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Absence of singularities in solutions for the compressible Euler equations with source terms in $\mathbb{R}^d$

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The present article is devoted to the study of global solution and large time behaviour of solution for the isentropic compressible Euler system with source terms in $\mathbb{R}^d$, $d \geqslant 1$, which extends and improves the results obtained by Sideris *et al.* in 'T.C. Sideris, B. Thomases, D.H. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, Comm. Partial Differential Equations 28 (2003) 795–816'. We first establish the existence and uniqueness of global smooth solution provided the initial datum is sufficiently small, which tells us that the damping terms can prevent the development of singularity in small amplitude. Next, under the additional smallness assumption, the large time behaviour of solution is investigated, we only obtain the algebra decay of solution besides the L^2-norm of ∇u is exponential decay.

Keywords: The isentropic compressible Euler equations; the global smooth solution; the large time behaviour of solution

2020 *Mathematics subject classification* Primary: 35Q31
Secondary: 35Q35

1. Introduction

In this article, we investigate the initial value problem (IVP) for the d-dimensional isentropic compressible Euler system with source term

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = -\nabla p - \mu \rho u, & \\ \rho(0, x) = \rho_0(x), u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

which governs the motion of a compressible inviscid fluid through a porous medium and describes the compressible gas flow passes a porous medium and the medium induces a friction force, where the functions $\rho(t, x)$, $u(t, x) = (u_1, u_2, \cdots, u_d)$ and $p(t, x)$ denote the density, velocity vector fluid and pressure, respectively. The symbol $u \otimes u$ denotes a matrix whose ij^{th} entry is $u_i u_j$, the constant $\mu > 0$. The first equation in system (1.1) is just the usual conservation of mass. The second equation in system (1.1) represents the Newton's law (or momentum conservation): the LHS denotes the acceleration of the fluid in Eulerian frame, whereas the RHS describes the force (where ρu denotes external forcing field).

In order to simplify the system (1.1), substitute $(1.1)_1$ into $(1.1)_2$ and $(1.1)_2$ into $(1.1)_3$, one has that

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla p + \mu \rho u = 0, & \\ \rho(0, x) = \rho_0(x), u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $u \cdot \nabla u = \sum_{i=1}^d u_i \partial_{x_i} u$, $\mathrm{div} u = \sum_{i=1}^d \partial_{x_i} u_i$.

The system (1.1) or (1.2) has (d+1) equations, (d+2) unknowns (ρ, $u_1, \cdots, u_d$, p) and thus is not formally self-consistent, however, when we introduce the equality of pressure and density, which is given by

$$p(t, x) = p(\rho) = A\rho^\gamma$$

with the adiabatic exponent $\gamma > 1$ and constant $A > 0$. Now the (d+2) unknowns of system (1.1) become (d+1) unknowns (ρ, $u_1, \cdots, u_d$), thus system (1.1) or (1.2) is formally self-consistent.

As is well known, the formation of singularities is a fundamental physical phenomenon manifested in solutions [**17**, **19**, **22**, **27**, **34**] for the compressible Euler equations (i.e., system (1.1) with $\mu = 0$), which are a prototypical example of hyperbolic systems of conservation laws. This phenomenon can be explained by mathematical analysis by showing the finite time formation of singularities in the solutions. Therefore, the blow-up phenomena for the multi-dimensional compressible Euler flows has attracted lots of interests and attentions because of its physical importance. However, it is a difficult problem to understand the blow-up behaviour of the general solutions of the compressible Euler equations. The earliest work of system (1.1) with $\mu = 0$ began with Taylor [**30**, **31**] finding the wave motion produced by an expanding sphere and preceded by a shock front. It is similar to the one-dimensional gas flow produced by a piston with constant speed. The progressing waves were also succeeded in finding some other types of spherical waves like detonation, deflagration, combustion and reflected shocks. In [**2**, **4**, **23**, **24**, **32**], the authors studied the global weak solution of the isentropic compressible Euler equations with spherical symmetry. Recently, Li and Wang [**18**] derived some special global and blow-up solutions of system (1.1) as $\mu = 0$ with spherical symmetry. Some other results of system (1.1) with $\mu = 0$ can be found in [**3**, **5**, **8**, **20**, **21**, **29**], as well as the references cited therein.

To our best knowledge, there are many mathematical works about system (1.1). For one dimensional case, assuming that the initial data are smooth enough and that the derivatives of the initial data are sufficiently small, the global existence and the large time behaviour of the smooth solutions of system (1.1) were studied by Hsiao, Liu and Luo in [**10**, **11**]. If this assumption is violated, the solutions eventually will develop singularities in general, hence it is necessary to consider the weak solutions. If the initial data belong to L^∞ and satisfy some conditions, then the equation admits a global entropy weak solution [**7**, **14**, **25**] and the solution converges to Barenblatt's profiles of the porous medium equation [**13**–**16**]. In [**6**, **12**], the global existence of BV solutions for the Cauchy problem of system (1.1) was investigated by using a fractional step version of the Glimm's scheme. For multi-dimensional case, if the initial data are sufficiently small, by analysing the Green function of the linearized system, Wang and Yang [**33**] obtained the global existence and pointwise estimates of the solutions by the energy estimates. When the initial data are near its equilibrium, Pan and Zhao [**26**] showed global existence and uniqueness of classical solutions to the initial boundary value problem for the 3D damped compressible Euler equations on bounded domain with slip boundary condition and showed that the classical solutions converge to steady state exponentially fast in time. In 2003, for 3 dimensional case, Sideris, Thomases and Wang [**28**] showed that if the initial data are sufficiently small in an appropriate norm, then damping term can prevent the development of singularities and the Cauchy problem of system (1.1) has a unique global smooth solution $u(t,\, x) \in \mathcal{C}(\mathbb{R}^+; H^3)$. Moreover, as the time t becomes large, they studied the long time behaviour of solutions to obtain the following algebra decay of solution $U(t,\, x)$ and exponential decay of vorticity $\omega(t,\, x)$

$$
\begin{aligned}
&\|U(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{3}{2}}, \quad \|U(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}},\\
&\|\nabla U(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{5}{4}}, \quad \|\omega(t,\cdot)\|_{L^2} \leqslant Ce^{-Ct}.
\end{aligned}
$$

Motivated by the article [**28**], in this paper, we investigate global existence and the large time behaviour of solution of the Cauchy problem for system (1.1) in $\mathbb{R}^d$. This tells us that if the initial data are sufficiently small in an appropriate norm, then source term can prevent the development of singularities. Compared with the results in [**28**], we consider the d-dimensional Euler equations with damping terms and obtain the unique global smooth solution $u(t,\, x) \in \mathcal{C}(\mathbb{R}^+; H^s) \cap \mathcal{C}^1(\mathbb{R}^+; H^{s-1}),\ \ s > 1 + \frac{d}{2}$. By a detailed analysis of the semigroup $S(t)$ of the linearized system, we show an important lemma 4.1. As the time t becomes large, we also have the following algebra decay of solution $v(t,\, x)$ and exponential decay of vorticity $\Omega(t,\, x)$

$$
\begin{aligned}
&\|v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d}{2}}, \quad \|v(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{d}{4}},\\
&\|\nabla v(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{d+2}{4}}, \quad \|\Omega(t,\cdot)\|_{L^2} \leqslant Ce^{-Ct},
\end{aligned}
$$

which is the same as the results in 3D derived in [**28**]. By introducing a new weight function $J^h_\infty(t)$ (see page 16 in § 4) and the Gagliardo–Nirenberg inequality, one has that the estimate of high order derivative about solutions

$$
\begin{aligned}
&\|\nabla v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d+1}{2}},\\
&\left\|\nabla^k v(t,\cdot)\right\|_{L^2} \leqslant C(1+t)^{-\frac{d+2k}{4}},
\end{aligned}
$$

where d denotes the dimension of space, $0 \leqslant k \leqslant 1 + \frac{d}{2}$, which is optimal in the linearized sense. These extend and improve the result obtained by Sideris *et al.* in [**28**].

The rest of the paper is organized as follows. In § 2, we rewrite the system (1.1) into a quasilinear symmetric system and state the local well-posedness which will be used in this article. In § 3, by virtue of a priori estimates, we establish the global smooth solution of the IVP for system (2.1) with small initial data. Finally, in § 4, we investigate the large time behaviour of solution to the isentropic compressible Euler system with source terms in $\mathbb{R}^d$, one only obtain the algebra decay of solution, besides the L^2-norm of ∇u is exponential decay.

2. Preliminaries

In this subsection, for the convenience of the readers, we first introduce some notations. Let $\|\cdot\|_X$ denote the norm of the Banach space X, such as, $\|\cdot\|_{H^s}$ and $(\cdot,\cdot)_s$ denote the norm and the inner product of $H^s(\mathbb{R}^d)$, $s\in\mathbb{R}$, respectively, where $L^r,\ H^s$ denotes $L^r(\mathbb{R}^d),\ H^s(\mathbb{R}^d)$ spaces, $r\geqslant 1$, $s\in\mathbb{R}$. Throughout the article, we will let c or C be a generic constant, which may assume different values in different formulas.

In order to achieve the aim, introduce the function

$$\pi = C_1 p^{\frac{\gamma-1}{2\gamma}} = \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}.$$

If the Cauchy problem of system (1.2) with the solution $(\rho,\, u)$ satisfies

$$\lim_{|x|\to\infty}\rho(t,x) = \tilde{\rho}(\text{constant}) > 0, \quad \lim_{|x|\to\infty} u(t,x) = 0,$$

let $\omega = \pi - \tilde{\pi}$, where $\tilde{\pi} = C_1\tilde{p}^{(\gamma-1)/2\gamma}$, $\tilde{p} = A\tilde{\rho}^{\gamma}$ and $\bar{\pi} = \tilde{\pi}\frac{\gamma-1}{2}$, then system (1.2) is equivalent to the following quasilinear symmetric system

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega + \bar{\pi}\mathrm{div}u + \dfrac{\gamma-1}{2}\omega\mathrm{div}u = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ \partial_t u + u\cdot\nabla u + \bar{\pi}\nabla\omega + \dfrac{\gamma-1}{2}\omega\nabla\omega + \mu u = 0, & \\ (\omega,u)|_{t=0} = (\omega_0(x), u_0(x)), & x\in\mathbb{R}^d. \end{cases} \tag{2.1}$$

Let $v = (\omega,\, u_1,\, \cdots,\, u_d)^{\top}$, $\top$ denotes transposition of matrix, the well-posedness of system (2.1) in Sobolev space is corollary of theorem 3.2 and 3.7 in [**9**].

