 3 component at one point. (This is a Kodaira fiber of type *IV* blown up at the triple point.) The fiber over $t = 1$ is two rational curves of multiplicity 1 meeting transversely at two points, i.e., a Kodaira fiber of type I_2. The two other singular fibers are of type I_1. It follows that $c_1(d) = d$ and

$$c_2(d) = \begin{cases} 4 & \text{if } 3 \nmid d \\ 6 & \text{if } 3 \mid d. \end{cases}$$

9.3. The curve X

The curve X, the smooth proper model of $\{f - tg\} \subset \mathbb{P}^1_K \times_K \mathbb{P}^1_K$, is an elliptic curve with invariants

$$c_4 = 16\left(a^2 - a + 1\right)^2 t^2,$$
$$c_6 = -216a^2 (a-1)^2 t^4 - 16 (a-2)^2 (a+1)^2 (2a-1)^2 t^3 - 216a^2 (a-1)^2 t^2,$$

and

$$\Delta = a^2 (a-1)^2 t^4 (t-1)^2 Q$$

where

$$Q = -27a^2 (a-1)^2 t^2 + (-16a^6 + 48a^5 - 42a^4 + 4a^3 - 42a^2 + 48a - 16)t - 27a^2 (a-1)^2 .$$

The discriminant of the quadratic Q is

$$64 (a-2)^2 (a+1)^2 (2a-1)^2 \left(a^2 - a + 1\right)^3$$

and so for $a \in U$, Δ has simple zeros at the roots of the quadratic. These roots are the 'two other points' referred to above.

9.4. Homomorphisms and endomorphisms

The curve $\mathcal{C}_d$ has genus $d - 1$ if $3 \nmid d$ and $d - 2$ if $3 \mid d$. The new part of its Jacobian has dimension $\phi(d)$ if $d > 3$ and 1 if $d = 2$ or 3.

Since $\mathcal{C}_d \cong \mathcal{D}_d$ compatibly with the μ_d action, we have

$$\operatorname{Hom}_{k-av}(J_{\mathcal{C}_d}, J_{\mathcal{D}_d})^{\mu_d} \cong \operatorname{End}_{k-av}(J_{\mathcal{C}_d})^{\mu_d}.$$

Thus our rank formula reads

$$\operatorname{Rank} X(K_d) = \operatorname{Rank} \operatorname{End}_{k-av}(J_{\mathcal{C}_d})^{\mu_d} - d + \begin{cases} 4 & \text{if } 3 \nmid d \\ 6 & \text{if } 3 \mid d. \end{cases}$$

Since the endomorphisms in μ_d commute with one another, Lemma 7.7.1 shows that the minimum value of Rank End $(J_{\mathcal{C}_d})^{\mu_d}$ is $d-1$. The upshot of the following result is that it is sometimes larger.

Theorem 9.5. *For $d \in \{2, 5, 7, 8, 9, 12, 14, 15, 16, 18, 22, 24\}$, there is a countably infinite subset $S_d \subset U$ (everywhere dense in the classical topology) such that if $a \in S_d$,*

$$\operatorname{Rank} X(K_d) \geqslant \phi(d) + \begin{cases} 3 & \text{if } 3 \nmid d \\ 5 & \text{if } 3 \mid d. \end{cases}$$

The largest rank guaranteed by the theorem occurs for $d = 15$, 22, and 24 where we have rank $\geqslant$13.

Proof. For any $d > 1$ and any $a \in U$ we write $\mathcal{C}_{d,a}$ for the curve with affine equation $z^d = x(x-1)(x-a)$ and $A_{d,a}$ for the new part of the Jacobian of $\mathcal{C}_{d,a}$. The dimension of $A_{d,a}$ is $\phi(d)$ if $d \neq 3$ and 1 if $d = 3$. By Lemma 7.7.1, the endomorphism algebra of $A_{d,a}$ contains the field $\mathbb{Q}(\mu_d)$. We claim that for d as in the theorem, there are infinitely many $a \in U$ such that $A_{d,a}$ is of CM type, and more precisely such that $\operatorname{End}^0(A_{d,a})$ contains a commutative subalgebra of rank 2 over $\mathbb{Q}(\mu_d)$. For such a, End $(J_{\mathcal{C}_{d,a}})^{\mu_d}$ has dimension at least $\phi(d) + d - 1$. Using the rank formula in Theorem 6.4 and the calculation of c_1 and c_2 above gives the desired result.

It remains to prove the claim. The cases $d = 5$ and 7 were treated by de Jong and Noot [7]. (See also [8] for a nice exposition of this and related material.) The other cases are quite similar and so we will just briefly sketch the argument.

First, we may construct $A_{d,a}$ in a family over U. In other words, there is an abelian scheme $\sigma : \mathcal{A}_d \to U$ whose fiber over $a \in U$ is $A_{d,a}$. This abelian scheme gives rise to a polarized variation of Hodge structures (PVHS) of weight 1 and rank $2\phi(d)$ with endomorphisms by $\mathbb{Q}(\mu_d)$. (Here and below we exclude $d = 3$ where the rank is 2 and the variation is locally constant.)

On the other hand, the corresponding period domain for such Hodge structures is the product of copies of the upper half-plane. The dimension is $r = \{i \mid (d, i) = 1, (d-1)/3 < i < d/2\}$. For the d in the statement, $r = 1$.

At any given point of U we can choose a trivialization of the local system $R^1\sigma_*\mathbb{Z}$ in a disk around the point and obtain a local period mapping. A Kodaira–Spencer calculation, which we omit, shows that these period maps are not constant. Since the period domain is one-dimensional, the images of the period maps must contain non-empty open subsets. But the set of CM points is an everywhere dense countable subset, and the corresponding values of a are values for which the rank of $X_{d,a}$ jumps to at least the value stated in the theorem. This establishes the lower bound stated for a countable, dense set of values of a. □

Remarks 9.6. (1) The points asserted to exist in this theorem are much less explicit than those of Theorem 8.1. Indeed, even the values of a for which the rank jumps are not explicit.

(2) The Andre–Oort conjecture would imply that for a fixed large d, the number of a for which the rank jumps as in the theorem is finite.

(3) A more interesting question is that of whether there are arbitrarily large d such that there exists at least one a for which the rank jumps. If so, we would have unbounded ranks over $\mathbb{C}(t)$. It seems likely to the author that this does not happen, in other words, that for all sufficiently large d and all $a \in U$, Rank End $(A_{d,a})^{\mu_d} = \phi(d)$.

(4) There are at least two other families of curves with countable, dense CM subsets that could be used in conjunction with the rank formula. See [11, p. 114]

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