A CLASSIFICATION OF SEMI-TRANSLATION PLANES

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1. Introduction. The classification of certain types of projective planes has recently been of considerable interest to both geometers and group theorists. Due in part to the current general interest in finite mathematics and the developments connecting group theory and finite geometry, the Lenz-Barlotti classification of finite projective planes (**2**; **10**), in particular, has generated a tremendous amount of research. A great deal of this research has been related to the construction of non-Desarguesian planes.

Fryxell (**6**), Hughes (**7**), Lüneburg (**11**), and Ostrom (**13**; **15**; **18**) have found examples of projective planes, all of which are of a general type that we call semi-translation planes. Many of these planes are of the same Lenz-Barlotti class I-1. (The Lüneburg planes are translation planes. However, the planes dual to the Lüneburg planes are semi-translation planes as well as dual translation planes.)

By Ostrom's method of "deriving" (**18**; **20**), semi-translation planes may be obtained from dual translation planes of order q^2 whose kernel is of order q. Ostrom has pointed out (**16**; **20**) that the possible number of translation planes obtained by net replacement in Desarguesian planes is apparently very large; see (**16**; **17**; **20**). Thus, a refinement of Lenz-Barlotti class I-1 seems to be warranted. A classification of semi-translation planes is a step in this direction.

An affine plane of order q^2 is a semi-translation plane if it admits a group H of translations such that each point orbit of H is the set of q^2 points of an affine subplane of order q. We shall say that the plane is a non-strict semi-translation plane if the translation group of the plane properly contains such a group H. Otherwise, we shall say that the plane is a strict semi-translation plane. If the affine plane is extended to a projective plane by adjoining a line L, the projective plane will be called a semi-translation plane with respect to the line L (non-strict or strict accordingly as its affine restriction is non-strict or strict). The translations now become elations with axis L. A projective plane will be called a dual (non-strict or strict) semi-translation plane with respect to a point p if and only if the plane is the dual version of a (non-strict or strict) semi-translation plane with respect to a line L. The elations with axis L and centres on L become elations with centre p and axes through p.

In (**9**) I showed that for non-strict semi-translation planes π with respect to a line L (with minor exception) L is invariant under the full collineation group of π. Here it will be shown that this result is also valid for one class of strict

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semi-translation planes (see (2.1)), but certainly is not generally true for all strict semi-translation planes.

Non-strict semi-translation planes have been classified previously (**9**), thus in this article we will concern ourselves only with strict semi-translation planes. Hence, the following assumption is made.

(1.1) *Assumption.* If π is a semi-translation plane (dual semi-translation plane) with respect to a line L (point p), then π is a *strict* semi-translation plane (dual semi-translation plane) with respect to the line L (point p).

The following notation is used:

(i) π is sst = π is a (strict) semi-translation plane;

(ii) π is Dsst = π is a (strict) dual semi-translation plane. This classification is contained in my Ph.D. thesis at Washington State University.

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The initial classification.

(1.2) Lemma. *If π is a projective plane of order q^2 satisfying* (1.1), *then π is sst* (*Dsst*) *with respect to* $1, q+1$ *or* q^2+q+1 *lines* (*points*). *Furthermore, the lines* (*points*) *are all lines* (*points*) *of a subplane of order q. If π is sst* (*Dsst*) *with respect to $q+1$ lines* (*points*), *the lines* (*points*) *are concurrent* (*collinear*). *If π is sst* (*Dsst*) *with respect to q^2+q+1 lines* (*points*), *the lines* (*points*) *are all of the lines* (*points*) *of the subplane of order q.*

Proof. Consider the following extension of (**18**, Lemma 5):

If π is a strict semi-translation plane with respect to three non-concurrent lines, then there exists a subplane π_0 of order q such that π is a strict semi-translation plane with respect to every line of π_0.

Proof of the extension. By (**18**, Lemmas 3, 4, and 5), π is sst with respect to one of the lines L_1 and is Dsst with respect to the intersection of the two remaining lines L_2 and L_3. By (**18**, Lemma 6), π admits a collineation moving L_1. It is also clear that $L_1 \cap L_2$ may be moved on L_2 onto any point of $\pi_0 \cap L_2 - L_2 \cap L_3$. By (**18**, Lemmas 4 and 6) and (**22**, p. 66, Theorem 7), it is easy to see that π is sst with respect to every line of π_0.

Both (**18**, Lemma 4) and this extension (and the dual arguments) yield the proof of (1.2). On the basis of (1.2), we formulate the following definition.

(1.3) *Definition.* Let π be a projective sst or Dsst. We define the following types:

Type 3. π is of Type 3 if π or $D\pi$ (the dual of π) is sst with respect to q^2+q+1 lines.

Type 2. π is of Type 2 if neither π nor $D\pi$ is of Type 3 and either π or $D\pi$ is sst with respect to $q + 1$ lines.

Type 1. π is of Type 1 if neither π nor $D\pi$ is of Type 2 or 3 and either π or $D\pi$ is sst with respect to one line.

(1.4) *Definition*. Let π be a projective plane of order q^2 and π_0 a subplane of order q. Let p be a point of π_0 and L a line of π such that $L \cap \pi_0$ is a line of π_0. π is said to be (p, L, π_0)-transitive if the stabilizer of π_0 in the group of all (p, L)-collineations of π induces a collineation group of π_0 such that π_0 is (p, L)-transitive.

Recall that if π is sst with respect to a line L, then the point orbits of the elation group with axis L are affine subplanes. The projective extensions of these subplanes all have the same points on L. Therefore, if π_0 is the projective extension of a point orbit of the elation group with axis L, then π is (p, L, π_0)-transitive for all points $p \in L \cap \pi_0$. Conversely, if π is (p, L, π_0)-transitive for all points $p \in L \cap \pi_0$ for some projective subplane π_0 of order q, then clearly π is a semi-translation plane with respect to L. We will use this characterization of semi-translation planes for our classification (the dual semi-translation planes are likewise characterized).

If π is sst (Dsst), choose a projective subplane π_0 of order q. We shall classify π by the configuration of point-line pairs (p, L) such that π is (p, L, π_0)-transitive for the fixed subplane π_0.

Clearly, the subplane π_0 is always a translation plane if π is sst. However, π_0 may have central collineations induced on it by collineations of π that are not central.

Thus, the classification of semi-translation planes depends on the choice of subplane π_0. However, it is possible to choose π_0 so that we obtain a "best" classification for π.

For example, any subplane π_0 which is invariant under the full collineation group of π yields a best classification for π and any two invariant subplanes give identical classifications.