Theorem 2.1. *For any initial data $v_0 = (\omega_0,\, u_0) \in H^s(\mathbb{R}^d)$, $s > 1 + \frac{d}{2}$, there exists a time $T > 0$ such that the initial value problem (2.1) has a unique solution $v = (\omega,\, u)$, which belongs to*

$$\mathcal{C}([0,T[;H^s(\mathbb{R}^d))\cap\mathcal{C}^1([0,T[;H^{s-1}(\mathbb{R}^d)).$$

Moreover, if the $v_0\in H^s(\mathbb{R}^d)$, then the solution map

$$\Phi:\ v_0\mapsto v:\ H^s(\mathbb{R}^d)\mapsto\mathcal{C}([0,T[;H^s(\mathbb{R}^d))\cap\mathcal{C}^1([0,T[;H^{s-1}(\mathbb{R}^d))$$

is continuous in the sense of Hadmard, and we have the following inequality

$$\|(v^n - v^\infty)(t)\|_{H^s} \leqslant C(\|v^n\|_{H^s}, \|v^\infty\|_{H^s})\|v_0^n - v_0^\infty\|_{H^s},$$

where sequence $\{v^n\}_{n\in N}$ is approximation solutions to system (2.1). In particular, let T_{v_0} be the lifespan of the solution v to system (2.1) with initial datum v_0, the *lifespan T_{v_0} satisfies*

$$T_{v_0} \geqslant \frac{1}{C\|v_0\|_{H^s}}.$$

If $T_{v_0} < \infty$, then for all $t \leqslant T_{v_0}$ we have

$$\log e\|v(t)\|_{H^s} \leqslant \log e\|v_0\|_{H^s} \exp(c\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau). \tag{2.2}$$

REMARK 2.2. The smoothness of solution $(\rho,\ u)$ of system (1.1) is equivalent to the smoothness of solution $(\omega,\ u)$ of system (2.1) by the definition of w. The positivity of the density ρ is guaranteed by the positivity of the initial density ρ_0, in fact, by the first equation in system (1.1), we have

$$\begin{aligned}\frac{d}{dt}\rho(t,\varphi(t,x)) &= \rho_t(t,\varphi(t,x)) + (u\cdot\nabla\rho)(t,\varphi(t,x))\\ &= -(\rho divu)(t,\varphi(t,x)),\end{aligned} \tag{2.3}$$

where we have applied the ordinary differential equation of the flow

$$\begin{cases}\dfrac{d}{dt}\varphi(t,x) = u(t,\varphi) & t>0,\ x\in\mathbb{R}^d,\\ \varphi(0,x) = x, & x\in\mathbb{R}^d.\end{cases}$$

For all time $t \in [0,\ T]$, by solving the equation (2.3) yields that

$$\rho(t,\varphi) = \rho_0(x)\exp(\int_0^t -\mathrm{div}u(\tau,\varphi(\tau,x))d\tau) > 0. \tag{2.4}$$

In view of $\|u(t,\ \varphi)\|_{L^\infty} = \|u(t,\ x)\|_{L^\infty}$ one has that

$$\rho(t,x) > 0,$$

for any $(t,\ x) \in \mathbb{R}^+ \times \mathbb{R}^d$, provided the initial density $\rho_0(x) > 0$.

3. The global existence of solution with small initial data

In this subsection, by showing a priori estimates of the IVP of system (2.1) by some lemmas, we shall establish the global smooth solution of system (2.1) with small initial data.

In order to distinguish time and space derivatives, let $\nabla = (\partial_{x_1},\ \cdots,\ \partial_{x_d})$ denote the space derivatives, $\partial = (\partial_t,\ \nabla)$ is all first time and space derivatives. For $s >$

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$1+\frac{d}{2}$, introduce the energy functions

$$\begin{aligned} Q(v)(t) &=: \sum_{|l|\leqslant s-1} \|\partial\nabla^l v(t,\cdot)\|_{L^2}^2 \\ &= \sum_{|l|\leqslant s-1} \|\partial_t\nabla^l v(t,\cdot)\|_{L^2}^2 + \sum_{0<|\delta|\leqslant s} \|\nabla^\delta v(t,\cdot)\|_{L^2}^2, \end{aligned} \tag{3.1}$$

$$\begin{aligned} I(v)(t) &=: Q(v)(t) + \|v\|_{L^2}^2 \\ &= \sum_{|l|\leqslant s-1} \|\partial_t\nabla^l v(t,\cdot)\|_{L^2}^2 + \sum_{|\delta|\leqslant s} \|\nabla^\delta v(t,\cdot)\|_{L^2}^2 \\ &= \|\partial_t v(t,\cdot)\|_{H^{s-1}}^2 + \|v(t,\cdot)\|_{H^s}^2, \end{aligned} \tag{3.2}$$

where for all $\sigma\in\mathbb{R}$, by the Fourier transformation yields that

$$\|v(t,\cdot)\|_{H^\sigma}^2 = \int_{\mathbb{R}^d} (1+|\xi|^2)^\sigma |\widehat{v}(t,\xi)|^2 d\xi.$$

Next, we state the following lemma of energy estimates.

LEMMA 3.1. *Assume the function $(\omega,\,u)\in\mathcal{C}([0,\,T[,\,H^s(\mathbb{R}^d))$ be a solution of system (2.1), for some $T>0$. Then the following energy inequalities hold:*

$$\frac{\partial}{\partial t}(\|\omega\|_{L^2}^2 + \|u\|_{L^2}^2) + 2\mu\|u\|_{L^2}^2 \leqslant C\|(\omega,u)\|_{L^\infty}\|u\|_{L^2}\|(\nabla\omega,\nabla u)\|_{L^2}, \tag{3.3}$$

$$\frac{\partial}{\partial t}(Q(\omega)(t)+Q(u)(t)) + 2\mu Q(u)(t) \leqslant C\|(\partial\omega,\partial u)\|_{L^\infty}(Q(\omega)(t)+Q(u)(t)). \tag{3.4}$$

Proof. Taking inner product of system $(2.1)_1$ with ω, and system $(2.1)_2$ with u, after integration by parts, adding them together, it yields that

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}(\|\omega\|_{L^2}^2 + \|u\|_{L^2}^2) + \mu\|u\|_{L^2}^2 &= -\int_{\mathbb{R}^d} (u\cdot\nabla\omega)\omega + (u\cdot\nabla u)u dx \\ &\quad - \frac{\gamma-1}{2}\int_{\mathbb{R}^d} \omega^2 \mathrm{div} u + (\omega\nabla\omega)u dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^d} u\cdot\nabla(\omega^2+u^2)dx + \frac{\gamma-1}{2}\int_{\mathbb{R}^d}(\omega\nabla\omega)u dx \\ &\leqslant C\|(\omega,u)\|_{L^\infty}\|u\|_{L^2}\|(\nabla\omega,\nabla u)\|_{L^2}, \end{aligned} \tag{3.5}$$

where we have used

$$\int_{\mathbb{R}^d} (\bar{\pi}\mathrm{div} u)\omega dx + \int_{\mathbb{R}^d} (\bar{\pi}\nabla\omega)u dx = 0,$$

in the first equality, the Hölder inequality in the last inequality, and $\|(\omega,\,u)\|_X = \|\omega\|_X + \|u\|_X$.

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Next, we will show inequality (3.4). Applying the operator $\partial\nabla^\delta$ on both sides of system (2.1), it follows that

$$\partial_t\partial\nabla^\delta\omega+\partial\nabla^\delta(\bar{\pi}\mathrm{div}u+u\cdot\nabla\omega)=-\frac{\gamma-1}{2}\partial\nabla^\delta(\omega\mathrm{div}u), \tag{3.6}$$

$$\partial_t\partial\nabla^\delta u+\partial\nabla^\delta(u\cdot\nabla u+\bar{\pi}\nabla\omega+\mu u)=-\frac{\gamma-1}{2}\partial\nabla^\delta(\omega\nabla\omega). \tag{3.7}$$

Multiplying equation (3.6) and equation (3.7) by $\partial\nabla^\delta\omega$ and $\partial\nabla^\delta u$ respectively, $|\delta|\leqslant s-1$. Adding them together and integration by parts, one has that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\sum_{|\delta|\leqslant s-1}(\|\partial\nabla^\delta\omega\|_{L^2}^2+\|\partial\nabla^\delta u\|_{L^2}^2)+\mu\sum_{|\delta|\leqslant s-1}\|\partial\nabla^\delta u\|_{L^2}^2 \\
&\quad=-\sum_{|\delta|\leqslant s-1}\int_{\mathbb{R}^d}\partial\nabla^\delta(u\cdot\nabla\omega+\frac{\gamma-1}{2}\omega\mathrm{div}u)\partial\nabla^\delta\omega dx \\
&\qquad-\sum_{|\delta|\leqslant s-1}\int_{\mathbb{R}^d}\partial\nabla^\delta(u\cdot\nabla u+\frac{\gamma-1}{2}\omega\nabla\omega)\partial\nabla^\delta u dx \\
&\quad=:-\sum_{|\delta|\leqslant s-1}(I_1+I_2+I_3+I_4).
\end{aligned} \tag{3.8}$$

where the first equality is guaranteed by

$$\int_{\mathbb{R}^d}\partial\nabla^\delta(\bar{\pi}\mathrm{div}u)\partial\nabla^\delta\omega dx+\int_{\mathbb{R}^d}\partial\nabla^\delta(\bar{\pi}\nabla\omega)\partial\nabla^\delta u dx=0.$$

