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Regularity of the extremal solutions associated with elliptic systems

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We examine the elliptic system given by

$$\begin{cases} -\Delta u = \lambda f(v) & \text{in } \Omega, \\ -\Delta v = \gamma f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ, γ are positive parameters, Ω is a smooth bounded domain in $\mathbb{R}^N$ and f is a C^2 positive, nondecreasing and convex function in $[0, \infty)$ such that $f(t)/t \to \infty$ as $t \to \infty$. Assuming

$$0 < \tau_- := \liminf_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \leqslant \tau_+ := \limsup_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \leqslant 2,$$

we show that the extremal solution (u^*, v^*) associated with the above system is smooth provided that $N < (2\alpha_*(2-\tau_+) + 2\tau_+)/(\tau_+)\max\{1, \tau_+\}$, where $\alpha_* > 1$ denotes the largest root of the second-order polynomial

$$P_f(\alpha, \tau_-, \tau_+) := (2-\tau_-)^2\alpha^2 - 4(2-\tau_+)\alpha + 4(1-\tau_+).$$

As a consequence, $u^*, v^* \in L^\infty(\Omega)$ for $N < 5$. Moreover, if $\tau_- = \tau_+$, then $u^*, v^* \in L^\infty(\Omega)$ for $N < 10$.

Keywords: extremal solution; stable solution; regularity of solutions; elliptic systems

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1. Introduction

In this short note, we examine the boundedness of the extremal solutions to the following system of equations:

$$(P)_{\lambda,\gamma}\begin{cases}-\Delta u=\lambda f(v) & \text{in }\Omega\\ -\Delta v=\gamma f(u) & \text{in }\Omega,\\ \qquad u=v=0 & \text{on }\partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $\lambda,\gamma>0$ are positive parameters. The nonlinearity f satisfies

(R) f is smooth, increasing and convex with $f(0)>0$ and superlinear at ∞.

Define $\mathcal{Q}:=\{(\lambda,\gamma),\ \lambda,\gamma>0\}$,

$$\mathcal{U}:=\{(\lambda,\gamma)\in\mathcal{Q}:\ \text{there exists a smooth solution }(u,v)\text{ of }(P)_{\lambda,\gamma}\},$$

and set $\Upsilon:=\partial\mathcal{U}\cap\mathcal{Q}$. M. Montenegro in [**11**] (for a more general system than $(P)_{\lambda,\gamma}$) showed that $\mathcal{U}\neq\emptyset$ and for every $(\lambda,\gamma)\in\mathcal{U}$ the problem $(P)_{\lambda,\gamma}$ has a minimal solution. Then, using monotonicity, for each $(\lambda^*,\gamma^*)\in\Upsilon$ one can define the extremal solution (u^*,v^*) as a pointwise limit of minimal solutions of $(P)_{\lambda,\sigma\lambda}$ with $\sigma:=\gamma^*/\lambda^*$, which is always a weak solution to $(P)_{\lambda^*,\gamma^*}$. Moreover, for a $(\lambda,\gamma)\in\mathcal{U}$, the minimal solution (u,v) of $(P)_{\lambda,\gamma}$ is stable in the sense that there is a constant $\eta>0$ and $0<\zeta,\chi\in H_0^1(\Omega)$ such that

$$-\Delta\zeta=\lambda f'(v)\chi+\eta\zeta,\quad -\Delta\chi=\gamma g'(v)\zeta+\eta\chi,\quad \text{in }\Omega. \tag{1}$$

For the proof see [**11**] (see also [**5**] for an alternative proof).

In [**11**] it is left open the question of the regularity of extremal solution (u^*,v^*). In the case when $f(t)=e^t$, in [**4**] Cowan proved the extremal solutions to $(P)_{\lambda,\sigma\lambda}$ are smooth for $1\leqslant N\leqslant 9$ under the further assumption $N-2/8<\gamma/\lambda<8/N-2$; Dupaigne, Farina and Sirakov in [**9**] then improved this result by removing this restriction. The same result is also obtained by Dávila and Goubet [**7**]. Furthermore, they proved that for $N\geqslant 10$, the singular set of any extremal solution of the system $(P)_{\lambda,\gamma}$ has Hausdorff dimension at most $N-10$. We now mention that some of the motivation for our proof of Theorem 3 in the current paper comes from the work of Dupaigne, Farina and Sirakov [**9**].

