Local realizability toposes and a modal logic for computability

S T E V E N   A W O D E Y[†],   L A R S   B I R K E D A L[‡§]   and   D A N A   S.   S C O T T[¶]

[†] *Department of Philosophy, Carnegie Mellon University, Pittsburgh, PA 15213, USA.*

[‡] *The IT University of Copenhagen, Glentevej 67, DK–2400 Copenhagen NV, Denmark.*

[¶] *School of Computer Science, Carnegie Mellon University, Pittsburgh, PA 15213, USA.*

This work is a step toward the development of a logic for types and computation that includes not only the usual spaces of mathematics and constructions, but also spaces from logic and domain theory. Using realizability, we investigate a configuration of three toposes that we regard as describing a notion of relative computability. Attention is focussed on a certain local map of toposes, which we first study axiomatically, and then by deriving a modal calculus as its internal logic. The resulting framework is intended as a setting for the logical and categorical study of relative computability.

1. Introduction

We report here on the current status of research on the Logic of Types and Computation at Carnegie Mellon University (Scott *et al.* 1998). The general goal of this research program is to develop a logical framework for the theories of types and computability that includes the standard mathematical spaces alongside the many constructions and spaces known from type theory and domain theory. One purpose of this goal is to facilitate the study of computable operations and maps on data that are not necessarily computable, such as the space of all real numbers.

Concretely, in the research described here we use the realizability topos over the graph model $P\mathbb{N}$ of the (untyped) lambda calculus, together with the sub-graph model given by the recursively enumerable subsets, to represent the classical and computable worlds, respectively. There results a certain configuration of toposes that can be regarded as describing a notion of *relative computability*[‖]. We study this configuration axiomatically, and derive a higher-order, modal logic in which to reason about it. The logic can then be applied to the original model to formalise reasoning about computability in that setting. Moreover, the resulting logical framework provides a general, categorical semantics and
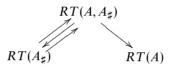

[§] The present work was carried out while this author was affiliated to the School of Computer Science, Carnegie Mellon University.

[‖] Not to be confused with the standard notion of computability relative to an oracle.

logical syntax for reasoning in a formal way about abstract computability, which, it is hoped, could also be useful for formally similar concepts, such as logical definability.

In somewhat more detail: Section 2 begins by recalling the standard realizability toposes $RT(A)$ and $RT(A_\sharp)$ resulting from a partial combinatory algebra A and a subalgebra $A_\sharp$. We then identify a third category $RT(A, A_\sharp)$ that plays a key role; very roughly speaking, it represents the world of all (continuous) objects, but with only computable maps between them. This category $RT(A, A_\sharp)$ is a topos, the *relative realizability topos* on A with respect to the subalgebra $A_\sharp$.

The toposes $RT(A)$ and $RT(A_\sharp)$ are not particularly well-related by themselves; the purpose of the relative realizability topos $RT(A, A_\sharp)$ is to remedy this defect. The *three* toposes are related to each other as indicated in the following diagram, in which the three functors on the left leg constitute a local geometric morphism, while the right leg is a logical morphism.

$$RT(A, A_\sharp)$$
$$RT(A_\sharp) \qquad\qquad RT(A)$$

The local geometric morphism on the left is our chief concern, and the focus of Section 3, which also mentions some examples and properties of these fairly well-understood maps of toposes. When we first encountered it, we were pleased to recognise our situation as an instance of one that F. W. Lawvere has already called attention to and dubbed an *adjoint cylinder* or, more colourfully, a *unity and identity of opposites* (Lawvere 1992; Lawvere 1989).

In Section 4 we present four sound and complete axioms for local maps of toposes. Actually, since the local map is localic in the situation we are mainly interested in – *viz*, realizability – we give the axioms in a form that implies this condition. We simply mention here that a modification of axiom 2 about generators will accomodate all (bounded) local maps. More information concerning the abstract axiomatisation of local maps (covering not only localic but also more general bounded local maps) can be found in Awodey and Birkedal (1999) and Birkedal (2000) – here we just recall the definitions and results needed in this paper. The axiomatisation has been found useful in working with the particular situation we have in mind, but its general utility for local maps of toposes remains to be seen.

One application, of sorts, of the axioms for local maps is the investigation of their logical properties. These are given in Section 5 in the form of a logical calculus involving two propositional operations, written $\sharp\varphi$ and $\flat\varphi$, with $\sharp$ left adjoint to $\flat$. It turns out that $\sharp$ satisfies the S4 modal logic postulates for the box-operation. We here term the $\sharp$-calculus a *modal logic for computability*, since that is the interpretation we have in mind; but, of course, this modal logic can be interpreted in any local topos. We intend to use it to investigate the logical relations that hold in the relative realizability topos; however, this aspect of our work is only just beginning.

In Section 6 we define and study *local triposes*, which are triposes that model the modal

logic for computability. Any localic local map arises from a local tripos and any local tripos gives rise to a local map of toposes.

Note that any local map also induces a closely related pair of adjoint operations on *logical types* (objects), in addition to the ones on formulas (subobjects) studied here, relating our work to Benaissa *et al.* (1999) and Benton (1995). The idea of a modal 'computability' operator $\sharp$ is due to the senior author (January 1998) and was the original impetus for this work, parts of which are from the second author's doctoral thesis (Birkedal, 2000). The final brief section of the paper spells out the intended interpretation of the $\sharp$-calculus in the relative realizability topos $RT(A, A_\sharp)$.

2. Realizability toposes for computability

Let $(A, \cdot, \mathbf{K}, \mathbf{S})$ be a partial combinatory algebra (PCA); we will often just denote it by its underlying set A. The binary operator $\cdot$ is the (partial) application and combinators $\mathbf{K}$ and $\mathbf{S}$ are taken to be part of the structure and not just required to exist.