If π is of Type 2 or 3, then the elation centres (or axes) form a unique invariant subplane. If π admits (p, L, π_0)-transitivity with p not on L and π sst with respect to L, then the L-homology centres form a unique invariant subplane. If π does not have an invariant subplane, then π is of Type 1.

Let π be of Type 1 (Type 2 or Type 3). We will consider the manner in which (p, L, π_0)-transitivity, for various point-line pairs (p, L) of π_0, can exist in π such that π remains of Type 1 (Type 2 or Type 3, respectively).

For each possible combination of such point-line pairs, the set of these pairs will be denoted by S(1-*kr*) if π is of Type 1, S(2-*kr*) if π is of Type 2 and S(3-*k*) if π is of Type 3 for some letter r and integer k. If π is (p, L, π_0)-transitive if and only if $(p, L) \in$ S(*j*-*kr*) ($r = 1$ if $j = 3$), we shall say that π is of Type *j*-*kr*. S(π) will denote the undetermined set of point-line pairs (p, L) such that π is (p, L, π_0)-transitive. Thus, S(π) = S(*j*-*kr*) if and only if π is of Type *j*-*kr*.

(1.5) *Definition.* A plane of Type j-k_1r_1, $j = 1, 2, 3$ ($r_1 = 1$ if $j = 3$), will be said to be *above* a plane of Type i-k_2r_2, $i = 1, 2, 3$ ($r_2 = 1$ if $i = 3$), if and only if $S(i\text{-}k_2r_2)$ is contained in $S(j\text{-}k_1r_1)$. In this case, we will also say that the Type j-k_1r_1 is above the Type i-k_2r_2.

There are many examples of semi-translation planes of Type 1 (**8**) and the Hughes planes (**7**; **18**) are examples of planes of Type 3. However, there are no known examples of planes of Type 2. This raises the following question: Are there projective planes of order q^2 that are semi-translation planes with respect to exactly $q + 1$ lines?

Hughes planes are (p, L, π_0)-transitive for all point-line pairs (p, L) of the Desarguesian subplane π_0 coordinatized by $GF(q)$. We raise the following question: Are there projective planes that are (p, L, π_0)-transitive for all incident point-line pairs of π_0 that are not, in fact, Hughes planes?

With one exception, the known strict semi-translation planes all have an invariant subplane of order q. Foulser (**4**) has shown that, with the exception of the Hall planes, the generalized André planes always have two points left fixed or interchanged by the full collineation group. André (**1**) has shown this same result for nearfield planes. In either of the above cases, the corresponding dual translation planes will have two lines that are either invariant or interchanged and which intersect in the special point of the dual translation plane. In order that the dual translation plane be represented in an affine form that is derivable, the line at infinity must go through the special point. If one of the two lines is chosen as the line at infinity, the other line must be invariant under the affine group. Upon derivation, this invariant line is converted into an invariant subplane. This means that semi-translation planes derived (**18**) from the duals of such planes must have an invariant subplane (square root subplane). In (**8**), it was shown that planes derived from the dual Hall planes also have an invariant subplane. We, therefore, raise the following question: Do semi-translation planes of order q^2 exist which contain no invariant subplane of order q?

2. Planes of Type 1.

(2.1) PROPOSITION. (i) *If π is sst (Dsst) of order q^2 with respect to a line L (point p) and π has no invariant subplane of order q, then L (p) is fixed by the full collineation group of π.*

(ii) *If π has no invariant subplane of order q and is sst with respect to a line L and Dsst with respect to a point k, then k I L.*

Proof. (i) If L (p) is moved, then (**18**, Theorem 2 (dual)) implies that π contains an invariant subplane of order q.

Note that if π is sst with respect to a line L and L is moved by the collineation group of π, then π contains an invariant subplane π_0 and furthermore π_0 is the unique subplane whose points are the elation centres of the lines in the orbit of L.

(ii) If $k \not\mathrm{I}\, L$, then, by (**18**, Lemma 6), L is moved. However, this is a contradiction by (i). Hence, (2.1) is proved.

Therefore, if π is sst and does not have an invariant subplane, then π is of Type 1.

For this section, let π be of Type 1. Let the unique line associated with π or Dπ be denoted by L_∞. Therefore, it is clear that L_∞ is fixed by the full collineation group of π (or Dπ). Furthermore, if β is any $(\tilde{p}, \tilde{L})$-collineation of π (or Dπ), then either $\tilde{p}\ \mathrm{I}\ L_\infty$ or $\tilde{L} = L_\infty$.

We will now label our subclasses of Type 1. We will collect the definitions for the subclasses into a theorem (2.16) and thus will not distinguish our initial definitions by number. For the following we shall assume that π is of Type 1 and π is sst. The classification for Dsst of Type 1 will be obtained by taking the dual of the point-line configuration which defines sst of Subtype 1-*kr*.

Recall that π is (p, L_∞, π_0)-transitive for all points $p\ \mathrm{I}\ L_\infty \cap \pi_0$. If π is not $(\tilde{p}, \tilde{L}, \pi_0)$-transitive for further point-line pairs $(\tilde{p}, \tilde{L})$, we shall say that π is of Type 1-1a.

If π is, however, also (p_∞, L, π_0)-transitive for all lines L of π_0 incident with p_∞ and is not $(\tilde{p}, \tilde{L}, \pi_0)$-transitive for further point-line pairs $(\tilde{p}, \tilde{L})$, π will be said to be of Type 1-2a. (Here p_∞ denotes some fixed point on L_∞.)

The proofs of the following lemmas (2.2) and (2.3) are routine and are left to the reader.

(2.2) LEMMA. *If π is sst and is (p, L_∞, π_0)-transitive for some line L_∞ and all points $p\ \mathrm{I}\ L_\infty \cap \pi_0$ and is also $(\tilde{p}, \tilde{L}, \pi_0)$-transitive for some point-line pair $(\tilde{p}, \tilde{L})$, $\tilde{L} \neq L_\infty$ and $\tilde{p}\ \mathrm{I}\ L_\infty$, then π is $(\tilde{p}, L, \pi_0)$-transitive for all lines L of π_0 such that $L\ \mathrm{I}\ L_\infty \cap \tilde{L}$.*

(2.3) LEMMA. *If π is above Type* 1-1a *and is (p', L_∞, π_0)-transitive for some point $p' \not\mathrm{I}\, L_\infty$, then π is (p, L_∞, π_0)-transitive for all points $p\ \mathrm{I}\ \pi_0$.*

(2.4) LEMMA. *If π is above Type* 1-2a (see (2.16)) *and is also (p', L', π_0)-transitive for some point $p'\ \mathrm{I}\ L_\infty$ and line L' such that $p'\ \mathrm{I}\ L'$, where $p' \neq p_\infty$, then π is of Type* 2 *or* 3.