It is also worth mentioning here that the second-order scalar analogue of (N_λ) with Dirichlet boundary conditions, that is,

$$(Q)_\lambda\begin{cases}-\Delta u=\lambda f(u) & \text{in }\Omega,\\ \qquad u=0 & \text{on }\partial\Omega,\end{cases}$$

is by now quite well understood whenever Ω is a bounded smooth domain in R^N and f is a nonlinearity of type (R), see for instance, [**1**–**3**, **8**, **12**, **15**, **16**]. In this case, the best-known result is due to X. Cabré [4] who showed that for $N<5$ the extremal solution u^* of (Q_λ) is smooth for arbitrary nonlinearity f if in addition Ω

is convex. In [15] Villegas proved the same replacing the condition that Ω is convex with f is convex. However, it is still an open problem to establish an L^∞ estimate in dimensions $5 \leqslant N \leqslant 9$, even in the case of convex domains Ω and convex nonlinearities satisfying (R).

Define

$$\tau_- := \liminf_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \leqslant \tau_+ := \limsup_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2}. \tag{2}$$

Our main result is the following.

Theorem 1. *Let f satisfy (R) with $0 < \tau_- \leqslant \tau_+ < 2$, and Ω an arbitrary bounded domain in $\mathbb{R}^N$ with smooth boundary and let (u^*, v^*) denote the extremal solution associated with $(P)_{\lambda,\gamma}$. Then $u^*, v^* \in L^\infty(\Omega)$ for*

$$N < N(f) := \frac{2\alpha_*(2-\tau_+) + 2\tau_+}{\tau_+} \max\{1, \tau_+\} \tag{3}$$

where $\alpha_* > 1$ denotes the largest root of the second-order polynomial

$$P_f(\alpha, \tau_-, \tau_+) := (2-\tau_-)^2\alpha^2 - 4(2-\tau_+)\alpha + 4(1-\tau_+). \tag{4}$$

As consequences,
i) $u^*, v^* \in L^\infty(\Omega)$ for $N < 5$.
i) If $\tau_- = \tau_+ := \tau$, then $u^*, v^* \in L^\infty(\Omega)$ for $N < 10$. Indeed, in this case, we have

$$N(f) = 2 + 4\frac{1+\sqrt{\tau}}{\tau} \geqslant 10.$$

For example consider problem $(P)_{\lambda,\gamma}$ with $f(t) = e^t$ or e^{t^α} $(\alpha > 0)$, then $\tau_+ = \tau_- = 1$, hence by theorem 1, $u^*, v^* \in L^\infty(\Omega)$ for $N < 10$. The same is true for $f(u) = (1+u)^p$ $(p > 1)$ as in this case, we have $\tau_+ = \tau_- = p - 1/p$. More precisely, in the latter case, we have $u^*, v^* \in L^\infty(\Omega)$ for

$$N < 2 + \frac{4}{p-1}\left(p + \sqrt{p^2 - p}\right).$$

This is exactly the same as the result obtained in [5] and [10] (corresponds to $p = \theta$ according to their notation).

In the examples above, we had $\tau_- = \tau_+$, here we also give an example of f with $0 < \tau_- \neq \tau_+ < 2$. Take arbitrary $a, b > 0$ with $0 < b < a \leqslant 1$, and define $f : [0,\infty) \to (0,\infty)$ as

$$f(t) = e^{\int_0^t ((\mathrm{d}s)/((1-a)s + b\sin s + 1))}, \quad t \geqslant 0.$$

Note that we have $f(0) = 1$, and since for every $s \in [0,\infty)$ we have $0 < 1 - b \leqslant (1-a)s + b\sin s + 1 \leqslant (1-a)s + b + 1$ then for $t > 0$, we have

$$f(t) \geqslant e^{\int_0^t ((\mathrm{d}s)/((1-a)s+b+1))} = \left(1 + \frac{1-a}{1+b}t\right)^{1/1-a}, \quad \text{when} \quad a < 1$$