Let $A_\sharp$ be a sub-PCA of A, that is, $A_\sharp$ is a subset of A containing $\mathbf{K}$ and $\mathbf{S}$ and closed under partial application. Intuitively, we are thinking of the realizers in A as 'continuous' realizers and of those in $A_\sharp$ as 'computable' realizers. This intuition comes from the main example, where A is $P\mathbb{N}$, the graph model on the powerset of the natural numbers, and $A_\sharp$ is RE, the recursively enumerable sub-graph-model. Note that the model $P\mathbb{N}$ has a continuum of (countable) sub-PCA's. As another example, one may consider Kleene's function realizability with $A = \mathbb{N}^\mathbb{N}$ and with $A_\sharp$ the set of total recursive functions. One may also consider van Oosten's combinatory algebra $\mathscr{B}$ for sequential computation and its effective subalgebra $\mathscr{B}_{eff}$, see van Oosten (1999) and Longley (1998).

The PCA's $A_\sharp$ and A give rise to two realizability toposes $RT(A_\sharp)$ and $RT(A)$ in the standard way (Hyland *et al.* 1980). One may think of $RT(A)$ as a universe where all objects and all maps are realized by continuous realizers. Likewise, $RT(A_\sharp)$ may be thought of as a universe where all objects and all maps are realized by computable realizers. Unfortunately, these two toposes are not very well related; in particular, it is not clear how to talk about computable maps operating on continuous objects, which is what one would like to do for the purposes of, for example, computable analysis (Pour-El and Richards 1989). Thus, one is led to introduce another realizability topos, $RT(A, A_\sharp)$, where, intuitively, equality on all objects is realized by continuous realizers and all maps are realized by computable realizers[†].

The topos $RT(A, A_\sharp)$ is constructed by modifying the underlying tripos for $RT(A)$ in the following way. The non-standard predicates φ, ψ on a set I are still functions $I \to PA$ into the powerset of A, and the Heyting pre-algebra operations are the same as in the tripos underlying $RT(A)$. The modification is in the definition of the entailment relation: we say $\varphi \vdash \psi$ over I iff there is a realizer a in $A_\sharp$ (not just in A) such that for all i in I, all $b \in \varphi(i)$, $a \cdot b$ is defined and $a \cdot b \in \psi(i)$. In the terminology of Pitts (1981), we have changed the 'designated truth values' to be those subsets of A that have a non-empty

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[†] We first learned about the topos $RT(A, A_\sharp)$ from Thomas Streicher in February 1998, but the construction has actually been known for a long time; see Pitts (1981, page 15, item (ii)).

intersection with $A_\sharp$. We use $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ to denote this new tripos. Then $RT(A, A_\sharp)$ is the topos $\mathbf{Set}[P]$ represented by P.

Explicitly, objects of $RT(A, A_\sharp)$ are pairs $(X, \approx_X)$ with X a set and $\approx_X : X \times X \to PA$ a non-standard equality predicate with computable realizers for symmetry and transitivity. Morphisms from $(X, \approx_X)$ to $(Y, \approx_Y)$ are equivalence classes of functional relations $F : X \times Y \to PA$ with computable realizers proving that F is a functional relation. Two such functional relations F and G are equivalent iff there are computable realizers showing them equivalent. We now see that, intuitively, it makes sense to think of objects of $RT(A, A_\sharp)$ as objects with *continuous* realizers for existence and equality elements, and of morphisms $f = [F]$ as *computable* maps, since the realizers for the functionality of F are required to be computable.

3. Geometry of the realizability toposes for computability

Let $Q : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ be the standard realizability tripos on $A_\sharp$, that is, the tripos underlying $RT(A_\sharp)$. We now define three $\mathbf{Set}$-indexed functors between Q and P:

$$\Delta : Q \to P \quad \text{and} \quad \Gamma : P \to Q \quad \text{and} \quad \nabla : Q \to P.$$

These are defined as follows. Over a set I, we have

$$
\begin{aligned}
\Delta_I(\psi : I \to P A_\sharp)(i) &= \psi(i) \\
\Gamma_I(\varphi : I \to P A)(i) &= A_\sharp \cap \varphi(i) \\
\nabla_I(\psi : I \to P A_\sharp)(i) &= \bigcup_{\varphi \in P A} \left(\varphi \wedge \left(A_\sharp \cap \varphi \supset \psi(i) \right) \right),
\end{aligned}
$$

where $\wedge$ and $\supset$ are calculated as in $P(1)$.

It is easy to see that Δ_I and Γ_I are well-defined functors; to show this for ∇_I, suppose that $\psi \vdash \psi'$ in $Q(I)$. Then there is a realizer $c \in A_\sharp$ such that

$$c \in \bigcap_{i \in I} \left(\psi(i) \supset \psi'(i) \right).$$

Let

$$d = \lambda x. \langle \pi x, \lambda y. c(\pi'(x)(y)) \rangle.$$

Then $d \in A_\sharp$ (since $c \in A_\sharp$), and it is easy to verify that

$$d \in \bigcap_{i \in I} \left(\nabla_I(\psi)(i) \supset \nabla_I(\psi')(i) \right),$$

as required.

Recall from Hyland *et al.* (1980) and Pitts (1981) that a geometric morphism f from a tripos P to a tripos Q is a pair of indexed functors (f^*, f_*) with f^* indexed left adjoint to f_* such that f_I^* is left exact, for all I in the base category.

Theorem 3.1. Under the foregoing definitions it follows that:
— (Δ, Γ) is a geometric morphism of triposes from P to Q.
— (Γ, ∇) is a geometric morphism of triposes from Q to P.
— For all $I \in \mathbf{Set}$, Δ_I and ∇_I are both full and faithful.

Proof. It is easy to see that Δ is (indexed) left adjoint to Γ using that $A_\sharp$ is closed under the partial application of A. Further, it is clear that Δ preserves finite limits and is full and faithful since the orderings in P and Q are defined in the same way (requiring computable realizers).

It remains to show that

$$\varphi \vdash_I^P \nabla\psi \iff \Gamma\varphi \vdash_I^Q \psi,$$

for all $\varphi \in P(I)$ and all $\psi \in Q(I)$ (where, of course, $\vdash_I^P$ is the ordering in $P(I)$ and $\vdash_I^Q$ is the ordering in $Q(I)$). To this end, suppose $\varphi \vdash_I^P \nabla\psi$, *via* a realizer $c \in A_\sharp$. Let

$$d = \lambda x.\,\pi(c(x))(\pi'(c(x))) \in A_\sharp.$$

It is easy to verify that d is a realizer for $\Gamma\varphi \vdash_I^Q \psi$.