For $|\delta|\geqslant 0$, we first deal with the term I_1 as follows

$$\begin{aligned}
|I_1|&=\int_{\mathbb{R}^d}\partial\nabla^\delta(u\cdot\nabla\omega)\partial\nabla^\delta\omega dx \\
&=\int_{\mathbb{R}^d}\left(\sum_{\alpha+\beta=\delta}\partial(\nabla^\alpha u\cdot\nabla^{1+\beta}\omega)\right)\partial\nabla^\delta\omega dx \\
&=\int_{\mathbb{R}^d}\partial(\nabla^\delta u\cdot\nabla\omega+u\cdot\nabla^\delta\nabla\omega)\partial\nabla^\delta\omega dx \\
&\quad+\int_{\mathbb{R}^d}\left(\sum_{\alpha,\beta\leqslant\delta-1,\alpha+\beta=\delta}(\partial\nabla^\alpha u\cdot\nabla^{1+\beta}\omega+\nabla^\alpha u\cdot\partial\nabla^{1+\beta}\omega)\right)\partial\nabla^\delta\omega dx \\
&=:I_1^{(1)}+I_1^{(2)}.
\end{aligned} \tag{3.9}$$

When $\delta=0$, thanks to the Hölder inequality, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d}u\partial\nabla\omega\partial\omega dx\right|&=\frac{1}{2}\left|\int_{\mathbb{R}^d}\mathrm{div}u(\partial\omega)^2dx\right| \\
&\leqslant C\|\mathrm{div}u\|_{L^\infty}\|\partial\omega\|_{L^2}^2.
\end{aligned} \tag{3.10}$$

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As $\delta > 0$, by virtue of the Gagliardo–Nirenberg inequality yields that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \nabla^\delta u \partial \nabla \omega \partial \nabla^\delta \omega dx\right| &\leqslant \|\partial \nabla \omega\|_{L^p} \|\nabla^\delta u\|_{L^q} \|\partial \nabla^\delta \omega\|_{L^2} \\ &\leqslant \|\partial \omega\|_{L^\infty}^\theta \|\partial \nabla^\delta \omega\|_{L^2}^{1-\theta} \|\nabla u\|_{L^\infty}^{1-\theta} \|\partial \nabla^\delta u\|_{L^2}^\theta \|\partial \nabla^\delta \omega\|_{L^2} \\ &\leqslant C(\|\partial \omega\|_{L^\infty} \|\partial \nabla^\delta u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\partial \nabla^\delta \omega\|_{L^2}) \|\partial \nabla^\delta \omega\|_{L^2},\end{aligned} \tag{3.11}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ for $p, q \in [2, \infty]$, and $\theta \in (0, 1)$, we also have used Young's inequality in the last inequality. In view of (3.10), (3.11) and the Hölder inequality, $I_1^{(1)}$ can be dealt with as

$$I_1^{(1)} \leqslant C\|(\partial \omega, \partial u)\|_{L^\infty} \|(\partial \nabla^\delta \omega, \partial \nabla^\delta u)\|_{L^2} \|\partial \nabla^\delta \omega\|_{L^2}. \tag{3.12}$$

On the other hand, in order to deal with $I_1^{(2)}$, in view of the Gagliardo–Nirenberg inequality, for $\alpha, \beta \leqslant \delta - 1, \ \alpha + \beta = \delta$, one has that

$$\begin{aligned}&\int_{\mathbb{R}^d} \Big(\partial \nabla^\alpha u \cdot \nabla^{1+\beta} \omega\Big) \partial \nabla^\delta \omega dx \leqslant C\|\partial \nabla^\alpha u\|_{L^p} \|\nabla^{1+\beta} \omega\|_{L^q} \|\partial \nabla^\delta \omega\|_{L^2} \\ &\quad \leqslant C\|\partial u\|_{L^\infty}^\vartheta \|\partial \nabla^\delta u\|_{L^2}^{1-\vartheta} \|\nabla \omega\|_{L^\infty}^{1-\vartheta} \|\nabla^{\delta+1} \omega\|_{L^2}^\vartheta \|\partial \nabla^\delta \omega\|_{L^2} \\ &\quad \leqslant C(\|\partial u\|_{L^\infty} \|\nabla^{\delta+1} \omega\|_{L^2} + \|\nabla \omega\|_{L^\infty} \|\partial \nabla^\delta u\|_{L^2}) \|\partial \nabla^\delta \omega\|_{L^2},\end{aligned} \tag{3.13}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ for $p, q \in [2, \infty]$, and $\vartheta \in (0, 1)$, we also have used Young's inequality in the last inequality. Similarly, we can show that

$$\begin{aligned}\int_{\mathbb{R}^d} (\nabla^\alpha u \cdot \partial \nabla^{1+\beta} \omega) \partial \nabla^\delta \omega dx &\leqslant C\|\nabla^\alpha u\|_{L^p} \|\partial \nabla^{1+\beta} \omega\|_{L^q} \|\partial \nabla^\delta \omega\|_{L^2} \\ &\leqslant C(\|\nabla u\|_{L^\infty} \|\partial \nabla^\delta \omega\|_{L^2} \\ &\quad + \|\partial \omega\|_{L^\infty} \|\nabla^{\delta+1} u\|_{L^2}) \|\partial \nabla^\delta \omega\|_{L^2}.\end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), it yields that

$$I_1^{(2)} \leqslant C\|(\partial \omega, \partial u)\|_{L^\infty} \|(\partial \nabla^\delta \omega, \partial \nabla^\delta u)\|_{L^2} \|\partial \nabla^\delta \omega\|_{L^2}. \tag{3.15}$$

In view of (3.9), (3.12) and (3.15), it gives that

$$I_1 =: I_1^{(1)} + I_1^{(2)} \leqslant C\|(\partial \omega, \partial u)\|_{L^\infty} (\|\partial \nabla^\delta \omega\|_{L^2}^2 + \|\partial \nabla^\delta u\|_{L^2}^2). \tag{3.16}$$

Similar to the process of proving (3.16), we can estimate $I_2, \ I_3$ and I_4 to derive

$$I_2 + I_3 + I_4 \leqslant C\|(\partial \omega, \partial u)\|_{L^\infty} (\|\partial \nabla^\delta \omega\|_{L^2}^2 + \|\partial \nabla^\delta u\|_{L^2}^2). \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.8), by virtue of (3.1) and (3.2), thus we consequently obtain (3.4). □

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LEMMA 3.2. *Let $\bar{\pi} > 0$ and $s > 1 + \frac{d}{2}$. Assume the function $(\omega, u) \in \mathcal{C}([0, T[, H^s(\mathbb{R}^d))$ be a solution of system (2.1), for some $T > 0$. Then the following inequality holds*:

$$Q(\omega)(t) \leqslant CI(\omega)(t)Q(\omega)(t) + C(1 + I(\omega))I(u). \tag{3.18}$$

Moreover, if the solution (ω, u) satisfies $I(\omega)(t) \ll 1$ for any time $t > 0$, then $Q(\omega)(t)$ can be controlled by $I(u)(t)$ and we have

$$Q(\omega)(t) \leqslant CI(u)(t). \tag{3.19}$$

Proof. Multiplying the operator ∇^δ on both sides of system (2.1), one has that

$$\partial_t \nabla^\delta \omega = -\nabla^\delta(\bar{\pi}\mathrm{div}u + u \cdot \nabla\omega) - \frac{\gamma - 1}{2}\nabla^\delta(\omega \mathrm{div}u), \tag{3.20}$$

$$\nabla^{\delta+1}\omega = \frac{-1}{\bar{\pi}}[\partial_t \nabla^\delta u + \nabla^\delta(u \cdot \nabla u + \mu u)] - \frac{\gamma - 1}{\bar{\pi}}\nabla^\delta(\omega\nabla\omega). \tag{3.21}$$

Taking L^2 norm of the equation (3.20) and (3.21) for $|\delta| \leqslant s - 1$, adding them together yields that

$$\begin{aligned}\sum_{|\delta|\leqslant s-1} \|\partial\nabla^\delta\omega\|_{L^2} &\leqslant C \sum_{|\delta|\leqslant s-1} \|\nabla^\delta(\mathrm{div}u + u \cdot \nabla\omega) + \nabla^\delta(\omega\mathrm{div}u)\|_{L^2} \\ &\quad + C \sum_{|\delta|\leqslant s-1} \|\partial_t\nabla^\delta u + \nabla^\delta(u \cdot \nabla u + u) + \nabla^\delta(\omega\nabla\omega)\|_{L^2} \\ &\leqslant C \sum_{|\delta|\leqslant s-1} \|\partial\nabla^\delta u\|_{L^2} + \|\nabla^\delta(u \cdot \nabla\omega + \omega\mathrm{div}u + u \cdot \nabla u + \omega\nabla\omega)\|_{L^2}.\end{aligned} \tag{3.22}$$

Similar to the method of dealing with (3.9), one can easily check that

$$\begin{aligned}\|\nabla^\delta(u \cdot \nabla\omega)\|_{L^2} &\leqslant C(\|\nabla\omega\|_{L^\infty}\|\nabla^\delta u\|_{L^2} + \|u\|_{L^\infty}\|\nabla^{\delta+1}\omega\|_{L^2}) \\ &\leqslant C(I(\omega)I(u))^{\frac{1}{2}},\end{aligned} \tag{3.23}$$

$$\begin{aligned}\|\nabla^\delta(\omega\mathrm{div}u)\|_{L^2} &\leqslant C(\|\nabla u\|_{L^\infty}\|\nabla^\delta\omega\|_{L^2} + \|\omega\|_{L^\infty}\|\nabla^{\delta+1}u\|_{L^2}) \\ &\leqslant C(I(\omega)I(u))^{\frac{1}{2}},\end{aligned} \tag{3.24}$$

$$\begin{aligned}\|\nabla^\delta(u \cdot \nabla u)\|_{L^2} &\leqslant C(\|\nabla u\|_{L^\infty}\|\nabla^\delta u\|_{L^2} + \|u\|_{L^\infty}\|\nabla^{\delta+1}u\|_{L^2}) \\ &\leqslant C(I(u)Q(u))^{\frac{1}{2}},\end{aligned} \tag{3.25}$$

$$\begin{aligned}\|\nabla^\delta(\omega\nabla\omega)\|_{L^2} &\leqslant C(\|\nabla\omega\|_{L^\infty}\|\nabla^\delta\omega\|_{L^2} + \|\omega\|_{L^\infty}\|\nabla^{\delta+1}\omega\|_{L^2}) \\ &\leqslant C(I(\omega)Q(\omega))^{\frac{1}{2}},\end{aligned} \tag{3.26}$$

where we have used for $s > 1 + \frac{d}{2}$ that

$$\|f\|_{L^\infty} \leqslant C\|f\|_{H^{s-1}} \leqslant CI^{\frac{1}{2}}(f),$$

$$\|\nabla f\|_{L^\infty} \leqslant C\|\nabla f\|_{H^{s-1}} \leqslant CQ^{\frac{1}{2}}(f).$$