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and

$$f(t) \geqslant e^{t/1+b}, \quad \text{when } a = 1,$$

thus f is superlinear. Also for every $t \geqslant 0$, we have $f'(t) = (f(t))/((1-a)t + b \sin t + 1) > 0$ and $f''(t) = (a - b\cos t)/(((1-a)t + b\sin t + 1)^2) f(t) > 0$, hence f is an increasing convex function, and

$$\frac{f(t)f''(t)}{f'(t)^2} = a - b\cos t, \quad t \geqslant 0.$$

Thus $\tau_- = a - b > 0$ and $\tau_+ = a + b < 2$. For example, take $a = 1$, then $\tau_- = 1 - b > 0$ and $\tau_+ = 1 + b < 2$. Now by theorem 1, we see that $u^*, v^* \in L^\infty(\Omega)$ for

$$N < 2\alpha_*(1-b) + 2(1+b), \quad \text{where} \quad \alpha_* = \frac{2(1-b) + 2\sqrt{(1-b)^2 + b(1+b)^2}}{(1+b)^2}.$$

Note also that if we tend $b \to 0$ in the above, then $\tau_-, \tau_+ \to 1$ and $\alpha_* \to 4$, then $2\alpha_*(1-b) + 2(1+b) \to 10$. Hence $u^*, v^* \in L^\infty(\Omega)$ for $N < 10$ for b close to 0, as we expected.

2. Preliminary estimates

To prove the main result, we use the following stability inequality. For the proof see lemma 1 in [**5**, **6**] and lemma 3 in [**9**].

LEMMA 1. *Let (u, v) denote a semistable solution of $(P)_{\lambda,\gamma}$. Then*

$$\sqrt{\lambda\gamma} \int_\Omega \sqrt{f'(u)f'(v)}\phi^2 \leqslant \int_\Omega |\nabla\phi|^2, \tag{5}$$

for all $\phi \in H_0^1(\Omega)$.

We also need the following lemmas.

LEMMA 2. *Assume $\lambda \geqslant \gamma$. Then for any smooth solution to the system $(P)_{\lambda,\gamma}$, we have*

$$v \leqslant u \leqslant \frac{\lambda}{\gamma} v.$$

Proof. Take $w = u - v$. Then $w = 0$ on $\partial\Omega$ and

$$-\Delta w = \lambda f(v) - \gamma f(u) \geqslant \lambda f(v) - \lambda f(u) = -\lambda \frac{f(u) - f(v)}{u - v} w := -\lambda a(x) w,$$

where $a(x) = (f(u) - f(v))/(u - v) \geqslant 0$ because f is increasing. Then by the maximum principle $w \geqslant 0$ in Ω. Now take $\tilde{w} = \lambda/\gamma v - u$. Then $\tilde{w} = 0$ on $\partial\Omega$ and using the above that $u \geqslant v$, we have

$$-\Delta\tilde{w} = \lambda f(u) - \lambda f(v) \geqslant 0,$$

hence $\tilde{w} \geqslant 0$ in Ω. □

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For the proof of the next lemma, we use the following standard regularity result, for the proof see theorem 3 of [**13**] and theorems 4.1 and 4.3 of [**14**].

THEOREM 2. *Let $u \in H_0^1(\Omega)$ be a weak solution of*

$$\begin{cases} \Delta u + c(x)u = g(x) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{6}$$

with $c, g \in L^p(\Omega)$ for some $p \geqslant 1$.
Then there exists a positive constant C independent of u such that if $p > N/2$ then

$$\|u\|_{L^\infty(\Omega)} \leqslant C(\|u\|_{L^1(\Omega)} + \|g\|_{L^p(\Omega)}).$$

LEMMA 3. *Assume for every semistable solution (u, v) of $(P)_{\lambda,\gamma}$ with $\lambda \geqslant \gamma$, we have*

$$\|v\|_{L^1(\Omega)} \leqslant C \quad \textit{and} \quad \|f'(v)\|_{L^p(\Omega)} \leqslant C,$$

for some $p > N/2$, where C is a constant independent of (u, v). Then $u^*, v^* \in L^\infty(\Omega)$.