For the other direction, suppose $d \in A_\sharp$ is a realizer for $\Gamma\varphi \vdash_I^Q \psi$. Then

$$c = \lambda x.\,\langle x, \lambda y.\,d(y)\rangle \in A_\sharp$$

is a realizer for $\varphi \vdash_I^P \nabla\psi$.

Since Δ is full and faithful, and since $\Delta \dashv \Gamma \dashv \nabla$, we also have ∇ is full and faithful, completing the proof of the theorem. $\qquad\square$

By Proposition 4.7 in Pitts (1981), these geometric morphisms lift to two geometric morphisms between the toposes, as in

$$RT(A_\sharp) \overset{\overset{\Delta}{\longrightarrow}}{\underset{\underset{\nabla}{\longrightarrow}}{\longleftarrow\Gamma-}} RT(A, A_\sharp), \qquad \Delta \dashv \Gamma \dashv \nabla.$$

(Here we do not distinguish notationally between the functors at the tripos level and at the topos level). In particular, Δ preserves finite limits. Moreover, $\Delta, \nabla\colon RT(A_\sharp) \to RT(A, A_\sharp)$ are easily shown to be both full and faithful also. The resulting geometric morphism $(\Delta, \Gamma)\colon RT(A, A_\sharp) \to RT(A)$ is therefore a (connected) surjection, while the one given by $(\Gamma, \nabla)\colon RT(A_\sharp) \to RT(A, A_\sharp)$ is an embedding. Note that $\Gamma \circ \nabla \cong 1 \cong \Gamma \circ \Delta$. It then follows by standard results that there is a Lawvere–Tierney topology j in $RT(A, A_\sharp)$ such that $RT(A_\sharp)$ is equivalent to the category $\mathrm{Sh}_j(RT(A, A_\sharp))$ of j-sheaves. We remark that one can show that, in general, the geometric morphism $(\Delta, \Gamma)\colon RT(A, A_\sharp) \to RT(A_\sharp)$ is *not* open (Birkedal, 2000).

The following theorem was known to Martin Hyland, but apparently has never been published. We include a proof here.

Theorem 3.2. Let $\mathbb{C}$ be a finitely complete category, and let P and Q be $\mathbb{C}$-triposes. Suppose $f = (f^*, f_*)\colon P \to Q$ is a geometric morphism of triposes. Then $\mathbb{C}[P]$ is localic over $\mathbb{C}[Q]$ *via* the induced geometric morphism $f = (f^*, f_*)\colon \mathbb{C}[P] \to \mathbb{C}[Q]$.

Proof. We want to prove that $\mathbb{C}[P]$ is equivalent to the category of $\mathbb{C}[Q]$-valued sheaves on the internal locale $f_*(\Omega_{\mathbb{C}[P]})$ in $\mathbb{C}[Q]$, where $\Omega_{\mathbb{C}[P]}$ is the subobject classifier in $\mathbb{C}[P]$. As usual (Johnstone 1977) it suffices to show that, for all $X \in \mathbb{C}[P]$, there exists a $Y \in \mathbb{C}[Q]$

and a diagram

$$S \rightarrowtail f^*Y$$
$$\downarrow$$
$$X$$

in $\mathbb{C}[P]$ presenting X as a subquotient of f^*Y for Y an object of $\mathbb{C}[Q]$. Write $\nabla_P : \mathbb{C} \to \mathbb{C}[P]$ for the functor $I \mapsto (I, \exists_{\delta_I}(T))$, where $\delta_I : I \mapsto I \times I$ is the diagonal map (the 'constant objects functor' (Pitts 1981)).

By a familiar property of realizability toposes, we have that for all $X \in \mathbb{C}[P]$, there exists an object $I \in \mathbb{C}$ and a diagram

$$S \rightarrowtail \nabla_P(I) = (I, \exists_{\delta_I}(T))$$
$$\downarrow \tag{1}$$
$$X$$

in $\mathbb{C}[P]$ presenting X as a subquotient of a constant object $\nabla_P(I)$. Now, since f^* is the inverse image of a geometric morphism of triposes, f^* preserves existential quantification (as an indexed left adjoint), so $f^*(\nabla_Q(I)) \cong \nabla_P(I)$, and the diagram in (1) is the required diagram. □

We summarise the consequences of Theorems 3.1 and 3.2 in the following corollary.

Corollary 3.3. $RT(A, A_\sharp)$ is localic over $RT(A_\sharp)$ *via* the geometric morphism $(\Delta, \Gamma) : RT(A, A_\sharp) \to RT(A_\sharp)$, which is a *localic local map of toposes*, since Γ has a right adjoint ∇, for which $\Gamma \circ \nabla \cong 1$.
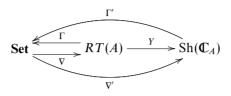
Local maps have been studied by Johnstone and Moerdijk (Johnstone and Moerdijk 1989), and provide an instance of what Lawvere (Lawvere 1992; Lawvere 1989) has called *unity and identity of opposites*. The idea is that the full subcategories $\Delta(RT(A_\sharp))$ and $\nabla(RT(A_\sharp))$ are each equivalent to $RT(A_\sharp)$, and yet are 'opposite' in the sense of being coreflective and reflective, respectively, in $RT(A, A_\sharp)$. We think of the objects in $\nabla(RT(A_\sharp))$ as sheaves, and here we think of those in $\Delta(RT(A_\sharp))$ as 'computable'.

Examples of local maps in addition to the basic ones mentioned in Johnstone and Moerdijk (1989) include the following:

(1) Let $RT(A)$ be a realizability topos, and let $i : \mathbb{C}_A \to RT(A)$ be a full subcategory of partitioned assemblies of suitably large, bounded cardinality so that $\mathbb{C}_A$ is a small generating subcategory of projectives. The covering families in $\mathbb{C}_A$ are to be those that are jointly epimorphic in $RT(A)$. Then the Grothendieck topos $\mathrm{Sh}(\mathbb{C}_A)$ is local; let us write its structure maps as $\Delta' \dashv \Gamma' \dashv \nabla' : \mathbf{Set} \to \mathrm{Sh}(\mathbb{C}_A)$. There is a restricted Yoneda embedding,

$$Y = RT(A)(i(\cdot), -) : RT(A) \to \mathrm{Sh}(\mathbb{C}_A),$$

for which the diagram below commutes, in the sense that $\Gamma \cong \Gamma' \circ Y$, $\nabla' \cong Y \circ \nabla$.