Combining (3.22)–(3.25) and (3.26), it yields that

$$Q(\omega)(t) \leqslant CI(\omega)(t)Q(\omega)(t) + C(1 + I(\omega))I(u).$$

If the solution ω satisfies $I(\omega)(t) \ll 1$ for any time $t > 0$, choosing $CI(\omega)(t) \leqslant \frac{1}{3}$ then we have

$$2Q(\omega)(t) \leqslant (3C + 1)I(u)(t),$$

which derives the inequality (3.19). □

Next, we will show the existence of global smooth solution for system (2.1).

THEOREM 3.3. *Assume the initial data* $(\omega_0, u_0) \in H^s(\mathbb{R}^d)$, $s > 1 + \frac{d}{2}$. *If the* (ω_0, u_0) *satisfies* $I(\omega_0, u_0) = \epsilon_0 \ll 1$, *then system* (2.1) *has a unique global solution* $(\omega, u) \in \mathcal{C}(\mathbb{R}^+, H^s(\mathbb{R}^d))$. *Moreover, there exists some* $\mu_0 > 0$, *for all* $t \in \mathbb{R}^+$, *we have the energy inequality*

$$I(\omega, u)(t) + \mu_0 \int_0^t I(u)(\tau)d\tau \leqslant I(\omega_0, u_0).$$

Proof. Combining (3.3) and (3.4) in lemma 3.1 yields that

$$\begin{aligned} \frac{\partial}{\partial t} I(\omega, u)(t) + 2\mu I(u)(t) \leqslant C(&\|(\omega, u)\|_{L^\infty}\|u\|_{L^2}\|\nabla(\omega, u)\|_{L^2} \\ &+ \|(\partial\omega, \partial u)\|_{L^\infty}(Q(\omega, u)(t))). \end{aligned} \tag{3.27}$$

Note that

$$\begin{aligned} \|(\omega, u)\|_{L^\infty}\|u\|_{L^2}\|\nabla(\omega, u)\|_{L^2} &\leqslant \|(\omega, u)\|_{H^s}\|u\|_{H^s}\|\nabla(\omega, u)\|_{H^{s-1}} \\ &\leqslant I^{\frac{1}{2}}(\omega, u)I^{\frac{1}{2}}(u)Q^{\frac{1}{2}}(\omega, u), \end{aligned} \tag{3.28}$$

$$\begin{aligned} \|(\partial\omega, \partial u)\|_{L^\infty}Q(\omega, u)(t) &\leqslant \|\partial(\omega, u)\|_{H^{s-1}}Q(\omega, u)(t) \\ &\leqslant I^{\frac{1}{2}}(\omega, u)(t)Q(\omega, u)(t), \end{aligned} \tag{3.29}$$

where we have used $s > 1 + \frac{d}{2}$, which guarantees

$$\|f\|_{L^\infty} < C\|f\|_{H^{s-1}}, \qquad \|\nabla f\|_{L^\infty} < C\|f\|_{H^s}.$$

Inserting (3.28) and (3.29) into (3.27), one has that

$$\begin{aligned} \frac{\partial}{\partial t} I(\omega, u)(t) + 2\mu I(u)(t) &\leqslant CI^{\frac{1}{2}}(\omega, u)[Q(\omega) + I(u)] + CI^{\frac{1}{2}}(\omega, u)Q(\omega, u) \\ &\leqslant CI^{\frac{1}{2}}(\omega, u)[Q(\omega) + I(u)] + CI^{\frac{1}{2}}(\omega, u)[Q(\omega) + I(u)] \\ &\leqslant CI^{\frac{1}{2}}(\omega, u)[Q(\omega) + I(u)]. \end{aligned} \tag{3.30}$$

Suppose the solution (ω, u) satisfies $I(\omega, u)(t) = \epsilon_0 \ll 1$ for any $t \in \mathbb{R}^+$, choosing $CI(\omega, u)(t) \leqslant \frac{1}{3}$, then it follows from lemma 3.2 that the solution satisfies (3.19),

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in view of (3.30) we have

$$\frac{\partial}{\partial t}I(\omega,u)(t)+2\mu I(u)(t)\leqslant CI^{\frac{1}{2}}(\omega,u)[Q(\omega)+I(u)]$$
$$\leqslant \frac{C}{2}(\epsilon_0)^{\frac{1}{2}}(3C+2)I(u)(t).$$

Since ϵ_0 is sufficiently small, we can choose ϵ_0 such that

$$\mu_0 =: 2\mu-\frac{C}{2}(\epsilon_0)^{\frac{1}{2}}(3C+2)>0.$$

Consequently, we have

$$\frac{\partial}{\partial t}I(\omega,u)(t)+\mu_0 I(u)(t)\leqslant 0. \tag{3.31}$$

Integrating the inequality (3.31) with respect to the time variable on interval $[0,\, t]$, it follows that

$$I(\omega,u)(t)+\mu_0\int_0^t I(u)(\tau)d\tau\leqslant I(\omega_0,u_0). \tag{3.32}$$

Thus if the initial data $I(\omega_0,\, u_0)=\epsilon_0$ is small enough, then the inequality (3.32) guarantees for all $t>0$ that

$$I(\omega,u)(t)\leqslant\epsilon_0\ll 1,$$

which completes the proof of theorem 3.3. □

4. The decay rates of solutions in large time

Base on the global existence of solutions of system (2.1) in § 3, in this subsection, as the time is sufficiently large, we shall derive the decay rates of the solution. In order to obtain the decay estimates, we first study the following linear system

$$\begin{cases}\partial_t\omega+\bar{\pi}\mathrm{div}u=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ \partial_t u+\bar{\pi}\nabla\omega+\mu u=0, & \\ (\omega,u)|_{t=0}=(\omega_0,u_0), & x\in\mathbb{R}^d.\end{cases} \tag{4.1}$$

Lemma 4.1. *Let the initial data $v_0(x)=(\omega_0,\, u_0)^\top\in L^1(\mathbb{R}^d)\cap H^s(\mathbb{R}^d)$, $s>1+\frac{d}{2}$. Then there exists a semigroup $S(t)$ such that the solution of system (4.1) is given by*

$$v(t,x)=S(t)v_0(x).$$

Moreover, the following estimates hold:

$$\begin{aligned}&\left\|\nabla^l S(t)v_0\right\|_{L^\infty}\leqslant C(1+t)^{-\frac{d+l}{2}}\|v_0\|_{L^1}+Ce^{-\beta t}\left\|\nabla^{((2l+d)/2)^+}v_0\right\|_{L^2},\\ &\left\|\nabla^k S(t)v_0\right\|_{L^2}\leqslant C(1+t)^{-(\frac{d}{4}+\frac{k}{2})}\|v_0\|_{L^1}+Ce^{-\beta t}\left\|\nabla^k v_0\right\|_{L^2},\end{aligned} \tag{4.2}$$

where d denotes the dimension of space, $l\geqslant 0$, $\beta>0$, $(2l+d)/2<((2l+d)/2)^+\leqslant s$, and $0\leqslant k\leqslant s$.

Proof. Note that the linear system (4.1) is equivalent to

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}\omega\\u\end{pmatrix} &= \begin{pmatrix}0 & -\bar{\pi}\nabla\\ -\bar{\pi}\nabla^{\top} & -\mu\mathbb{I}_d\end{pmatrix}\begin{pmatrix}\omega\\u\end{pmatrix},\\ &=: \mathcal{A}\begin{pmatrix}\omega\\u\end{pmatrix}, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\end{aligned} \tag{4.3}$$

with initial data $v_0(x)\in L^1(\mathbb{R}^d)\cap H^s(\mathbb{R}^d)$, where $\top$ denotes the transposition of vector, $\bar{\pi}$ and μ are positive constants, and $\mathbb{I}_d$ is $d\times d$ unit matrix. In view of the Fourier transformation and $v=(\omega,\,u)^{\top}$ we have

$$\partial_t\widehat{v}(t,\xi)=\mathcal{A}(\xi)\widehat{v}(t,\xi)$$

with the $(d+1)\times(d+1)$ matrix

$$\mathcal{A}(\xi)=\begin{pmatrix}0 & -i\bar{\pi}\xi\\ -i\bar{\pi}\xi^{\top} & -\mu\mathbb{I}_d\end{pmatrix}.$$

By computing the determinant $|\lambda I-\mathcal{A}(\xi)|=0$, we derive that the eigenvalues of the matrix $\mathcal{A}(\xi)$ are $\lambda_1=\cdots=\lambda_{d-1}=-\mu$, and

$$\begin{cases}\lambda_d=-\dfrac{1}{2}(\mu+\sqrt{\mu^2-4\bar{\pi}^2|\xi|^2}),\\[2mm] \lambda_{d+1}=-\dfrac{1}{2}(\mu-\sqrt{\mu^2-4\bar{\pi}^2|\xi|^2}).\end{cases} \tag{4.4}$$