Proof. We rewrite the first equation in $(P)_{\lambda,\gamma}$ as

$$\Delta u + \lambda \frac{f(v) - f(0)}{u} u = -\lambda f(0).$$

Taking $c(x) := \lambda(f(v(x)) - f(0))/(u(x))$ then using lemma 2 and the convexity of f, we have

$$0 \leqslant c(x) \leqslant \lambda \frac{f(v(x)) - f(0)}{v(x)} \leqslant \lambda^* f'(v).$$

Thus by the assumption and theorem 2, we get $u^* \in L^\infty(\Omega)$, and by lemma 2 we also get $v^* \in L^\infty(\Omega)$. □

3. Proof of the main result

Proof of Theorem 3. Fix an $\alpha > 1$ such that $P_f(\alpha, \tau_-, \tau_+) < 0$. Such an α exists since we have $P_f(1, \tau_-, \tau_+) = (2 - \tau_-)^2 - 4 < 0$ and $P_f(+\infty, \tau_-, \tau_+) = +\infty$. Hence we can take positive numbers $\tau_1 \in (0, \tau_-)$ and $\tau_2 \in (\tau_+, 2)$ such that

$$P_f(\alpha, \tau_1, \tau_2) < 0. \tag{7}$$

Now let (u, v) be a semistable solution of $(P)_{\lambda,\gamma}$ and take $\phi(x) = (\tilde{f}(u)^\alpha)/(f'(u)^{\alpha/2})$ in the semistability inequality (5), where $\tilde{f}(u) := f(u) - f(0)$. Note that here for simplicity, we assumed that $f'(0) > 0$, this does not cause any

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problem, as in what follows we need only the behaviour of f and f' at infinity. We have

$$|\nabla\phi(x)|^2 = \alpha^2 \tilde{f}(u)^{2\alpha-2} f'(u)^{2-\alpha} \left(1 - \frac{\tilde{f}(u)f''(u)}{2f'(u)^2}\right)^2 |\nabla u|^2,$$

and then taking

$$\theta(t) := \alpha^2 \int_0^t \tilde{f}(s)^{2\alpha-2} f'(s)^{2-\alpha} \left(1 - \frac{\tilde{f}(s)f''(s)}{2f'(s)^2}\right)^2 \mathrm{d}s,$$

we can write

$$|\nabla\phi(x)|^2 = \theta'(u)|\nabla u|^2 = \nabla\theta(u)\cdot\nabla u.$$

Thus, by the integration of part formula and the first equation of $(P)_{\lambda,\gamma}$, we compute

$$\int_\Omega |\nabla\phi(x)|^2 \mathrm{d}x = \int_\Omega \nabla\theta(u).\nabla u \mathrm{d}x = \int_\Omega \theta(u)(-\Delta u) = \lambda \int_\Omega \theta(u) f(v),$$

and using the above in the semistability inequality (5), we arrive at

$$\sqrt{\lambda\gamma} \int_\Omega f'(u)^{1/2-\alpha} f'(v)^{1/2} \tilde{f}(u)^{2\alpha} \leqslant \lambda \int_\Omega \theta(u) f(v). \tag{8}$$

A first step in using (8) to obtain L^p estimates on u and v is to obtain an upper bound for θ, which we now do. By the definitions of $\tau_\pm$ there exists a $T > 0$ such that $\tau_1 \leqslant (\tilde{f}(t)f''(t))/(f'(t)^2) \leqslant \tau_2$ for $t > T$ implies that

$$0 < 1 - \frac{\tau_2}{2} \leqslant 1 - \frac{\tilde{f}(t)f''(t)}{2f'(t)^2} \leqslant 1 - \frac{\tau_1}{2}, \quad \text{for } t > T. \tag{9}$$

Using (9), we get

$$\theta(t) \leqslant \theta(T) + \alpha^2 \left(1 - \frac{\tau_1}{2}\right)^2 \int_T^t \tilde{f}(s)^{2\alpha-2} f'(s)^{2-\alpha} \mathrm{d}s, \quad \text{for } t > T. \tag{10}$$