Proof. By (2.2), π is (p', L, π_0)-transitive for all lines L of π_0 such that $L\ \mathrm{I}\ p'$. We may move p' onto any point of $L_\infty - p_\infty$ by the (p_∞, L, π_0)-elations. Therefore, it is clear that π is (p, L, π_0)-transitive for all points $p\ \mathrm{I}\ L_\infty \cap \pi_0$ and lines L of π_0 such that L is incident with p_∞. That is, π is of Type 2 or 3 since Dπ is sst with respect to $q + 1$ or $q^2 + q + 1$ lines.

Therefore, if π is above Type 1-2a and of Type 1 and $(p, L) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}2\mathrm{a})$, then $p \not\mathrm{I}\, L$.

We now consider two sequences of planes:

(1) sst that are above Type 1-1a but not above Type 1-2a and

(2) sst above Type 1-2a and still of Type 1.

Planes above Type 1-1a *but not above Type* 1-2a. If π is above Type 1-1a but not above Type 1-2a and $(p', L') \in S(\pi) - S(1\text{-}1a)$ with $L' = L_\infty$, then clearly $(p, L_\infty) \in S(\pi)$ for all points p I π_0. If, on the other hand, p' I L_∞ and $L' \neq L_\infty$, then $(p', L) \in S(\pi)$ for all lines L of π_0 that are incident with a fixed point $k = L' \cap L_\infty$ of $L_\infty \cap \pi_0$.

If $S(\pi) = \{(p, L_\infty)$ for all points p I $\pi_0\}$, we shall say that π is of Type 1-1b.

If $S(\pi) = S(1 - 1a) \cup \{(p', L)$ for all lines L of $\pi_0 - L_\infty$ such that L I k, for some fixed point k of $L_\infty \cap \pi_0$, $p' \neq k\}$, we shall say that π is of Type 1-4a.

If $S(\pi) = S(1\text{-}4a) \cup S(1\text{-}1b)$, we shall say that π is of Type 1-4b.

(2.5) LEMMA. *If π is above Type* 1-4b *and* $(\tilde{p}, \tilde{L}) \in S(\pi) - S(1\text{-}4b)$ *such that either* $\tilde{p} = p'$ *or* $\tilde{L}$ I k (p' *and* k *as above*), *then π is above Type* 1-2a.

Proof. If $p = p'$, then $\tilde{L} \not\mathrm{I}\, k$, and thus k may be moved onto any point of $L_\infty - p'$ by the $(\tilde{p}, \tilde{L}, \pi_0)$-collineations. Thus, by (**22**, p. 66, Theorem 7), π is (p', L, π_0)-transitive for all lines L of π_0 incident with a point p where p I $(L_\infty - p') \cap \pi_0$. Therefore, by the dual of (**1**, *Satz* 1), π is also (p, L, π_0)-transitive for p I $L_\infty \cap \pi_0$. Clearly then, π is above Type 1-2a.

The argument for the case $\tilde{L}$ I k is similar to the above and is left to the reader. Note that this argument depends on the fact that p' may be moved by collineations of π which fix k.

Now suppose that π is strictly above Type 1-4b but not above Type 1-2a. Then, if $(\tilde{p}, \tilde{L}) \in S(\pi) - S(1\text{-}4b)$, $\tilde{p} \neq p'$ and $\tilde{L} \not\mathrm{I}\, k$. We have the following possibilities:

(1) $\tilde{p} = k$, $\tilde{L}$ I p',
(2) $\tilde{p} = k$, $\tilde{L} \not\mathrm{I}\, p'$,
(3) $\tilde{p} \neq k$, $\tilde{L}$ I p', or
(4) $\tilde{p} \neq k$, $\tilde{L} \not\mathrm{I}\, p'$.

In cases (2) and (3) we have p' or k moved by collineations which fix k or p', respectively, which is a contradiction by the argument of (2.5).

If we have case (1), then $(k, L) \in S(\pi)$ for all lines of π_0 such that L I p' (see (2.2)). We shall say, in this case, that π is of Type 1-5b.

Case (4). By (2.5) and the remarks immediately thereafter, we can have only the following situation: $p' \neq \tilde{p}$ and $\tilde{L} \not\mathrm{I}\, k$.

Let G_1 (G_2) denote the group of central collineations with centre p' ($\tilde{p}$) and axis L ($\tilde{L}$) for some line L of π_0 such that L I k. By following the argument in (**19**, Theorem 16), we can establish that either the group generated by G_1 and G_2 induces a permutation group G on $L_\infty \cap \pi_0$ which is doubly-transitive or the set of points on $L_\infty \cap \pi_0$ can be divided into pairs (p_i, q_i) such that π is (p_i, L_i, π_0)-transitive for some line L_i of π_0 such that L_i I q_i, $p_i \neq q_i$. Moreover, every collineation of π_0 fixing p_i also fixes q_i. Furthermore, in the latter case, it follows that π is also (q_i, R_i, π_0)-transitive for some line R_i of π_0 such that R_i I p_i. Thus, in this case, π is above Type 1-5b.

If R is a finite group, let R_Q denote the stabilizer of Q in the group R. If G is doubly transitive on $L_\infty \cap \pi_0$, then G_P, for $P \in L_\infty \cap \pi_0$, is transitive on the

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remaining points of $L_\infty \cap \pi_0 - P$. Thus, by arguing as in (2.5), π is above Type 1-2a, contrary to our assumptions.

Thus, assume that we have the second alternative and hence the order q of π_0 is odd. Foulser has pointed out (written communication) that π_0 is Desarguesian or the nearfield plane of order 9 (see also **19**, Theorem 16 (iii)). If π_0 is Desarguesian, we note that G cannot contain an element of order p, where $p^r = q$. Let $\mathrm{PGL}_2(q)$ denote the projective general linear group (linear fractional group). Clearly, $G \subseteq \mathrm{PGL}_2(q)$. Note also that $\mathrm{PGL}_2(q)_Q$ is a doubly-transitive Frobenius group for $Q \in L_\infty \cap \pi_0$. If $\beta \in G$ such that $|\beta| = p$, then clearly (recall that q is odd) β fixes exactly one point of $L_\infty \cap \pi_0$. However, this implies that G is doubly-transitive. Hence, assume that π_0 is Desarguesian, G is not doubly-transitive, $p \nmid |G|$ and q is odd. G is clearly transitive on $L_\infty \cap \pi_0$ and the stabilizer of a point of $L_\infty \cap \pi_0$ in G is transitive on $q - 1$ points. Hence, $(q^2 - 1) \,|\, |G|$.