Thus we can regard $\Gamma \colon RT(A) \to \mathbf{Set}$ as what *would be* the direct image of a local map, if only $RT(A)$ had enough colimits.

(2) Let $\mathbb{C}$ be a small category with finite limits and $i \colon \mathbb{D} \longrightarrow \mathbb{C}$ a full subcategory, closed under the same. The geometric morphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ with direct image the restriction i^* along i is then a local map. The image of $\widehat{\mathbb{D}}$ under the full embedding $i_!$ (where $i_! \dashv i^*$) then consists of those presheaves P on $\mathbb{C}$ for which

$$P C \;\cong\; \operatorname*{colim}_{D,h \colon C \to D} P D.$$

These are the objects that we are interested in as candidates for being 'computable', when $\mathbb{D}$ represents the computable subcategory. They are the ones termed 'discrete' in the rest of this paper.

Regarding this choice of terminology, we use the term 'discrete' by analogy to topological examples. We would have liked to call these objects 'cosheaves' since they are the objects that are *co*orthogonal to the morphisms inverted by $\mathbf{a}$, and sheaves are those that are orthogonal. However, 'cosheaf' has already been used to describe something else, namely a 'covariant sheaf'.

4. Axioms for localic local maps

In this section we present a set of axioms for localic local maps. The axioms are sound and complete, in the sense that whenever a given topos satisfies the axioms, it gives rise to a localic local map of toposes and, moreover, any localic local map of toposes satisfies the axioms. See Awodey and Birkedal (1999) and Birkedal (2000) for more information about the abstract axiomatisation of (not only localic but also more general bounded) local maps. Here we just recall the definitions and results needed in the following to describe a *modal logic for computability*. First we need a couple of definitions.

For the remainder of this section let $\mathscr{E}$ be an elementary topos and j a Lawvere–Tierney topology in $\mathscr{E}$. The axioms we will present are on $\mathscr{E}$ and j. We write $V \mapsto \overline{V}$ for the associated closure operation on subobjects $V \rightarrowtail X$. We say that j is **principal** if, for all $X \in \mathscr{E}$, the closure operation on $Sub(X)$ has a left adjoint $U \mapsto \overset{\circ}{U}$, called **interior**, that is,

$$\overset{\circ}{U} \leqslant V \;\Longleftrightarrow\; U \leqslant \overline{V} \qquad \text{in } Sub(X).$$

The interior operation is *not* assumed to be natural; that is, it is not assumed to commute with pullbacks. In other words, the interior operation is not assumed to be induced by an internal map on the subobject classifier Ω in $\mathscr{E}$, and in that sense it is not a *logical operation* (in the internal logic of $\mathscr{E}$).

The interior operation extends to a functor $\mathscr{E} \to \mathscr{E}$, since, whenever $f: X \to Y$, we have $\overset{\circ}{X} \leqslant f^*(\overset{\circ}{Y})$ (because: $\overset{\circ}{X} \leqslant f^*(\overset{\circ}{Y}) \iff X \leqslant \overline{f^*(\overset{\circ}{Y})} = f^*(\overset{\circ}{Y}) = f^*(\overline{Y})$). We say that an object $X \in \mathscr{E}$ is **open** if $\overset{\circ}{X} \cong X$. An object is open iff the interior of its diagonal equals its diagonal. An object $C \in \mathscr{E}$ is called **discrete** if it is **coorthogonal** to all morphisms inverted by the j-associated sheaf functor **a**; that is, C is discrete if for all $e: X \to Y$ such that $\mathbf{a}(e)$ is an isomorphism, for all $f: C \to Y$, there exists a unique $f': C \to X$ such that

$$
\begin{array}{ccc}
 & & X \\
 & \overset{f'}{\nearrow} & \downarrow e \\
C & \underset{f}{\longrightarrow} & Y
\end{array}
$$

commutes.

Recall, for example, from Johnstone (1977), that a sheaf can be characterised as an object that is orthogonal to all morphisms inverted by **a**, and that it suffices to test orthogonality just with respect to the dense monomorphisms. For discrete objects there is a similar simplification: an object is discrete iff it is coorthogonal to all **codense epimorphisms**, where an epimorphism $e: X \twoheadrightarrow Y$ is codense iff the interior of its kernel is included in the diagonal of X. We write $\mathrm{D}_j\mathscr{E}$ for the full subcategory of $\mathscr{E}$ of the discrete objects.

Now we propose the following **axioms for a localic local map** on a topos $\mathscr{E}$ with topology j.

Axiom 1. The topology j is principal.

Axiom 2. For all $X \in \mathscr{E}$, there exists a discrete object C and a diagram

$$
\begin{array}{ccc}
S & \rightarrowtail & C \\
\downarrow & & \\
X & &
\end{array}
$$

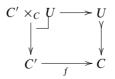in $\mathscr{E}$, presenting X as a subquotient of C.

Axiom 3. For all discrete $C \in \mathscr{E}$, if $X \rightarrowtail C$ is open, then X is also discrete.

Axiom 4. For all discrete $C, C' \in \mathscr{E}$, the product $C \times C'$ is again discrete.

Let $\mathscr{E}$ be a topos with a topology j satisfying the Axioms 1–4 for localic local maps. We can then prove (Awodey and Birkedal 1999; Birkedal, 2000) the following theorem.

Theorem 4.1. The category of discrete objects $\mathrm{D}_j\mathscr{E}$ is coreflective in $\mathscr{E}$, that is, the inclusion $\Delta: \mathrm{D}_j\mathscr{E} \hookrightarrow \mathscr{E}$ has a right adjoint. Moreover, $\mathrm{D}_j\mathscr{E}$ is a topos, Δ is left exact, and $(\Delta, \Gamma): \mathscr{E} \to \mathrm{D}_j\mathscr{E}$ is a localic local map.