Take $h(t) := \tilde{f}(t)^{2\alpha-1} f'(t)^{1-\alpha}$. Then we have

$$\begin{aligned} h'(t) &= (2\alpha-1)\tilde{f}(t)^{2\alpha-2} f'(t)^{2-\alpha} \left(1 - \frac{\alpha-1}{2\alpha-1} \frac{\tilde{f}(s)f''(s)}{f'(s)^2}\right) \\ &\geqslant (2\alpha-1)\left(1 - \frac{\alpha-1}{2\alpha-1}\tau_2\right) \tilde{f}(t)^{2\alpha-2} f'(t)^{2-\alpha}, \quad \text{for } t > T, \end{aligned}$$

and integrating from T to t yields,

$$h(t) - h(T) \geqslant (2\alpha-1)\left(1 - \frac{\alpha-1}{2\alpha-1}\tau_2\right) \int_T^t \tilde{f}(s)^{2\alpha-2} f'(s)^{2-\alpha} \mathrm{d}s, \quad \text{for } t > T.$$

Now using the above inequality in (10), we obtain

$$\theta(t) \leqslant C + A\tilde{f}(t)^{2\alpha-1} f'(t)^{1-\alpha}, \quad \text{where } A := \frac{\alpha^2}{(2\alpha-1)} \frac{(1-\tau_1/2)^2}{(1-\alpha-1/2\alpha-1\tau_2)}$$
$$\text{and} \quad C := \theta(T) - Ah(T). \tag{11}$$

Note that in the above we also used that $1-\alpha-1/2\alpha-1\tau_2 > 0$ which holds since $\tau_2 < 2$. Now, the fact that the inequality $\tilde{f}(t)f''(t)/f'(t)^2 \leqslant \tau_2$ for $t > T$ is equivalent to $d/dt(f'(t)/\tilde{f}(t)^{\tau_2}) \leqslant 0$ for $t > T$, gives

$$f'(t) \leqslant C_1 \tilde{f}(t)^{\tau_2} \quad \text{for } t > T. \tag{12}$$

From (12) we obtain, for $t > T$

$$\tilde{f}(t)^{2\alpha-1} f'(t)^{1-\alpha} \geqslant f'(t)^{(2\alpha-1)/(\tau_2)-(\alpha-1)} \to \infty, \quad \text{as } t \to \infty.$$

Now take an $\epsilon > 0$. From the inequality above and (11), there exists $T_\epsilon > T$ such that

$$\theta(t) \leqslant (A+\epsilon)\tilde{f}(t)^{2\alpha-1} f'(t)^{1-\alpha}, \quad \text{for } t > T_\epsilon. \tag{13}$$

Also, we can find $T'_\epsilon > 0$ such that

$$f(t) \leqslant (1+\epsilon)\tilde{f}(t), \quad \text{for } t > T'_\epsilon. \tag{14}$$

Without the loss of generality, we assume $\lambda \geqslant \gamma$, then from lemma 2, we get $v \leqslant u \leqslant \lambda/\gamma v$. Using this and taking $T''_\epsilon := \max\{T_\epsilon, T'_\epsilon\}$ then from (13) and (14), we obtain

$$\int_\Omega \theta(u) f(v) = \int_{v \leqslant T''_\epsilon} \theta(u) f(v) + \int_{v > T''_\epsilon} \theta(u) f(v)$$

$$\leqslant C_\varepsilon + (A+\epsilon)(1+\epsilon) \int_{v > T''_\epsilon} \tilde{f}(u)^{2\alpha-1} f'(u)^{1-\alpha} \tilde{f}(v),$$

where

$$C_\varepsilon = C_\varepsilon(\lambda, \gamma) := \theta\left(\frac{\lambda}{\gamma} T''_\epsilon\right) f(T''_\epsilon)|\Omega|.$$

