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Ripple modifications to alpha transport in tokamaks

Peter J. Catto†

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Magnetic field ripple is inherent in tokamaks since the toroidal magnetic field is generated by a finite number of toroidal field coils. The field ripple results in departures from axisymmetry that cause radial transport losses of particles and heat. These ripple losses are a serious concern for alphas near their birth speed v_0 since alpha heating of the background plasma is required to make fusion reactors into economical power plants. Ripple in tokamaks gives rise to at least two alpha transport regimes of concern. As the slowing down time τ_s is much larger than the time for an alpha just born to make a toroidal transit, a regime referred to as the $1/\nu \propto \tau_s$ regime can be encountered, with ν the appropriate alpha collision frequency. In this regime the radial transport losses increase as $v_0\tau_s/R$, with R the major radius of the tokamak. The deleterious effect of ripple transport is mitigated by electric and magnetic drifts within the flux surface. When drift tangent to the flux surface becomes significant another ripple regime, referred to as the $\sqrt{\nu}$ regime, is encountered where a collisional boundary layer due to the drift plays a key role. We evaluate the alpha transport in both regimes, taking account of the alphas having a slowing down rather than a Maxwellian distribution function and their being collisionally scattered by a collision operator appropriate for alphas. Alpha ripple transport is found to be in the $\sqrt{\nu}$ regime where it will be a serious issue for typical tokamak reactors as it will be well above the axisymmetric neoclassical level and can be large enough to deplete the alpha slowing down distribution function unless toroidal rotation is strong.

Key words: fusion plasma, plasma confinement

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1. Introduction

Tokamak fusion reactors desire to operate such that the alpha particles born at high energy slow down by electron and then ion drag and thereby deposit nearly all their energy in the background plasma before being lost. However, near the edge of a tokamak, magnetic field ripple, $\delta \ll 1$, due to the finite number, $N \gg 1$, of toroidal field coils can result in trapping in small localized ripple wells that can lead to collisional ripple losses (larger than the axisymmetric collisional losses) by mechanisms considered by Galeev *et al.* (1969), Stringer (1972), Connor & Hastie (1973), and Ho & Kulsrud (1987). These collisional ripple loss calculations did not consider alpha particles which have a slowing down tail background distribution

† Email address for correspondence: catto@psfc.mit.edu

function rather than a Maxwellian background distribution. In the following sections we formulate and solve for collisional transport in a ripple tokamak by considering both the $1/\nu$ and $\sqrt{\nu}$ regimes, where ν refers to the relevant collision frequency and the regime is characterized by the ν dependence of its diffusivity.

In the $1/\nu$ regime the radial magnetic drift departure or step, primarily due to the gradient of the magnetic field (that is, the ∇B magnetic drift), is determined by collisions. The evaluation of ripple transport of alphas in the $1/\nu$ regime when pitch angle scattering by the ions dominates is a re-application of the method of Stringer (1972) as substantially improved by Connor & Hastie (1973). When electron drag dominates a similar, but approximate, calculation is performed. Ripple transport of alphas in the $\sqrt{\nu}$ regime is based on a careful collisional boundary layer analysis as recently put forth by Calvo *et al.* (2017), rather than the approximate treatments of Galeev *et al.* (1969) and Ho & Kulsrud (1987). In the $\sqrt{\nu}$ regime the $\boldsymbol{E}\times\boldsymbol{B}$ drift in a flux surface reduces the radial step so it is no longer determined by collisions where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields. In this regime the $\boldsymbol{E}\times\boldsymbol{B}$ drift is large enough that a narrow velocity space boundary layer arises so that only pitch angle scattering collisions need be considered. Our evaluation in the $1/\nu$ regime allows general ripple, $qN\delta \sim \varepsilon \gg \delta$; while that in the $\sqrt{\nu}$ regime assumes strong ripple, $qN\delta \gg \varepsilon \gg \delta$, where $\varepsilon \simeq a/R$ is the inverse aspect ratio with a the minor radius near the separatrix, R the major radius, and q the safety factor. We do not consider the weak ripple limit ($qN\delta \ll \varepsilon$) of Linsker & Boozer (1982) that evaluates $1/\nu$ ripple transport due to small radial magnetic drift steps causing small poloidal angle phase shifts in banana turning point locations after many bounces. The transport associated with this effect differs from the weak ripple ($qN\delta \ll \varepsilon$) limit of our general $1/\nu$ regime evaluation because we do not consider ripple effects on the turning points of alphas with banana trapped orbits in the nearly axisymmetric tokamak magnetic field. Indeed, based on the Linsker & Boozer (1982) estimate of their equation (5) or (26) such effects are expected to be rather small. In the pedestal, if $q \sim 5$–7, $\varepsilon \sim 1/3$, and $\delta \sim 1/100$, with $N \sim 16$–20 (see Paul *et al.* (2017) for more specific numbers), then $\varepsilon/qN\delta \sim 1/3 - 1/4$. Toroidal field coil ripple can be somewhat smaller (by adding coils, for example) so we will order $\varepsilon/qN\delta \sim 1$ when it is possible to do so and still obtain simple analytic results that are useful for insight and numerical checks.

Sections 2 and 3 present some background on alpha particle behaviour and the drift kinetic formalism we use to consistently retain the various collisional transport processes. These sections also serve to introduce most of the notation. In § 4 we give phenomenological estimates for all the transport processes we evaluate in the latter sections. In § 5 we briefly review axisymmetric tokamak collisional transport as evaluated in greater detail by Hsu, Catto & Sigmar (1990). We then go on to § 6 to consider $1/\nu$ transport in the pitch angle scattering and electron drag dominated limits, and give a brief explanation of the difficulty of treating both at the same time, even with a boundary layer analysis about the ripple trapped–passing boundary. Section 7 presents the evaluation of transport in the $\sqrt{\nu}$ regime for $qN\delta \gg \varepsilon \gg \delta$. The analysis is performed by assuming the $\boldsymbol{E}\times\boldsymbol{B}$ drift in a flux surface is strong enough to create a boundary layer narrower than that due to ripple, $\delta^{1/2}$. Due to the careful boundary layer analysis the $\sqrt{\nu}$ regime picks up a logarithmic correction that depends on ripple, collisions, and $\boldsymbol{E}\times\boldsymbol{B}$ drift as first pointed out by Calvo *et al.* (2017). Our results are summarized in § 8, where we indicate that the two ripple transport mechanisms considered are a serious concern for tokamak reactors. Appendix A gives more details on the procedure of Stringer (1972) and Connor & Hastie (1973) as needed to extend their analysis to handle particle transport.

2. Alpha background

The isotropic slowing down tail solution $f_s = f_s(\boldsymbol{r}, v)$ satisfies

$$\frac{Ze}{Mc}\boldsymbol{v}\times\boldsymbol{B}\cdot\nabla_v f_s = C\{f_s\} + \frac{S\delta(v-v_0)}{4\pi v^2}, \tag{2.1}$$

where the coefficient of the $\delta(v-v_0)$ function is the fusion birth rate for alphas:

$$S = S(\psi) = n_T(\psi)n_D(\psi)\langle\sigma v\rangle_{DT}. \tag{2.2}$$

The alphas of mass M and charge number Z are born with birth speed v_0. The alpha collision operator is C, with c the speed of light, e the charge on a proton and $\boldsymbol{B}$ the magnetic field.

We expect alphas to be born on surfaces of constant pressure so that they have the usual slowing down distribution function on a flux surface to lowest order for small poloidal gyroradius and ripple ($\delta \sim \varepsilon/qN$):

$$f_s = f_s(\psi, v) = \frac{S(\psi)\tau_s(\psi)H(v_0 - v)}{4\pi[v^3 + v_c^3(\psi)]}, \tag{2.3}$$

with ψ the poloidal flux function, $H(v_0 - v)$ the Heaviside step function, τ_s the alpha slowing down time

$$\tau_s = \tau_s(\psi) = \frac{3MT_e^{3/2}(\psi)}{4(2\pi m)^{1/2}Z^2e^4n_e(\psi)\ell n\Lambda} \tag{2.4}$$

and v_c the critical speed defined by summing over background ions

$$v_c^3 = v_c^3(\psi) = \frac{3\pi^{1/2}T_e^{3/2}(\psi)}{(2m)^{1/2}n_e(\psi)}\sum_i \frac{Z_i^2 n_i(\psi)}{M_i}. \tag{2.5}$$

Electron and bulk ion densities and temperatures are denoted by n_j and T_j, with m the electron mass and M_i the mass of a bulk ion of charge Z_i. The density of slowing down alphas is

$$n_s = \int \mathrm{d}^3v f_s = S\tau_s\int_0^{v_0}\frac{v^2\,\mathrm{d}v}{(v^3+v_c^3)} = \frac{S\tau_s}{3}\ell n[1 + (v_0^3/v_c^3)] \simeq S\tau_s\ell n(v_0/v_c), \tag{2.6}$$

where we assume $v_0^3 \gg v_c^3$ as is the case for the deuterium–tritium (D–T) fusion reaction. Self-collisions are unimportant for the alphas. They are effectively a trace population since $n_s/n_e \sim T_i/Mv_0^2 \ll 1$, based on the alpha heating estimate $n_sMv_0^2 \sim n_eT_i$.

To obtain the slowing down distribution function the usual collision operator for alphas is employed (see Cordey 1976), namely,

$$C\{f\} = \frac{1}{\tau_s}\nabla_v\cdot\left[\left(\frac{v^3+v_c^3}{v^3}\right)\boldsymbol{v}f + \frac{v_\lambda^3}{2v^3}(v^2\boldsymbol{I} - \boldsymbol{vv})\cdot\nabla_v f\right], \tag{2.7}$$

where

$$v_\lambda^3 = v_\lambda^3(\psi) = \frac{3\pi^{1/2}T_e^{3/2}(\psi)}{(2m)^{1/2}M_\alpha n_e(\psi)}\sum_i Z_i^2 n_i(\psi). \tag{2.8}$$

To obtain this form and generalizations of it expansions of the collision operator are employed for $v_e \gg v \gg v_i$, where $v_e = \sqrt{2T_e/m}$ and $v_i = \sqrt{2T_i/M_i}$ are the electron and typical bulk ion thermal speeds. Typically, $v_0 \gtrsim v_c \sim v_\lambda$, with $v_0/v_c \sim 3$ for D–T fusion. Equation (2.3) is the solution of (2.1) with (2.7) inserted since a delta function sink resides in $\tau_s^{-1}v_c^3\nabla_v\cdot(v^{-3}\boldsymbol{v}f) \to -4\pi\tau_s^{-1}v_c^3 f(\psi, v=0)\delta(\boldsymbol{v})$ as $v \to 0$.

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3. Drift kinetic formulation

For the Vlasov operator we use the drift kinetic equation of Hazeltine (1973) in $\boldsymbol{r}$, v or

$$E = v^2/2 + Ze\Phi(\psi)/M, \tag{3.1}$$
$$\mu = v_\perp^2/2B, \tag{3.2}$$

and gyrophase φ variables, with an electrostatic electric field

$$\boldsymbol{E} = -\nabla\Phi(\psi) - \nabla\phi(\boldsymbol{r}), \tag{3.3}$$

where the total electrostatic potential is $\Phi(\psi) + \phi(\boldsymbol{r})$ with $\phi(\boldsymbol{r})$ a periodic function of the poloidal, ϑ, and toroidal, ζ, angles that also depends on the poloidal flux function ψ. Then

$$\begin{aligned}(v_\parallel \boldsymbol{b} + \boldsymbol{v}_d)\boldsymbol{\cdot}\left(\nabla f - \frac{Ze}{Mv}\frac{\partial f}{\partial v}\nabla\phi\right) &= \frac{v_\parallel}{B}\nabla\boldsymbol{\cdot}\left\{f\left[\boldsymbol{B} + \frac{Mc}{Ze}\nabla\times(v_\parallel\boldsymbol{b})\right]\right\} \\ &\quad - \frac{Ze}{Mv}v_\parallel\boldsymbol{b}\boldsymbol{\cdot}\nabla\phi\frac{\partial f}{\partial v} = C\{f\} + \frac{S\delta(v-v_0)}{4\pi v^2},\end{aligned} \tag{3.4}$$

where f is the alpha distribution function, $C\{f\}$ is the linear collision operator (2.7) for alphas, and

$$\boldsymbol{v}_d = \frac{v_\parallel}{\Omega}\nabla\times(v_\parallel\boldsymbol{b}) = \frac{c}{B^2}\boldsymbol{B}\times\nabla\Phi + \frac{\mu}{\Omega}\boldsymbol{b}\times\nabla B + \frac{v_\parallel^2}{\Omega}\boldsymbol{b}\times(\boldsymbol{b}\boldsymbol{\cdot}\nabla\boldsymbol{b}) + \frac{v_\parallel^2}{\Omega}\boldsymbol{b}\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b}. \tag{3.5}$$

The preceding form of the drift kinetic equation is adequate for our purposes as it gives the correct electric and magnetic drifts. It does not give the proper small parallel velocity correction $(v_\perp^2/2\Omega)\boldsymbol{b}\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b}$, to the parallel streaming term, as discussed in Boozer (1980), Parra & Catto (2008), and Landreman & Catto (2013). We neglect this small parallel streaming correction from here on as it is of no importance, and we assume $Ze\phi \ll Ze\Phi \sim T_i \sim T_e$. In addition, we may view $v = v(E, \psi)$ by using

$$v = \sqrt{2[E - Ze\Phi(\psi)/M]} \tag{3.6}$$

whenever we need to keep non-radial components of the $(c/B)\boldsymbol{b}\times\nabla\Phi$ drift. Moreover, we use

$$v_\parallel^2 = v^2 - 2\mu B = 2[E - (Ze\Phi/M) - \mu B]. \tag{3.7}$$

The turning points of a trapped particle are always at the same value of B as it moves on a flux surface even in the presence of precession.

In the preceding, the magnetic field is

$$\boldsymbol{B} = \boldsymbol{B}(\psi, \vartheta, \zeta) = B\boldsymbol{b} = |\boldsymbol{B}|\boldsymbol{b}, \tag{3.8}$$

and the alpha gyrofrequency is

$$\Omega = ZeB/Mc. \tag{3.9}$$

Using ψ, ϑ, and ζ as the variables, with

$$\alpha = \zeta - q\vartheta, \tag{3.10}$$

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and $q = q(\psi)$ the safety factor, the Clebsch and Boozer (1981) representations for the magnetic field are

$$\boldsymbol{B} = \nabla\alpha \times \nabla\psi = K(\psi, \vartheta, \zeta)\nabla\psi + G(\psi)\nabla\vartheta + I(\psi)\nabla\zeta, \tag{3.11}$$

with $K(\psi, \vartheta, \zeta)$ periodic in ϑ and ζ. The preceding give

$$\boldsymbol{B}\cdot\nabla\vartheta = \nabla\alpha \times \nabla\psi \cdot \nabla\vartheta = \nabla\psi \times \nabla\vartheta \cdot \nabla\zeta \tag{3.12}$$

and

$$\boldsymbol{B}\cdot\nabla\zeta = q\nabla\psi \times \nabla\vartheta \cdot \nabla\zeta = q\boldsymbol{B}\cdot\nabla\vartheta, \tag{3.13}$$

as well as $\boldsymbol{B}\cdot\nabla\alpha = 0 = \boldsymbol{B}\cdot\nabla\psi$. In addition,

$$B^2 = (G + qI)\boldsymbol{B}\cdot\nabla\vartheta, \tag{3.14}$$

with $G/qI \sim rB_p/qRB_t \sim \varepsilon^2/q^2 \ll 1$, ε the inverse aspect ratio, and B_p and B_t the poloidal and toroidal magnetic fields. In addition, we are ignoring finite gyroradius corrections, but will retain finite poloidal gyroradius effects by assuming $q/\varepsilon \gg 1$.

The ripple δ due to N toroidal field coils is defined as

$$\delta = (B_{\max} - B_{\min})/(B_{\max} + B_{\min}), \tag{3.15}$$

with $\delta \ll \varepsilon$. We consider the ripple trapping wells as being due to the toroidal magnetic field that is produced by the N toroidal field coils. We will often make use of the simple form

$$B = B_0[1 - \varepsilon(\psi)\cos\vartheta - \delta(\psi)\cos(N\zeta)], \tag{3.16}$$

to obtain explicit results. This simple form allows magnetic field minima away from $\vartheta = 0$. For ripple extending to the magnetic axis a form such as $\delta \simeq \delta_0 + (\delta_a - \delta_0)(r/a)^p$ can be used with p an integer and poloidal variation of the ripple ignored. Forms with poloidally dependent ripple are not considered in order to obtain carefully evaluated analytic results. However, the evaluations in the $1/\nu$ are performed quite generally, with the simple form of (3.16) only being used to get explicit results in the final steps. Following Stringer (1972) and Connor & Hastie (1973), the extrema in the magnetic field are found from $\boldsymbol{B}\cdot\nabla B = 0$ at fixed α to be given by $\varepsilon\sin\vartheta\boldsymbol{B}\cdot\nabla\vartheta + N\delta\sin(N\zeta)\boldsymbol{B}\cdot\nabla\zeta = 0$ or

$$\varepsilon\sin\vartheta + qN\delta\sin(N\zeta) = 0. \tag{3.17}$$

If $qN\delta \ll \varepsilon$ then the ripple wells are only near $\vartheta = 0$ and $\vartheta = \pm\pi$. However, for $qN\delta > \varepsilon$ ripple wells exist for all ϑ. To see this more clearly we note that because $N \gg 1$ the ripples have a short toroidal extent compared to the tokamak circumference so the variation ϑ of a trapped alpha during its bounce motion in a ripple is small while $N\zeta$ varies by a full bounce period. To introduce magnetic field minima containing ripple trapped particles at all ϑ requires $qN\delta > \varepsilon|\sin\vartheta| > \varepsilon$. Consequently, we will assume the ordering $qN\delta \sim \varepsilon$ when possible, and use $qN\delta \gg \varepsilon$ to further simplify expressions. Retaining a slow poloidally varying ripple introduces a small order δ correction to (3.17). We do not expect it to result in a qualitative change in our results.