Now $|\mathrm{PGL}_2(q)| = q(q^2 - 1)$ (see, e.g., **3**, § 239), $|G| = (q^2 - 1)s$ for some integer s such that $s|q$. Since $p \nmid |G|$, we have $|G| = q^2 - 1$. By (**5**, (12.2)) (see also (**3**, §§ 256–260)), $q^2 - 1 = 12$, 24, or 60. Hence, $q^2 = 25$ and thus $q = 5$ is the only possibility. In this case, G is isomorphic to S_4 (symmetric group on four letters) which is transitive but not doubly transitive on six elements.

We therefore have the two possible exceptions for $q = 5$ and 9. We will not, however, include this case (4) for $q = 5$ or 9 in the general classification and thus, for the following, we will assume that $q \neq 5$ or 9.

(2.6) LEMMA. *If π is strictly above Type* 1-4a, *but not above Types* 1-1b *or* 1-2a, *then $(k, L) \in S(\pi)$ for all lines L of π_0 such that L I p' (p' and k as in* (2.5)).

Proof. The proof of (2.6) is virtually identical to the previous case where π is above Type 1-4b and is left to the reader.

If π satisfies (2.6), we shall say that π is of Type 1-5a.

Note that the possible exceptional types when $q = 5$ or 9 and which satisfy (2.6) are above Type 1-5a. Note also that planes of Type 1-5b are strictly above planes of Type 1-5a, and Type 1-5b is strictly above Type 1-4b.

From (2.5), we clearly have the following result.

(2.7) LEMMA. *If π is strictly above Type* 1-5b, *then π is above Type* 1-2a.

Therefore, we have considered (up to duality) all of the possible types of planes that are above Type 1-1a but not above Type 1-2a.

Planes of Type 1 *above Type* 1-2a. If π is above Type 1-2a, let p_∞ denote the unique point of L_∞ such that π is (p_∞, L, π_0)-transitive for all lines L of π_0 such that L I p_∞.

(2.8) LEMMA. *If π is above Type* 1-2a *and $(\tilde{p}, \tilde{L}) \in S(\pi) - S(1\text{-}2a)$ such that $\tilde{p}$ I̸ L_∞, then $(p, L_\infty) \in S(\pi)$ for all p I π_0.*

Proof. (2.8) is immediate from (2.3).

If π satisfies (2.8), we shall say that π is of Type 1-2b.

(2.9) LEMMA. *If π is above Type* 1-2b *and* $(\tilde{p}, \tilde{L}) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}2\mathrm{b})$ *such that* $\tilde{p} = p_\infty$ *and* $\tilde{L} \not\mathrm{I}\, p_\infty$, *then* $(p_\infty, L) \in \mathrm{S}(\pi)$ *for all lines L of* π_0.

Proof. The (p_∞, L, π_0)-elations move $L_\infty \cap \tilde{L}$ onto any point of $L_\infty - p_\infty$. By applying (2.2) and (**22**, p. 66, Theorem 7), we have the proof of (2.9). (Note that if π is above Type 1-2a and $\tilde{p}\ \mathrm{I}\ L_\infty$ and $\tilde{p} = p_\infty$, then π is above D1-2b.)

If π satisfies (2.9), we shall say that π is of Type 1-2c.

It should now be clear that if π is strictly above Type 1-2c and

$$(p_0, L_0) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}2\mathrm{c}),$$

then $p_0\ \mathrm{I}\ L_\infty$ and $p_0 \neq p_\infty$. Now suppose that $L_0 \not\mathrm{I}\, p_\infty$. Then, by the dual of (2.4), π is of Type 2 or 3. Now p_0 may be moved onto any point of $(L_\infty - p_\infty) \cap \pi_0$ by the (p_∞, L, π_0)-collineations. We thus have the following lemma.

(2.10) LEMMA. *If π is strictly above Type* 1-2c *and* $(p_0, L_0) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}2\mathrm{c})$, *then* $(p, L) \in \mathrm{S}(\pi)$ *for all points $p\ \mathrm{I}\ L_\infty \cap \pi_0$ and for all lines L of π_0 such that $L\ \mathrm{I}\ p_\infty$.*

π will, in this case, be said to be of Type 1-3c.

(2.11) LEMMA. *If π is strictly above Type* 1-3c, *then π is not of Type* 1.

Proof. If π is strictly above Type 1-3c, then there is a pair $(\tilde{p}, \tilde{L}) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}3\mathrm{c})$ and $\tilde{p}$ may be moved onto any point of $L_\infty \cap \pi_0$ by the existing collineations. And, since $\tilde{L} \not\mathrm{I}\, p_\infty$ (see (2.10)), we have the situation that π is at least of Type 2. Hence, (2.11) is proved.

The following two results are clear from (2.2) and (2.3).

(2.12) LEMMA. *If π is above Type* 1-2a *and* $(\tilde{p}, \tilde{L}) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}2\mathrm{a})$, $\tilde{p} \not\mathrm{I}\, L_\infty$, *and* $\tilde{L}\ \mathrm{I}\ p_\infty$, *then* $(p, L) \in \mathrm{S}(\pi)$ *for all points $p\ \mathrm{I}\ L_\infty \cap \pi_0$ and for all lines L of π_0 such that $L\ \mathrm{I}\ p_\infty$.*

We will, in the case of (2.12), say that π is of Type 1-3a.

(2.13) LEMMA. *If π is above Type* 1-3a *and* $(\tilde{p}, \tilde{L}) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}3\mathrm{a})$ *with* $\tilde{p} \not\mathrm{I}\, L_\infty$, *then* $(p, L_\infty) \in \mathrm{S}(\pi)$ *for all points $p\ \mathrm{I}\ \pi_0$.*

If π satisfies (2.13), we shall say that π is of Type 1-3b. That is,

$$\mathrm{S}(1\text{-}3\mathrm{b}) = \mathrm{S}(1\text{-}3\mathrm{a}) \cup \mathrm{S}(1\text{-}1\mathrm{b})$$

(see (2.16)).

(2.14) LEMMA. *If π is strictly above Type* 1-3a *and not above Type* 1-2b, *then the dual of π (L_∞ interchanged with p_∞) is of Type* 1-3b.

