Relative vertex asphericity

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Abstract. Diagrammatic reducibility DR and its generalization, vertex asphericity VA, are combinatorial tools developed for detecting asphericity of a 2-complex. Here we present tests for a relative version of VA that apply to pairs of 2-complexes (L, K), where K is a subcomplex of L. We show that a relative weight test holds for injective labeled oriented trees, implying that they are VA and hence aspherical. This strengthens a result obtained by the authors in 2017 and simplifies the original proof.

1 Introduction

A 2-complex L is *vertex aspherical* VA, if every combinatorial map from a 2-sphere into L contains a pair of faces with a vertex in common so that the faces are mapped mirrorwise across this vertex to the same 2-cell of L. Vertex asphericity implies topological asphericity. The closely related concept of *diagrammatic reducibility* DR was introduced by Sieradski [14] in 1983. See also Gersten [8]. The weaker notion of VA was first considered by Huck–Rosebrock [11]. Recent developments concerning combinatorial asphericity can be found in Barmak–Minian [1] and Blufstein–Minian [2].

Relative vertex asphericity for pairs of 2-complexes $K \subseteq L$ already appeared in a previous article [9] by the authors, where it was used to establish asphericity of injective LOT-complexes.

Other and related notions of relative combinatorial asphericity are in the literature. Diagrammatic reducibility for relative presentations was considered by Bogley–Pride [3] in 1992 and has found many applications over the years. See Bogley–Edjvet–Williams [4] for a good overview. Very recently the idea of directed diagrammatic reducibility was introduced and studied by the authors in [10].

A powerful method for showing that a 2-complex L is DR is by showing that it satisfies a weight test. This test appeared first in work of Sieradski [14], who called it a coloring test, and was later generalized by Gersten [8] who saw it as a combinatorial version of the Gauss–Bonet Theorem. Sieradski allowed only colors (now called weights) 0 and 1, whereas any real number could be used for weights in Gersten's version. Wise [15] showed that if L satisfies the coloring test then it has nonpositive sectional curvature and hence $\pi_1(L)$ is locally indicable. This is not true in the context of Gersten's weight test. In this paper we give a relative weight test for pairs (L, K) that implies relative VA.

The Whitehead conjecture, which states that subcomplexes of aspherical 2-complexes are aspherical, has motivated the development of combinatorial versions of asphericity, such as DR and VA, and labeled oriented trees have long been known to be an important testing ground for the conjecture. In [12] Huck and Rosebrock proved that prime injective labeled oriented trees satisfy Sieradski's coloring test. In [9] the authors showed that injective labeled oriented trees are aspherical. We strengthen this result here by showing that injective labeled oriented trees satisfy a relative weight test with weights 0 and 1 and hence are VA. The proof is a direct generalization of the proof given in the prime case.

We thank the referee for several helpful comments and suggestions.

2 Relative Vertex Asphericity

A map $f: X \to Y$ between complexes is *combinatorial* if f maps open cells of X homeomorphically to open cells of Y. Here a 2-complex will always mean a 2-dimensional cell complex with combinatorial attaching maps. A *spherical diagram* over a 2-complex L is a combinatorial map $C \to L$, where C is a 2-sphere with a cell structure. If a 2-complex L is nonaspherical, then there exists a spherical diagram which realizes a nontrivial element of $\pi_2(L)$. In fact, $\pi_2(L)$ is generated by spherical diagrams. So in order to check whether a 2-complex is aspherical or not it is enough to check spherical diagrams. We also study *surface diagrams*. These are combinatorial maps $F \to L$, where F is an orientable surface with or without boundary.