In an axisymmetric tokamak of inverse aspect ratio ε there is a single well and the fraction of particles that are trapped is $\varepsilon^{1/2}$. In a tokamak with N field ripples the trapped fraction in each ripple well is $\delta^{1/2}$. Consequently, transit averages in simple

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ripple wells such as $B \simeq B_0[1 - \delta \cos(N\zeta)]$ are very similar to those in standard axisymmetric tokamak wells $B \simeq B_0(1 - \varepsilon \cos\vartheta)$.

To retain axisymmetric and ripple transport systematically it is convenient to separate f into two terms by introducing one function f_* that only depends on what would be the constants of motion if we were in the axisymmetric limit and a remainder function h. Therefore, we let

$$f = f_*(\psi_*, v) + h(\boldsymbol{r}, v, \mu, \sigma), \tag{3.18}$$

where $f_* \simeq f_s \gg h$,

$$\sigma = v_\parallel / |v_\parallel|, \tag{3.19}$$

and a variable that is not the drift kinetic constant of the motion in the presence of ripple,

$$\psi_* \equiv \psi - I(\psi) v_\parallel / \Omega. \tag{3.20}$$

Because our tokamak is rippled $(v_\parallel \boldsymbol{b} + \boldsymbol{v}_d) \cdot \nabla \psi_* \neq 0$. Then we may write

$$\begin{aligned}(v_\parallel \boldsymbol{b} + \boldsymbol{v}_d) \cdot \nabla f_* + \frac{v_\parallel}{B} \nabla \cdot \left\{ h \left[\boldsymbol{B} + \frac{Mc}{Ze} \nabla \times (v_\parallel \boldsymbol{b}) \right] \right\} \\ - \frac{Ze}{Mv} v_\parallel \boldsymbol{b} \cdot \nabla \phi \frac{\partial f}{\partial v} = C\{f\} + \frac{S\delta(v - v_0)}{4\pi v^2},\end{aligned} \tag{3.21}$$

where in going from (3.4) to (3.21) only a small term proportional $\boldsymbol{v}_d \cdot \nabla \phi$ to is neglected.

Recall that in an axisymmetric tokamak

$$\boldsymbol{B} \to I(\psi) \nabla \zeta + \nabla \zeta \times \nabla \psi, \tag{3.22}$$

and the alphas try to move on a surface of constant canonical angular momentum

$$\psi_* \to \psi - (Mc/Ze) R^2 \nabla \zeta \cdot \boldsymbol{v} = \psi - I(\psi) v_\parallel / \Omega + \Omega^{-1} \boldsymbol{b} \times \nabla \psi \cdot \boldsymbol{v} \simeq \psi - I(\psi) v_\parallel / \Omega, \tag{3.23}$$

where we used $R^2 \nabla \zeta \to B^{-1}(I\boldsymbol{b} - \boldsymbol{b} \times \nabla \psi)$ and neglect gyromotion in the last form on the right.

To rewrite the kinetic equation we use

$$(v_\parallel \boldsymbol{b} + \boldsymbol{v}_d) \cdot \nabla \psi_* = \boldsymbol{v}_d \cdot \nabla (\psi - I v_\parallel / \Omega) - v_\parallel \boldsymbol{b} \cdot \nabla (I v_\parallel / \Omega). \tag{3.24}$$

In an axisymmetric tokamak $\boldsymbol{v}_d \cdot \nabla \psi \to v_\parallel \boldsymbol{b} \cdot \nabla (I v_\parallel / \Omega)$ and $\boldsymbol{v}_d \cdot \nabla (I v_\parallel / \Omega) \to 0$ (see appendix B of Parra & Catto (2010)). Due to ripple we must keep the difference $\boldsymbol{v}_d \cdot \nabla \psi - v_\parallel \boldsymbol{b} \cdot \nabla (I v_\parallel / \Omega)$. However, we can neglect $\boldsymbol{v}_d \cdot \nabla (I v_\parallel / \Omega)$ as small since it can only depend on the departure from axisymmetry and it is also small in the drift. As a result, we find

$$\begin{aligned}v_\parallel \boldsymbol{b} \cdot \nabla h + \left[\boldsymbol{v}_d \cdot \nabla \psi - v_\parallel \boldsymbol{b} \cdot \nabla \left(\frac{I v_\parallel}{\Omega} \right) \right] \frac{\partial f_*}{\partial \psi_*} + \frac{v_\parallel}{\Omega} \nabla \cdot [h \nabla \times (v_\parallel \boldsymbol{b})] \\ - \frac{Ze}{Mv} v_\parallel \boldsymbol{b} \cdot \nabla \phi \frac{\partial f_*}{\partial v} = C\{f\} + \frac{S\delta(v - v_0)}{4\pi v^2},\end{aligned} \tag{3.25}$$

where we have also neglected $v_\parallel \boldsymbol{b} \cdot \nabla \phi \partial h / \partial v$ as small since we assume both h and ϕ are small.

It is convenient to be able to relate f_s and f_* by Taylor expansion by defining

$$f_* = f_s(\psi \to \psi_*, v) = \frac{S(\psi_*)\tau_s(\psi_*)H(v_0 - v)}{4\pi[v^3 + v_c^3(\psi_*)]} = f_s(\psi, v) + (\psi_* - \psi)\frac{\partial f_s}{\partial \psi} + \cdots, \tag{3.26}$$

from which it follows that our kinetic equation becomes

$$\begin{aligned} v_\parallel \boldsymbol{b} \cdot \nabla h + \left[\boldsymbol{v}_d \cdot \nabla \psi - v_\parallel \boldsymbol{b} \cdot \nabla \left(\frac{I v_\parallel}{\Omega} \right) \right] \frac{\partial f_s}{\partial \psi} + \frac{v_\parallel}{\Omega} \nabla \cdot [h \nabla \times (v_\parallel \boldsymbol{b})] \\ - \frac{Ze}{Mv} v_\parallel \boldsymbol{b} \cdot \nabla \phi \frac{\partial f_s}{\partial v} = C\{f_* - f_s + h\}, \end{aligned} \tag{3.27}$$

where we use the approximations

$$\partial f_* / \partial \psi_* \simeq \partial f_s / \partial \psi \tag{3.28}$$

and

$$\partial f_* / \partial v \simeq \partial f_s / \partial v, \tag{3.29}$$

that are consistent with assuming $(I v_\parallel / \Omega) f_s^{-1} \partial f_s / \partial \psi \ll 1$. In an axisymmetric tokamak the left side of (3.27) reduces to

$$v_\parallel \boldsymbol{b} \cdot \nabla \left(h - \frac{Ze\phi}{Mv} \frac{\partial f_s}{\partial v} \right) \simeq v_\parallel \boldsymbol{b} \cdot \nabla h, \tag{3.30}$$

as the streaming term is larger than the drift term and the ϕ term is unimportant. We always assume that to lowest order the slowing distribution function f_s holds. In the next section and in the summary we will discuss that the consistency of this assumption requires that the radial diffusion is weak compared to slowing down and results in an important constraint.

We will often make use of ψ, ϑ, and ζ independent variables. In these variables, the divergence of an arbitrary vector $\boldsymbol{A}$ can be written as

$$\nabla \cdot \boldsymbol{A} = \boldsymbol{B} \cdot \nabla \vartheta \left[\frac{\partial}{\partial \psi} \left(\frac{\boldsymbol{A} \cdot \nabla \psi}{\boldsymbol{B} \cdot \nabla \vartheta} \right) + \frac{\partial}{\partial \theta} \left(\frac{\boldsymbol{A} \cdot \nabla \vartheta}{\boldsymbol{B} \cdot \nabla \vartheta} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\boldsymbol{A} \cdot \nabla \zeta}{\boldsymbol{B} \cdot \nabla \vartheta} \right) \right]. \tag{3.31}$$

Then, for example,

$$\nabla \psi \cdot \nabla \times (v_\parallel \boldsymbol{b}) = \nabla \cdot (v_\parallel \boldsymbol{b} \times \nabla \psi) = \boldsymbol{B} \cdot \nabla \vartheta \left[\frac{\partial}{\partial \vartheta} \left(\frac{I v_\parallel}{B} \right) - \frac{\partial}{\partial \zeta} \left(\frac{G v_\parallel}{B} \right) \right]. \tag{3.32}$$

Using these variables and assuming streaming dominates for the alphas gives the lowest order equation to be

$$v_\parallel \boldsymbol{b} \cdot \nabla \bar{h} = v_\parallel \boldsymbol{b} \cdot \nabla \vartheta \left(\frac{\partial \bar{h}}{\partial \vartheta} + q \frac{\partial \bar{h}}{\partial \zeta} \right) = 0. \tag{3.33}$$

For the passing alphas, $h \to h_p$ and it must be a periodic flux function to lowest order as the passing trace out a flux surface. Therefore, we write

$$h_p = \bar{h}_p(\psi, v, \mu, \sigma) + \tilde{h}_p(\psi, \vartheta, \zeta, v, \mu, \sigma). \tag{3.34}$$

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The lowest order trapped alpha distribution function, $h \to \bar{h}_t$, must be even in $v_\|$ and need not be periodic in ϑ and ζ. For it we must allow

$$h_t = \bar{h}_t(\psi, \alpha, v, \mu) + \tilde{h}_t(\psi, \alpha, \vartheta, v, \mu, \sigma). \tag{3.35}$$

To annihilate the streaming term to next order we introduce the transit average along the exact $\boldsymbol{B}$:

$$\bar{A} = \frac{\oint_\alpha \mathrm{d}\ell\, A/v_\|}{\oint_\alpha \mathrm{d}\ell/v_\|} = \frac{\oint_\alpha \mathrm{d}\tau\, A}{\oint_\alpha \mathrm{d}\tau} = \frac{\oint_\alpha \mathrm{d}\vartheta\, A/v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\vartheta}{\oint_\alpha \mathrm{d}\vartheta\, /v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\vartheta} = \frac{\oint_\alpha \mathrm{d}\zeta\, A/v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\zeta}{\oint_\alpha \mathrm{d}\zeta\, /v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\zeta}, \tag{3.36}$$

with A arbitrary, $\mathrm{d}\tau = \mathrm{d}\ell/v_{||} = \mathrm{d}\vartheta/v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\vartheta = \mathrm{d}\zeta/v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\zeta$, and $q\,\mathrm{d}\vartheta = \mathrm{d}\zeta$ for α fixed (as denoted by the subscript on the loop integral) for the trapped. Here the integrals are over a full bounce for trapped particles and over all ϑ or ζ for passing particles as they trace out a flux surface. We transit average by allowing $B = B(\psi, \vartheta, \zeta)$ and $v_{||} = v_\|(\psi, \vartheta, \zeta)$ with ϑ and ζ at fixed α related by $\alpha = \zeta - q\vartheta$.

Then, using

$$\overline{v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla h} = 0, \tag{3.37}$$

$$\overline{v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\left(\frac{Iv_\|}{\Omega}\right)} = 0, \tag{3.38}$$

and

$$\overline{v_\|\boldsymbol{b}\boldsymbol{\cdot}\nabla\phi} = 0, \tag{3.39}$$

the transit averaged equation to the requisite order becomes

$$\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}\frac{\partial f_s}{\partial\psi} + \overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\alpha}\left.\frac{\partial\bar{h}_t}{\partial\alpha}\right|_\zeta = \overline{C\{f_* - f_s + \bar{h}_p + \bar{h}_t\}}, \tag{3.40}$$

for the trapped and the passing. Our transit averaged equation retains all the ripple neoclassical transport effects we are interested in as well as the standard axisymmetric neoclassical banana regime transport. In the axisymmetric limit $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi} \to 0$ and $\partial/\partial\alpha \to 0$ giving $\bar{h}_t \to 0$. The linearity of the collision operator allows (3.40) to be split into the axisymmetric/passing and non-axisymmetric/trapped equations:

$$\overline{C\{f_* - f_s + \bar{h}_p\}} = 0, \tag{3.41}$$

and

$$\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}\frac{\partial f_s}{\partial\psi} + \overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\alpha}\left.\frac{\partial\bar{h}_t}{\partial\alpha}\right|_\zeta = \overline{C\{\bar{h}_t\}}. \tag{3.42}$$

The axisymmetric equation for the passing contains only a drive $\overline{f_* - f_s}$ and response $\bar{h}_p$ odd in $v_{||}$, while the non-axisymmetric equation has a drive $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}$ and response $\bar{h}_t$ even in $v_\|$. When non-axisymmetric effects are present $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi} = 0$ for the passing along with $\partial\bar{h}_p/\partial\alpha = 0$. Consequently, there is no drive term for the passing and we are left with the homogeneous equation $\overline{C\{\bar{h}_p\}} = 0$ with ripple response $\bar{h}_p = 0$. For the trapped, however, $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi} \neq 0$, so there is a drive term to make $\bar{h}_t \neq 0$. For the trapped it is convenient to use ψ, α, and ζ variables for $\bar{h}_t$. As a result, we employ

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$$\nabla\cdot\boldsymbol{A}=\boldsymbol{B}\cdot\nabla\zeta\left[\frac{\partial}{\partial\psi}\left(\frac{\boldsymbol{A}\cdot\nabla\psi}{\boldsymbol{B}\cdot\nabla\zeta}\right)+\left.\frac{\partial}{\partial\alpha}\right|_{\zeta}\left(\frac{\boldsymbol{A}\cdot\nabla\alpha}{\boldsymbol{B}\cdot\nabla\zeta}\right)+\left.\frac{\partial}{\partial\zeta}\right|_{\alpha}\left(\frac{\boldsymbol{A}\cdot\nabla\zeta}{\boldsymbol{B}\cdot\nabla\zeta}\right)\right]. \tag{3.43}$$

We have also assumed $\partial f_s/\partial\psi\gg\partial\bar{h}_t/\partial\psi$.

For the passing it is useful to introduce the flux surface average

$$\langle A\rangle=\frac{\displaystyle\oint \mathrm{d}\vartheta\,\mathrm{d}\zeta\,A/\boldsymbol{B}\cdot\nabla\vartheta}{\displaystyle\oint \mathrm{d}\vartheta\,\mathrm{d}\zeta/\boldsymbol{B}\cdot\nabla\vartheta}=\frac{\displaystyle\oint \mathrm{d}\vartheta\,\mathrm{d}\zeta\,A/B^2}{\displaystyle\oint \mathrm{d}\vartheta\,\mathrm{d}\zeta/B^2} \tag{3.44}$$

to rewrite the passing kinetic equation as

$$\left\langle\frac{B}{v_\parallel}C\{f_*-f_s+\bar{h}_p\}\right\rangle=0. \tag{3.45}$$

The passing collisional constraint determines the velocity space dependence required of the transit averaged $\bar{h}_p$ in response to the poloidal as well as velocity space variation of the radial drift drive term

$$f_*-f_s=(\psi_*-\psi)\partial f_s/\partial\psi+\cdots\simeq-(I\,v_\parallel/\Omega)\partial f_s/\partial\psi. \tag{3.46}$$

In this regime the radial magnetic drift of the passing across flux surfaces is disrupted by collisions, but the collisional fluxes in velocity space across the trapped–passing boundary must balance to maintain a vanishing perturbed axisymmetric trapped distribution function $\bar{h}_t=0$. The trapped response vanishes because $\overline{v_\parallel/B}=0$, thereby removing the drive term

$$\overline{C\{v_\parallel f_s/B\}}=C\{\overline{(v_\parallel/B)}f_s\}=0. \tag{3.47}$$

In (3.42) collisions cause radial steps that prevent the ripple trapped from returning to their starting point making $\overline{\boldsymbol{v}_d\cdot\nabla\psi}\neq 0$ (unless the departure from axisymmetry is omnigeneous – see Landreman & Catto (2012), Calvo *et al.* (2014) and Helander (2014), for example). The radial steps randomized by collisions generate the trapped response $\bar{h}_t$. When drifts within a flux surface are negligible ($\overline{\boldsymbol{v}_d\cdot\nabla\alpha}\to 0$) the $1/\nu$ radial diffusivity regime is recovered. When the drifts within a flux surface become significant the radial drift decreases and the radial diffusivity is reduced to roughly a $\sqrt{\nu}$ dependence. Normally only the $\boldsymbol{E}\times\boldsymbol{B}$ is retained in the $\sqrt{\nu}$ regime.