Plugging the above inequality in (8), we obtain

$$\sqrt{\lambda\gamma} \int_\Omega f'(u)^{1/2-\alpha} f'(v)^{1/2} \tilde{f}(u)^{2\alpha}$$
$$\leqslant \lambda(C_\epsilon + (A+\epsilon)(1+\epsilon) \int_{v \geqslant T''_\epsilon} \tilde{f}(u)^{2\alpha-1} f'(u)^{1-\alpha} \tilde{f}(v)), \tag{15}$$

Letting

$$I = I(u, v) := \int_\Omega f'(u)^{1/2-\alpha} f'(v)^{1/2} \tilde{f}(u)^{2\alpha},$$

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and replacing the integral on the right-hand side of inequality (15) with integral over the full region Ω, we get

$$\sqrt{\frac{\gamma}{\lambda}} I \leqslant C_\epsilon + (A+\epsilon)(1+\epsilon) \int_\Omega \tilde{f}(u)^{2\alpha-1} f'(u)^{1-\alpha} \tilde{f}(v), \tag{16}$$

By symmetry, taking

$$J = J(u,v) := \int_\Omega f'(v)^{1/2-\alpha} f'(u)^{1/2} \tilde{f}(v)^{2\alpha},$$

we then also get

$$\sqrt{\frac{\lambda}{\gamma}} J \leqslant C'_\epsilon + (A+\epsilon)(1+\epsilon) \int_\Omega \tilde{f}(v)^{2\alpha-1} f'(v)^{1-\alpha} \tilde{f}(u), \tag{17}$$

where the constant

$$C'_\epsilon = C'_\epsilon(\lambda,\gamma) := f\left(\frac{\lambda}{\gamma} T''_\epsilon\right) \theta(T''_\epsilon)|\Omega|$$

is independent of u, v.

Now we write

$$\tilde{f}(u)^{2\alpha-1} f'(u)^{1-\alpha} \tilde{f}(v) = \left(f'(u)^{1/2-\alpha} f'(v)^{1/2} \tilde{f}(u)^{2\alpha}\right)^{(2\alpha-1)/(2\alpha)} \times \left(f'(v)^{1/2-\alpha} f'(u)^{1/2} \tilde{f}(v)^{2\alpha}\right)^{1/2\alpha}.$$

Then, by the Hölder inequality, we obtain

$$\int_\Omega \tilde{f}(u)^{2\alpha-1} f'(u)^{1-\alpha} \tilde{f}(v) \leqslant I^{(2\alpha-1)/(2\alpha)} J^{1/2\alpha},$$

and using this in (16), we get

$$\sqrt{\frac{\gamma}{\lambda}} I \leqslant C_\epsilon + (A+\epsilon)(1+\epsilon) I^{(2\alpha-1)/(2\alpha)} J^{1/2\alpha}. \tag{18}$$

Similarly, from (17), we obtain

$$\sqrt{\frac{\lambda}{\gamma}} J \leqslant C'_\epsilon + (A+\epsilon)(1+\epsilon) J^{(2\alpha-1)/(2\alpha)} I^{1/2\alpha}. \tag{19}$$

Multiplying inequalities (18) and (19), we get

$$(1-(A+\epsilon)^2(1+\epsilon)^2) IJ \leqslant C''_\epsilon \left(1 + I^{(2\alpha-1)/(2\alpha)} J^{1/2\alpha} + J^{(2\alpha-1)/(2\alpha)} I^{1/2\alpha}\right), \tag{20}$$

where

$$C''_\varepsilon = C''_\varepsilon(\lambda,\gamma) := \max\{C_\varepsilon C'_\varepsilon, C_\varepsilon(A+\epsilon)(1+\epsilon), C'_\varepsilon(A+\epsilon)(1+\epsilon)\}$$

is independent of u, v.