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Proof. By (2.13), if $(\tilde{p}, \tilde{L}) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}3a)$, then $\tilde{p} \mathrm{I} L_\infty$ and $\tilde{L} \not\mathrm{I}\, p_\infty$ since otherwise π would be of Type 1-3b or above Type 1-2c. π is of Type 2 or 3 unless $\tilde{p} = p_\infty$. Therefore, $\tilde{p} = p_\infty$ and by the dual of (2.3), $(p_\infty, L) \in \mathrm{S}(\pi)$ for all lines L in π_0. If we interchange p_∞ and L_∞ in π and its dual, we obtain $\mathrm{S}(\mathrm{Dual}\ \pi) = \mathrm{S}(1\text{-}3b)$.

(2.15) LEMMA. *If π is strictly above Type* 1-3b *and is of Type* 1, *then π is of Type* 1-3c.

Proof. We see, by (2.4) and the remarks immediately following, that if π is of Type 1, p_∞ is fixed by any additional $(\tilde{p}, \tilde{L}, \pi_0)$-transitivities such that $\tilde{p} \not\mathrm{I}\, \tilde{L}$. Therefore, if there exists a pair $(p_0, L_0) \in \mathrm{S}(\pi) - \mathrm{S}(1\text{-}3b)$, then clearly $p_0 = p_\infty$ and $L_0 \neq L_\infty$. By the argument of (2.10), $(p, L) \in \mathrm{S}(\pi)$ for all points $p \mathrm{I} L_\infty \cap \pi_0$ and for all lines L of π_0 such that $L \mathrm{I} p_\infty$. That is, π is of Type 1-3c.

The previous lemmas and remarks establish the following result for planes of order q^2 and $q \neq 5$ or 9.

(2.16) THEOREM. *If π is a semi-translation plane of Type* 1 *with respect to L_∞ (or its dual), then π is of one and only one of the following types:*

Type 1-1a:	$(p, L_\infty) \in \mathrm{S}(\pi)$ *for all* $p \mathrm{I} L_\infty \cap \pi_0$;
Type 1-2a:	$\mathrm{S}(1\text{-}2a) = \mathrm{S}(1\text{-}1a) \cup \{(p_\infty, L)$ *for all lines L of π_0 such that $L \mathrm{I} p_\infty$ and p_∞ a fixed point of L_∞*$\}$;
Type 1-3a:	$\mathrm{S}(1\text{-}3a) = \mathrm{S}(1\text{-}2a) \cup \{(p, L)$ *for all points $p \mathrm{I} L_\infty \cap \pi_0$ for all lines of π_0 incident with p_∞*$\}$;
Type 1-4a:	$\mathrm{S}(1\text{-}4a) = \mathrm{S}(1\text{-}1a) \cup \{(p', L)$ *for all lines L of π_0 such that $L \mathrm{I} k$, $p' \not\mathrm{I}\, L$, k and $p' \mathrm{I} L_\infty$*$\}$;
Type 1-5a:	$\mathrm{S}(1\text{-}5a) = \mathrm{S}(1\text{-}4a) \cup \{(k, L)$ *for all lines L of π_0 such that $L \mathrm{I} p'$, $k \not\mathrm{I}\, L$, k and $p' \mathrm{I} L_\infty$*$\}$;
Type 1-1b:	$\mathrm{S}(1\text{-}1b) = \mathrm{S}(1\text{-}1a) \cup \{(p, L)$ *for all* $p \mathrm{I} \pi_0 - L_\infty\}$;
Type 1-2b:	$\mathrm{S}(1\text{-}2b) = \mathrm{S}(1\text{-}2a) \cup \mathrm{S}(1\text{-}1b)$;
Type 1-3b:	$\mathrm{S}(1\text{-}3b) = \mathrm{S}(1\text{-}3a) \cup \mathrm{S}(1\text{-}1b)$;
Type 1-4b:	$\mathrm{S}(1\text{-}4b) = \mathrm{S}(1\text{-}4a) \cup \mathrm{S}(1\text{-}1b)$;
Type 1-5b:	$\mathrm{S}(1\text{-}5b) = \mathrm{S}(1\text{-}5a) \cup \mathrm{S}(1\text{-}1b)$;
Type 1-2c:	$\mathrm{S}(1\text{-}2c) = \mathrm{S}(1\text{-}2b) \cup \{(p_\infty, L)$ *for all lines* $L \mathrm{I} \pi_0\}$;
Type 1-3c:	$\mathrm{S}(1\text{-}3c) = \mathrm{S}(1\text{-}3b) \cup \mathrm{S}(1\text{-}2c)$.

D1-1a, D1-4a, D1-5a, and D1-*i*b (i = 1, 2, 3, 4, or 5) *are the duals of the above corresponding types with the exception that* 1-2a, 1-3a, 1-2c, *and* 1-3c *are self-dual forms.*

3. Planes of Type 2. Let L_∞ be a line of π such that π is sst with respect to L_∞. Clearly (see (1.2)), if π is sst of Type 2 and not sst of Type 3, then L_∞ is not invariant under the full collineation group of π but there is a unique point $p_\infty \mathrm{I} L_\infty \cap \pi_0$ which is so invariant (p_∞ is the intersection of L_∞ and an image of L_∞ under a collineation which displaces L_∞).

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If π is sst of Type 2 such that $(p, L) \in S(\pi)$ for all incident point-line pairs such that L is a line of π_0 incident with p_∞, for the unique point $p_\infty \mathrm{I} L_\infty$, we will say that π is of Type 2-1a.

We shall assume for this section that π is sst of Type 2. The classification for Dsst of Type 2 will be obtained in the usual manner (see remarks prior to (2.2)).

(3.1) LEMMA. *If π is above Type* 2-1a *and* $(p_0, L_0) \in S(\pi) - S(2\text{-}1a)$ *and if either*

(a) $p_0 \mathrm{I} L_0$, *or*

(b) $L_0 \mathrm{I} p_\infty$, *or*

(c) $p_0 \neq p_\infty$ *and* $L_0 \not\mathrm{I}\, p_\infty$,

then π is of Type 3.

Proof. (a) If $p_0 \mathrm{I} L_0$, then since $(p_0, L_0) \notin S(2\text{-}1a)$, it must be that $p_0 \neq p_\infty$. Therefore, p_∞ may be moved. However, this implies that π is of Type 3.

(b) Since the Hughes planes are self-dual (see **18**, Theorem 15), we have, by the dual of (**12**, Theorem 1), the following theorem.

THEOREM. *If π is (p, L, π_0)-transitive for all lines L of π_0 such that $L \mathrm{I} p_\infty$, for all points $p \mathrm{I} \pi_0$, then π is a Hughes plane.*

Ostrom (**18**) has pointed out that the Hughes planes are of Type 3.