An alternate form of (3.42) can be found by introducing the second adiabatic invariant J,

$$J=J(\psi,\alpha,v,\mu)=\frac{Mc}{Ze}\oint_\alpha \mathrm{d}\ell\,v_\parallel=\frac{Mc}{Ze}\oint_\alpha \mathrm{d}\tau\,v_\parallel^2=\frac{Mc}{Ze}\oint_\alpha\frac{\mathrm{d}\zeta\,v_\parallel}{\boldsymbol{b}\cdot\nabla\zeta}=\frac{Mc}{Zeq}(G+qI)\oint_\alpha\frac{\mathrm{d}\zeta\,v_\parallel}{B}, \tag{3.48}$$

with the last form using Boozer coordinates. Then for $\bar{f}=\bar{f}(\psi,\alpha,v,\mu)$

$$\begin{aligned}\overline{\frac{v_\parallel}{\Omega}\nabla\cdot[\bar{f}\,\nabla\times(v_\parallel\boldsymbol{b})]}&=\frac{Mc}{Ze\displaystyle\oint_\alpha \mathrm{d}\tau}\left[\frac{\partial\bar{f}}{\partial\alpha}\left.\frac{\partial}{\partial\psi}\right|_\alpha\left(\oint_\alpha\frac{\mathrm{d}\zeta\,v_\parallel}{\boldsymbol{b}\cdot\nabla\zeta}\right)-\frac{\partial\bar{f}}{\partial\psi}\left.\frac{\partial}{\partial\alpha}\right|_\psi\left(\oint_\alpha\frac{\mathrm{d}\zeta\,v_\parallel}{\boldsymbol{b}\cdot\nabla\zeta}\right)\right]\\&=\frac{1}{\displaystyle\oint_\alpha \mathrm{d}\tau}\left(\frac{\partial\bar{f}}{\partial\alpha}\left.\frac{\partial J}{\partial\psi}\right|_\alpha-\frac{\partial\bar{f}}{\partial\psi}\left.\frac{\partial J}{\partial\alpha}\right|_\psi\right).\end{aligned} \tag{3.49}$$

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The J integral is to be performed at fixed α such that $J = J(\psi, \alpha, v, \mu)$. Neglecting the $\partial \bar{h}_t/\partial\psi$ term as small compared to the $\partial f_s/\partial\psi$ term leads to our trapped kinetic equation with

$$\overline{\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\psi} = -\frac{1}{\displaystyle\oint_\alpha \mathrm{d}\tau} \left.\frac{\partial J}{\partial \alpha}\right|_\psi \tag{3.50}$$

and

$$\overline{\boldsymbol{v}_d \cdot \boldsymbol{\nabla}\alpha} = \frac{1}{\displaystyle\oint_\alpha \mathrm{d}\tau} \left.\frac{\partial J}{\partial \psi}\right|_\alpha . \tag{3.51}$$

The α dependence of J acts as the drive for the ripple transport, or said another way, it is responsible for the departure from omnigenity.

In the next section we will use the preceding expressions to make estimates of the neoclassical and ripple transport levels.

4. Phenomenological transport estimates

In the following subsections we modify the standard diffusivity estimates to make them appropriate for alphas in the axisymmetric and ripple transport regimes. In making these estimates we treat $\ell n(v_0/v_c) \sim 1$ for simplicity.

4.1. *Axisymmetric regime*

For an axisymmetric tokamak, collisional alpha transport is evaluated by solving the passing equation $\overline{C\{f_* - f_s + \bar{h}_p\}} = 0$ with $\bar{h}_t = 0$ for the trapped. For the passing $f_* - f_s \sim (\rho_{p0}/a_\alpha) f_s$, with $\rho_{p0} \simeq q\rho_0/\varepsilon$ the poloidal gyroradius at birth, $\rho_0 = v_0/\Omega$ the gyroradius at birth, $\varepsilon \sim a/R$ in the ripple region of minor radius a, and $\partial f_s/\partial r \sim f_s/a_\alpha$, where the radial scale of the alpha density a_α is allowed to be less than the minor radius a for near axis burn or within the pedestal. Assuming pitch angle scattering dominates gives the trapped fraction as $\varepsilon^{1/2}$, the step size as a banana width $q\rho_0/\varepsilon^{1/2}$, so that $\bar{h}_p \sim f_s q\rho_0/\varepsilon^{1/2} a_\alpha$, and an effective collision time of $\varepsilon\tau_s$. The diffusivity ($D \sim \mathscr{F}\Delta^2/\tau$ with $\mathscr{F}$ the trapped fraction, Δ the step, τ the correlation time) is then

$$D_{\mathrm{axi}}^{\mathrm{pas}} \sim (q\rho_0)^2/\varepsilon^{3/2}\tau_s = \varepsilon^{1/2}\rho_{p0}^2/\tau_s, \tag{4.1}$$

as in Catto (1988). The pitch angle scattering pre-factor $(v_\lambda/v)^3$ plays no role because in this limit it only enters as a multiplier in the solubility constraint.

However, if electron drag dominates then this estimate must be modified in accordance with the Nocentini, Tessarotto & Engelmann (1975) evaluation. In this case the trapped fraction and banana width step remain unchanged, but the effective collision time is just the slowing down time, giving the ε smaller result

$$D_{\mathrm{axi}}^{\mathrm{drag}} \sim (q\rho_0)^2/\varepsilon^{1/2}\tau_s = \varepsilon^{3/2}\rho_{p0}^2/\tau_s. \tag{4.2}$$

The $v_\| > 0$ passing alphas are moving to larger ψ surfaces to keep ψ_* constant as v is reduced by drag, while the $v_\| < 0$ passing alphas move to smaller ψ surfaces. More alphas move out then in, resulting in transport due to electron drag.

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Although the ratio

$$D_{\mathrm{axi}}^{\mathrm{drag}}/D_{\mathrm{axi}}^{\mathrm{pas}} \sim \varepsilon \tag{4.3}$$

is small, in practice the neoclassical diffusivity is between $D_{\mathrm{axi}}^{\mathrm{pas}}$ and $D_{\mathrm{axi}}^{\mathrm{drag}}$ as shown by Hsu *et al.* (1990). Their plots compare their exact results with the Nocentini *et al.* (1975) and Catto (1988) results, given by their equations (33a) and (33b), and in rough agreement with the preceding estimates.

To maintain a slowing down distribution function we need the slowing down time to be short compared to the time for diffusive losses to takes to place, that is, we need $\tau_s D_{\mathrm{axi}}^{\mathrm{pas}}/a_\alpha^2 \ll 1$ or

$$\varepsilon^{1/2}\rho_{p0}^2 \ll a_\alpha^2. \tag{4.4}$$

Consequently, we need to keep the alpha poloidal gyroradius at birth comparable or less than the radial scale length of the alpha density.

4.2. *The* $1/\nu$ *regime*

To evaluate transport in the $1/\nu$ regime we solve the trapped equation with no toroidal rotation

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\psi}\,\frac{\partial f_s}{\partial\psi} = \overline{C\{\bar{h}_t\}}, \tag{4.5}$$

when the passing response is zero, $\bar{h}_p = 0$. Pitch angle scattering dominates when $\delta(v_0^3/v_\lambda^3) \ll 1$, where the boundary layer width due to ripple is of order $\delta^{1/2}$. Using $\overline{C\{\bar{h}_t\}} \sim \bar{h}_t v_\lambda^3/v_0^3\tau_s\delta$ this equation gives the estimate

$$\frac{\bar{h}_t}{f_s} \sim \frac{\rho_0 v_0^4\tau_s\delta}{a_\alpha R v_\lambda^3} = \frac{\rho_*\delta}{\nu_{*p}} \ll 1, \tag{4.6}$$

with $\rho_* = \rho_0/a_\alpha \ll 1$, $\boldsymbol{v}_d \sim v_0\rho_0/R$, and a normalized pitch angle scattering frequency ν_{*p} of

$$\nu_{*p} = R v_\lambda^3/\tau_s v_0^4 \ll 1. \tag{4.7}$$

The N toroidal field coils result in ripple wells of depth δ resulting in a trapped fraction of $\mathscr{F} \sim \delta^{1/2}$. The pitch angle scattering time is $(v_0^3/v_\lambda^3)\tau_s$ is larger than the slowing down time, but the effective ripple trapped collision time $\tau \sim (v_0^3/v_\lambda^3)\tau_s\delta$ is smaller. The radial magnetic drift estimate for the alphas, $v_0\rho_0/R$, is insensitive to the number of coils since it is due to the nearly axisymmetric toroidal field. It gives the radial ∇B drift as $V \sim v_0\rho_0/R$ (the poloidal gyroradius does not enter in the absence of axisymmetric banana motion). Then a smaller step $\varDelta \sim V\tau \sim \delta\,(v_0^3/v_\lambda^3)\,\rho_0 v_0\tau_s/R$ occurs because $\delta\,(v_0^3/v_\lambda^3) \ll 1$. As a result, we expect an alpha diffusivity $D \sim \mathscr{F}\varDelta^2/\tau$ of

$$D_{1/\nu}^{\mathrm{pas}} \sim \mathscr{F}\varDelta^2/\tau \sim \delta^{3/2}(v_0^3/v_\lambda^3)\,\tau_s v_0^2\,\rho_0^2/R^2 \tag{4.8}$$

due to pitch angle scattering.

Our estimate assumes large ripple $\delta \gg \varepsilon/qN$ and is the same as Galeev *et al.* (1969), and also the same as the Ho & Kulsrud (1987) estimate for electrons when their power of $\varepsilon_h = \delta$ in (3) is corrected to read $\varepsilon_h^{1/2}$ as would be expected by their argument. This estimate is consistent with the large ripple tokamak limit of Stringer (1972) and the more general treatment of Connor & Hastie (1973). Interestingly, it is smaller than the very near quasisymmetry stellarator estimate from equation (7) of Calvo *et al.* (2014)

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that implies $D_{1/\nu}^{\rm CPAV} \sim \delta(v_0^3/v_\lambda^3)\tau_s\rho_0^2 v_0^2/R^2$, but larger than the Calvo *et al.* (2013) less quasisymmetric stellarator estimate that suggests $D_{1/\nu}^{CPVA} \sim \delta^2(v_0^3/v_\lambda^3)\tau_s\rho_0^2 v_0^2/R^2$.

To keep $1/\nu$ ripple losses due to pitch angle scattering from depleting the lowest order slowing down distribution during a slowing down time we need the ripple to be small enough to keep $\tau_s D_{1/\nu}^{\rm pas}/a_\alpha^2 \ll 1$. Needing $\bar{h}_t \ll f_s$ with $\delta(v_0^3/v_\lambda^3) \ll 1$ this gives a condition that is difficult to satisfy,

$$\delta^{3/2}(v_\lambda^3/v_0^3)\,\rho_*^2 \ll \nu_{*p}^2 \ll 1. \tag{4.9}$$

The ratio of pitch angle scattering $1/\nu$ ripple transport to neoclassical,

$$D_{1/\nu}^{\rm pas}/D_{\rm axi}^{\rm pas} \sim \delta^{3/2}(v_\lambda^3/v_0^3)(\varepsilon^{3/2}/\nu_{*p}^2 q^2), \tag{4.10}$$

implies that $1/\nu$ regime ripple transport dominates at small collisionalities when $D_{1/\nu}^{\rm pas} > D_{\rm axi}^{\rm pas}$ or

$$q\nu_{*p} < \varepsilon^{3/4}\delta^{3/4}(v_\lambda^3/v_0^3)^{1/2}. \tag{4.11}$$

Our estimate does not account for electron drag, which does not trap and de-trap alphas, but does change their radial step size as they slow. Electron drag dominates when $\delta\,(v_0^3/v_\lambda^3) \gg 1$. Using $\tau \sim \tau_s$, $V \sim v_0\rho_0/R$, $\varDelta \sim V\tau \sim \rho_0 v_0\tau_s/R$, and $\mathscr{F} \sim \delta^{1/2}$ gives the estimate

$$D_{1/\nu}^{\rm drag} \sim \mathscr{F}\varDelta^2/\tau \sim \delta^{1/2}\tau_s v_0^2\,\rho_0^2/R^2. \tag{4.12}$$

Electron drag ripple loss and pitch angle scattering of ripple trapped are comparable when

$$D_{1/\nu}^{\rm pas}/D_{1/\nu}^{\rm drag} \sim \delta(v_0^3/v_\lambda^3) \sim 1. \tag{4.13}$$

For D–T fusion $v_0/v_c \sim 3$ with $v_\lambda \sim v_c$, indicating that for a ripple $\delta > 1/30$, electron drag could dominate.

Keeping $1/\nu$ ripple losses due to electron drag from depleting the lowest order slowing down distribution during a slowing down time requires $\tau_s D_{1/\nu}^{\rm drag}/a_\alpha^2 \ll 1$ or

$$\delta\,\rho_*^2 \ll R^2/v_0^2\tau_s^2 = \nu_{*d}^2 \ll 1. \tag{4.14}$$

This inequality is difficult to satisfy in this electron drag dominated limit since it requires

$$\frac{\bar{h}_t}{f_s} \sim \frac{\rho_0 v_0\tau_s}{a_\alpha R} = \frac{\rho_*}{\nu_{*d}} \ll 1 \tag{4.15}$$

to find a self-consistent solution.

Electron drag ripple loss compares and axisymmetric neoclassical compare as

$$D_{1/\nu}^{\rm drag}/D_{\rm axi}^{\rm drag} \sim \delta^{1/2}\,\varepsilon^{1/2}\,(v_0\tau_s/qR)^2 = \delta^{1/2}\,\varepsilon^{1/2}/q^2\nu_{*d}^2. \tag{4.16}$$

Consequently, $D_{1/\nu}^{\rm drag} > D_{\rm axi}^{\rm drag}$ when

$$q\nu_{*d} < \delta^{1/4}\varepsilon^{1/4}, \tag{4.17}$$

which differs by $\delta^{1/2}\varepsilon^{1/2}v_0^{3/2}/v_\lambda^{3/2}$ from the pitch angle scattering inequality.

Once the ripple is large enough to satisfy $q\nu_{*d} < \delta^{1/4}\varepsilon^{1/4}$ electron drag transport enters the $1/\nu$ ripple loss regime only if (4.15) is satisfied. Similarly, when $q\nu_* < \varepsilon^{3/4}\delta^{3/4}(v_\lambda^3/v_0^3)^{1/2}$ the ripple is large enough for pitch angle scattering to be in the $1/\nu$ ripple loss regime provided (4.6) is satisfied.

4.3. The $\sqrt{\nu}$ regime

As the toroidal rotation increases we enter the $\sqrt{\nu}$ regime. Once again only the trapped matter since $\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\psi} = 0$ provides no drive for the passing. For the trapped we must solve

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\psi}\frac{\partial f_s}{\partial\psi} + \overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\alpha}\frac{\partial \bar{h}_t}{\partial\alpha} = \overline{C\{\bar{h}_t\}}, \tag{4.18}$$

where we expect

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\alpha}\,\partial\bar{h}_t/\partial\alpha \sim \omega\bar{h}_t/q, \tag{4.19}$$

with $\omega = \overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\alpha}$ the toroidal $\boldsymbol{E}\times\boldsymbol{B}$ drift frequency in a flux surface. In the $\sqrt{\nu}$ regime $\overline{C\{\bar{h}\}} \ll \overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\alpha}\,\partial\bar{h}_t/\partial\alpha$ except in a narrow boundary layer requiring the toroidal drift frequency to be fast compared to the effective pitch angle scattering frequency,

$$\omega\delta\tau_s v_0^3/v_\lambda^3 \gg q. \tag{4.20}$$

Moreover, away from the boundary layer

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\psi}\frac{\partial f_s}{\partial\psi} \sim \overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\alpha}\frac{\partial \bar{h}_t}{\partial\alpha} \tag{4.21}$$

gives

$$\bar{h}_t/f_s \sim q v_0 \rho_0/\omega R\, a_\alpha \ll 1. \tag{4.22}$$

A boundary layer narrower than $\delta^{1/2}$ is only possible if pitch angle scattering is the dominant collisional process. Therefore, in the boundary layer we must balance

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\alpha}\frac{\partial \bar{h}_t}{\partial\alpha} \sim \overline{C\{\bar{h}_t\}}. \tag{4.23}$$

This balance between the strong $\boldsymbol{E}\times\boldsymbol{B}$ drift within the flux surface and collisions, further reduces the width w in pitch angle λ of the boundary layer by enhancing the pitch angle scattering time $\tau_s v_0^3/v_\lambda^3$:

$$\overline{C\{\bar{h}\}} \sim \frac{v_\lambda^3}{\tau_s v_0^3}\frac{\partial^2\bar{h}}{\partial\lambda^2} \sim \frac{v_\lambda^3\bar{h}}{\tau_s v_0^3 w^2}. \tag{4.24}$$

The balance between collisions and $\boldsymbol{E}\times\boldsymbol{B}$ drift then gives the normalized width of the boundary layer w to be

$$w \sim (q/\omega\tau_s)^{1/2}(v_\lambda/v_0)^{3/2} \ll \delta^{1/2}, \tag{4.25}$$

indicating that the alphas must $\boldsymbol{E}\times\boldsymbol{B}$ drift on a flux surface faster than they pitch angle scatter off the ions. The effective trapped fraction is estimated from this boundary layer width to be

$$\mathscr{F} \sim w, \tag{4.26}$$

with $w \ll \delta^{1/2}$ giving (4.20). Then the effective correlation time to move the alphas out of the ripple traps is then the pitch angle scattering time multiplied by the fraction squared,

$$\tau \sim \mathscr{F}^2(v_0/v_\lambda)^3\tau_s \sim q/\omega. \tag{4.27}$$

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The effective step size in the presence of $\boldsymbol{E}\times\boldsymbol{B}$ drift is

$$\Delta \sim v_{d0}\tau \sim q\rho_0 v_0/\omega R, \tag{4.28}$$

as the toroidal $\boldsymbol{E}\times\boldsymbol{B}$ drift now limits the radial step. The narrower boundary layer results in $\mathscr{F}$, τ, and Δ being independent of δ. Therefore, the pitch angle scattering diffusivity in the $\sqrt{\nu}$ regime is

$$D^{\mathrm{pas}}_{\sqrt{\nu}} \sim \mathscr{F}\Delta^2/\tau = (qv_\lambda/v_0)^{3/2}(\rho_0 v_0/\omega R)^2(\omega/\tau_s)^{1/2}. \tag{4.29}$$

These estimates are consistent with the Galeev *et al.* (1969) and Ho & Kulsrud (1987) estimates for ions except they did not keep q. Like them, we assume large ripple, $\delta \gg \varepsilon/qN$. A more detailed boundary layer analysis presented later modifies this estimate slightly because of boundary layer subtleties found by Calvo *et al.* (2017) that introduce a $[\ell n(\delta^{1/2}/w)]^{1/2}$ factor in the denominator of $D^{\mathrm{pas}}_{\sqrt{\nu}}$.