By virtue of the eigenvalue $\lambda_i=-\mu$ of matrix $\mathcal{A}(\xi)$, $i=1,\,\cdots,\,d-1$, one has that the unit orthonormal eigenvectors $b_i=(0,\,y_i)^{\top}=(0,\,y_{i1},\,y_{i2},\,\cdots,\,y_{id})^{\top}$, such that for every $i=1,\,\cdots,\,d-1$

$$\xi\cdot y_i=\xi_1y_{i1}+\xi_2y_{i2}+\cdots\xi_dy_{id}=0.$$

Similarly, it is easy to obtain the unit eigenvectors b_j of the eigenvalue λ_j, $j=d,\ \ d+1$ satisfying

$$b_d=\frac{(i\lambda_{d+1},\bar{\pi}\xi)^{\top}}{\sqrt{\lambda_{d+1}^2+|\bar{\pi}\xi|^2}}\qquad\text{and}\qquad b_{d+1}=\frac{(i\lambda_d,\bar{\pi}\xi)^{\top}}{\sqrt{\lambda_d^2+|\bar{\pi}\xi|^2}},\ i^2=-1.$$

Thus we can choose the unitary matrix $\mathcal{B}(\xi)=(b_1,\,\cdots,\,b_d,\,\widetilde{b}_{d+1})$ such that

$$\mathcal{A}(\xi)\mathcal{B}(\xi)=\mathcal{B}(\xi)\begin{pmatrix}-\mu\mathbb{I}_{d-1} & 0 & 0\\ 0 & \lambda_d & \eta\\ 0 & 0 & \lambda_{d+1}\end{pmatrix},$$

where $b_1,\,\cdots,\,b_d$ and $\widetilde{b}_{d+1}$ are unit orthonormal eigenvectors in $\mathbb{R}^{d+1}$, the function η satisfies

$$\eta=\begin{cases}-(\mu+\sqrt{\mu^2-4\bar{\pi}^2|\xi|^2}), & \text{if}\quad \mu^2-4\bar{\pi}^2|\xi|^2\geqslant 0,\\ -\mu, & \text{if}\quad \mu^2-4\bar{\pi}^2|\xi|^2<0.\end{cases} \tag{4.5}$$

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Consequently, the solutions of system (4.1) are given by $v(t,\, x) = S(t)v_0(x)$, where

$$\widehat{S}(t) = \exp(t\mathcal{A}(\xi)) = \mathcal{B}(\xi)\mathcal{D}(t,\xi)\mathcal{B}^{-1}(\xi)$$
$$=: \mathcal{B}(\xi)\begin{pmatrix} e^{-\mu t}\mathbb{I}_{d-1} & 0 & 0 \\ 0 & e^{\lambda_d t} & \kappa\eta \\ 0 & 0 & e^{\lambda_{d+1}t} \end{pmatrix}\mathcal{B}^{-1}(\xi) \tag{4.6}$$

with the function

$$\kappa = \frac{e^{\lambda_d t} - e^{\lambda_{d+1}t}}{\lambda_d - \lambda_{d+1}}.$$

Next, in order to show (4.2), we first estimate every element of the matrix $\mathcal{D}(t,\, \xi)$.
Case 1: if $\mu^2 - 4\bar{\pi}^2|\xi|^2 < 0$, then we have

$$\kappa = \frac{e^{\lambda_d t} - e^{\lambda_{d+1}t}}{\lambda_d - \lambda_{d+1}} = e^{-\frac{1}{2}\mu t}\frac{2\sin(\frac{t}{2}\sqrt{4\bar{\pi}^2|\xi|^2 - \mu^2})}{\sqrt{4\bar{\pi}^2|\xi|^2 - \mu^2}}$$
$$\leqslant \begin{cases} Cte^{-\dfrac{1}{2}\mu t} \leqslant Ce^{-\dfrac{1}{3}\mu t}, & \text{if } \ t\sqrt{4\bar{\pi}^2|\xi|^2 - \mu^2} \ll 1, \\ Ce^{-\dfrac{1}{3}\mu t}, & \text{if } \ t\sqrt{4\bar{\pi}^2|\xi|^2 - \mu^2} > 1. \end{cases}$$

Case 2: if $\mu^2 - 4\bar{\pi}^2|\xi|^2 \geqslant 0$, then we have

$$\kappa = \frac{e^{\lambda_d t} - e^{\lambda_{d+1}t}}{\lambda_d - \lambda_{d+1}} = e^{-\frac{1}{2}\mu t}\frac{2\sinh(\frac{t}{2}\sqrt{\mu^2 - 4\bar{\pi}^2|\xi|^2})}{\sqrt{\mu^2 - 4\bar{\pi}^2|\xi|^2}}.$$

As $0 \leqslant \sqrt{\mu^2 - 4\bar{\pi}^2|\xi|^2} \ll 1$, one has that

$$\kappa \leqslant Cte^{-\frac{1}{2}\mu t} \leqslant Ce^{-\frac{1}{3}\mu t}.$$

Otherwise, if $\sqrt{\mu^2 - 4\bar{\pi}^2|\xi|^2} \in (\delta_0,\, \frac{\mu}{2})$, for some $\delta_0 > 0$, then it is easy to check that

$$\frac{t}{2}\delta_0 < \frac{t}{2}\sqrt{\mu^2 - 4\bar{\pi}^2|\xi|^2} < \frac{t}{4}\mu,$$

thus we have

$$\kappa \leqslant Ce^{-\frac{1}{5}\mu t}.$$

On the other hand, if $\sqrt{\mu^2 - 4\bar{\pi}^2|\xi|^2} \in [\frac{\mu}{2},\, \mu]$, then one can easily check that

$$-\frac{3}{16}\mu \leqslant -\frac{|\bar{\pi}\xi|^2}{\mu} \leqslant 0,$$

this implies that

$$\kappa \leqslant Ce^{-\frac{|\bar{\pi}\xi|^2}{\mu}t}.$$

Note that

$$|\eta| \leqslant \max\{\mu, 2|\lambda_{d-1}|\} \leqslant 2\mu.$$

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Thus it follows that

$$|\kappa\eta| \leqslant \begin{cases} Ce^{-\dfrac{|\bar{\pi}\xi|^2}{\mu}t}, & \text{if} \quad |\bar{\pi}\xi| \leqslant \dfrac{\sqrt{3}}{4}\mu, \\ Ce^{-\dfrac{1}{5}\mu t}, & \text{if} \quad |\bar{\pi}\xi| > \dfrac{\sqrt{3}}{4}\mu. \end{cases} \tag{4.7}$$

In fact, in a similar way, the bound of all diagonal elements of the matrix $\mathcal{D}(t, \xi)$ satisfies (4.7).

Now, we prove the inequality (4.2). Let $(2l+d)/2 < ((2l+d)/2)^+ \leqslant (2l+d)/2+\varepsilon$ for any $\varepsilon > 0$. In view of the Fourier transformation and (4.7), for $((2l+d)/2)^+ \leqslant s$, it follows that

$$\begin{aligned} \left\|\nabla^l S(t)v_0\right\|_{L^\infty} &\leqslant C\int_{\mathbb{R}^d}\left|e^{ix\cdot\xi}(i\xi)^l\widehat{S}(t)\widehat{v}_0(\xi)\right|d\xi \\ &\leqslant C\int_{|\bar{\pi}\xi|\leqslant\frac{\sqrt{3}}{4}\mu} e^{-\frac{|\bar{\pi}\xi|^2}{\mu}t}|\xi|^l|\widehat{v}_0(\xi)|d\xi + C\int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu} e^{-\frac{t}{5}\mu}|\xi|^l|\widehat{v}_0(\xi)|d\xi \\ &\leqslant C\|\widehat{v}_0\|_{L^\infty}\int_0^{\frac{\sqrt{3}}{4\bar{\pi}}\mu} r^{l+d-1}e^{-\frac{(\bar{\pi}r)^2}{\mu}t}dr + Ce^{-\frac{t}{5}\mu}\int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu}|\xi|^l|\widehat{v}_0(\xi)|d\xi \\ &\leqslant C(1+t)^{-(l+d)/2}\|v_0\|_{L^1} + Ce^{-\frac{1}{5}\mu t}\left\|\nabla^{((2l+d)/2)^+}v_0\right\|_{L^2}, \end{aligned} \tag{4.8}$$

where the last inequality is guaranteed by

$$\begin{aligned} \left|\int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu}|\xi|^l|\widehat{v}_0(\xi)|d\xi\right|^2 &\leqslant \int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu}|\xi|^{-(d)^+}d\xi\int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu}|\xi|^{2l+(d)^+}|\widehat{v}_0(\xi)|^2d\xi \\ &\leqslant C\left\|\nabla^{((2l+d)/2)^+}v_0\right\|_{L^2}^2. \end{aligned}$$

On the other hand, by virtue of the Fourier transformation and (4.7), for $0 \leqslant k \leqslant s$, one has that

$$\begin{aligned} \left\|\nabla^k S(t)v_0\right\|_{L^2}^2 &= \left\||\xi|^k\widehat{S}(t)\widehat{v}_0\right\|_{L^2}^2 \\ &\leqslant C\int_{|\bar{\pi}\xi|\leqslant\frac{\sqrt{3}}{4}\mu} e^{-2t\frac{|\bar{\pi}\xi|^2}{\mu}}|\xi|^{2k}|\widehat{v}_0(\xi)|^2d\xi + C\int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu} e^{-\frac{2}{5}\mu t}|\xi|^{2k}|\widehat{v}_0(\xi)|^2d\xi \\ &\leqslant C\|\widehat{v}_0\|_{L^\infty}^2\int_0^{\frac{\sqrt{3}}{4\bar{\pi}}\mu} e^{-2t\frac{(\bar{\pi}r)^2}{\mu}}r^{2k+d-1}dr + Ce^{-\frac{2}{5}\mu t}\int_{|\bar{\pi}\xi|>\frac{\sqrt{3}}{4}\mu}|\xi|^{2k}|\widehat{v}_0(\xi)|^2d\xi \\ &\leqslant C(1+t)^{-(k+\frac{d}{2})}\|v_0\|_{L^1}^2 + Ce^{-\frac{2}{5}\mu t}\left\|\nabla^k v_0\right\|_{L^2}^2, \end{aligned} \tag{4.9}$$

which concludes the proof of lemma 4.1. □

THEOREM 4.2. *Assume the initial data* $v_0(x) = (\omega_0,\, u_0)^\top \in L^1(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)$, $s > 1 + \frac{d}{2}$, *and* $\|v_0\|_{L^1} + I(v_0) = \epsilon_0 \ll 1$, *then system* (2.1) *has a unique global solution*