Corollary 4.2. For any discrete $C, C' \in \mathscr{E}$ and any $f: C' \to C$, and all open subobjects $U \rightarrowtail C$, the pullback $C' \times_C U$ indicated in

$$
\begin{array}{ccc}
C' \times_C U & \longrightarrow & U \\
\downarrow & & \downarrow \\
C' & \underset{f}{\longrightarrow} & C
\end{array}
$$

is open.

Theorem 4.3. Every localic local map of toposes satisfies Axioms 1-4 for localic local maps.

Proof. See Awodey and Birkedal (1999) and Birkedal (2000). □

5. A modal logic for computability

Let $\mathscr{E}$ be a topos with a topology j satisfying the axioms set out in the previous section. In this section our goal is to describe a *logic* with which one can reason about *both* of the two toposes $\mathscr{E}$ and $\mathrm{D}_j\mathscr{E}$. This will then apply to $RT(A, A_\sharp)$ and $RT(A_\sharp)$, see Section 7.

As a first attempt, one may consider the ordinary internal logic of $\mathscr{E}$ extended with a closure operator induced by the topology j and try to extend it further with a logical operator corresponding to the interior operation. But since interior does not commute with pullback in general, it is not a logical operation on all subobjects of objects of $\mathscr{E}$. However, since the interior of an object X is the least dense subobject of X, one may instead add a new atomic predicate U_X for each type X and write down axioms expressing that it is the least dense subobject. This is straightforward. But, as yet, we do not have a convenient internal logical characterisation in these terms for the discrete objects.

Instead, we shall describe another approach in which types and terms are objects and morphisms of $\mathrm{D}_j\mathscr{E}$, and predicates are all the predicates in $\mathscr{E}$ on objects from $\mathrm{D}_j\mathscr{E}$. More precisely, we consider the internal logic of the fibration

$$\begin{array}{c} Pred \\ \downarrow \\ \mathrm{D}_j\mathscr{E} \end{array}$$

obtained by change-of-base as indicated in

$$\begin{array}{ccc} Pred & \longrightarrow & Sub(\mathscr{E}) \\ \downarrow & & \downarrow \\ \mathrm{D}_j\mathscr{E} & \xrightarrow{\ \Delta\ } & \mathscr{E}. \end{array}$$

Thus a predicate on an object $X \in \mathrm{D}_j\mathscr{E}$ is a subobject of ΔX in $\mathscr{E}$. Since

$$Sub_{\mathscr{E}}(\Delta X) \cong \mathscr{E}(\Delta X, \Omega_{\mathscr{E}}) \cong \mathrm{D}_j\mathscr{E}(X, \Gamma\Omega_{\mathscr{E}}), \tag{2}$$

the internal locale $\Gamma\Omega_{\mathscr{E}}$ is a generic object for the fibration

$$\begin{array}{c} Pred \\ \downarrow \\ \mathrm{D}_j\mathscr{E} \end{array}.$$

Hence this internal logic is many-sorted higher-order intuitionistic logic.

Note that, since $\mathscr{E}$ is localic over $\mathrm{D}_j\mathscr{E}$, we can completely describe $\mathscr{E}$ in this internal logic in the standard way (Fourman and Scott 1979) as partial equivalence relations and functional relations between such. In other words, applying the tripos-to-topos construction to the tripos

$$\begin{array}{c} Pred \\ \downarrow \\ \mathrm{D}_j\mathscr{E} \end{array}$$

results in a topos equivalent to $\mathscr{E}$.

By Corollary 4.2, the interior operation is a logical operation on predicates; we denote it by $\sharp$. Note that the ordinary logic of $D_j\mathscr{E}$ is obtained by restricting our attention to predicates of the form $\sharp\varphi$ – in fact it suffices to restrict attention to predicates with a $\sharp$ in front of every $\supset$ and $\forall$ subformula. The topology j in $\mathscr{E}$ induces a closure operator, which we are pleased to denote $\flat$, on predicates in this internal logic.

We now describe how the $\sharp$ and $\flat$ operations can be axiomatised. Logical entailment is written $\Gamma \mid \varphi \vdash \psi$, where Γ is a context of the form $x_1 : \sigma_1, \dots, x_n : \sigma_n$ giving types σ_i to variables x_i, and where φ and ψ are formulas with free variables in Γ. There are the usual rules of many-sorted higher-order intuitionistic logic plus the following axioms and rules:

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi} \text{ (ML-1)} \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi} \text{ (ML-2)}$$

$$\frac{}{\Gamma \mid \top \vdash \sharp(\top)} \text{ (ML-3)} \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)} \text{ (ML-4)}$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\Gamma \mid \varphi \vdash \flat\psi} \text{ (ML-5)} \qquad \frac{}{x : \sigma, y : \sigma \mid x = y \vdash \sharp(x = y)} \text{ (ML-6)}$$

One can then show that $\sharp$ has the formal properties of the box operator in the modal logic S4, that is, for formulas φ and ψ in context Γ,

$$\vdash \sharp\varphi \supset \varphi$$
$$\vdash \sharp(\varphi \supset \psi) \supset (\sharp\varphi \supset \sharp\psi)$$
$$\vdash \sharp\varphi \supset \sharp\sharp\varphi$$

and

$$\frac{\vdash \varphi}{\vdash \sharp\varphi} \text{ (ML-7)}$$

We therefore refer to this logic as a **modal logic for computability**.

We remark that the following principles of inference for quantifiers can be derived from ML-1–ML-6:

$$\frac{\Gamma \mid \top \vdash \sharp\forall x{:}X.\, \varphi}{\Gamma \mid \top \vdash \forall x{:}X.\, \sharp\varphi} \text{ (ML-8)}$$

$$\frac{}{\Gamma \mid \sharp\exists x{:}X.\, \varphi \dashv\vdash \exists x{:}X.\, \sharp\varphi} \text{ (ML-9)}$$

One can also show that, for any formula φ in context Γ,

$$\sharp(\sharp\varphi \supset \psi) \dashv\vdash \sharp\varphi \supset \sharp\psi. \tag{3}$$

Quite generally, the modal logic of any local map of toposes

$$\Gamma : \mathscr{E} \to \mathscr{F}, \qquad \Delta \dashv \Gamma \dashv \nabla$$

can be used to compare the internal logic of $\mathscr{E}$ with that of $\mathscr{F}$, since the types are then the discrete objects E in $\mathscr{E}$, that is, those for which $E \cong \Delta\Gamma E$, and

$$Sub_{\mathscr{F}}(\Gamma E) \cong OpenSub_{\mathscr{E}}(E),$$

where $OpenSub_{\mathscr{E}}(E) \subseteq Sub_{\mathscr{E}}(E)$ is the subposet of open subobjects of E in $\mathscr{E}$. Observe, for example, that the natural numbers object N is among the discrete objects, and that the identity relation on any discrete object is open.