To avoid depleting the slowing down distribution function during $\sqrt{\nu}$ regime ripple transport requires $\tau_s D^{\mathrm{pas}}_{\sqrt{\nu}}/a_\alpha^2 \ll 1$ or very strong toroidal rotation satisfying

$$\omega R/qv_\lambda \gg (\rho_0/a_\alpha)^{4/3}(v_0\tau_s/R)^{1/3}. \tag{4.30}$$

As $v_\lambda \sim v_c \gg v_i$, sonic rotation seems required even though $\rho_0/a_\alpha \ll 1$ since $v_0\tau_s/R \gg 1$.

Comparing $\sqrt{\nu}$ to axisymmetric banana regime neoclassical transport yields

$$D^{\mathrm{pas}}_{\sqrt{\nu}}/D^{\mathrm{pas}}_{\mathrm{axi}} \sim (v_0\tau_s/qR)^{1/2}(v_0/\omega R)^{3/2}(\varepsilon v_\lambda/v_0)^{3/2}, \tag{4.31}$$

which will be larger than unity as we expect

$$(v_0\tau_s/qR)^{1/3} > \omega R/\varepsilon v_\lambda. \tag{4.32}$$

Normally the toroidal rotation frequency will be strong enough that the weaker $\sqrt{\nu}$ regime transport replaces the stronger $1/\nu$ regime transport. Even when $\delta(v_0^3/v_\lambda^3) \ll 1$, the transition occurs at about $D^{\mathrm{pas}}_{\sqrt{\nu}} \sim D^{\mathrm{pas}}_{1/\nu}$ or

$$\omega\tau_s\delta v_0^3/v_\lambda^3 \sim q, \tag{4.33}$$

as might be expected from (4.20). As $\tau_s \sim 1$ s, we expect there will normally be enough toroidal rotation to make $\omega\tau_s\delta v_0^3/v_\lambda^3 \gg q$ unless δ is tiny.

If $\delta(v_0^3/v_\lambda^3) \gg 1$, then drag dominates in the $1/\nu$ regime and the transition condition becomes $D^{\mathrm{pas}}_{\sqrt{\nu}} \sim D^{\mathrm{drag}}_{1/\nu}$, giving a lower toroidal rotation for the transition to the pitch angle scattering dominated $\sqrt{\nu}$ regime of

$$\omega\tau_s(\delta v_0^3/v_\lambda^3)^{1/3} \sim q. \tag{4.34}$$

In the $\sqrt{\nu}$ regime we assume the $\boldsymbol{E}\times\boldsymbol{B}$ dominates over the $\boldsymbol{\nabla}B$ by taking $Ze\partial\Phi/\partial\psi > M\mu\partial B/\partial\psi$ or

$$\frac{\omega R}{v_0} > \frac{q\rho_0}{a}, \tag{4.35}$$

where $\omega \sim c\partial\Phi/\partial\psi \sim c|\boldsymbol{E}|/RB_p$. Consequently, to enter the $\sqrt{\nu}$ regime with $\boldsymbol{E}\times\boldsymbol{B}$ the dominant drift requires a large electric field and/or the birth alpha gyroradius to be small compared to the minor radius. In H mode pedestals (Kagan & Catto 2008)

$$\omega R \sim c|\boldsymbol{E}|/B_p \sim v_i, \tag{4.36}$$

so that using $v_0/v_\lambda \sim 3$ and $v_i/v_\lambda \sim (m/M)^{1/6} \sim 1/4$, with $v_i = (2T_i/M_i)^{1/2}$, we expect $q\rho_0/a < 1/10$ or less to be required. If the $\boldsymbol{E} \times \boldsymbol{B}$ and ∇B drifts compete such that $\overline{\boldsymbol{v}_d \cdot \nabla\alpha}$ vanishes at isolated flux surfaces for $\mu \simeq \lambda v_0^2/2B_0$ with $\lambda \simeq 1$ to be in the narrow boundary layer, then superbanana-plateau transport can occur in small regions of phase space as discussed in detail in Calvo *et al.* (2017). We ignore such effects. Also, we do not consider direct orbit losses or collisionless stochastic diffusion in the vicinity of the separatrix since the difficult issue of the collisional filling of the unconfined loss regions would need to be evaluated. Instead, we simply assume the birth flux surface departure is small compared to the alpha radial scale. This assumption also avoids the issue of stochastic diffusion in the vicinity of the separatrix.

5. Neoclassical alpha transport simplifications and notation

For H mode operation the plasma density and ion temperature profiles are rather flat until they approach the pedestal, where a rapid drop in the density occurs with an often weaker drop in the ion temperature. Moreover, the radial variation of v_c^3 is unimportant since $v_c^3 \ll v_0^3$ for D–T ($v_c^3 \sim v_0^3$ for D–He^3). The ion and electron densities enter as ratios in v_c^3, with the electron and ion temperatures equilibrated. As a result, for D–T we can assume the strongest radial variation of the alphas is due to $S\tau_s$ and we may approximate f_{s*} by

$$f_* \simeq \frac{n_s(\psi_*)}{n_s(\psi)} f_s(\psi, v). \tag{5.1}$$

If we write the ψ dependence of the slowing down density as an exponential

$$n_s(\psi) \propto \mathrm{e}^{-\kappa(\psi)}, \tag{5.2}$$

then for κ a slowly varying function we may use

$$\psi_* = \psi - Iv_\parallel/\Omega \tag{5.3}$$

to write

$$n_s(\psi_*)/n_s(\psi) = \mathrm{e}^{-\kappa(\psi_*)+\kappa(\psi)} \simeq \mathrm{e}^{-(\psi_*-\psi)\,\mathrm{d}\kappa/\mathrm{d}\psi} = \mathrm{e}^{Q} \tag{5.4}$$

to retain finite orbit effects, by defining

$$Q = \kappa' I v_\parallel/\Omega \ \sim \rho_p/a_\alpha \lesssim 1, \tag{5.5}$$

with $RB_p \partial n_s/\partial\psi \sim n_s/a_\alpha$ and a_α the radial scale length of the alpha density profile. Based on the preceding, for D–T we can use

$$f_* - f_s = (\mathrm{e}^{Q} - 1)f_s. \tag{5.6}$$

We assume $Q \ll 1$ so we may use

$$n_s(\psi_*)/n_s(\psi) = 1 + Q + Q^2/2 + \cdots . \tag{5.7}$$

We are, of course, assuming that gyroradius corrections from the full expression for the canonical angular momentum will be small, that is, we assume that at birth

$$\rho_0/a_\alpha \ll 1. \tag{5.8}$$

The drag and pitch angle scattering collision operators are

$$C_{\mathrm{drag}}\{g\} = \frac{1}{\tau_s}\nabla_v \cdot \left[\left(\frac{v^3+v_c^3}{v^3}\right)\boldsymbol{v}g\right] = \frac{1}{\tau_s v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)g] \tag{5.9}$$

and

$$C_{\mathrm{pas}}\{g\} = \frac{v_\lambda^3}{2\tau_s}\nabla_v \cdot \left[\frac{1}{v^3}(v^2\overleftrightarrow{I} - \boldsymbol{vv})\cdot\nabla_v g\right] = \frac{2v_\lambda^3 B_0}{\tau_s v^3 B}\xi\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial g}{\partial\lambda}\right), \tag{5.10}$$

where $\lambda = B_0 v_\perp^2/v^2 B$ and $\xi = v_{||}/v$, with B_0 any convenient normalization constant or flux function such as the on axis value or $\sqrt{\langle B^2\rangle}$. Transit averaging gives

$$\overline{C_{\mathrm{drag}}\{g\}} = \frac{1}{\tau_s v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)\bar{g}] \tag{5.11}$$

and

$$\overline{C_{\mathrm{pas}}\{g\}} = \frac{2v_\lambda^3 B_0}{\tau_s v^3 \oint_\alpha \mathrm{d}\zeta/\xi B}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \frac{\mathrm{d}\zeta\,\xi}{B^2}\frac{\partial g}{\partial\lambda}\right)\right], \tag{5.12}$$

with the subscribe a reminder that α and ψ are to be held fixed when transit averaging.

6. Axisymmetric neoclassical alpha transport: a brief summary

The $v_0^3 \gg v_c^3 \sim v_\lambda^3$ limit of axisymmetric neoclassical alpha transport is briefly summarized to indicate how it relates to the neoclassical ripple transport of alphas.

Keeping pitch angle scatter as well as drag, and expanding for $Q \ll 1$, we must solve

$$\overline{C\{\bar{h}_p + Qf_s\}} = 0. \tag{6.1}$$

However, $\bar{h}_t = 0$ for the axisymmetric trapped response so we may employ flux surface averages to obtain the passing equation

$$\frac{1}{\tau_s v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)(\bar{h}_p+\bar{Q}f_s)] + \frac{2v_\lambda^3 B_0}{\tau_s v^5\langle B/v_{\|}\rangle}\frac{\partial}{\partial\lambda}\left[\lambda\left\langle v_{\|}\frac{\partial}{\partial\lambda}(\bar{h}_p+Qf_s)\right\rangle\right] = 0. \tag{6.2}$$

For this axisymmetric portion of the solution the ripples may be assumed small since $\delta \ll \varepsilon \ll 1$ and the flux surface average simplifies to

$$\langle A\rangle \to \frac{\oint \mathrm{d}\vartheta\, A/\boldsymbol{B}\cdot\nabla\vartheta}{\oint \mathrm{d}\vartheta/\boldsymbol{B}\cdot\nabla\vartheta} = \frac{\oint \mathrm{d}\vartheta\, A/B^2}{\oint \mathrm{d}\vartheta/B^2}. \tag{6.3}$$

To relate this equation to that of Hsu *et al.* (1990) we continue to use the odd function

$$Q=\sigma v\xi Q_0B_0/Bv_0 \tag{6.4}$$

and define the poloidal angle independent dimensionless function P via

$$\bar{h}_p=(\sigma Q_0vf_s/2v_0)P(\psi,v,\lambda), \tag{6.5}$$

with

$$Q_0=v_0I\kappa'/\Omega_0=v_0I\Omega_0^{-1}\,\partial\kappa/\partial\psi, \tag{6.6}$$

then

$$\bar{h}_p+Qf_s=(\sigma Q_0vf_s/2v_0)[P+2\xi(B_0/B)], \tag{6.7}$$

and our axisymmetric passing equation becomes

$$\begin{aligned}&\frac{(v^3+v_c^3)}{2v_\lambda^3}\frac{\partial}{\partial v}[v(P+2B_0\overline{\xi/B})H(v_0-v)]\\&\quad+\frac{B_0H(v_0-v)}{\langle B/\xi\rangle}\frac{\partial}{\partial\lambda}\left[\lambda\left\langle\xi\frac{\partial}{\partial\lambda}(P+2\xi B_0/B)\right\rangle\right]=0,\end{aligned} \tag{6.8}$$

where $\overline{\xi/B}\,\langle B/\xi\rangle=1$. Using $\xi^2=1-\lambda B/B_0$ gives $2\xi\,\partial\xi/\partial\lambda=-B/B_0$ so this becomes

$$\frac{(v^3+v_c^3)}{v_\lambda^3}\frac{\partial}{\partial v}\left[v\left(1-P\frac{\partial\langle\xi\rangle}{\partial\lambda}\right)H(v_0-v)\right]+H(v_0-v)\frac{\partial}{\partial\lambda}\left[\lambda\left(\langle\xi\rangle\frac{\partial P}{\partial\lambda}-1\right)\right]=0, \tag{6.9}$$

since

$$\langle B/\xi\rangle=1/(\overline{\xi/B})=-2B_0\partial\langle\xi\rangle/\partial\lambda. \tag{6.10}$$

The preceding equation for P is in agreement with equation (7) of Hsu *et al.* (1990) in the large drag limit. For the passing in the large aspect limit

$$\langle\xi\rangle=\frac{\displaystyle\oint \mathrm{d}\theta\,\xi/B^2}{\displaystyle\oint \mathrm{d}\theta/B^2}\simeq\frac{\displaystyle\oint \mathrm{d}\theta\,\xi}{2\pi}=\frac{4\sqrt{2\varepsilon}E(k)}{2\pi\sqrt{(1-\varepsilon)k^2+2\varepsilon}}, \tag{6.11}$$

with $\varepsilon=r/R$ and $k^2=2\varepsilon\lambda/[1-(1-\varepsilon)\lambda]$.

The drag only result of Nocentini *et al.* (1975) is

$$P(\psi,v,\lambda)|_{\mathrm{drag}}\rightarrow 1/(\partial\langle\xi\rangle/\partial\lambda), \tag{6.12}$$

while the pitch angle scattering only solution of Catto (1988) is

$$\partial P(\psi,v,\lambda)/\partial\lambda|_{\mathrm{pas}}\rightarrow 1/\langle\xi\rangle, \tag{6.13}$$

with $P=0$ for the trapped. Hsu *et al.* (1990) point out that the drag solution fails at the trapped–passing separatrix, and the pitch angle scattering solution does not satisfy the jump condition obtained by integrating (6.8) across the delta function alpha birth source.

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Hsu *et al.* (1990) solve the general problem for arbitrary aspect ratio concentric circular flux surfaces by separation of variables and a Sturm–Liouville eigenvalue procedure similar to that of Cordey (1976) by writing

$$P(\psi, v, \lambda) = \sum_{k=1}^{\infty} \Lambda_k(\psi, \lambda) V_k(\psi, v). \tag{6.14}$$

The eigenfunctions Λ_k satisfy the eigenvalue equation (Cordey 1976; Hsu *et al.* 1990)

$$\frac{\partial}{\partial \lambda}\left(\lambda \langle \xi \rangle \frac{\partial \Lambda_k}{\partial \lambda}\right) = \kappa_n \frac{\partial \langle \xi \rangle}{\partial \lambda} \Lambda_k, \tag{6.15}$$

with κ_n the nth eigenvalue associated with the normalization

$$\Lambda_k(\psi, \lambda = 0) = 1 \tag{6.16}$$

and the trapped–passing separatrix $(\lambda = \lambda_c)$ boundary condition

$$\Lambda_k(\psi, \lambda = \lambda_c) = 0. \tag{6.17}$$

The associated orthogonality condition is

$$\int_0^{\lambda_c} \mathrm{d}\lambda \Lambda_k \Lambda_j \partial \langle \xi \rangle / \partial \lambda = 0 \tag{6.18}$$

for $j \neq k$. Continuing to follow Hsu *et al.* (1990) gives for our simplified version of their (12a),

$$\frac{\partial}{\partial v}[v(\sigma_k - V_k) H(v_0 - v)] = \frac{v_\lambda^3(\sigma_k - \kappa_k V_k)}{(v^3 + v_c^3)} H(v_0 - v), \tag{6.19}$$

where integrating across v_0 using

$$V_k(\psi, v > v_0) = 0 \tag{6.20}$$

yields the jump condition

$$V_k(\psi, v_0) = \sigma_k \equiv \frac{\displaystyle\int_0^{\lambda_c} \mathrm{d}\lambda\, \Lambda_k}{\displaystyle\int_0^{\lambda_c} \mathrm{d}\lambda\, \Lambda_k^2 \partial \langle \xi \rangle / \partial \lambda}. \tag{6.21}$$

For the D–T case considered here the limiting solution for $v_0^3 \gg v_c^3 \sim v_\lambda^3$ is simply

$$V_k \simeq \sigma_k \left[1 - (\kappa_k - 1) \frac{v_\lambda^3 (v_0^2 - v^2)}{2 v_0^3 v^2}\right], \tag{6.22}$$

with the more general solution given in Hsu *et al.* (1990). The first eigenvalue κ_1 is typically order unity and the other κ_n are typically large.

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The axisymmetric neoclassical alpha radial particle ($d=0$) and radial heat ($d=1$) fluxes are evaluated as in Hsu *et al.* (1990) from

$$\Gamma_d^{neo} = -(I/\Omega_0)\left\langle \int \mathrm{d}^3 v (Mv^2/2)^d v_\parallel C\{Qf_s + \bar{h}_p\} \right\rangle . \tag{6.23}$$

The results are typically above the Nocentini *et al.* (1975) and below the Catto (1988) results.

Notice that for the axisymmetric neoclassical particle fluxes ambipolarity is automatically satisfied because if we sum over all species momentum conservation gives

$$\left\langle \sum_{\text{all}} M \int \mathrm{d}^3 v\, v_\parallel C\{f\} \right\rangle = 0. \tag{6.24}$$

As a result,

$$\sum_{\text{all}} Z\, \Gamma_0^{neo} = 0, \tag{6.25}$$

where here Γ_0^{neo}, Z, and M are the species particle flux, charge number and mass.

7. $1/\nu$ ripple transport of alphas

In the $1/\nu$ regime $\bar{h}_p = 0$ for the passing, while for the trapped we must solve

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\psi}\frac{\partial f_s}{\partial \psi} = \overline{C\{\bar{h}_t\}} \tag{7.1}$$

for $\bar{h}_t = \bar{h}_t(\psi, \alpha, v, \mu)$. We consider the two limits of pitch angle scattering and electron drag dominating. In all situations considered for $1/\nu$ transport both sets of turning points ($v_\parallel = 0$) for a charge of fixed μ are at the same value of B as the motion is on a fixed field line of constant ψ and α. For the simple form of (3.16) this means the poloidal angle of the two turning points must be at the same value of $\varepsilon \cos\vartheta + \delta \cos[N(\alpha - q\vartheta)]$. In the $1/\nu$ regime trapped alphas in local wells cannot be detrapped or trapped without collisions.