Now we can move p_0 onto any point of $\pi_0 - p_\infty$ by the existing elation group. Therefore, applying (2.2), it follows that $(p, L) \in S(\pi)$ for all points $p \mathrm{I} \pi_0 - p_\infty$ and all lines L of π_0 such that $L \mathrm{I} p_\infty$ where $p \not\mathrm{I}\, L$. Clearly then, π is a Hughes plane and thus of Type 3.

(c) If $p_0 \neq p_\infty$ and $L_0 \not\mathrm{I}\, p_\infty$, p_∞ is not fixed by the full group, and thus π is of Type 3.

Thus, (3.1) is proved.

(3.2) COROLLARY. *If π is strictly above Type* 2-1a *and not of Type* 3, *then* $(p', L') \in S(\pi) - S(2\text{-}1a)$ *implies* $p' = p_\infty$. *In this case,* $(p_\infty, L) \in S(\pi)$ *for all lines L of π_0, where $L \not\mathrm{I}\, p_\infty$.*

If π satisfies the hypothesis of (3.2), we shall say that π is of Type 2-2a. It also follows that if π is strictly above Type 2-2a, then π is of Type 3.

Hence, we have shown the following result.

(3.3) THEOREM. *If π is a strict semi-translation plane of Type* 2 (*or its dual*), *then π is of one and only one of the following types:*

Type 2-1a: $S(2\text{-}1a) = \{(p, L)$ *for all incident point-line pairs of π_0 such that* $L \mathrm{I} p_\infty\}$, *where p_∞ is the intersection of the $q + 1$ lines L_i, $1 \leqq i \leqq q + 1$, such that π is sst with respect to L_i for all i;*

Type 2-2a: $S(2\text{-}2a) = S(2\text{-}1a) \cup \{(p_\infty, L)$ *for all lines L of π_0 such that* $L \not\mathrm{I}\, p_\infty\}$.

D2-1a *and* D2-2a *are the duals of the corresponding classes.*

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4. Planes of Type 3. Let π be sst with respect to L_∞. If π is of Type 3, it follows that neither L_∞ nor any point of L_∞ is left invariant under the full collineation group of π.

If π is of Type 3 such that $(p, L) \in \mathrm{S}(\pi)$ for all incident point-line pairs of π_0, we shall say that π is of Type 3-1.

(4.1) LEMMA. *If π is strictly above Type 3-1, then π is a Hughes plane.*

Proof. Let $(\tilde{p}, \tilde{L}) \in \mathrm{S}(\pi) - \mathrm{S}(3\text{-}1)$. Clearly then $\tilde{p} \not{I} \tilde{L}$. The existing (p, L, π_0)-elations move $\tilde{p}$ onto any point of $\pi_0 - \tilde{L}$. Also, we may clearly move $\tilde{L}$ to any line of π_0. Thus, $(p, L) \in \mathrm{S}(\pi)$ for all point-line pairs of π_0. By (**12**, Theorem 1), π is a Hughes plane.

If $(p, L) \in \mathrm{S}(\pi)$ for all point-line pairs of π_0, we shall say that π is of Type 3-2. Thus, the Hughes planes are examples and the only possible examples of planes of Type 3-2.

Thus, we have proved the following theorem.

(4.2) THEOREM. *If π is a strict semi-translation plane of Type* 3 (*or its dual*), *then π is of one and only one of the following types:*
Type 3-1: $\mathrm{S}(3\text{-}1) = \{(p, L)$ *for all incident point-line pairs of* $\pi_0\}$;
Type 3-2: $\mathrm{S}(3\text{-}2) = \{(p, L)$ *for all point-line pairs of* $\pi_0\}$.

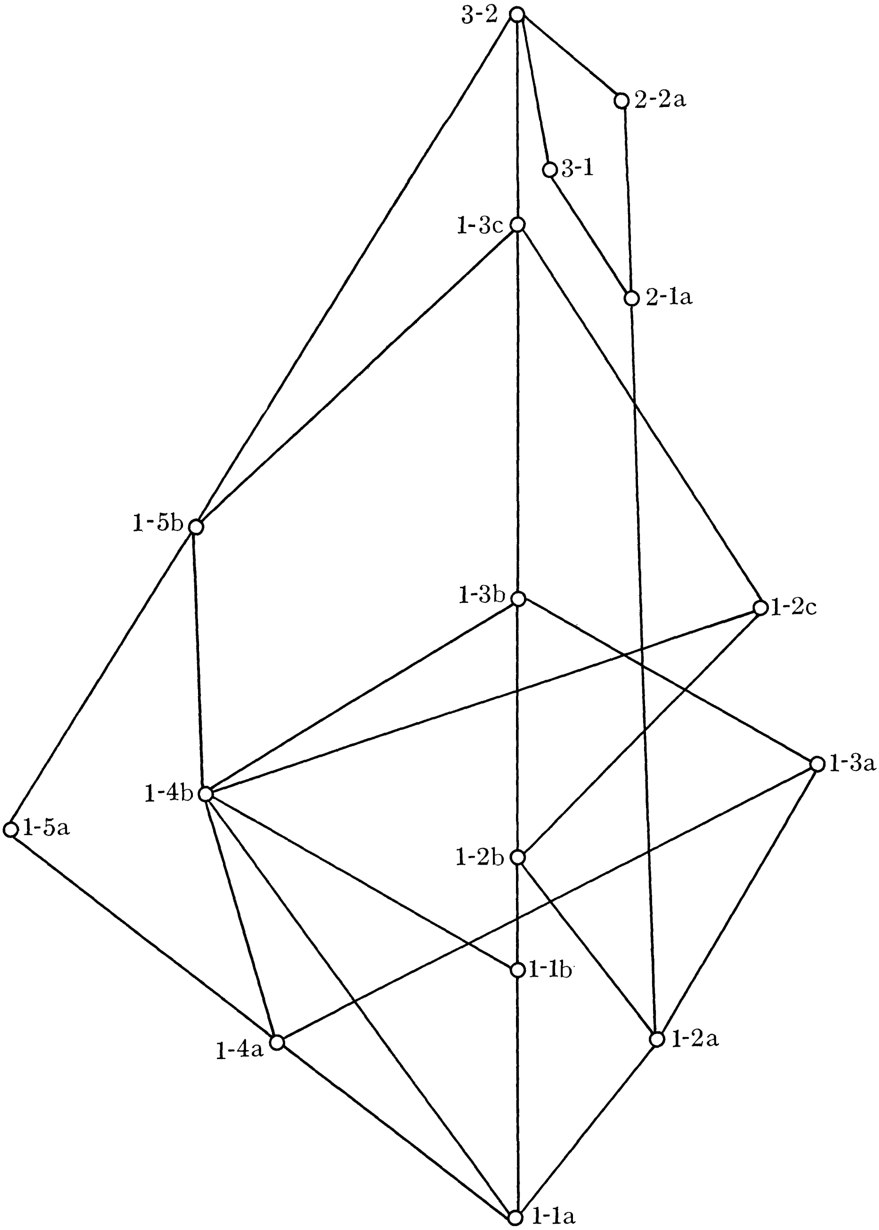
In the following diagram, Type j-$k_1 r_1$, $j = 1, 2, 3$, is above Type i-$k_2 r_2$, $i = 1, 2, 3$ ($r_1, r_2 = 1$ if j or $i = 3$, respectively), if and only if the symbol referring to the first type is higher on the page than the symbol referring to the second type and one can travel from the second symbol to the first on line segments.