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$v=(\omega,\,u)^\top\in\mathcal{C}(\mathbb{R}^+,\,H^s(\mathbb{R}^d))$, which is guaranteed by theorem 3.3. In particular, for all time $t>0$ the solution $v(t,\,x)$ of system (2.1) satisfies

$$\begin{aligned}&\|v(t,\cdot)\|_{L^2}\leqslant C(1+t)^{-\frac{d}{4}},\quad \|v(t,\cdot)\|_{L^\infty}\leqslant C(1+t)^{-\frac{d}{2}},\\&\|\nabla v(t,\cdot)\|_{L^2}\leqslant C(1+t)^{-\frac{d+2}{4}},\quad \|\nabla v(t,\cdot)\|_{L^\infty}\leqslant C(1+t)^{-\frac{d}{2}}.\end{aligned}\tag{4.10}$$

Furthermore, we have

$$\left\|\nabla^k v(t,\cdot)\right\|_{L^2}\leqslant C\max\left\{(1+t)^{-\frac{d+2k}{4}},(1+t)^{-\frac{d}{2}}\right\},\tag{4.11}$$

where d denotes the dimension of space, $0\leqslant k\leqslant 1+\frac{d}{2}$.

Proof. In view of the linear system (4.1) and $v=(\omega,\,u)^\top$, the system (2.1) can be transformed into

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}\omega\\u\end{pmatrix}&=\begin{pmatrix}0&-\bar\pi\nabla\\-\bar\pi\nabla^\top&-\mu\mathbb{I}_d\end{pmatrix}\begin{pmatrix}\omega\\u\end{pmatrix}-\begin{pmatrix}u\cdot\nabla\omega+\dfrac{\gamma-1}{2}\omega\mathrm{div}u\\u\cdot\nabla u+\dfrac{\gamma-1}{2}\omega\nabla\omega\end{pmatrix},\\&=:\mathcal{A}v+F(v,\nabla v),\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\end{aligned}\tag{4.12}$$

with initial data $v_0(x)\in L^1(\mathbb{R}^d)\cap H^s(\mathbb{R}^d)$. By the Duhamel principle, the solutions of system (4.12) are given by

$$v(t,x)=S(t)v_0(x)+\int_0^t S(t-\tau)F(v,\nabla v)(\tau,x)d\tau.\tag{4.13}$$

By virtue of the assumption in theorem 4.2, inequality (4.2) and equation (4.13), for $(l+\frac{d}{2})^+\leqslant s$ we have

$$\begin{aligned}\left\|\nabla^l v(t,\cdot)\right\|_{L^\infty}&\leqslant\left\|\nabla^l S(t)v_0\right\|_{L^\infty}+\int_0^t\left\|\nabla^l S(t-\tau)F(v,\nabla v)(\tau)\right\|_{L^\infty}d\tau\\&\leqslant C(1+t)^{-(l+d)/2}\|v_0\|_{L^1}+Ce^{-\beta t}\left\|\nabla^{((2l+d)/2)^+}v_0\right\|_{L^2}\\&\quad+C\int_0^t(1+t-\tau)^{-(l+d)/2}\|F\|_{L^1}+Ce^{-\beta(t-\tau)}\left\|\nabla^{((2l+d)/2)^+}F\right\|_{L^2}d\tau\\&\leqslant C(1+t)^{-(l+d)/2}\epsilon_0+C\int_0^t(1+t-\tau)^{-(l+d)/2}\|F(\tau,\cdot)\|_{L^1}d\tau\\&\quad+C\int_0^t e^{-\beta(t-\tau)}\left\|\nabla^{((2l+d)/2)^+}F(\tau,\cdot)\right\|_{L^2}d\tau.\end{aligned}\tag{4.14}$$

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In order to derive the results, introducing the following four functions

$$\begin{aligned}
\mathcal{H}(t) &= \sup_{\tau\in[0,t]} \|v(\tau,\cdot)\|_{H^s},\\
J_0(t) &= \sup_{\tau\in[0,t]} (1+\tau)^{\frac{d}{4}}\|v(\tau,\cdot)\|_{L^2},\\
J_1(t) &= \sup_{\tau\in[0,t]} (1+\tau)^{\frac{d+2}{4}}\|\nabla v(\tau,\cdot)\|_{L^2},\\
J_\infty^h(t) &= \sup_{\tau\in[0,t]} (1+\tau)^{\frac{d+h}{2}}\|\nabla^h v(\tau,\cdot)\|_{L^\infty}.
\end{aligned}$$

Then for all $\tau\in[0,\,t]$ and $(l+1+\frac{d}{2})^+\leqslant s$, the nonlinear term $F(v,\,\nabla v)$ in (4.14) can be dealt with

$$\begin{aligned}
\|F(v,\nabla v)(\tau)\|_{L^1} &\leqslant C\|(\omega,u)\|_{L^2}\|\nabla(\omega,u)\|_{L^2}\\
&=: C\|v\|_{L^2}\|\nabla v\|_{L^2}\\
&\leqslant C(1+\tau)^{-\frac{d+1}{2}}J_0(t)J_1(t),
\end{aligned}\tag{4.15}$$

$$\begin{aligned}
\|F(v,\nabla v)(\tau)\|_{L^2} &\leqslant C\|v\|_{L^\infty}\|\nabla v\|_{L^2}\\
&\leqslant C\|v\|_{L^\infty}\|v\|_{H^s}\\
&\leqslant C(1+\tau)^{-\frac{d}{2}}J_\infty^0(t)\mathcal{H}(t),
\end{aligned}\tag{4.16}$$

$$\begin{aligned}
\|\nabla^{(l+\frac{d}{2})^+}F(v,\nabla v)(\tau)\|_{L^2} &\leqslant C\|v\|_{L^\infty}\|\nabla^{(l+1+\frac{d}{2})^+}v\|_{L^2}\\
&\leqslant C(1+\tau)^{-\frac{d}{2}}J_\infty^0(t)\mathcal{H}(t).
\end{aligned}\tag{4.17}$$

Plugging (4.15) and (4.17) into (4.14) yields that

$$\begin{aligned}
\|v(t,\cdot)\|_{L^\infty} &\leqslant C(1+t)^{-d/2}\epsilon_0 + CJ_\infty^0(t)\mathcal{H}(t)\int_0^t e^{-\beta(t-\tau)}(1+\tau)^{-\frac{d}{2}}\,d\tau\\
&\quad + CJ_0(t)J_1(t)\int_0^t (1+t-\tau)^{-\frac{d}{2}}(1+\tau)^{-\frac{d+1}{2}}\,d\tau\\
&\leqslant C(1+t)^{-d/2}(\epsilon_0 + J_\infty^0(t)\mathcal{H}(t))\\
&\quad + CJ_0(t)J_1(t)\int_0^t (1+t-\tau)^{-\frac{d}{2}}(1+\tau)^{-\frac{d+1}{2}}\,d\tau\\
&\leqslant C(1+t)^{-\frac{d}{2}}(\epsilon_0 + J_0(t)J_1(t) + J_\infty^0(t)\mathcal{H}(t)),
\end{aligned}\tag{4.18}$$

where we have used for $t\gg 1$

$$\int_0^t (1+t-\tau)^{-\frac{d}{2}}(1+\tau)^{-\frac{d+1}{2}}\,d\tau \leqslant C(1+t)^{-\frac{d}{2}}.$$

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On the other hand, thanks to (4.2), one can easily check that

$$
\begin{aligned}
\|v(t,\cdot)\|_{L^2} &\leqslant \|S(t)v_0\|_{L^2} + \int_0^t \|S(t-\tau)F(v,\nabla v)(\tau)\|_{L^2}d\tau \\
&\leqslant C(1+t)^{-\frac{d}{4}}\epsilon_0 + C\int_0^t e^{-\beta(t-\tau)}\|F(\tau,\cdot)\|_{L^2}d\tau \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{d}{4}}\|F(\tau,\cdot)\|_{L^1}d\tau \\
&\leqslant C(1+t)^{-\frac{d}{4}}\epsilon_0 + CJ^0_\infty(t)\mathcal{H}(t)\int_0^t e^{-\beta(t-\tau)}(1+\tau)^{-\frac{d}{2}}d\tau \\
&\quad + CJ_0(t)J_1(t)\int_0^t (1+t-\tau)^{-\frac{d}{4}}(1+\tau)^{-\frac{d+1}{2}}d\tau \\
&\leqslant C(1+t)^{-\frac{d}{4}}(\epsilon_0 + J_0(t)J_1(t) + J^0_\infty(t)\mathcal{H}(t)),
\end{aligned}
\tag{4.19}
$$

where the last inequality comes from for $d \geqslant 2$ and $t \gg 1$

$$
\Phi(t) =: \int_0^t (1+t-\tau)^{-\frac{d}{4}}(1+\tau)^{-\frac{d+1}{2}}d\tau \leqslant C(1+t)^{-\frac{d}{4}},
$$

which is guaranteed by

$$
\begin{aligned}
\Phi(t) &\leqslant (1+t)^{-\frac{d}{4}}\int_0^t \left(\frac{1}{1+t-\tau} + \frac{1}{1+\tau}\right)^{\frac{d}{4}}(1+\tau)^{-\frac{d+2}{4}}d\tau \\
&\leqslant C(1+t)^{-\frac{d}{4}}\int_0^t [(1+t-\tau)^{-\frac{d}{4}} + (1+\tau)^{-\frac{d}{4}}](1+\tau)^{-\frac{d+2}{4}}d\tau \\
&\leqslant C(1+t)^{-\frac{d}{4}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|\nabla v(t,\cdot)\|_{L^2} &\leqslant \|\nabla S(t)v_0\|_{L^2} + \int_0^t \|\nabla S(t-\tau)F(v,\nabla v)(\tau)\|_{L^2}d\tau \\
&\leqslant C(1+t)^{-(\frac{d+2}{4})}\epsilon_0 + C\int_0^t e^{-\beta(t-\tau)}\|\nabla F(\tau,\cdot)\|_{L^2}d\tau \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{d+2}{4}}\|F(\tau,\cdot)\|_{L^1}d\tau \\
&\leqslant C(1+t)^{-\frac{d+2}{4}}\epsilon_0 + CJ^0_\infty(t)\mathcal{H}(t)\int_0^t e^{-\beta(t-\tau)}(1+\tau)^{-\frac{d}{2}}d\tau \\
&\quad + CJ_0(t)J_1(t)\int_0^t (1+t-\tau)^{-\frac{d+2}{4}}(1+\tau)^{-\frac{d+1}{2}}d\tau \\
&\leqslant C(1+t)^{-\frac{d+2}{4}}(\epsilon_0 + J_0(t)J_1(t) + J^0_\infty(t)\mathcal{H}(t)),
\end{aligned}
\tag{4.20}
$$