7.1. *Pitch angle scattering dominates:* $\delta(v_0^3/v_\lambda^3) \ll 1$

Our pitch angle scattering evaluation ignores drag in the alpha collision operator so that

$$\overline{C\{\bar{h}_t\}} = \frac{2v_\lambda^3 B_0}{\tau_s v^3 \left(\oint_\alpha \mathrm{d}\zeta/\xi B\right)} \frac{\partial}{\partial \lambda}\left[\lambda \left(\oint_\alpha \mathrm{d}\zeta\, \xi/B^2\right) \frac{\partial \bar{h}_t}{\partial \lambda}\right] \tag{7.2}$$

and uses

$$\overline{\boldsymbol{v}_d \boldsymbol{\cdot} \nabla\psi} = -\frac{B_0 v^2 (\partial/\partial\alpha)\left(\oint_\alpha \mathrm{d}\zeta\, \xi/B\right)}{\Omega_0 \left(\oint_\alpha \mathrm{d}\zeta/\xi B\right)}. \tag{7.3}$$

This limit is valid when $\delta(v_0^3/v_\lambda^3) \ll 1$.

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As a result of the preceding, we must solve

$$\frac{3v_\lambda^3}{\tau_s v^3}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\zeta\,\xi/B^2\right)\frac{\partial\bar{h}_t}{\partial\lambda}\right]\simeq\frac{B_0 v^2}{\varOmega_0}\frac{\partial f_s}{\partial\psi}\frac{\partial}{\partial\lambda}\left[\frac{\partial}{\partial\alpha}\left(\oint_\alpha \mathrm{d}\zeta\,\xi^3/B^2\right)\right], \tag{7.4}$$

where we use

$$\frac{\partial\xi^3}{\partial\lambda}=\frac{3\xi}{2}\frac{\partial\xi^2}{\partial\lambda}=-\frac{3B\xi}{2B_0}. \tag{7.5}$$

To keep $\partial\bar{h}_t/\partial\lambda$ well behaved in the deeply trapped limit, the constant of integration must vanish. This also avoids a jump at the trapped–passing boundary where we make $\bar{h}_t=0$ to match $\bar{h}_p=0$ for the passing that must satisfy $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}=0$. Integrating once gives

$$\lambda\left(\oint_\alpha \mathrm{d}\zeta\,\xi/B^2\right)\frac{\partial\bar{h}_t}{\partial\lambda}\simeq\frac{B_0\tau_s v^5}{3v_\lambda^3\varOmega_0}\frac{\partial f_s}{\partial\psi}\frac{\partial}{\partial\alpha}\left(\oint_\alpha \mathrm{d}\zeta\,\xi^3/B^2\right). \tag{7.6}$$

Integrating again from the trapped–passing boundary associated with the maximum ripple magnetic field $\widehat{B}$ (where $\lambda=B_0/\widehat{B}$) to λ we obtain the pitch angle scattering solution

$$\bar{h}_t=\frac{B_0\tau_s v^5}{3\varOmega_0 v_\lambda^3}\frac{\partial f_s}{\partial\psi}\int_{B_0/\widehat{B}}^{\lambda}\mathrm{d}\lambda\frac{(\partial/\partial\alpha)\left(\oint_\alpha \mathrm{d}\zeta\,\xi^3/B^2\right)}{\lambda\left(\oint_\alpha \mathrm{d}\zeta\,\xi/B^2\right)}. \tag{7.7}$$

We will continue to work with this general expression for now, rather than ignoring curvature drift and using (3.16) to find the approximate results

$$\left.\frac{\partial B}{\partial\alpha}\right|_\zeta=-\frac{B_0\varepsilon}{q}\sin[(\zeta-\alpha)/q]\simeq\frac{B_0\varepsilon}{q}\sin(\alpha/q) \tag{7.8}$$

and

$$\partial\xi/\partial\alpha|_\zeta\simeq-(\varepsilon\lambda/2q\xi)\sin(\alpha/q). \tag{7.9}$$

These approximate results if used here give a slight difference with a final result smaller by a factor 3/4. The approximate results of (7.8) and (7.9) follow the procedure of Stringer (1972) and Connor & Hastie (1973) that take advantage of the slow variation of ϑ while ζ varies between its two values at the lower of the magnetic field maximums $\widehat{B}$ in $\vartheta=(\zeta-\alpha)/q$ for fixed α. Their procedure notes that $-\pi<N\zeta<\pi$ implies $-\pi/Nq<\vartheta+(\alpha/q)<\pi/Nq$ so that ϑ varies very little along a ripple trapped orbit at fixed α for $N\gg 1$. It will be used shortly to simplify (7.20).

The alpha particle ($d=0$) and heat ($d=1$) ripple fluxes are calculated from

$$\begin{aligned}\varGamma_d^{\mathrm{rip}}&=\left\langle\int \mathrm{d}^3v(Mv^2/2)^d\bar{h}_t\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi\right\rangle\\&\simeq(1/2\varOmega_0)\left\langle\int \mathrm{d}^3v(Mv^2/2)^d v^2\lambda\bar{h}_t\nabla\psi\times\boldsymbol{b}\boldsymbol{\cdot}\nabla\ell nB\right\rangle,\end{aligned} \tag{7.10}$$

where $\boldsymbol{v}_m$ is the magnetic drift and we use $\langle\int \mathrm{d}^3v f_s\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi\rangle=\langle\nabla\boldsymbol{\cdot}[\varOmega^{-1}\boldsymbol{b}\times\nabla\psi\int \mathrm{d}^3v f_s v_\parallel^2]\rangle=0$. The last form of the ripple flux is obtained by using $v_\parallel$ small to ignore the curvature drift compared to the ∇B drift. The $\boldsymbol{E}\times\boldsymbol{B}$ drift does not enter even though it has been implicitly retained.

Using the Boozer representation for the field

$$\nabla\psi\times\boldsymbol{b}\boldsymbol{\cdot}\nabla\ell nB=(G+qI)^{-1}(G\partial B/\partial\zeta-I\partial B/\partial\vartheta) \tag{7.11}$$

and

$$\mathrm{d}^3v\rightarrow 2\pi(Bv^2/B_0\xi)\,\mathrm{d}v\,\mathrm{d}\lambda, \tag{7.12}$$

where we sum over both signs of σ, yields

$$\Gamma_d^{\mathrm{rip}}=\frac{\pi}{\Omega_0(G+qI)}\left\langle\left(G\frac{\partial B}{\partial\zeta}-I\frac{\partial B}{\partial\vartheta}\right)\frac{B}{B_0}\int_0^{v_0}\mathrm{d}v\,v^4\left(\frac{Mv^2}{2}\right)^d\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\,\lambda\frac{\bar{h}_t}{\xi}\right\rangle. \tag{7.13}$$

To evaluate the fluxes we use $\partial\xi/\partial\lambda=-B/2\xi B_0$ to integrate by parts to find

$$\begin{aligned}
&\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\frac{\lambda}{\xi}\left[\int_{B_0/\widehat{B}}^{\lambda}\mathrm{d}\lambda\frac{(\partial/\partial\alpha)\left(\oint_\alpha\mathrm{d}\zeta\,\xi^3/B^2\right)}{\lambda\left(\oint_\alpha\mathrm{d}\zeta\,\xi/B^2\right)}\right]\\
&=\frac{2B_0}{B}\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\,\xi\frac{\partial}{\partial\lambda}\left[\lambda\int_{B_0/\widehat{B}}^{\lambda}\mathrm{d}\lambda\frac{(\partial/\partial\alpha)\left(\oint_\alpha\mathrm{d}\zeta\,\xi^3/B^2\right)}{\lambda\left(\oint_\alpha\mathrm{d}\zeta\,\xi/B^2\right)}\right]\\
&=\frac{2B_0}{B}\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\,\xi\left[\int_{B_0/\widehat{B}}^{\lambda}\mathrm{d}\lambda\frac{(\partial/\partial\alpha)\left(\oint_\alpha\mathrm{d}\zeta\,\xi^3/B^2\right)}{\lambda\left(\oint_\alpha\mathrm{d}\zeta\,\xi/B^2\right)}\right]\\
&\quad+\left[\frac{(\partial/\partial\alpha)\left(\oint_\alpha\mathrm{d}\zeta\,\xi^3/B^2\right)}{\left(\oint_\alpha\mathrm{d}\zeta\,\xi/B^2\right)}\right].
\end{aligned} \tag{7.14}$$

Integrating the first term on the right side by parts again using $\partial\xi^3/\partial\lambda=-3B\xi/2B_0$ yields

$$\begin{aligned}
&\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\frac{\lambda}{\xi}\left[\int_{B_0/\widehat{B}}^{\lambda}\mathrm{d}\lambda\frac{(\partial/\partial\alpha)\left(\oint_\alpha\mathrm{d}\zeta\,\xi^3/B^2\right)}{\lambda\left(\oint_\alpha\mathrm{d}\zeta\,\xi/B^2\right)}\right]\\
&=\frac{2B_0}{B}\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\,\xi\left(1+\frac{2B_0\xi^2}{3\lambda B}\right)\frac{(\partial/\partial\alpha)\left(\oint_\alpha\mathrm{d}\zeta\,\xi^3/B^2\right)}{\left(\oint_\alpha\mathrm{d}\zeta\,\xi/B^2\right)}.
\end{aligned} \tag{7.15}$$

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Neglecting curvature drift and other ξ^2 terms compared to order unity terms and assuming $(\oint_\alpha \mathrm{d}\zeta\, \xi\partial B/\partial\alpha)/(\oint_\alpha \mathrm{d}\zeta\, \xi)$ is a slow function of λ, we integrate by parts one final time to obtain

$$\int_{B_0/\widehat{B}}^{B_0/B} \mathrm{d}\lambda \frac{\lambda}{\xi}\left[\int_{B_0/\widehat{B}}^{\lambda} \mathrm{d}\lambda \frac{(\partial/\partial\alpha)\left(\oint_\alpha \mathrm{d}\zeta\, \xi^3/B^2\right)}{\lambda\left(\oint_\alpha \mathrm{d}\zeta\, \xi/B^2\right)}\right] \simeq \frac{4B_0^2}{3B^2}\widehat{\xi}^3 \frac{(\partial/\partial\alpha)\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}^3\right)}{\left(\oint_\alpha \mathrm{d}\zeta \widehat{\xi}\right)}$$
$$+\cdots \simeq -2\widehat{\xi}^3 \frac{\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}\partial B/\partial\alpha|_\zeta\right)}{\widehat{B}\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}\right)}, \tag{7.16}$$

with $\widehat{\xi} \equiv \sqrt{1-B/\widehat{B}}$.

Inserting the preceding, noticing $G/I \simeq rB_p/RB_t \simeq \varepsilon^2/q$, and using $\partial(B/B_0)/\partial\theta \simeq \varepsilon\sin\theta$ and $\partial(B/B_0)/\partial\zeta \simeq N\delta\sin(N\zeta)$, gives the ripple driven radial particle $(d=0)$ and heat $(d=1)$ fluxes associated with pitch angle scattering to be

$$\Gamma_d^{\mathrm{pas}} \simeq -\frac{2\pi B_0^2\varepsilon\tau_s}{3\Omega_0^2 q v_\lambda^3}\left(\int_0^{v_0}\mathrm{d}v\, v^9\left(\frac{Mv^2}{2}\right)^d \frac{\partial f_s}{\partial\psi}\right)\left\langle \sin\vartheta\,\widehat{\xi}^3 \frac{\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}\partial B/\partial\alpha|_\zeta\right)}{\widehat{B}\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}\right)}\right\rangle. \tag{7.17}$$

Using $v_0 \gg v_c$ to integrate we find

$$\int_0^{v_0}\mathrm{d}v\, v^9\left(\frac{Mv^2}{2}\right)^d \frac{\partial f_s}{\partial\psi} \simeq \frac{\partial}{\partial\psi}\left(\frac{S\tau_s}{4\pi}\right)\int_0^{v_0}\mathrm{d}v\, v^6\left(\frac{Mv^2}{2}\right)^d$$
$$= \frac{(Mv_0^2/2)^d v_o^7}{4\pi(7+2d)}\frac{\partial}{\partial\psi}\left[\frac{n_s}{\ell n(v_0/v_c)}\right]. \tag{7.18}$$

Assuming

$$\frac{\ell n(v_0/v_c)}{n_s}\frac{\partial n_s}{\partial\psi} \gg \frac{1}{v_c}\frac{\partial v_c}{\partial\psi} \sim \frac{1}{n_i}\frac{\partial n_i}{\partial\psi}, \tag{7.19}$$

we obtain our most general form for the $1/\nu$ fluxes due to pitch angle scattering to be

$$\Gamma_d^{\mathrm{pas}} \simeq -\frac{(Mv_0^2/2)^d\varepsilon^2 B_0^2\tau_s v_0^7(\partial n_s/\partial\psi)}{6(7+2d)q^2\Omega_0^2 v_\lambda^3 \ell n(v_0/v_c)}\left\langle \sin\vartheta\,\widehat{\xi}^3 \frac{\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}\partial B/\partial\alpha|_\zeta\right)}{(\varepsilon B_0/q)\left(\oint_\alpha \mathrm{d}\zeta\, \widehat{\xi}\right)}\right\rangle. \tag{7.20}$$

To simplify we now use approximation (7.8) and assume the poloidal variation of $\partial B/\partial\alpha$ is slow so we can use $\sin(\vartheta - q^{-1}\zeta) \simeq \sin\vartheta$. The details are presented in the appendix A and yield

$$\left\langle \sin^2\vartheta\sqrt{[1-(B/\widehat{B})]^3}\right\rangle = \frac{1}{4\pi^2}\int_{\text{all}} \mathrm{d}\vartheta\,\sin^2\theta\int_{\text{all}}\mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^3} \equiv \frac{4\sqrt{2}}{3\pi}\delta^{3/2}W(\varepsilon/qN\delta), \tag{7.21}$$

with the ripple scattering coefficient defined as a function of a single parameter by

$$W(\gamma)=\frac{3}{16\pi\sqrt{2}\,\gamma^2}\int_0^U\frac{\mathrm{d}X\cos X\sin^2 X}{\sqrt{\gamma^2-\sin^2 X}} \times\int_X^{Y_1(X)}\mathrm{d}Y[\cos X-\cos Y+(X-Y)\sin X]^{3/2}\rightarrow\begin{cases}1-3\gamma & \gamma\ll 1\\ 0.02/\gamma^3 & \gamma\gg 1.\end{cases} \tag{7.22}$$

The upper limit U of the X integral is $\pi/2$ when $\gamma=\varepsilon/qN\delta>1$, while for $\gamma=\varepsilon/qN\delta<1$ it is $\sin^{-1}(\varepsilon/qN\delta)$. The upper limit of the Y integral is the zero of $\cos X-\cos Y_1+(X-Y_1)\sin X=0$ adjacent to $Y=X$. This monotonically decreasing function is plotted as G in figure 1 in Connor & Hastie (1973), where it is compared to the similar, but somewhat less accurate result of Stringer (1972). Only accessible values of ϑ and ζ are to be integrated over in the flux surface average. In particular, $B>\widehat{B}$ are not accessible bythe ripple trapped so $\widehat{\zeta}$ is the value of ζ for which $B=\widehat{B}$. For weak ripple, $\varepsilon/qN\delta>1$, some values of ϑ will not be accessible by the ripple trapped. The relevant details of the Stringer (1972) and Connor & Hastie (1973) treatments are given in appendix A.

Using the preceding we obtain

$$\Gamma_d^{\text{pas}}\simeq-\frac{2\sqrt{2}\,\delta^{3/2}(Mv_0^2/2)^d\varepsilon^2\tau_s B_0^2 v_0^7 W(\varepsilon/qN\delta)}{9\pi(7+2d)q^2 v_\lambda^3\Omega_0^2\ell n(v_0/v_c)}\frac{\partial n_s}{\partial\psi}, \tag{7.23}$$

yielding an alpha particle diffusivity due to pitch angle scattering of

$$D_0^{\text{pas}}\simeq\frac{2\sqrt{2}\,\delta^{3/2}}{63\pi}\left(\frac{\tau_s v_0^3}{v_\lambda^3}\right)\frac{\rho_0^2 v_0^2 W(\varepsilon/qN\delta)}{R^2\ell n(v_0/v_c)}, \tag{7.24}$$

with $W(\varepsilon/qN\delta)\leqslant 1$. The preceding is in agreement with our phenomenological estimate when $W/\ell n(v_0/v_c)\sim 1$, but with a rather small coefficient that seems to be due to the slowing down distribution based on (7.18).

The weak ripple limit reduces D_0^{pas} by order $(qN\delta/\varepsilon)^3\ll 1$. In this limit, when $(\delta/\varepsilon)^{5/8}\sim 1/qN$, D_0^{pas} becomes comparable to the result of Linsker & Boozer (1982) who find a diffusivity of roughly $(\delta^2/\varepsilon^{1/2}qN)(\tau_s v_0^3/v_\lambda^3)(\rho_0 v_0/R)^2$ due to the effect of very weak ripple on the turning points (at $\vartheta\simeq\pm\pi$) of the non-ripple trapped banana regime alphas in the nearly axisymmetric limit (also see Davidson (1976) and Tsang (1977)). By assuming $\varepsilon N\delta/q\ll 1$, that is, $I\partial B/\partial\vartheta\gg G\partial B/\partial\zeta$, we are ignoring the radial drift effect evaluated by Davidson (1976), Tsang (1977), and Linsker & Boozer (1982). This radial drift is of order $\delta Na\rho_0 v_0/qR^2$ and appears to be negligible unless $(\delta/\varepsilon)^{5/8}<1/qN$. Stochastic effects due to ripple may play a role at low collisionalities and were considered by Goldston, White & Boozer (1981).

The ripple departure from axisymmetry means that ambipolarity,

$$\sum_{\text{all}} Z\,\Gamma_0^{\text{rip}}=0, \tag{7.25}$$

is no longer intrinsic. However, the alpha density n_s is so small they may be treated as a trace population. Consequently, the radial electric field adjusts the ion and electron particle transport to maintain ambipolarity as discussed for a plasma without alphas by Galeev *et al.* (1969).