5. The choice of subplane π_0 in the classification.

(5.1) PROPOSITION. *Let π be sst of order q^2 and β a non-trivial (p, L)-collineation of π. If π_0 is any subplane of order q which is invariant under β, then both p and L are in π_0.*

Proof. Suppose that $p \not{I} \pi_0$. Every line through p meets π_0 in one or $q + 1$ points. If $\tilde{L}$ is on p and meets π_0 in one point, then $\tilde{L} \cap \pi_0$ is fixed by β. Now since $p \not{I} \pi_0$, there is exactly one line of π_0 through p. There are $q^2 + 1$ lines through p so that we have q^2 fixed points in π_0 under β. Thus, there are at least $q^2 - q$ points on L and not in π_0 that are fixed by β. Hence, β fixes at least $2q^2 - q > q^2 + q$ points of π_0 (note that $q^2 > 4$ for otherwise π would be Desarguesian, contrary to assumption). However, this implies that β is the identity collineation (see **21**, pp. 101–102, theorem 19). However, this is a contradiction. The dual argument may be used to show that L is also in π_0. Hence, (5.1) is proved.

The possible types 1-, 2-, and 3- shown schematically.



(5.2) COROLLARY. *Let π be sst and π_1, π_2 subplanes of order q. Let π be classified by (p, L, π_1)-transitivity and then, independently, by (p, L, π_2)-transitivity. Let π be in class C_1 with respect to π_1 and class C_2 with respect to π_2. If both π_1 and π_2 are invariant, then $C_1 = C_2$.*

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Proof. Let $(\tilde{p}, \tilde{L}) \in S(\pi$ with respect to $\pi_1)$. By (5.1), $(\tilde{p}, \tilde{L}) \in S(\pi$ with respect to $\pi_2)$ since π_2 is invariant. Since also π_1 is invariant, we may interchange π_1 and π_2 in the above argument. Thus, (5.2) is proved.

Therefore, if we are classifying planes containing invariant subplanes of order q, we may choose any invariant subplane of order q in order to make our classification, and the class into which a particular plane is placed is independent of the choice of invariant subplane. Also, recall (see remarks following (2.1)(i)) that if π is of Type 2 or 3, there is a unique invariant subplane whose points are the elation centres of the lines in the orbit of L_∞.

We will now consider the situation where π is sst and π does not have an invariant subplane of order q.

We will, in the following, consider π classified with respect to (p, L, π_i)-transitivity for certain subplanes π_i of order q. It should be noted that in order for π to be classified with respect to (p, L, π_i)-transitivity for a particular subplane π_i, it is assumed that π is (p, L, π_i)-transitive for all points $p \text{ I } L_\infty \cap \pi_i$ and $L = L_\infty$. That is, π will always be assumed to be above Type 1-1a with respect to each subplane π_i. One could, of course, make a classification of planes which are not semi-translation planes without the above restriction.

If π_0 and π_1 are subplanes of order q such that π is (p, L, π_i)-transitive for all points $p \text{ I } L_\infty \cap \pi_i$, $i = 0, 1$, then it is easily seen that π_0 and π_1 are disjoint as affine subplanes and have the same points on L_∞.

(5.3) *Definition.* Let π_0 and π_1 be subplanes of π and of order q and H a group of collineations of π and $\beta \in H$ such that $\pi_0\beta = \pi_1$. We shall say, in this case, that π_1 and π_0 are in the same subplane orbit under H. If H is the full group of collineations of π, we will say that π_0 and π_1 are in the same subplane orbit.

(5.4) PROPOSITION. *Let π_1 and π_2 be subplanes of π of order q. Let π be classified by (p, L, π_i)-transitivity for $i = 1, 2$. If π_1 and π_2 are in the same subplane orbit, then π is in the same class regardless of the choice of classification with respect to π_1 or with respect to π_2.*

Proof. Suppose that there exists a collineation β of π such that $\pi_1\beta = \pi_2$ and π is $(\tilde{p}, \tilde{L}, \pi_1)$-transitive. Let α be a $(\tilde{p}, \tilde{L}, \pi_1)$-collineation. Then, $\beta^{-1}\alpha\beta$ is a $(\tilde{p}\beta, \tilde{L}\beta, \pi_1\beta) = (\tilde{p}\beta, \tilde{L}\beta, \pi_2)$-collineation. Therefore, π is $(\tilde{p}\beta, \tilde{L}\beta, \pi_2)$-transitive. Since we can interchange π_1 and π_2 in the above argument, we have proved (5.4).

Now suppose that π_0 and π_1 are in different subplane orbits and suppose that we classify π by both (p, L, π_0)-transitivity and (p, L, π_1)-transitivity. If π_0 is fixed by the $(\tilde{p}, \tilde{L}, \pi_1)$-collineations, then by the argument of (5.1), $\tilde{p}$ and $\tilde{L}$ would be in π_0. Thus, π would be in a "higher" class classified with respect to π_0 than classified with respect to π_1. That is, if π is of Type T_0 with respect to π_0 and of Type T_1 with respect to π_1, then Type T_0 is above Type T_1.

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Therefore, if, under any two classifications by π_0 and π_1, either π_0 is fixed by the (p, L, π_1)-collineations or π_1 is fixed by the (p, L, π_0)-collineations (these particular (p, L)-collineations are assumed to be other than the translations), then we can always choose the subplane that will give us the "best" classification. That is, we can pick the subplane that will place the plane in the "highest" class.