Now let $(\lambda^*, \gamma^*) \in \Upsilon$, $\sigma = \lambda^*/\gamma^*$ and suppose that (u,v) denotes a minimal solution of $(P)_{\lambda,\gamma}$ on the ray $\Gamma_\sigma := \{(\lambda, \sigma\lambda);\ \lambda^*/2 < \lambda < \lambda^*$. Then by the definition of

$C''_{\varepsilon}(\lambda,\gamma)$, we see that it is uniformly bounded on the ray Γ_σ independent of λ. Then from (20), we deduce that if both of I and J are unbounded then we must have $1-(A+\epsilon)^2(1+\epsilon)^2 \leqslant 0$, and since $\epsilon>0$ was arbitrary, we get $A \geqslant 1$. But $A \geqslant 1$ is exactly equivalent to $P_f(\alpha,\tau_1,\tau_2) \geqslant 0$, which contradicts (7). Hence, either I or J must be bounded with a bound independent of λ, implies that

$$f'(u)^{1/2-\alpha}\tilde{f}(u)^{2\alpha}f'(v)^{1/2} \in L^1(\Omega) \quad \text{or} \quad f'(v)^{1/2-\alpha}\tilde{f}(v)^{2\alpha}f'(u)^{1/2} \in L^1(\Omega), \tag{21}$$

with a uniform bound in $L^1(\Omega)$ independent of λ.

Now note that by our choice of α and the assumption that $\tau_+ < 2$, the function $y(t) := f'(t)^{1/2-\alpha}\tilde{f}(t)^{2\alpha}$ is an increasing function for t large. Indeed, we have

$$\begin{aligned} y'(t) &= \left(\alpha-\frac{1}{2}\right)\tilde{f}(t)^{2\alpha-1}f'(t)^{3/2-\alpha}\left(\frac{4\alpha}{2\alpha-1}-\frac{f(t)f''(t)}{f'(t)^2}\right) \\ &\geqslant \left(\alpha-\frac{1}{2}\right)\tilde{f}(t)^{2\alpha-1}f'(t)^{3/2-\alpha}\,(2-\tau_+) > 0, \end{aligned}$$

for t sufficiently large. Hence, from (21) and our assumption that $u \geqslant v$, we get

$$f'(v)^{1-\alpha}\tilde{f}(v)^{2\alpha} \in L^1(\Omega), \tag{22}$$

with a uniform bound in $L^1(\Omega)$ independent of λ.

Now note that from the inequality (12) and $\alpha > 1$, we also get

$$f'(t)^{1-\alpha}\tilde{f}(t)^{2\alpha} \geqslant \tilde{f}(t)^{(2-\tau_2)\alpha+\tau_2}, \quad t > T,$$

and

$$f'(t)^{1-\alpha}\tilde{f}(t)^{2\alpha} \geqslant f'(t)^{((2-\tau_2)\alpha+\tau_2)/(\tau_2)}, \quad t > T.$$

Hence, from (22) together with the above two inequalities, we deduce that $\tilde{f}(v)^{(2-\tau_2)\alpha+\tau_2} \in L^1(\Omega)$ and also $f'(v)^{((2-\tau_2)\alpha+\tau_2)/(\tau_2)} \in L^1(\Omega)$, again with a uniform bound in $L^1(\Omega)$ independent of λ. Now with the help of lemma 3 and the standard elliptic regularity, we get $u^*, v^* \in L^\infty(\Omega)$ for

$$N < \max\left\{2\alpha(2-\tau_2)+2\tau_2, \frac{2\alpha(2-\tau_2)+2\tau_2}{\tau_2}\right\} = \frac{2\alpha(2-\tau_2)+2\tau_2}{\tau_2}\max\{1,\tau_2\}. \tag{23}$$

And since we can choose τ_2 arbitrary close to τ_+ and α near to the largest root of the polynomial P_f, then (23) completes the proof of the first part.

To see the second part, first note that we always have (since $\alpha_* > 1$)

$$N(f) > 2\alpha_*(2-\tau_+) + 2\tau_+ > 2(2-\tau_+) + 2\tau_+ = 4.$$

Also, if $\tau_- = \tau_+ := \tau$ then

$$\alpha^* = \frac{2+2\sqrt{\tau}}{2-\tau}.$$

Hence, $N(f) = 2 + 4(1+\sqrt{\tau})/(\tau) \geqslant 2 + 8(\sqrt[4]{\tau})/(\tau)$. Thus, using the fact that $\tau \leqslant 1$ (since we always have $\tau_- \leqslant 1$ by the assumptions on f), we get $N(f) \geqslant 10$, and the proof is complete. □

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