7.2. *Electron drag dominates:* $\delta(v_0^3/v_\lambda^3) \gg 1$

To make the electron drag estimate we ignore pitch angle scatter and ion drag by keeping only

$$\overline{C\{\bar{h}_t\}} \rightarrow \frac{1}{\tau_s v^2}\frac{\partial}{\partial v}(v^3\bar{h}_t). \tag{7.26}$$

In this limit $\delta(v_0^3/v_\lambda^3) \gg 1$ and we must solve

$$\frac{1}{\tau_s v^2}\frac{\partial}{\partial v}(v^3\bar{h}_t) = \overline{\boldsymbol{v}_d\boldsymbol{\cdot}\boldsymbol{\nabla}\psi}\frac{\partial f_s}{\partial\psi} = -\frac{B_0 v^2(\partial/\partial\alpha)\left(\displaystyle\oint_\alpha \mathrm{d}\zeta\,\xi/B\right)}{\varOmega_0\left(\displaystyle\oint_\alpha \mathrm{d}\zeta/\xi\,B\right)}\frac{\partial f_s}{\partial\psi}. \tag{7.27}$$

By performing the bounce averages in ζ at fixed α we will again be able to take advantage of the slow variation θ of while ζ varies between its two values at the lower of the magnetic field maxima. Again, the $\boldsymbol{E}\times\boldsymbol{B}$ does not enter even though it is implicitly retained.

When solving for $\bar{h}_t$ we must avoid a step at v_0. Consequently, we integrate from v to v_0, where $\bar{h}_t(v > v_0) = 0$, to find the piecewise continuous electron drag solution

$$\bar{h}_t = \frac{B_0\tau_s(v_0^2 - v^2)}{2\varOmega_0}\frac{\partial f_s}{\partial\psi}\frac{(\partial/\partial\alpha)\left(\displaystyle\oint_\alpha \mathrm{d}\zeta\,\xi/B\right)}{\left(\displaystyle\oint_\alpha \mathrm{d}\zeta/\xi\,B\right)}\{H[\lambda - (B_0/\widehat{B})] - H[\lambda - (B_0/B)]\}, \tag{7.28}$$

where we insert Heaviside step functions as a reminder that the trapped must satisfy $(B_0/B) \geqslant \lambda \geqslant (B_0/\widehat{B})$. In this case the radial alpha flux occurs because electron drag acts to reduce flux surface departure of the larger (smaller) number of alphas inside (outside) the flux surface. The result is outward particle and heat fluxes. The boundary layer about $\lambda = B_0/\widehat{B}$ is not expected to appreciably change this result. A non-trivial boundary layer analysis is required to make $\bar{h}_t(\lambda = B_0/\widehat{B}) = 0$ and will be discussed in the next subsection. The procedure is more involved than one used in a later section since a separable solution of the homogeneous equation that vanishes above the birth speed does not exist.

Neglecting curvature drift corrections by taking $\xi^2 \ll 1$, and assuming slow poloidal variation of $\partial B/\partial\alpha$ so we can use (7.9), gives

$$\bar{h}_t = \frac{B_0\varepsilon\tau_s(v_0^2 - v^2)\lambda}{4q\varOmega_0}\frac{\partial f_s}{\partial\psi}\sin(\vartheta - q^{-1}\zeta)\{H[\lambda - (B_0/\widehat{B})] - H[\lambda - (B_0/B)]\}, \tag{7.29}$$

Using $\sin\vartheta\,\sin(\vartheta - q^{-1}\zeta) \simeq \sin^2\vartheta$ in the flux expression (7.13) gives the electron drag result

$$\varGamma_d^{\mathrm{drag}} \simeq -\frac{\pi\varepsilon^2\tau_s B_0^2}{4q^2\varOmega_0^2}\left[\int_0^{v_0}\mathrm{d}v\,v^4(v_0^2 - v^2)\left(\frac{Mv^2}{2}\right)^d\frac{\partial f_s}{\partial\psi}\right]\left\langle\sin^2\vartheta\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\frac{B\lambda^2}{B_0\xi}\right\rangle. \tag{7.30}$$

Then

$$\int_0^{v_0} \mathrm{d}v\, v^4(v_0^2 - v^2)\left(\frac{Mv^2}{2}\right)^d \frac{\partial f_s}{\partial \psi} \simeq \frac{v_0^4(Mv_0^2/2)^d}{8\pi(1+d)(2+d)\ell n(v_0/v_c)}\frac{\partial n_s}{\partial \psi}, \tag{7.31}$$

and

$$\begin{aligned}\frac{B}{B_0}\int_{B_0/\widehat{B}}^{B_0/B} \mathrm{d}\lambda \frac{\lambda^2}{\xi} &= -2\int_{B_0/\widehat{B}}^{B_0/B} \mathrm{d}\lambda\, \lambda^2 \frac{\partial \xi}{\partial \lambda}\\ &= 2\frac{B_0^2}{\widehat{B}^2}\left\{\sqrt{1 - B/\widehat{B}} + \frac{2+3B/\widehat{B}}{15}\sqrt{(1-B/\widehat{B}^3)}\right\} \simeq 2\sqrt{1-B/\widehat{B}}\end{aligned} \tag{7.32}$$

give

$$\Gamma_d^{\mathrm{drag}} \simeq -\frac{(Mv_0^2/2)^d \varepsilon^2 \tau_s B_0^2 v_0^4}{16(1+d)(2+d)q^2\Omega_0^2 \ell n(v_0/v_c)}\frac{\partial n_s}{\partial \psi}\left\langle \sin^2\vartheta \sqrt{[1-(B/\widehat{B})]}\right\rangle, \tag{7.33}$$

where

$$\begin{aligned}\left\langle \sin^2\vartheta\sqrt{[1-(B/\widehat{B})]}\right\rangle &= \frac{1}{4\pi^2}\int_{\mathrm{all}} \mathrm{d}\vartheta\, \sin^2\theta \int_{\mathrm{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]}\\ &\equiv \frac{4\sqrt{2}}{3\pi}\delta^{1/2} D(\varepsilon/qN\delta)\end{aligned} \tag{7.34}$$

is the drag coefficient derived in appendix A to be

$$\begin{aligned}D(\gamma) &= \frac{\sqrt{2}}{\pi\gamma^2}\int_0^U \frac{\mathrm{d}X \cos X \sin^2 X}{\sqrt{\gamma^2 - \sin^2 X}}\int_X^{Y_0} \mathrm{d}Y[\cos X - \cos Y + (X-Y)\sin X]^{1/2}\\ &\to \begin{cases} 1 - \dfrac{\pi\gamma}{3}\left[\ell n\left(\dfrac{2}{\pi\gamma}\right) + \dfrac{17}{6}\right] & \gamma \ll 1\\ O(1/\gamma^3) & \gamma \gg 1.\end{cases}\end{aligned} \tag{7.35}$$

Using this result we obtain the flux due to drag to be

$$\Gamma_d^{\mathrm{drag}} \simeq -\frac{\sqrt{2}\,\delta^{1/2}(Mv_0^2/2)^d\varepsilon^2\tau_s B_0^2 v_0^4 D(\varepsilon/qN\delta)}{12\pi(1+d)(2+d)q^2\Omega_0^2\ell n(v_0/v_c)}\frac{\partial n_s}{\partial \psi}, \tag{7.36}$$

giving the alpha particle diffusivity due to drag of

$$D_0^{\mathrm{drag}} \simeq \frac{\sqrt{2}\,\delta^{1/2}\tau_s}{24\pi}\frac{\rho_0^2 v_0^2 D(\varepsilon/qN\delta)}{R^2\ell n(v_0/v_c)}, \tag{7.37}$$

with $D(\varepsilon/qN\delta) \leqslant 1$. The diffusivity due to drag is consistent with our earlier phenomenological estimate when $D/\ell n(v_0/v_c) \sim 1$ and again has a small coefficient due to (7.31).

As expected, the ratio of the pitch angle scattering to electron drag results is

$$\frac{\Gamma_d^{\mathrm{pas}}}{\Gamma_d^{\mathrm{drag}}} \simeq \frac{4(1+d)(2+d)\delta v_0^3 W(\varepsilon/qN\delta)}{(7+2d)v_\lambda^3 D(\varepsilon/qN\delta)} \sim \frac{\delta v_0^3}{v_\lambda^3}, \tag{7.38}$$

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which is larger (smaller) than unity when electron drag (pitch angle scatter) ripple loss dominates.

The pitch angle scattering and electron drag solutions in the $1/\nu$ regime are not exact since in reality both effects will enter in combination while we have calculated them separately. However, these results are a useful indication of when ripple starts to become a problem.

7.3. *Pitch angle scattering and drag*

With both pitch angle scattering and drag the full equation must be solved:

$$\begin{aligned}
&\frac{2v_\lambda^3 B_0}{\tau_s v^3\left(\oint_\alpha \mathrm{d}\zeta/\xi\, B\right)}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\zeta\ \xi/B^2\right)\frac{\partial \bar{h}_t}{\partial\lambda}\right]+\frac{1}{\tau_s v^2}\frac{\partial}{\partial v}[(v^3+v_c^3)\bar{h}_t]\\
&\quad=\frac{2B_0^2v^2}{3\Omega_0}\frac{(\partial^2/\partial\lambda\partial\alpha)\left(\oint_\alpha \mathrm{d}\zeta\ \xi^3/B^2\right)}{\left(\oint_\alpha \mathrm{d}\zeta/\xi B\right)}\simeq-\frac{B_0\varepsilon v^2}{q\Omega_0}\frac{\partial f_s}{\partial\psi}\sin\left(\frac{\alpha}{q}\right)\\
&\quad\quad\times\frac{(\partial/\partial\lambda)\left(\lambda\oint_\alpha \mathrm{d}\zeta\ \xi\right)}{\left(\oint_\alpha \mathrm{d}\zeta/\xi\right)}.
\end{aligned}\tag{7.39}$$

However, this general $1/\nu$ does not seem to be an analytically tractable problem. To see the reason we seek a separable solution to the homogeneous equation

$$\frac{2B_0}{\left(\oint_\alpha \mathrm{d}\zeta/\xi\, B\right)}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\zeta\ \xi/B^2\right)\frac{\partial H}{\partial\lambda}\right]+\frac{v}{v_\lambda^3}\frac{\partial}{\partial v}[(v^3+v_c^3)H]=0,\tag{7.40}$$

by taking

$$H=\Lambda(\lambda)V(v),\tag{7.41}$$

then for the separation constant σ we would find

$$\frac{v}{v_\lambda^3 V}\frac{\partial}{\partial v}[(v^3+v_c^3)V]=\sigma=-\frac{2B_0}{\left(\oint_\alpha \mathrm{d}\zeta/\xi\, B\right)\Lambda}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\zeta\ \xi/B^2\right)\frac{\partial\Lambda}{\partial\lambda}\right].\tag{7.42}$$

But demanding $V(v>v_0)=0$ means there is only the trivial solution $V=0$.

8. $\surd\nu$ ripple transport of alphas

In the $\surd\nu$ regime $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}=0$ and $\partial\bar{h}_p/\partial\alpha=0$ for the passing, giving $\overline{C\{\bar{h}_p\}}=0$ and therefore $\bar{h}_p=0$. Here we must be careful to keep the $v\partial v/\partial\psi$ term to retain the $\boldsymbol{E}\times\boldsymbol{B}$ drift within a flux surface. This toroidal drift results in a very narrow collisional boundary layer in which alphas detrap and retrap.

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For the ripple trapped we must solve

$$\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}\frac{\partial f_s}{\partial\psi}+\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\alpha}\frac{\partial\bar{h}_t}{\partial\alpha}=\overline{C\{\bar{h}_t\}} \tag{8.1}$$

for the transit average trapped response $\bar{h}_t=\bar{h}_t(\psi,\alpha,v,\mu)$. For simplicity we consider only the case when the $\boldsymbol{E}\times\boldsymbol{B}$ drift dominates and define its associated toroidal rotation frequency ω via

$$\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\alpha}=\frac{Mc}{Ze}\frac{(\partial/\partial\psi)\left(\oint_\alpha \mathrm{d}\zeta\ v_\parallel\right)}{\left(\oint_\alpha \mathrm{d}\zeta/v_\parallel\right)}\simeq -c\frac{\partial\Phi}{\partial\psi}\equiv-\omega. \tag{8.2}$$

In a rippled tokamak the $\boldsymbol{E}\times\boldsymbol{B}$ drift in a flux surface is mostly toroidal. If ∇B drifts dominate then $\omega\to-(Mc\mu/Ze)\partial B/\partial\psi\simeq(q\lambda v^2/2\Omega aR)\cos(\alpha/q)$ and a similar procedure can be employed. If $\boldsymbol{E}\times\boldsymbol{B}$ and ∇B drifts cancel superbanana-plateau transport must retained (Calvo *et al.* 2017).

We also use the more complete expressions for

$$\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}=\frac{2B_0^2v^2}{3\Omega_0}\frac{(\partial/\partial\lambda)(\partial/\partial\alpha)\left(\oint_\alpha \mathrm{d}\zeta\ \xi^3/B^2\right)}{\left(\oint_\alpha \mathrm{d}\zeta/\xi\ B\right)}, \tag{8.3}$$

and

$$\overline{C\{\bar{h}_t\}}=\frac{2v_\lambda^3B_0}{\tau_s v^3\left(\oint_\alpha \mathrm{d}\zeta/\xi\ B\right)}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\zeta\ \xi/B^2\right)\frac{\partial\bar{h}_t}{\partial\lambda}\right]. \tag{8.4}$$

As a result, we are led to consider

$$\begin{aligned}-\omega\frac{\partial\bar{h}_t}{\partial\alpha}&=\frac{2v_\lambda^3B_0}{\tau_s v^3\left(\oint_\alpha \mathrm{d}\zeta/\xi B\right)}\frac{\partial}{\partial\lambda}\\&\quad\times\left\{\lambda\left[\left(\oint_\alpha \mathrm{d}\zeta\ \xi/B^2\right)\frac{\partial\bar{h}_t}{\partial\lambda}-\frac{B_0\tau_s v^5}{3\Omega_0 v_\lambda^3}\frac{\partial f_s}{\partial\psi}\left(\frac{\partial}{\partial\alpha}\oint_\alpha \mathrm{d}\zeta\ \frac{\xi^3}{B^2}\right)\right]\right\}.\end{aligned} \tag{8.5}$$

Notice that this form reduces to the solution in the $1/\nu$ regime when $\boldsymbol{E}\times\boldsymbol{B}$ drift is unimportant. We begin with this form to insure that the $1/\nu$ regime is being properly retained.

Neglecting curvature drift and assuming weak ϑ variation during a ripple trapped bounce leads to the simplified form

$$-\omega\frac{\partial\bar{h}_t}{\partial\alpha}=\frac{2v_\lambda^3}{\tau_s v^3\left(\oint_\alpha \mathrm{d}\zeta/\xi\right)}\frac{\partial}{\partial\lambda}\left\{\left(\lambda\oint_\alpha \mathrm{d}\zeta\ \xi\right)\left[\frac{\partial\bar{h}_t}{\partial\lambda}+\frac{B_0\varepsilon\tau_s v^5}{2q\Omega_0 v_\lambda^3}\frac{\partial f_s}{\partial\psi}\sin\left(\frac{\alpha}{q}\right)\right]\right\}. \tag{8.6}$$

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Ho & Kulsrud (1987) assume any field line between two mirror points is at fixed poloidal angle (see their comments below equation (13)). Consequently, they are assuming large ripple with $\varepsilon/qN\delta \ll 1$ based on the Stringer (1972) and Connor & Hastie (1973) treatments. Using this same assumption we can replace (3.16) by

$$B = \tilde{B}_0[1 - \delta\cos(N\zeta)], \tag{8.7}$$

with

$$\tilde{B}_0 \simeq B_0(1 - \varepsilon\cos\vartheta) \simeq B_0, \tag{8.8}$$

and then let $\varepsilon \to \delta$ and $\vartheta \to N\zeta$ in the usual tokamak bounce averages (see, for example, the Appendix of Catto, Parra & Pusztai (2017)). Introducing the new independent variable κ by

$$\lambda = 1/(1 - \delta + 2\delta\kappa^2), \tag{8.9}$$

with $\kappa = 1$ the ripple trapped–passing boundary and $\kappa = 0$ the deeply ripple trapped, we use

$$\mathrm{d}\lambda/\lambda^2 = -4\delta\kappa\,\mathrm{d}\kappa \tag{8.10}$$

to obtain

$$\frac{(\partial/\partial\lambda)\left(\lambda \oint_\alpha \mathrm{d}\zeta\,\xi\right)}{\left(\oint_\alpha \mathrm{d}\zeta/\xi\right)} \simeq -\frac{1}{2\kappa K(\kappa)}\frac{\partial}{\partial\kappa}[E(\kappa) - (1-\kappa^2)K(\kappa)] \underset{\kappa\to 1}{\longrightarrow} -\frac{1}{2}, \tag{8.11}$$

where we use the full bounce results

$$N\oint_\alpha \mathrm{d}\zeta\,\xi = \frac{8\sqrt{2\delta}}{\sqrt{1-\delta+2\delta\kappa^2}}[E(\kappa) - (1-\kappa^2)K(\kappa)] \tag{8.12}$$

and

$$N\oint_\alpha \mathrm{d}\zeta/\xi = \frac{8}{\sqrt{2\delta}}\sqrt{1-\delta+2\delta\kappa^2}\,K(\kappa), \tag{8.13}$$

with $E(\kappa) \to 1 + (1-\kappa^2)[\ell n(4/\sqrt{1-\kappa^2}) - (1/2)]/2 + \cdots$ and $K(\kappa) \to \ell n(4/\sqrt{1-\kappa^2}) + \cdots$. As a result, using

$$\begin{aligned}\frac{1}{\left(\oint_\alpha \mathrm{d}\zeta/\xi\right)}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\zeta\,\xi\right)\frac{\partial}{\partial\lambda}\right] &\simeq \frac{1}{8\delta\kappa K(\kappa)}\frac{\partial}{\partial\kappa}\\ &\times\left\{\frac{[E(\kappa) - (1-\kappa^2)K(\kappa)]}{\kappa}\frac{\partial}{\partial\kappa}\right\} \underset{\kappa\to 1}{\longrightarrow} \frac{1}{4\delta\ell n\left(\dfrac{16}{1-\kappa^2}\right)}\frac{\partial^2}{\partial\kappa^2}\end{aligned} \tag{8.14}$$

the equation for $\bar{h}_t$ becomes

$$\frac{B_0\varepsilon v^2}{2q\Omega_0}\frac{\partial f_s}{\partial\psi}\sin\left(\frac{\alpha}{q}\right) - \omega\frac{\partial\bar{h}_t}{\partial\alpha} \simeq \frac{(v_\lambda^3/\tau_s v^3)}{2\delta\ell n[16/(1-\kappa^2)]}\frac{\partial^2\bar{h}_t}{\partial\kappa^2}. \tag{8.15}$$

Continuing to take $\kappa \rightarrow 1$ and defining

$$\eta = (1-\kappa)/8, \tag{8.16}$$

$$k \equiv \frac{64\delta\, \omega\, \tau_s v^3}{q v_\lambda^3} \gg 1, \tag{8.17}$$

and

$$L \equiv \frac{32\delta\, B_0 \varepsilon\, \tau_s v^5}{q \Omega_0 v_\lambda^3} \frac{\partial f_s}{\partial \psi}, \tag{8.18}$$

gives the boundary layer form of the equation to be

$$\frac{1}{\ell n(\eta)} \frac{\partial^2 \bar{h}_t}{\partial \eta^2} - 2kq \frac{\partial \bar{h}_t}{\partial \alpha} \simeq -2L \sin\left(\frac{\alpha}{q}\right) = -2L\mathrm{Im}(\mathrm{e}^{\mathrm{i}\alpha/q}), \tag{8.19}$$

which is valid when $k \gg 1$. Here Im denotes the imaginary part.