(5.5) LEMMA. *If* $S(1\text{-}1b) \subseteq S(\pi)$, *then* π *has a unique invariant subplane* π_0 *consisting of* L_∞*-homology centres.*

Proof. If π_0 is not fixed, then for all points $p \text{ I } \pi_0 - L_\infty$ there exists a collineation β of π which moves p to a point $p\beta$ not in π_0. Therefore, by (**1**), there is a translation moving p to $p\beta$, contrary to π being a strict semi-translation plane.

Thus, if π does not have an invariant subplane, the only possible types are 1-1a, 1-2a, 1-3a, 1-4a, 1-5a.

(5.6) LEMMA. *If* π *does not have an invariant subplane and is above Type* 1-2a *then the special point* p_∞ *is invariant under all collineations of* π *and the set(s) of* p_∞ *elation axes is also invariant under all collineations of* π.

Proof. p_∞ is clearly fixed (**18**, Lemma 7). If π is above Type 1-2a with respect to π_0 and the set of p_∞-elation axes is not fixed, then there is a line $\tilde{L} \text{ I } p_\infty$ and $\tilde{L} \neq L_\infty$ (clearly L_∞ is always fixed) such that $\tilde{L}$ is moved onto a line not in π_0. Each two lines of the orbit of $\tilde{L}$ on p_∞ generate a distinct translation with centre p_∞. Since the length of the orbit of $\tilde{L}$ on p_∞ is strictly larger than q, we clearly can generate a translation group of order strictly larger than q with centre p_∞. However, this is a contradiction to π being strict.

(5.7) LEMMA. *Let* π *be classified by both* (p, L, π_0)*-transitivity and* (p, L, π_1)*-transitivity, where* π_0 *and* π_1 *are in different subplane orbits. Suppose that there exists a* (p_1, L_1, π_1)*-transitivity that does not fix* π_0 *and a* (p_0, L_0, π_0)*-transitivity that does not fix* π_1*; then* π *is not of Type* 1-2a.

Proof. Suppose that π is of Type 1-2a with respect to π_0 with special point p_∞. Then $p_0 = p_\infty$, $p_1 = p_0$, and $p_1 \not\mathrm{I}\, L_1$ (see (5.5)).

Thus, L_1 is moved onto a line of the pencil of lines on $L_1 \cap L_\infty$ which is not in π_1 by the (p_0, L_0, π_0)-collineations. (If L_1 is moved onto a line of π_1, then the points $L_1 \cap \pi_1$ are moved onto points of π_1, contrary to the assumption that π_1 is moved by the (p_0, L_0, π_0)-collineations.)

Thus, L_1 is in an orbit of lines on $L_1 \cap L_\infty$ of length strictly larger than q. And, all of these lines are axes of central collineations with centre p_1. By (**1**), L_1 and every line $L_1\beta$ of this orbit generate a translation with centre p_1 which moves L_1 onto $L_1\beta$. Thus, we clearly have a translation group of order strictly larger than q with centre p_1, contrary to the assumption that π is a strict semi-translation plane.

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(5.8) THEOREM. *If π is a strict semi-translation plane of order q^2 with no invariant subplane of order q, then there always exists a subplane π_0 such that a classification by (p, L, π_0)-transitivity places π in a type above that which we obtain by using any other subplane of order q.*

Proof. By (5.4) and (5.7), either we can make such a choice or π is classified by (p, L, π_0)-transitivity and (p, L, π_1)-transitivity and there exists a (p_1, L_1, π_1)-transitivity moving π_0 and a (p_0, L_0, π_0)-transitivity moving π_1.

By (5.5), (5.6), and (5.7) we have the following possibilities: π is of Types 1-3a and 1-4a, of Types 1-3a and 1-5a, of Types 1-4a and 1-4a, or of Types 1-4a and 1-5a with respect to π_0 and π_1, respectively.

Clearly, either we can choose a subplane to give us a "best" classification for π, or π is of Types 1-3a and 1-5a with respect to π_0 and π_1, respectively.

In the latter case, $p_0 = p_1 = p_\infty$ (see (5.6)). Suppose that $L_0 \not{I}\, p_0$. If $L_1 \cap L_0 \mathrm{\,I\,} L_\infty$, then the (p_0, L_0, π_0)-collineations must move L_1 onto a line not of π_1. By an argument similar to that of (5.7), we can use André's results to generate a translation group with centre p_1 of order strictly larger than q, which is a contradiction.

Now also, if $L_0 \mathrm{\,I\,} p_0$, the argument of (5.7) yields a contradiction.

Hence, $L_1 \cap L_0 \not{I}\, L_\infty$. By (**18**, Lemma 1), we can choose a subplane π_2 to include $L_1 \cap L_0$ such that π is (p, L_∞, π_2)-transitive for all $p \mathrm{\,I\,} L_\infty \cap \pi_2$. Furthermore, by (5.1), p_0, L_1, and L_0 are all in π_2.

Each of the (p_0, L_0, π_0)-collineations move L_1 onto a distinct line not in π_1 and fixes p_1. Each two groups with centre p_1 and axis one of these lines (all of which intersect at $L_1 \cap L_0$) generate an elation with centre p_1 and axis $(L_1 \cap L_0)p_1$. Hence, p_1 is the centre of an elation group of order strictly larger than q. However, this again is contrary to our assumptions, since this elation group fixes π_2 (see (5.1)). (Alternately, π is clearly above D1-2b with respect to π_2, contrary to (5.5).)

Remarks. In (**8**), I gave examples of strict semi-translation planes of Types 1-1a, 1-3a, 1-4a, 1-1b, and 3-2 (see **8**, (2.16) and (4.2)). (In addition, I found an example of a plane of Type 1-2a. This result will be reported in a later paper.)

There exist examples of planes which contain an invariant subplane of all of the above types. The planes derived from the dual Ostrom-Rosati planes are of Type 1-3a. It is not known whether these planes have an invariant subplane of order q.

For $q = 5$ or 9, we have seen that there might exist exceptional cases of planes of types strictly above 1-5a or 1-5b but not above 1-2a or 1-2b, respectively. For $q = 9$, π_0 could then be the nearfield plane of order 9. If $q = 5$ and π_0 is Desarguesian, there might exist a plane of order 25 such that the group induced on $L_\infty \cap \pi_0$ by the group generated by the existing homology (p, L, π_0)-transitivity groups is isomorphic to the symmetric group on four letters.

With the exception of the Hughes planes, for every example of a plane of a certain type, there is an example of a plane of the same type where the plane is derived from a dual translation plane. In (**8**) I showed that the possible types for planes so derived are: 1-1a, 1-2a, 1-3a, 1-4a, and 1-1b.

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