We now describe two example applications of the modal logic.

Example 1. As in Awodey and Birkedal (1999), call a formula ϑ **stable** if it is built up from atomic predicates (including equations) and first-order logic, and if for every subformula of the form $\varphi \supset \psi$, the formula φ has no $\forall$ or $\supset$. For any sentence ϑ, we write $\mathscr{F} \vDash \vartheta$ to mean that ϑ holds in the standard internal logic of $\mathscr{F}$ with basic types σ interpreted by objects X_σ of $\mathscr{F}$, and atomic formulas R on type σ interpreted as subobjects $S_R \rightarrowtail X_\sigma$. We then write $\mathscr{E} \vDash \vartheta$ to mean that ϑ holds in the standard logic of $\mathscr{E}$ with basic types σ interpreted by objects ΔX_σ and atomic formulas R interpreted by $\Delta S_R \rightarrowtail \Delta X_\sigma$.

Proposition 5.1. For any stable sentence ϑ, $\mathscr{F} \vDash \vartheta$ iff $\mathscr{E} \vDash \vartheta$.

Proof (sketch). There are the interpretations $[\![-]\!]_{\mathscr{F}}$ and $[\![-]\!]_{\mathscr{E}}$ for which one shows by induction that for any stable formula ϑ

$$\Delta[\![\vartheta]\!]_{\mathscr{F}} = [\![\sharp\vartheta]\!]_{\mathscr{E}},$$

using the fact that Γ preserves $\forall$ along maps between discrete objects. Thus, for any stable sentence ϑ

$$
\begin{aligned}
\mathscr{F} \vDash \vartheta \quad &\text{iff} \quad 1 = [\![\vartheta]\!]_{\mathscr{F}} \\
&\text{iff} \quad 1 = \Delta 1 = \Delta[\![\vartheta]\!]_{\mathscr{F}} = [\![\sharp\vartheta]\!]_{\mathscr{E}} \\
&\text{iff} \quad \mathscr{E} \vDash \sharp\vartheta \\
&\text{iff} \quad \mathscr{E} \vDash \vartheta. \qquad\qquad\qquad\qquad \square
\end{aligned}
$$

The proposition can be used to show that, for example, if $\mathscr{F}$ has choice for functions from N to N in the external sense, then so does $\mathscr{E}$. Indeed, let R be any relation (not necessarily open) on N in $\mathscr{E}$, and suppose that

$$\mathscr{E} \vDash \forall n{:}N.\, \exists m{:}N.\, R(n,m).$$

Then we reason informally as follows:

$$
\begin{aligned}
&\mathscr{E} \vDash \forall n{:}N.\, \exists m{:}N.\, R(n,m), \\
&\mathscr{E} \vDash \sharp\forall n{:}N.\, \exists m{:}N.\, R(n,m) && \text{by ML-7,} \\
&\mathscr{E} \vDash \forall n{:}N.\, \sharp\exists m{:}N.\, R(n,m) && \text{by ML-8,} \\
&\mathscr{E} \vDash \forall n{:}N.\, \exists m{:}N.\, \sharp R(n,m) && \text{by ML-9,} \\
&\mathscr{F} \vDash \forall n{:}N.\, \exists m{:}N.\, \sharp R(n,m) && \text{by stability,} \\
&\mathscr{F} \vDash \forall n{:}N.\, \sharp R(n, f(n)) && \text{for some } f : N \to N \text{ by AC in } \mathscr{F}, \\
&\mathscr{E} \vDash \forall n{:}N.\, \sharp R(n, f(n)) && \text{by stability,} \\
&\mathscr{E} \vDash \forall n{:}N.\, R(n, f(n)) && \text{by ML-1.}
\end{aligned}
$$

Example 2. As in Birkedal (2000), we show that if the topos of discrete objects satisfies the arithmetic form of Church's Thesis (in the sense of, for instance, Troelstra and van Dalen (1988) and Troelstra (1973), then a $\sharp$'ed version is satisfied by $\mathscr{E}$.

Observe that $\mathscr{E}$ has a natural numbers object if and only if $D_j\mathscr{E}$ does, because both

$\Delta : D_j \mathscr{E} \to \mathscr{E}$ and $\Gamma : \mathscr{E} \to D_j \mathscr{E}$ are inverse images of geometric morphisms, which preserve the natural numbers object (Johnstone 1977, Proposition 6.12). Let N be a natural numbers object in $D_j \mathscr{E}$.

Recall from Kleene's Normal Form Theorem that the basic predicates of recursion theory can be defined from Kleene's T-predicate and output function $U : N \to N$, see, for example, Troelstra and van Dalen (1988). The predicates $T \rightarrowtail N \times N \times N$ and $U(-) = (-)$ are both primitive recursive. Hence their interpretation is preserved by the inclusion $D_j \mathscr{E} \longrightarrow \mathscr{E}$ (see Birkedal (2000) for a detailed argument).

Recall from Troelstra and van Dalen (1988, Section 4.3) that the **arithmetical form of Church's Thesis** is the schema

$$\mathrm{CT}_0 \qquad \forall n. \exists m. \varphi(n,m) \supset \exists k. \forall n. \exists m. \big(\varphi(n, Um) \wedge T(k,n,m)\big),$$

where all the variables range over the type of natural numbers N and where φ is a formula.

For a first-order formula φ, we use $|\varphi|$ to denote the formula that is like φ except that there is a $\sharp$ in front of every $\forall$ and $\supset$ subformula.