To solve the preceding equation in the boundary layer we take

$$\bar{h}_t = \mathrm{Im}[H(\eta)\mathrm{e}^{\mathrm{i}\alpha/q}] \tag{8.20}$$

to obtain

$$\frac{1}{\ell n(\eta)} \frac{\partial^2 H}{\partial \eta^2} - 2\mathrm{i}kH = -2L, \tag{8.21}$$

which has the particular solution

$$H|_p = -\mathrm{i}L/k. \tag{8.22}$$

As this solution does not vanish at $\eta = 0$ we must use the homogeneous solutions to satisfy this boundary condition.

For $1 \ll \ell n(1/\eta) \ll 1/k\eta^2$ one homogeneous solution starts off as

$$H|_{\mathrm{he}} \simeq \mathrm{i}\,(L/k)\{1 + \mathrm{i}k\,\eta^2[\ell n(\eta) - \tfrac{3}{2}] + \cdots\}, \tag{8.23}$$

where the coefficient is chosen to cancel the particular solution at $\eta = 0$. The other begins as

$$H|_{\mathrm{ho}} \simeq C\eta\{1 + \mathrm{i}\tfrac{32}{3}k\eta^2[\ell n(\eta) - \tfrac{5}{6}] + \cdots\}, \tag{8.24}$$

with C an unknown constant (see appendix D of Calvo *et al.* (2017)). These represent possible inner solutions. Defining a new variable

$$z \equiv \eta\sqrt{2k} \sim 1, \tag{8.25}$$

the homogeneous equation becomes

$$\frac{1}{\ell n(z/\sqrt{2k})} \frac{\partial^2 H|_h}{\partial z^2} \simeq \mathrm{i}H|_h, \tag{8.26}$$

which slightly away from $\eta = 0$ can be approximated by

$$\frac{\partial^2 H|_h}{\partial z^2} + \mathrm{i}\ell n(\sqrt{2k})H|_h \simeq 0 \tag{8.27}$$

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since $k \gg 1$. The decaying solution to this equation is

$$H|_h \simeq \mathrm{i}(L/k)\mathrm{e}^{-\sqrt{-\mathrm{i}\ell n(\sqrt{2k})}\,z} \simeq \mathrm{i}(L/k)\mathrm{e}^{-(1-\mathrm{i})\sqrt{2k\ell n(\sqrt{2k})}\,\eta}, \tag{8.28}$$

where the coefficient is chosen to cancel the particular solution as $\eta \to 0$ to satisfy the boundary condition. Taking the inner limit of this outer solution gives

$$H|_h \simeq \mathrm{i}(L/k)[1-(1-\mathrm{i})\sqrt{2k\ell n(\sqrt{2k})}\,\eta + \cdots], \tag{8.29}$$

which allows us to match the terms linear in η to determine the constant C to be

$$C = -(1+\mathrm{i})L\sqrt{\frac{2\ell n(\sqrt{2k})}{k}}. \tag{8.30}$$

Consequently, a adequate matched asymptotic solution for the trapped alphas up to the trapped–passing boundary and away from it is

$$H = \frac{\mathrm{i}L}{k}[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1], \tag{8.31}$$

where both k and L are proportional to τ_s. Forming $\bar{h}_t$ gives

$$\bar{h}_t = \frac{B_0\varepsilon v^2}{2\Omega_0\omega}\frac{\partial f_s}{\partial \psi}\mathrm{Re}\{[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1]\,\mathrm{e}^{\mathrm{i}\alpha/q}\}, \tag{8.32}$$

with Re denoting the real part is to be taken.

Next we evaluate the particle and heat fluxes

$$\Gamma_d^{\mathrm{rip}} \simeq \frac{-\pi}{\Omega_0 q}\left\langle \frac{\partial B}{\partial \vartheta}\int_0^{v_0}\mathrm{d}v\, v^4\left(\frac{Mv^2}{2}\right)^d \int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\,\lambda\frac{\bar{h}_t}{\xi}\right\rangle. \tag{8.33}$$

To perform the pitch angle integral we let

$$\sin(N\zeta/2) = \kappa\sin\varphi \tag{8.34}$$

so that

$$\xi = \sqrt{1-\lambda[1-\delta\cos(N\zeta)]} = \kappa\sqrt{2\lambda\delta}\cos\varphi \tag{8.35}$$

and

$$\kappa\cos\varphi\,\mathrm{d}\varphi = \cos(N\zeta/2)\mathrm{d}(N\zeta/2) = \sqrt{1-\kappa^2\sin^2\varphi}\,\mathrm{d}(N\zeta/2). \tag{8.36}$$

Recalling that $\alpha/q \simeq -\vartheta$ and noting that only $\sin(\alpha/q) \simeq -\sin\vartheta$ terms contribute when evaluating flux surface averages, we find

$$\begin{aligned}\mathrm{Re}&\left\langle \sin\vartheta\,\mathrm{e}^{-\mathrm{i}\vartheta}\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\frac{\lambda}{\xi}[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1]\right\rangle \\ &\simeq -\frac{1}{2}\mathrm{Re}\left\langle \mathrm{i}\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\frac{\lambda}{\xi}[\mathrm{e}^{-8(1-\mathrm{i})(1-\kappa)\sqrt{2k\ell n(\sqrt{2k})}} - 1]\right\rangle.\end{aligned} \tag{8.37}$$

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Then using

$$\langle \xi^{-1}\rangle = \frac{\displaystyle\int_0^{\sin^{-1}\kappa} \xi^{-1}\mathrm{d}(N\zeta/2)}{\displaystyle\int_0^{\pi/2} \mathrm{d}(N\zeta/2)} = \frac{\sqrt{2}}{\pi\sqrt{\delta}}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}} \\ = \frac{\sqrt{2}K(\kappa)}{\pi\sqrt{\delta}} \simeq \frac{\ell n[16/(1-\kappa^2)]}{\pi\sqrt{2\delta}}, \tag{8.38}$$

we obtain

$$\mathrm{Re}\left\langle \sin\vartheta\, \mathrm{e}^{-\mathrm{i}\vartheta}\int_{B_0/\widehat{B}}^{B_0/B}\mathrm{d}\lambda\frac{\lambda}{\xi}[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}}-1]\right\rangle \\ \simeq -\frac{\sqrt{2\delta}}{\pi}\mathrm{Re}\left\{\mathrm{i}\int_0^1 \mathrm{d}\kappa\,\kappa[\mathrm{e}^{-8(1-\mathrm{i})(1-\kappa)\sqrt{2k\ell n(\sqrt{2k})}}-1]\ln\left(\frac{8}{1-\kappa}\right)\right\} \\ \simeq -\frac{\sqrt{\delta}\ell n[64\sqrt{2k\ell n(\sqrt{2k})}]}{8\pi\sqrt{k\ell n(\sqrt{2k})}}\mathrm{Re}\left[\mathrm{i}\int_0^{8\sqrt{2k\ell n(\sqrt{2k})}}\mathrm{d}\chi\,\mathrm{e}^{-(l-\mathrm{i})\chi}\right] \\ = \sqrt{\delta}\frac{\ell n[64\sqrt{2k\ell n(\sqrt{2k})}]}{16\pi\sqrt{k\ell n(\sqrt{2k})}}, \tag{8.39}$$

where the imaginary integral does not contribute and $\chi = 8(1-\kappa)\sqrt{2k\ell n(\sqrt{2k})}$.

We also make use of

$$\int_0^{v_0}\mathrm{d}v\, v^{9/2}(Mv^2/2)^d\frac{\ell n[64\sqrt{2k\ell n(\sqrt{2k})}]}{\sqrt{\ell n(\sqrt{2k})}}\frac{\partial f_s}{\partial\psi} \simeq \frac{2v_0^{5/2}(Mv_0^2/2)^d\ell n[64\sqrt{2k_0\ell n(\sqrt{2k_0})}]}{(5+4d)\sqrt{\ell n(\sqrt{2k_0})}\,\ell n(v_0/v_c)}\frac{\partial n_s}{\partial\psi}, \tag{8.40}$$

where

$$k_0 \equiv \frac{64\delta\,\omega\,\tau_s v_0^3}{qv_\lambda^3} \gg 1. \tag{8.41}$$

From the preceding we obtain

$$\varGamma_d^{\sqrt{\nu}} \simeq -\frac{\varepsilon^2 B_0^2\tau_s v_\lambda^{3/2}v_0^{5/2}(Mv_0^2/2)^d[\ell n(64\sqrt{2k_0\ell n(\sqrt{2k_0})})]}{128(5+4d)\,q^{1/2}\varOmega_0^2\omega\sqrt{\omega\tau_s\ell n(2k_0)}\,\ell n(v_0/v_c)}\frac{\partial n_s}{\partial\psi}, \tag{8.42}$$

which is valid in the limit of $\varepsilon/qN\delta \ll 1$. Notice that the diffusivity is of order

$$D_0^{\sqrt{\nu}} \simeq \frac{(qv_\lambda/v_0)^{3/2}[\ell n(64\sqrt{k_0\ell n(2k_0)})](\rho_0 v_0/R)^2}{640\,\omega\sqrt{\omega\tau_s\ell n(2k_0)}\,\ell n(v_0/v_c)}. \tag{8.43}$$

Within logarithmic factors as in Calvo *et al.* (2017) this result agrees with our phenomenological estimate, but notice that the coefficient is small due to the collisional boundary layer analysis. Interestingly, $D_0^{\sqrt{\nu}}$ only depends logarithmically on the ripple δ because (8.17) or (8.41) require a very narrow boundary layer of width $(qv_\lambda^3/\omega\,\tau_s v_0^3)^{1/2} \ll \delta^{1/2}$. These boundary layers become comparable, $(qv_\lambda^3/\omega\,\tau_s v_0^3)^{1/2} \sim \delta^{1/2}$, at the transition between the $\sqrt{\nu}$ and $1/\nu$ regimes.

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9. Summary

We formulate and solve for the neoclassical transport of alphas in a rippled tokamak by evaluating the $1/\nu$ and $\sqrt{\nu}$ regime modifications of Galeev *et al.* (1969), Stringer (1972), Connor & Hastie (1973), and Ho & Kulsrud (1987) associated with alpha birth and slowing. The formulation also retains the standard axisymmetric neoclassical effects of Nocentini *et al.* (1975), Catto (1988), and Hsu *et al.* (1990). The ripple transport of alphas differs from that of the bulk ions and electrons because of its slowing down tail distribution function and the need to consider electron drag as well as pitch angle scatter. Moreover, the approximate variational treatment of Galeev *et al.* (1969) and boundary layer analysis of Ho & Kulsrud (1987) in the $\sqrt{\nu}$ regime is avoided by a more complete and rigorous boundary layer analysis similar to that of Calvo *et al.* (2017). Our $\sqrt{\nu}$ regime evaluation assumes large ripple ($\varepsilon/qN\delta \gg 1$) and the results are given by (8.42) and (8.43). In this $\sqrt{\nu}$ regime drag is unimportant. The $1/\nu$ regime evaluations presented here allow general ripple ($\varepsilon/qN\delta \sim 1$) and the results with only pitch angle scattering ($\delta v_0^3/v_\lambda^3 \ll 1$) are given by equations (7.23) and (7.24), while those keeping only electron drag ($\delta v_0^3/v_\lambda^3 \gg 1$) are given by equations (7.36) and (7.37). When pitch angle scatter and electron drag compete in the $1/\nu$ regime ($\delta v_0^3/v_\lambda^3 \sim 1$) and the ripple is strong ($\varepsilon/qN\delta \ll 1$), the transition from the axisymmetric banana regime to the $1/\nu$ occurs when the time for an birth alpha to travel a connection length is substantially smaller than the slowing down time, roughly $qR/\tau_s v_0 \sim \delta^{1/4}\varepsilon^{1/2}$, or more precisely

$$qR/\tau_s v_0 \simeq 0.3\delta^{1/4}\varepsilon^{1/2}(\delta v_0^3/v_\lambda^3)^{1/2} \sim 0.3\delta^{1/4}\varepsilon^{1/2} \ll 1. \tag{9.1}$$

For the axisymmetric banana regime we simplify by using $\delta v_0^3/v_\lambda^3 \sim 1$ and taking

$$D_{\mathrm{axi}}^{\mathrm{ban}} \simeq 0.25 q^2 \rho_0^2/\varepsilon\tau_s \ell n(v_0/v_c) \tag{9.2}$$

to obtain an estimate between the Nocentini *et al.* (1975) and Catto (1988) limits to be in better agreement with the precise results of Hsu *et al.* (1990). Our $1/\nu$ calculation assumes small alpha birth gyroradius,

$$\delta^{1/4}\rho_0/a_\alpha \ll R/\tau_s v_0 \ll 1, \tag{9.3}$$

to keep the alpha distribution function near the slowing down distribution that is assumed to have a radial scale length of a_α. To keep $\tau_s D_{\mathrm{axi}}^{\mathrm{ban}}/a_\alpha^2 \ll 1$ we need $\rho_0/a_\alpha \ll \varepsilon^{1/2}/q \ll 1$, which is typically less restrictive than (9.3).

It is unlikely that (9.3) can be satisfied for any feasible δ. However, the inability to satisfy (9.3) is good news! It means that for all practical purposes there is no significant $1/\nu$ regime so that axisymmetric neoclassical banana regime transport transitions almost directly into less dangerous $\sqrt{\nu}$ regime for alpha ripple transport!

Moreover, for $\delta v_0^3/v_\lambda^3 \sim 1$ and $\varepsilon/qN\delta \ll 1$, the transition from the $1/\nu$ to $\sqrt{\nu}$ regime happens when a significant fraction of a full toroidal rotation happens in a slowing down time, roughly $\omega\tau_s \sim q$, or more carefully

$$\frac{\omega\tau_s}{q} \sim \left[\frac{\ell n(64\sqrt{k_0 \ell n(2k_0)})}{40\sqrt{\ell n(2k_0)}}\right]^{2/3}, \tag{9.4}$$

where $2k_0 \simeq 128\,\omega\,\tau_s\delta v_0^3/qv_\lambda^3 \sim 128\omega\,\tau_s/q \gg 1$ is required for our boundary layer analysis to be valid. Normally, $\omega\,\tau_s \gg q$.

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To remain near a slowing down distribution the $\sqrt{\nu}$ regime requires the alpha birth gyroradius to satisfy the much stricter condition of

$$\frac{\omega R}{q v_\lambda} \gg \left[\frac{\ell n(64\sqrt{k_0 \ell n(2k_0)})}{640 \ell n(v_0/v_c)\sqrt{\ell n(2k_0)}}\right]^{2/3} \left(\frac{\rho_0}{a_\alpha}\right)^{4/3} \left(\frac{v_0 \tau_s}{R}\right)^{1/3}. \tag{9.5}$$

Even with the small coefficient, rotation larger than the ∇B drift level ($\omega R > \rho_0 v_0/R$) and small ρ_0/a_α may be required to avoid depletion of the alpha slowing down tail distribution function.