Combining (4.18), (4.19) with (4.20), by the definition of $J_i(t),\ \ i=0,\,1,\,\infty$, we end up with

$$J_0(t)+J_1(t)+J_\infty^0(t)\leqslant C(\epsilon_0+J_0(t)J_1(t)+J_\infty^0(t)\mathcal{H}(t)). \tag{4.21}$$

If the initial data satisfy $\|v_0\|_{L^1}+I(v_0)=\epsilon_0\ll 1$, thanks to theorem 3.3, then it implies that

$$\mathcal{H}(t)\leqslant I(v)(t)\leqslant \epsilon_0\ll 1.$$

Define $f(t)=J_0(t)+J_1(t)+J_\infty^0(t)$, in view of inequality (4.21), one has for all $t\in\mathbb{R}^+$ that

$$f(t)\leqslant C\epsilon_0+Cf^2(t). \tag{4.22}$$

If we choose $\epsilon_0\ll 1$ such that $4C^2\epsilon_0<1$, then the equation

$$Cy^2-y+C\epsilon_0=0$$

has two differential roots

$$0<y_1=\frac{1-\sqrt{1-4C^2\epsilon_0}}{2C}<y_2=\frac{1+\sqrt{1-4C^2\epsilon_0}}{2C}.$$

Thanks to

$$f(0)=J_0(0)+J_1(0)+J_\infty^0(0)\leqslant C\epsilon_0\ll 1,$$

in order to ensure inequality (4.22) hold for all $t\geqslant 0$, thus we deduce that $f(t)<y_1$ is bound, this implies that

$$\begin{aligned}
&\|v(t,\cdot)\|_{L^2}\leqslant C(1+t)^{-\frac{d}{4}},\\
&\|\nabla v(t,\cdot)\|_{L^2}\leqslant C(1+t)^{-\frac{d+2}{4}},\\
&\|v(t,\cdot)\|_{L^\infty}\leqslant C(1+t)^{-\frac{d}{2}}.
\end{aligned} \tag{4.23}$$

In view of the above inequality (4.23), we can consequently estimate

$$\begin{aligned}
\left\|\nabla^k v(t,\cdot)\right\|_{L^2}&\leqslant \left\|\nabla^k S(t)v_0\right\|_{L^2}+\int_0^t\left\|\nabla^k S(t-\tau)F(v,\nabla v)(\tau)\right\|_{L^2}d\tau\\
&\leqslant C(1+t)^{-(\frac{d+2k}{4})}\epsilon_0+C\int_0^t e^{-\beta(t-\tau)}\left\|\nabla^k F(\tau,\cdot)\right\|_{L^2}d\tau\\
&\quad+C\int_0^t(1+t-\tau)^{-\frac{d+2k}{4}}\|F(\tau,\cdot)\|_{L^1}d\tau\\
&\leqslant C(1+t)^{-\frac{d+2k}{4}}\epsilon_0+CJ_\infty^0(t)\mathcal{H}(t)\int_0^t e^{-\beta(t-\tau)}(1+\tau)^{-\frac{d}{2}}d\tau
\end{aligned}$$

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$$
\begin{aligned}
&+ CJ_0(t)J_1(t)\int_0^t (1+t-\tau)^{-\frac{d+2k}{4}}(1+\tau)^{-\frac{d+1}{2}}d\tau \\
&\leqslant C\max\left\{(1+t)^{-\frac{d+2k}{4}}, (1+t)^{-\frac{d}{2}}\right\}(\epsilon_0 + J_0(t)J_1(t) + J_\infty^0(t)\mathcal{H}(t)) \\
&\leqslant C\max\left\{(1+t)^{-\frac{d+2k}{4}}, (1+t)^{-\frac{d}{2}}\right\}, \qquad (4.24)
\end{aligned}
$$

where we have used

$$\left\|\nabla^k F(\tau,\cdot)\right\|_{L^2} \leqslant C\|v\|_{L^\infty}\left\|\nabla^{k+1}v\right\|_{L^2} \leqslant C(1+\tau)^{-\frac{d}{2}}J_\infty^0(t)\mathcal{H}(t),$$

and for $0 \leqslant k \leqslant 1+\frac{d}{2}$

$$\left|\int_0^t (1+t-\tau)^{-\frac{d+2k}{4}}(1+\tau)^{-\frac{d+1}{2}}d\tau\right| \leqslant C(1+t)^{-\frac{d+2k}{4}}.$$

On the other hand, thanks to (4.2) and (4.23), similar to estimate (4.18) it follows that

$$
\begin{aligned}
\|\nabla v(t,\cdot)\|_{L^\infty} &\leqslant C(1+t)^{-\frac{d}{2}}(\epsilon_0 + J_0(t)J_1(t) + J_\infty^0(t)\mathcal{H}(t)) \\
&\leqslant C(1+t)^{-\frac{d}{2}}.
\end{aligned} \qquad (4.25)
$$

This completes the proof of theorem 4.2. □

Remark 4.3. In fact, we can show the following estimates of high order derivative of solution

$$
\begin{aligned}
&\|\nabla^\delta v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d}{2}}, \ \textit{for } \delta \geqslant 1, \\
&\|\nabla^\sigma v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d}{2}}, \ \textit{for } \delta > 1+\frac{d}{2}.
\end{aligned}
$$

Because the result of (4.24) and (4.25) is not optimal in the sense of linearization, if the solution is sufficiently smooth, we can improve the result of theorem 4.2 and have the following result.

Corollary 4.4. *Under the assumptions of theorem 4.2, the system (2.1) has a unique global solution $v = (\omega, u)^\top \in \mathcal{C}(\mathbb{R}^+, H^s(\mathbb{R}^d))$. Moreover, for all time $t > 0$ and $s > \frac{d}{2}+2+\frac{1}{d}$, the decay rate of solution $v(t, x)$ of system (2.1) satisfies*

$$\|\nabla v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d+1}{2}},$$

and for $0 < k \leqslant 1+\frac{d}{2}$ we have

$$\left\|\nabla^k v(t,\cdot)\right\|_{L^2} \leqslant C(1+t)^{-\frac{d+2k}{4}},$$

where d denotes the dimension of space.

Proof. Taking advantage of the Gagliardo–Nirenberg inequality, it follows that

$$\begin{aligned}\left\|\nabla^{((2+d)/2)^+}F(\tau,\cdot)\right\|_{L^2} &\leqslant \|v(\tau,\cdot)\|_{L^\infty}\left\|\nabla^{((4+d)/2)^+}v(\tau,\cdot)\right\|_{L^2}\\ &\leqslant \|v(\tau,\cdot)\|_{L^\infty}\|\nabla v(\tau,\cdot)\|_{L^\infty}^{1-\theta_1}\|\nabla^s v(\tau,\cdot)\|_{L^2}^{\theta_1}\\ &\leqslant C(1+\tau)^{-(\frac{d}{2}+(1-\theta_1)\frac{d+1}{2})}J_\infty^0(t)(J_\infty^1(t))^{1-\theta_1}\mathcal{H}^{\theta_1}(t)\\ &\leqslant C(1+\tau)^{-\frac{d+1}{2}}J_\infty^0(t)(J_\infty^1(t))^{1-\theta_1}\mathcal{H}^{\theta_1}(t),\end{aligned} \tag{4.26}$$

where the last inequality is guaranteed by $\frac{d}{2}+2+\frac{1}{d}<s$, the constant

$$\theta_1=\frac{(1)^+}{s-(1+\frac{d}{2})}\in(0,1).$$

Using inequality (4.26), we can estimate

$$\begin{aligned}\|\nabla v(t,\cdot)\|_{L^\infty} &\leqslant C(1+t)^{-(d+1)/2}\epsilon_0+C\int_0^t(1+t-\tau)^{-(d+1)/2}\|F(\tau,\cdot)\|_{L^1}d\tau\\ &\quad+C\int_0^t e^{-\beta(t-\tau)}\left\|\nabla^{((2+d)/2)^+}F(\tau,\cdot)\right\|_{L^2}d\tau\\ &\leqslant C(1+t)^{-(d+1)/2}(\epsilon_0+J_0(t)J_1(t)+J_\infty(t)^0(J_\infty^1(t))^{1-\theta_1}\mathcal{H}(t)^{\theta_1}).\end{aligned} \tag{4.27}$$