Proposition 5.2. Let φ be a formula and suppose that the φ instance of CT_0 holds in $\begin{smallmatrix} Sub\,D_j \mathscr{E} \\ \downarrow \\ D_j \mathscr{E} \end{smallmatrix}$. Then the following formula holds in $\begin{smallmatrix} Pred \\ \downarrow \\ D_j \mathscr{E} \end{smallmatrix}$:

$$\sharp(\forall n. \exists m. |\varphi|(n,m)) \supset \exists k. \forall n. \exists m. \big(|\varphi|(n, Um) \wedge T(k,n,m)\big).$$

Proof. By the fact that $D_j \mathscr{E} \vDash \varphi$ iff $|\varphi|$ holds in $\begin{smallmatrix} Pred \\ \downarrow \\ D_j \mathscr{E} \end{smallmatrix}$, we get that

$$\sharp(\forall n. \exists m. |\varphi|(n,m)) \supset \exists k. \sharp\forall n. \exists m. \big(|\varphi|(n, Um) \wedge T(k,n,m)\big)$$

holds in $\begin{smallmatrix} Pred \\ \downarrow \\ D_j \mathscr{E} \end{smallmatrix}$. The statement then follows by the modal logic using ML-9 and (3). $\qquad\square$

Thus, in particular, if all geometric instances (that is, those with φ built up using only atomic relations, and $\top, \wedge, \bot, \vee, \exists$) of CT_0 hold in $D_j \mathscr{E}$, then all geometric instances of

$$\mathrm{CT}_0^{\sharp} \qquad \sharp(\forall n. \exists m. \varphi(n,m)) \supset \exists k. \forall n. \exists m. \big(\varphi(n, Um) \wedge T(k,n,m)\big)$$

hold in $\begin{smallmatrix} Pred \\ \downarrow \\ D_j \mathscr{E} \end{smallmatrix}$.

6. Local triposes

In this section we introduce the notion of a *local tripos* – we will assume familiarity with the basic theory of triposes as set out in Pitts (1981).

The model of the modal logic for computability given by the fibration

$$\begin{smallmatrix} Pred \\ \downarrow \\ D_j \mathscr{E} \end{smallmatrix}$$

defined above is in fact a tripos, namely the standard tripos on the internal locale $\Gamma\Omega_{\mathscr{E}}$, see (2) above. Indeed, we can give the following general definition.

Definition 6.1. Let $P = \mathscr{F}(-, \Sigma)$ be a canonically-presented tripos on an object Σ in a topos $\mathscr{F}$. The tripos P is said to be **local** if it comes together with maps $I, J : \Sigma \to \Sigma$ in $\mathscr{F}$ satisfying:

1    $p : \Sigma \mid Ip \vdash p$
2    $p : \Sigma \mid Ip \vdash IIp$
3    $\varnothing \mid \top \vdash I(\top)$
4    $p, q : \Sigma \mid Ip \wedge Iq \vdash I(p \wedge q)$

5    $\dfrac{p, q : \Sigma \mid Ip \vdash q}{p, q : \Sigma \mid p \vdash Jq}$

all in the logic of P.

The axioms and rules that have to hold for I and J are of course just as for $\sharp$ and $\flat$ in the modal logic for local maps in Section 5[†]. In other words, a local tripos models the modal logic for local maps. The following proposition is clear.

Proposition 6.2. Let $P = \mathscr{F}(-, \Sigma)$ be a local tripos *qua* $I, J : \Sigma \to \Sigma$. Then J is a Lawvere–Tierney topology on P (as defined in Pitts (1981)).

Let $P = \mathscr{F}(-, \Sigma)$ be a local $\mathscr{F}$-tripos *qua* $I, J : \Sigma \to \Sigma$. We define a new canonically presented $\mathscr{F}$-tripos P_I as follows (in the following we prove that P_I so defined is indeed a tripos). Let $I\Sigma$ be the image of I in $\mathscr{F}$. Tripos P_I is canonically presented on $I\Sigma$. The ordering is defined as in P, that is, for $\varphi, \psi \in \mathscr{F}(X, I\Sigma)$ (P_I's fibre over $X \in \mathscr{F}$), we have $\varphi \vdash^{P_I} \psi$ iff $\varphi \vdash^{P} \psi$.

Since J is a topology by Proposition 6.2, we have a well-defined $\mathscr{F}$-tripos P_J as in Pitts (1981, Chapter 5). We recall that P_J is canonically presented on Σ and that entailment is defined as $\varphi \vdash^{P_J} \psi$ iff $\varphi \vdash^{P} J\psi$.

It is easy to verify that composing with $I : \Sigma \to \Sigma$ gives an indexed functor, also denoted I, from P_J to P_I over $\mathscr{F}$. Likewise, composing with $J : \Sigma \to \Sigma$ gives a fibred functor, also denoted J, from P_I to P_J over $\mathscr{F}$. The following lemma is straightforward.

Lemma 6.3. Functor I is a fibred left adjoint to J and the triposes P_I and P_J are equivalent, as indexed categories over I, *via* the functors I and J.

By the lemma, it follows that P_I has all the first-order structure required in the definition of a tripos (since it is defined categorically and thus preserved by equivalence functors). It is clear that $id : I\Sigma \to I\Sigma$ is a generic object for P_I, and thus P_I is indeed a tripos as claimed.

Theorem 6.4. Let $P = \mathscr{F}(-, \Sigma)$ be a local $\mathscr{F}$-tripos *qua* $I, J : \Sigma \to \Sigma$. Then P gives rise to a localic local map of toposes from $\mathscr{F}[P]$ to $\mathscr{F}[P_I]$.

[†] The only exception is that in the definition of local tripos we have left out the rule for equality – the rule for equality follows since equality in a tripos is given using existential quantification and truth $\top$, and I commutes with existential quantification as a left adjoint and with $\top$ by item (3).

Proof. We define three indexed functors over $\mathscr{F}$, as in

$$P_I \underset{J}{\overset{\Delta}{\underset{\xleftarrow{\hspace{1em}} I \xrightarrow{\hspace{1em}}}{\rightleftarrows}}} P \qquad \Delta \dashv I \dashv J.$$

Functor Δ is simply the inclusion functor. Functor I is induced by composing with $I : \Sigma \to \Sigma$, and functor J is induced by composing with $J : \Sigma \to \Sigma$. It is easy to see that all three functors are indexed, since P and P_I are canonically presented, and that Δ is left adjoint to I and that I is left adjoint to J. The functor Δ is left exact since $I\Sigma$ is closed under finite limits in Σ by items (3) and (4) in the definition of a local tripos. Hence (Δ, I) is a geometric morphism of triposes, as (I, J) is also. Functor Δ is clearly full and faithful and thus J is full and faithful also (it is also straightforward to verify directly that J is full and faithful).