To estimate the size of quantities for D–T fusion we use $v_0 \simeq 1.3 \times 10^9$ cm s^{-1} for a 3.5 MeV alpha, and $\tau_s = \tau_{ee} M/Z^2 m \simeq 0.63$ s for $T_e \simeq 10$ keV and $n_e \simeq 10^{14}$ cm^{-3}, with τ_{ee} the electron collision time. We note that even if the alpha pressure was comparable to the plasma pressure that the slowing down alpha density n_s would be small compared to the plasma density $n_s/n_e \sim 1/300$. Consequently, the alphas are not expected to play a role in ambipolarity so any radial electric field due to ripple is due to the balance of $1/\nu$ regime electron transport and $\sqrt{\nu}$ regime ion transport or the radial electric field associated with an axisymmetric tokamak and determined by convservation of toroidal angular momentum (see Parra & Catto 2008, 2009, 2010, must be smaller in this situation), giving an inadequate rotation level of $\omega R \sim q\rho_i v_i/\varepsilon a$. Then the ∇B drift will dominant and is expected to play a role similar to the $\boldsymbol{E} \times \boldsymbol{B}$. The assumptions used to derive the alpha collision operator are satisfied since $v_0 \ll v_e = (2T_e/m)^{1/2} \simeq 6 \times 10^9$ cm s^{-1} and $v_0 \gg v_i = (2T_i/M_i)^{1/2} \simeq 10^8$ cm s^{-1}. Moreover, for equal amounts of D and T, $v_0/v_c \simeq 3.25$ and $v_\lambda^3/v_c^3 \simeq 3/5$, so our assumptions that $v_0^3 \gg v_c^3 \sim v_\lambda^3$ are satisfied. In addition, we note that for $B_0 = 5$ T, $\Omega_0 \simeq 2.4 \times 10^8$ rad s^{-1}, giving $\rho_0 \simeq 5.4$ cm, which must be small compared to the radial scale length a_α of the alphas.

From the preceding numbers and $R \simeq 10$ m, we obtain $R/\tau_s v_0 \simeq 1.2 \times 10^{-6}$. Consequently, based on (9.3), we expect that tokamaks will operate with the alphas well into the $\sqrt{\nu}$ ripple transport regime with $\omega\tau_s \gg q$. Indeed, if we assume $\omega R \simeq v_i$, then we find $\omega\tau_s \simeq 6.3 \times 10^4$, giving $\sqrt{\nu}$ regime ripple transport. Then the key issue becomes avoiding the depletion of the alpha slowing down distribution by satisfying (9.5).

In summary, ripple transport of alphas will be in the $\sqrt{\nu}$ regime, rather than the $1/\nu$ regime, thereby suppressing ripple transport well below the $1/\nu$ level. However, alpha ripple transport in the $\sqrt{\nu}$ regime will be a serious issue for tokamak reactors as it will be well above the axisymmetric neoclassical level and can deplete the alpha slowing down distribution function unless toroidal rotation is strong.

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Appendix A. Stringer (1972) and Connor & Hastie (1973)

The extremal along the magnetic field

$$B = B_0[1 - \varepsilon\cos\vartheta - \delta\cos(N\zeta)] \tag{A 1}$$

are found for fixed α from

$$\varepsilon\sin\vartheta + qN\delta\sin(N\zeta) = 0. \tag{A 2}$$

If $qN\delta \ll \varepsilon$, then even if $\sin(N\zeta) \to \pm 1$, the minima must be near $\vartheta = 0$. In this weak ripple limit the well centres are near the equatorial plane and $N\zeta$ varies over less than 2π. In the $qN\delta \gg \varepsilon$ the strong ripple can localize the ripple well minima away from the equatorial plane.

More precisely, the minima occur at

$$N\breve{\zeta} = 2m\pi - \sin^{-1}[(\varepsilon/qN\delta)\sin\breve{\vartheta}] \simeq 2m\pi - \sin^{-1}[(\varepsilon/qN\delta)\sin\vartheta], \tag{A 3}$$

where m is an integer or zero satisfying $0 \leqslant m \leqslant N$ to keep $2\pi \geqslant \breve{\zeta} \geqslant 0$ and we will soon see that ϑ variation is weak for $N \gg 1$. The adjacent maxima occur at

$$N\widehat{\zeta} = (2m \pm 1)\pi + \sin^{-1}[(\varepsilon/qN\delta)\sin\widehat{\vartheta}] \simeq (2m \pm 1)\pi + \sin^{-1}[(\varepsilon/qN\delta)\sin\vartheta]. \tag{A 4}$$

The minima, maxima, and lower maxima are found by assuming that ϑ varies only slightly during a bounce when $N \gg 1$. The minima and lower maxima must satisfy

$$|N(\widehat{\zeta} - \breve{\zeta})| \simeq |\pm\pi + 2\sin^{-1}[(\varepsilon/qN\delta)\sin\vartheta]| = \pi - 2\sin^{-1}[(\varepsilon/qN\delta)|\sin\vartheta|], \tag{A 5}$$

giving

$$|\widehat{\vartheta} - \breve{\vartheta}| \simeq \{\pi - 2\sin^{-1}[(\varepsilon/qN\delta)|\sin\vartheta|]\}/qN \ll 1. \tag{A 6}$$

Even for large ripple, $qN\delta \gg \varepsilon$, the variation in $|\widehat{\vartheta} - \breve{\vartheta}|$ is small, even though $|N(\widehat{\zeta} - \breve{\zeta})|$ varies over almost a full 2π.

Only the lower maximum of the magnetic field,

$$\widehat{B} = B_0[1 - \varepsilon\cos\widehat{\vartheta} - \delta\cos(N\widehat{\zeta})] \simeq B_0[1 - \varepsilon\cos\widehat{\vartheta} + \delta\sqrt{1 - (\varepsilon/qN\delta)^2\sin^2\vartheta}], \tag{A 7}$$

enters in our integrals, where we must choose the smaller of the two possible values of $\varepsilon\cos\widehat{\vartheta}$. Using the minimum field

$$\breve{B} = B_0[1 - \varepsilon\cos\breve{\vartheta} - \delta\cos(N\breve{\zeta})] \simeq B_0[1 - \varepsilon\cos\breve{\vartheta} - \delta\sqrt{1 - (\varepsilon/qN\delta)^2\sin^2\vartheta}] \tag{A 8}$$

and expanding $\cos\breve{\vartheta}$ about $\cos\widehat{\vartheta}$ at fixed α gives the lower maximato be

$$\begin{aligned}\widehat{B} \simeq B_0 \bigg\{ & 1 - \varepsilon\cos\breve{\vartheta} - (\varepsilon/qN)(\pi - 2\sin^{-1}[(\varepsilon/qN\delta)|\sin\vartheta|])|\sin\vartheta| \\ & + \delta\sqrt{1 - (\varepsilon/qN\delta)^2\sin^2\vartheta}\bigg\},\end{aligned} \tag{A 9}$$

where $\breve{\vartheta} \simeq \vartheta$ except in $\cos\breve{\vartheta}$,

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In addition, we can form

$$\begin{aligned}\widehat{B}-\breve{B} &= B_0\{\varepsilon(\cos\breve{\vartheta}-\cos\widehat{\vartheta})+\delta[\cos(N\breve{\zeta})-\cos(N\widehat{\zeta})]\} \\ &\simeq 2B_0\delta\left\{\sqrt{1-(\varepsilon/qN\delta)^2\sin^2\vartheta}\right. \\ &\quad\left.-(\varepsilon/qN\delta)[(\pi/2)-\sin^{-1}[(\varepsilon/qN\delta)|\sin\vartheta|]]|\sin\vartheta|\right\}\end{aligned} \tag{A 10}$$

as in Stringer (1972) and Connor & Hastie (1973), where here the distinction between $\widehat{\vartheta}$, $\breve{\vartheta}$, and ϑ no longer matters on the right side.

Similarly, we can also form $\widehat{B}-B$

$$(\widehat{B}-B)/B_0 \simeq \varepsilon(\cos\vartheta-\cos\widehat{\vartheta})+\delta\left[\cos(N\zeta)+\sqrt{1-(\varepsilon/qN\delta)^2\sin^2\vartheta}\right]. \tag{A 11}$$

Expanding using $\widehat{\vartheta}\simeq\vartheta$ gives

$$\cos\widehat{\vartheta}\simeq\cos\vartheta-(\widehat{\vartheta}-\vartheta)\sin\vartheta=\cos\vartheta-|\widehat{\vartheta}-\vartheta||\sin\vartheta| \tag{A 12}$$

and

$$(\widehat{B}-B)/B_0\simeq\varepsilon(\widehat{\vartheta}-\vartheta)\sin\vartheta+\delta[\cos(N\zeta)-\cos(N\widehat{\zeta})], \tag{A 13}$$

where

$$\cos(N\widehat{\zeta})\simeq-\sqrt{1-(\varepsilon/qN\delta)^2\sin^2\vartheta}. \tag{A 14}$$

Along the magnetic field $q(\widehat{\vartheta}-\vartheta)=\widehat{\zeta}-\zeta$ giving for $p=1$ and 3,

$$\begin{aligned}\int_{\text{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^p} &\simeq \delta^{p/2}\int_{\text{all}}\mathrm{d}(N\zeta) \\ &\times\left\{(\varepsilon/qN\delta)(N\widehat{\zeta}-N\zeta)\sin\vartheta+[\cos(N\zeta)-\cos(N\widehat{\zeta})]\right\}^{p/2},\end{aligned} \tag{A 15}$$

where we can view $\widehat{\zeta}=\widehat{\zeta}(\vartheta)$.

To get the same signs and form as Connor & Hastie (1973) and conveniently keep track of signs, we let $Y=N\zeta+\pi$ so that $\cos(N\zeta)=-\cos Y$ and $\zeta=0$ is at π, and $X=N\widehat{\zeta}(\vartheta)+\pi$ so that $\cos(N\widehat{\zeta})=-\cos X$ and we may take

$$\sin X=(\varepsilon/qN\delta)\sin\vartheta. \tag{A 16}$$

Then

$$\int_{\text{all}}\mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^p}\simeq\delta^{p/2}\int_{\text{all}}\mathrm{d}Y[\cos X-\cos Y+(X-Y)\sin X]^{p/2}. \tag{A 17}$$

Continuing, using

$$\mathrm{d}X\cos X=(\varepsilon/qN\delta)\cos\vartheta\,\mathrm{d}\vartheta=\sqrt{(\varepsilon/qN\delta)^2-\sin^2X}\,\mathrm{d}\vartheta, \tag{A 18}$$

and the up–down symmetry of the ϑ integral, and then noting that the 0 to $\pi/2$ and $\pi/2$ to π contributions to the ϑ integral are equal, gives

$$\begin{aligned}\int_{\text{all}} \mathrm{d}\vartheta \sin^2\theta \int_{\text{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^p} &= \frac{4\delta^{p/2}}{(\varepsilon/qN\delta)^2}\int_0^U \frac{\mathrm{d}X\cos X\sin^2 X}{\sqrt{(\varepsilon/qN\delta)^2-\sin^2 X}} \\ &\quad \times \int_X^{Y_1(X)} \mathrm{d}Y[\cos X - \cos Y + (X-Y)\sin X]^{p/2}. \end{aligned} \tag{A 19}$$

The lower limit $Y=X$ corresponds to $B=\widehat{B}$ (the lower maximum), while the upper limit U also corresponds to $B=\widehat{B}$ but not at a maximum, instead it satisfies $\cos X - \cos Y_1 + (X-Y_1)\sin X = 0$ (note for $X\rightarrow -X$, $Y_1\rightarrow -Y_1$). Consequently, like Connor & Hastie (1973) we integrate between two maxima. The upper limit U of the X integral is $\pi/2$ when $\varepsilon/qN\delta > 1$, while for $\varepsilon/qN\delta < 1$ it is $\sin^{-1}(\varepsilon/qN\delta)$.

To get the maximum value for normalization we consider $\varepsilon/qN\delta \ll 1$

$$\begin{aligned}\int_{\text{all}} \mathrm{d}\vartheta \sin^2\theta \int_{\text{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^p} &\simeq \frac{4\delta^{p/2}}{(\varepsilon/qN\delta)^2} \\ \times \int_0^{\varepsilon/qN\delta} \frac{\mathrm{d}X\,X^2}{\sqrt{(\varepsilon/qN\delta)^2-X^2}} &\int_X^{Y_1} \mathrm{d}Y(1-\cos Y - XY)^{p/2}, \end{aligned} \tag{A 20}$$

where for $X\ll 1$, Y_1 satisfies $1-\cos Y_1 \simeq Y_1 X$ giving $Y_1 \simeq 2\pi - 2\sqrt{\pi X}$. Letting $Y=2y$ and making use of $X\ll 1$ to neglect order X^2 corrections gives

$$\int_X^{Y_1} \mathrm{d}Y(1-\cos Y - XY)^{p/2} = 2\sqrt{2^p}\int_{X/2}^{Y_1/2} \mathrm{d}y(\sin^2 y - Xy)^{p/2}. \tag{A 21}$$

For the $p=3$ case

$$\begin{aligned}\int_X^{Y_1} \mathrm{d}Y(1-\cos Y - XY)^{3/2} &\simeq 4\sqrt{2}\left(\int_{X/2}^{Y_1/2} \mathrm{d}y\sin^3 y - \frac{3X}{2}\int_0^{\pi} \mathrm{d}y\,y\sin y\right) \\ &\simeq \frac{16}{3}\sqrt{2}\left(1-\frac{9\pi}{8}X+\cdots\right). \end{aligned} \tag{A 22}$$

Then letting $X=(\varepsilon/qN\delta)x$ and $\gamma=\varepsilon/qN\delta$ gives

$$\int_{\text{all}} \mathrm{d}\vartheta \sin^2\theta \int_{\text{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^3} \simeq \frac{16\pi\sqrt{2}\,\delta^{3/2}}{3}(1-3\gamma+\cdots), \tag{A 23}$$

which for $p=3$ agrees with Connor & Hastie (1973) and gives $16\pi\sqrt{2}\,\delta^{3/2}/3$ as the normalization coefficient.

The case $p=1$ is more involved. We split the integral

$$\int_{X/2}^{Y_1/2} \mathrm{d}y(\sin^2 y - Xy)^{1/2} = \int_{X/2}^{\pi/2} \mathrm{d}y(\sin^2 y - Xy)^{1/2} + \int_{\pi/2}^{Y_1/2} \mathrm{d}y(\sin^2 y - Xy)^{1/2}. \tag{A 24}$$

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For the first term we use

$$\int_{X/2}^{\pi/2} \mathrm{d}y(\sin^2 y - Xy)^{1/2} \simeq \int_{X/2}^{\pi/2} \mathrm{d}y \sin y - \frac{X}{2}\int_0^{\pi} \frac{\mathrm{d}y\, y}{\sin y} = 1 - CX + \cdots, \tag{A 25}$$

with $C = 0.916$ Catalan's constant. For the second integral we let $z = \pi - y$ to obtain

$$\begin{aligned}\int_{\pi/2}^{Y_1/2} \mathrm{d}y\sqrt{\sin^2 y - Xy} &= \int_{\sqrt{\pi X}}^{\pi/2} \mathrm{d}z\sqrt{\sin^2 z + X(z-\pi)} \\ &\simeq \int_{\sqrt{\pi X}}^{\pi/2} \mathrm{d}z \sin z + \frac{X}{2}\int_{\sqrt{\pi X}}^{\pi/2} \mathrm{d}z\frac{z-\pi}{\sin z} \simeq 1 + X\left\{C + \frac{\pi}{4}\left[\ell n\left(\frac{\pi X}{4}\right) - 2\right]\right\}.\end{aligned} \tag{A 26}$$

Consequently,

$$\int_{X/2}^{Y_1/2} \mathrm{d}y(\sin^2 y - Xy)^{1/2} \simeq 2 + \frac{\pi X}{4}\left[\ell n\left(\frac{\pi X}{4}\right) - 2\right]. \tag{A 27}$$

Then again letting $X = (\varepsilon/qN\delta)x$ we obtain

$$\begin{aligned}\int_{\mathrm{all}} \mathrm{d}\vartheta \sin^2\theta \int_{\mathrm{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]} &\simeq 16\sqrt{2}\,\delta^{1/2} \\ &\times \int_0^1 \frac{\mathrm{d}x\, x^2}{\sqrt{1-x^2}}\left\{1 + \frac{\pi\gamma x}{8}\left[\ell n\left(\frac{\pi\gamma x}{4}\right) - 2\right]\right\} \\ &\simeq 4\pi\sqrt{2}\,\delta^{1/2}\left\{1 + \frac{\pi\gamma}{3}\left[\ell n\left(\frac{\pi\gamma}{2}\right) - \frac{17}{6}\right]\right\},\end{aligned} \tag{A 28}$$

where we use 4.241.2 of Gradshteyn & Ryzhik (2007),

$$\int_0^1 \frac{\mathrm{d}x\, x^3 \ell n x}{\sqrt{1-x^2}} = \frac{2}{3}\left(\ell n 2 - \frac{5}{6}\right). \tag{A 29}$$

As a result, the normalization coefficient for $p = 1$ is $2\pi\sqrt{2}\,\delta^{1/2}$.

When $\varepsilon/qN\delta \gg 1$ then $U = \pi/2$ and we must evaluate

$$\begin{aligned}\int_{\mathrm{all}} \mathrm{d}\vartheta \sin^2\theta \int_{\mathrm{all}} \mathrm{d}(N\zeta)\sqrt{[1-(B/\widehat{B})]^p} &\simeq \frac{4\delta^{1/2}}{\gamma^3}\int_0^{\pi/2} \mathrm{d}X \cos X \sin^2 X\left[1 + \frac{\sin^2 X}{2\gamma^2} + \cdots\right] \\ &\times \int_X^{Y_1} \mathrm{d}Y[\cos X - \cos Y + (X-Y)\sin X]^{p/2},\end{aligned} \tag{A 30}$$

where $\vartheta \simeq 0$ implies $\sin X \simeq \gamma\vartheta \leqslant 1$. Notice that for $X = 0$, $Y_1 = 2\pi$ (or $N\zeta = \pi$), and for $X = \pi/2$, $Y_1 = \pi/2$ (or $N\zeta = -\pi/2$); while for $X = 0 = Y$, $N\zeta = -\pi$.

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