Note that $J_0(t)$, $J_1(t)$ and $J_\infty^0(t)$ are bounded, in view of Young's inequality yields that

$$J_\infty^1(t)\leqslant C,$$

which implies for $s>\frac{d}{2}+2+\frac{1}{d}$ that

$$\|\nabla v(t,\cdot)\|_{L^\infty}\leqslant C(1+t)^{-(d+1)/2}.$$

Finally, thanks to theorem 4.2, we only need to show the last inequality holds for $\frac{d}{2}<k\leqslant 1+\frac{d}{2}$. Similar to estimate (4.26), one has that

$$\begin{aligned}\left\|\nabla^k F(\tau,\cdot)\right\|_{L^2} &\leqslant \|v(\tau,\cdot)\|_{L^\infty}\left\|\nabla^{k+1}v(\tau,\cdot)\right\|_{L^2}\\ &\leqslant \|v(\tau,\cdot)\|_{L^\infty}\|\nabla v(\tau,\cdot)\|_{L^\infty}^{\vartheta_1}\|\nabla^s v(\tau,\cdot)\|_{L^2}^{1-\vartheta_1}\\ &\leqslant C(1+\tau)^{-(\frac{d}{2}+\vartheta_1\frac{d+1}{2})}J_\infty^0(t)(J_\infty^1(t))^{\vartheta_1}\mathcal{H}^{1-\vartheta_1}(t)\\ &\leqslant C(1+\tau)^{-\frac{d+2k}{4}}J_\infty^0(t)(J_\infty^1(t))^{\vartheta_1}\mathcal{H}^{1-\vartheta_1}(t),\end{aligned} \tag{4.28}$$

where the last inequality is guaranteed by $k\leqslant[(\frac{3d}{2}+1)(s-1)-\frac{d^2}{4}]/(s+\frac{d}{2})$, the constant

$$\vartheta_1=\frac{s-(k+1)}{s-(1+\frac{d}{2})}\in(0,1).$$

By virtue of $J^1_\infty(t) \leqslant C$ and (4.28), it follows that

$$\begin{aligned}\left\|\nabla^k v(t,\cdot)\right\|_{L^2} &\leqslant C(1+t)^{-(\frac{d+2k}{4})}\epsilon_0 + C\int_0^t e^{-\beta(t-\tau)}\left\|\nabla^k F(\tau,\cdot)\right\|_{L^2} d\tau \\ &\quad + C\int_0^t (1+t-\tau)^{-\frac{d+2k}{4}}\|F(\tau,\cdot)\|_{L^1} d\tau \\ &\leqslant C(1+t)^{-\frac{d+2k}{4}}(\epsilon_0 + J_0(t)J_1(t)) \\ &\quad + CJ^0_\infty(t)(J^1_\infty(t))^{\vartheta_1}(\mathcal{H}(t))^{1-\vartheta_1}\int_0^t e^{-\beta(t-\tau)}(1+\tau)^{-\frac{d+2k}{4}} d\tau \\ &\leqslant C(1+t)^{-\frac{d+2k}{4}}(\epsilon_0 + J_0(t)J_1(t) + J^0_\infty(t)(J^1_\infty(t))^{\vartheta_1}(\mathcal{H}(t))^{1-\vartheta_1}) \\ &\leqslant C(1+t)^{-\frac{d+2k}{4}}, \end{aligned} \tag{4.29}$$

where we have applied $0 < k \leqslant 1 + \frac{d}{2}$ and $\frac{d}{2} + 2 + \frac{1}{d} < s$, which is equivalent to

$$k < [(\frac{3d}{2}+1)(s-1) - \frac{d^2}{4}]/(s+\frac{d}{2}).$$

This completes the proof of corollary 4.4. □

REMARK 4.5. In view of lemma 4.1, for sufficiently large time $t > 0$, the algebra decay rate of solution $v(t, x)$ for linear equation (4.1) satisfies

$$\begin{aligned}\left\|\nabla^l v(t,\cdot)\right\|_{L^\infty} &=: \left\|\nabla^l S(t)v_0\right\|_{L^\infty} \leqslant C(1+t)^{-\frac{d+l}{2}}, \\ \left\|\nabla^k v(t,\cdot)\right\|_{L^2} &=: \left\|\nabla^k S(t)v_0\right\|_{L^2} \leqslant C(1+t)^{-(\frac{d}{4}+\frac{k}{2})},\end{aligned} \tag{4.30}$$

where d denotes the dimension of space, $l \geqslant 0$, $d < 2(s-l)$ and $0 \leqslant k \leqslant s$. However, by virtue of theorem 4.2 and corollary 4.1, we only show that the algebra decay rate of L^∞ -norm of v, ∇v and L^2-norm of $\nabla^\sigma v$ satisfies

$$\|v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d}{2}}, \quad \|\nabla v(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{d+1}{2}}$$

and

$$\|\nabla^\sigma v(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{d+2\sigma}{4}},$$

which is optimal in the linearized sense (4.30), where $0 \leqslant \sigma \leqslant 1 + \frac{d}{2}$. How to estimate the high order derivative of the solution $v(t, x)$ in L^2 and L^∞ norm is an open problem.

REMARK 4.6. For the smooth initial data $(\rho_0, u_0) \in H^s(\mathbb{R}^d)$, $s \geqslant 1 + \frac{d}{2}$ with small amplitude, there exists a unique global smooth solution of the Cauchy problem for system (1.1). As the time t becomes large, theorem 4.2 tells us that the smooth solution $v(t, x)$ is algebra decay which extends and improves the following result

$$\begin{aligned}&\|U(t,\cdot)\|_{L^\infty} \leqslant C(1+t)^{-\frac{3}{2}}, \quad \|U(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}}, \\ &\|\nabla U(t,\cdot)\|_{L^2} \leqslant C(1+t)^{-\frac{5}{4}}, \qquad \|\omega(t,\cdot)\|_{L^2} \leqslant Ce^{-Ct}.\end{aligned}$$

derived by Sideris, Thomases and Wang in [28].

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In addition, if the initial data belong to $L^1(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)$, then one shows that the following algebra decay rate of smooth solution in L^p norm.

Corollary 4.7. *Under the assumptions of theorem 4.2, the system (2.1) has a unique global solution $v = (\omega, u)^\top \in \mathcal{C}(\mathbb{R}^+, H^s(\mathbb{R}^d))$. Moreover, for all time $t > 0$ and $2 \leqslant p \leqslant \infty$, the decay rate of solution $v(t, x)$ of system (2.1) satisfies*

$$\|v(t,\cdot)\|_{L^p} \leqslant C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})},$$
$$\|\nabla^\alpha v(t,\cdot)\|_{L^p} \leqslant C(1+t)^{-\frac{1}{2}(d+\alpha-\frac{d}{p})},$$

where d denotes the dimension of space, $0 \leqslant \alpha < 1 + \frac{d}{2}$. In particular, we have

$$\left\|\nabla^{(1+\frac{d}{p})} v(t,\cdot)\right\|_{L^p} \leqslant C(1+t)^{-\frac{d+1}{2}}.$$

Proof. In view of interpolation inequality and theorem 4.2 yields that

$$\|v(t,\cdot)\|_{L^p} \leqslant \|v(t,\cdot)\|_{L^2}^{\frac{2}{p}} \|v(t,\cdot)\|_{L^\infty}^{1-\frac{2}{p}} \leqslant C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}.$$

By the Gagliardo–Nirenberg inequality and corollary 4.1, for $0 \leqslant \alpha < 1 + \frac{d}{2}$ one shows that

$$\|\nabla^\alpha v(t,\cdot)\|_{L^p} \leqslant \|v(t,\cdot)\|_{L^\infty}^{\theta} \left\|\nabla^{(1+\frac{d}{2})} v(t,\cdot)\right\|_{L^2}^{1-\theta} \leqslant C(1+t)^{-\frac{1}{2}(d+\alpha-\frac{d}{p})},$$

where $1-\theta = \alpha - \frac{d}{p}$ and $\theta \in (0, 1)$. Note that

$$\left\|\nabla^{((1+\frac{d}{p})} v(t,\cdot)\right\|_{L^p} \leqslant \|\nabla v(t,\cdot)\|_{L^\infty}^{1/2} \left\|\nabla^{(1+\frac{d}{2})} v(t,\cdot)\right\|_{L^2}^{1/2} \leqslant C(1+t)^{-\frac{d+1}{2}},$$

which includes the proof of corollary 4.7. □

Corollary 4.8. *Under the additional assumptions of theorem 4.2, the derivative of velocity decays exponentially in Sobolev space $L^2(\mathbb{R}^d)$, i.e.,*

$$\|\nabla u(t,\cdot)\|_{L^2} \leqslant C\|\nabla u_0\|_{L^2} \exp(-\frac{\mu}{2}t).$$

Proof. If we define the vorticity $\Omega = Du - \nabla u$, where Du stands for the Jacobian matrix of velocity u, and ∇u stands for its transposed matrix, then the vorticity plays a fundamental role in the compressible fluid mechanics. Indeed, by system

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(2.1), Ω takes the form of a quasi-linear evolution equation of hyperbolic type

$$\partial_t \Omega + u \cdot \nabla \Omega + \Omega \cdot Du + \nabla u \cdot \Omega + \mu \Omega = 0. \tag{4.31}$$

Multiplying Ω on both sides of equation (4.31), integration by parts, it follows that

$$\begin{aligned} \frac{\partial}{\partial_t} \int_{\mathbb{R}^d} |\Omega|^2 dx + 2\mu \int_{\mathbb{R}^d} |\Omega|^2 dx &\leqslant C\|\nabla u\|_{L^\infty} \int_{\mathbb{R}^d} |\Omega|^2 dx \\ &\leqslant \mu \int_{\mathbb{R}^d} |\Omega|^2 dx, \end{aligned} \tag{4.32}$$

where we have applied $I(v_0) \ll 1$, which guarantees that

$$C\|\nabla u\|_{L^\infty} \leqslant CI^{\frac{1}{2}}(v)(t) \leqslant CI^{\frac{1}{2}}(v_0) \leqslant \mu.$$

In view of Gronwall's inequality to (4.32) one has that

$$\|\Omega(t, \cdot)\|_{L^2} \leqslant \|\Omega_0\|_{L^2} e^{-\frac{\mu}{2} t}.$$

Thanks to $\|\Omega\|_{L^2} \leqslant C\|\nabla u\|_{L^2}$, and $\|\nabla u\|_{L^2} \leqslant C\|\Omega\|_{L^2}$ (see proposition 7.5 on page 294 in [**1**]), therefore, we have

$$\|\nabla u(t, \cdot)\|_{L^2} \leqslant C\|\nabla u_0\|_{L^2} e^{-\frac{\mu}{2} t}.$$

□

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