It follows now, in the same way as in Section 3, that the geometric morphism from $\mathscr{F}[P]$ to $\mathscr{F}[Q]$ induced by (Δ, I) is a localic local map. $\qquad\square$

Conversely, one can easily see the following theorem.

Theorem 6.5. Every localic local map of toposes arises from a local tripos (in the way given by the proof of Theorem 6.4).

Example 6.6. The relative realizability tripos P from Section 2 is a local tripos. The maps I and J are given by $\Delta\Gamma$ and $\nabla\Gamma$, respectively, see Section 3.

In this realizability example, the topos $\mathscr{E}$ is the topos $RT(A_\sharp)$ and the topos $D_j\mathscr{E}$ is the topos $RT(A, A_\sharp)$. Note that if we were just given the topos $RT(A_\sharp)$ with the internal locale $\Gamma\Omega_{RT(A, A_\sharp)} = (PA, \approx)$ with $(p \approx q) = (p \rightleftharpoons q) \cap A_\sharp$, it would be quite hard to recognise the canonical tripos on this locale as being local, because it is complicated to calculate with internal adjoints, *etc.*, in $RT(A_\sharp)$. That is one reason why it can be advantageous to describe a localic local map of toposes *via* a local tripos (rather than just in terms of internal locale theory).

Example 6.7 (Extensional realizability). Let A be a PCA and let P be the standard **Set**-realizability tripos over A. Let **PER**(A) denote the category of partial equivalence relations over A. Define a tripos R over **Set** by taking predicates over sets I to be elements of **PER**$(A)^I$, that is, I-indexed families of PER's. For two such families φ and ψ, we define the ordering over I to be $\varphi \vdash \psi$ iff there is an $a \in A$ such that, for all $i \in I$, a is in the domain of the PER $\psi(i)^{\varphi(i)}$ (the exponential in the category of PER's). See Pitts (1981, Section 1.6) and van Oosten (1997) for more details. Then the tripos R is local over P, since the forgetful functor mapping a PER to its domain has both left and right (fibred) full and faithful adjoints. Over 1, the left adjoint maps a subset of A to the discrete PER on the subset and the right adjoint maps a subset of A to the PER with only one equivalence class. See Pitts (1981, Example 4.9(iii)) and van Oosten (1997) for more details.

7. Interpretation of the modal logic in $RT(A_\sharp)$ and $RT(A, A_\sharp)$

Finally, we briefly describe in concrete terms how the modal logic for computability is interpreted in $RT(A_\sharp)$ and $RT(A, A_\sharp)$.

Types and terms are interpreted by objects and morphisms of $RT(A_\sharp)$ in the standard way. A predicate φ on a type $(X, \approx_X) \in RT(A_\sharp)$ is an equivalence class of a strict, extensional relation in $P(X \times X)$ (recall P is the tripos underlying $RT(A, A_\sharp)$), that is, φ is an equivalence class of set-theoretic functions $X \times X \to PA$ that are strict and extensional *via* computable realizers, two such functions being equivalent iff they are isomorphic as objects of $P(X \times X)$.

On such a predicate φ on an object $(X, \approx)$, $\sharp\varphi$ is just $x \mapsto \varphi(x) \cap A_\sharp$ and $\flat\varphi$ is

$$x \mapsto \bigcup_{\psi \in PA} (\psi \wedge (\psi \cap A_\sharp \supset \varphi(x) \cap A_\sharp)).$$

Thus we can think of $\sharp\varphi$ as φ being computably true, that is, realized *via* computable realizers.

Objects of $RT(A, A_\sharp)$ are then described as pairs $((X, \approx), \varphi)$, with $(X, \approx) \in RT(A_\sharp)$ and φ a partial equivalence in $Pred$ on $(X, \approx) \times (X, \approx)$. Likewise, morphisms are described as functional relations in the standard way.

In this realizability model, we have the following further principle for $\sharp$:

$$\Gamma \mid \neg\varphi \dashv\vdash \sharp\neg\varphi,$$

because the types are the objects of $RT(A_\sharp)$. From this it follows that

$$\Gamma \mid \neg\neg\varphi \dashv\vdash \sharp\neg\neg\varphi,$$

which accords with the intuition that double-negation closed formulas have no computational content.

Note also that since $\Gamma : RT(A_\sharp) \to \mathbf{Set}$ has a right adjoint, the same is true for the global sections functor $\Gamma : RT(A, A_\sharp) \to \mathbf{Set}$. Thus in $RT(A, A_\sharp)$ too, 1 is indecomposable and projective; so $RT(A, A_\sharp)$ has the logical disjunction and existence properties.

Example 7.1. Let $A = P\mathbb{N}$ be the graph model and let $A_\sharp = RE$ be the recursively enumerable sub-graph-model. The arithmetical form of Church's Thesis holds in $RT(RE)$, but not in $RT(P\mathbb{N})$ (since arithmetic in $RT(P\mathbb{N})$ is classical). Since there is a logical functor from $RT(A, A_\sharp) = RT(P\mathbb{N}, RE)$ to $RT(A) = RT(P\mathbb{N})$ (Birkedal, 2000), Church's Thesis does not hold in $RT(A, A_\sharp) = RT(P\mathbb{N}, RE)$ either. However, using the modal logic for computability, we can express that a form of Church's Thesis does hold in $RT(P\mathbb{N}, RE)$: by Proposition 5.2, for every arithmetical formula φ,

$$\sharp(\forall n. \exists m. |\varphi|(n, m)) \supset \exists k. \forall n. \exists m. \big(|\varphi|(n, Um) \wedge T(k, n, m)\big)$$

holds in the modal logic.

More information about the relative realizability topos $RT(A, A_\sharp)$ can be found in the second author's Ph. D. thesis (Birkedal, 2000).

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