The link of a vertex u, $\mathrm{lk}(L, u)$, is the boundary of a regular neighborhood of u in L. So $\mathrm{lk}(L, u)$ is a graph whose edges are the corners of 2-cells at u. Suppose L is a standard 2-complex with a single vertex u and oriented edge set X. Then the vertices of $\mathrm{lk}(L, u) = \mathrm{lk}(L)$ are $\{x^+, x^- \mid x \in X\}$, where x^+ is a point of the oriented edge x close to the beginning, and x^- is a point close to the ending of that edge. The *positive link* $\mathrm{lk}^+(L)$ is the *full subgraph* on the vertex set $\{x^+ \mid x \in X\}$ and the *negative link* $\mathrm{lk}^-(L)$ is the full subgraph on the vertex set $\{x^- \mid x \in X\}$. The positive and negative links were called right and left graph, respectively, in [12]. Let $f: C \to L$ be a spherical diagram and let v be a vertex in C. Restricting to the link we obtain a combinatorial map $f|_{\mathrm{lk}(C,v)}: \mathrm{lk}(C, v) \to \mathrm{lk}(L, f(v))$ for every vertex $v \in C$ and we let $z(v) = c_1 \ldots c_q$ be the image, which is a closed edge path in $\mathrm{lk}(L, f(v))$.

Definition 2.1   Let Γ be a graph and Γ_0 be a subgraph. Let $z = e_1 \ldots e_q$ be a cycle (closed edge path) in Γ. We say

(1) z is *homology reduced* if it contains no pair of edges e_i, e_j such that $e_i = \bar{e}_j$ (the bar indicates opposite orientation and we read z cyclically);

(2) z is *homology reduced relative to* Γ_0 if any pair of edges e_i, e_j such that $e_i = \bar{e}_j$ is contained in Γ_0.

Let $f: C \to L$ be a spherical diagram. A vertex $v \in C$ is called a *folding vertex* if $z(v) = c_1 \ldots c_q \in \mathrm{lk}(L, f(v))$ is not homology reduced. In that case the pair of 2-cells (d_i, d_j) of C containing the preimages of c_i and c_j, respectively, satisfying $c_i = \bar{c}_j$ is

called a *folding pair*. We call f *vertex reduced* if it does not have a folding vertex. A 2-complex L is called *vertex aspherical* VA if each spherical diagram over L has a folding vertex. Clearly VA implies asphericity.

Definition 2.2   Let K be a subcomplex of the 2-complex L. We say that L *is VA relative to K* if every spherical diagram $f: C \to L$, $f(C) \nsubseteq K$, has a folding vertex with folding pair of 2-cells in $L - K$.

We can phrase relative VA also in the following way. L is VA relative to K if in every spherical diagram $f: C \to L$, $f(C) \nsubseteq K$, there is a vertex $v \in C$ so that $z(v) = c_1 \ldots c_q \subseteq$ $\mathrm{lk}(L)$ is not contained in $\mathrm{lk}(K)$ and is not homology reduced relative to $\mathrm{lk}(K)$.

Theorem 2.3   *Let L be a 2-complex and K a subcomplex. If K is VA and L is VA relative to K then L is VA.*

Proof   Assume $f: C \to L$ is a vertex reduced spherical diagram. Since L is VA relative to K we have that $f(C) \subseteq K$. So $f: C \to K$ is a vertex reduced spherical diagram, contradicting the assumption that K is VA.   ∎

Theorem 2.4   *If L is VA relative to K, then $\pi_2(L)$ is generated, as $\pi_1(L)$-module, by the image of $\pi_2(K)$ under the map induced by inclusion. In particular, if K is aspherical, then so is L.*

Proof   Every vertex reduced spherical diagram $f: C \to L$ has its image $f(C)$ in K. Thus f represents an element in $\pi_2(K)$. Since $\pi_2(L)$ is generated by vertex reduced spherical diagrams, it follows that $\pi_2(L)$ is generated by the image of $\pi_2(K)$.   ∎

3   Tests for Relative Vertex Asphericity

Let $K \subseteq L$ be 2-complexes. We say a spherical diagram $f: C \to L$ is *K-thin* if for every vertex $v \in C$ there is a 2-cell in C containing v which is mapped to a 2-cell in $L - K$. Thus if $f(\mathrm{lk}(C, v)) = z(v) = c_1 \ldots c_k$, then at least one corner $c_i \in \mathrm{lk}(L) - \mathrm{lk}(K)$. We can apply a "reversed subdivision" to C to turn a spherical diagram into a thin one. The idea is to collect material in C that forms an open disc in C which is mapped to K and make it into a single 2-cell in C. For this to work we need to attach additional 2-cells to K.

Definition 3.1   Given a pair (L, K) of 2-complexes, where K is a subcomplex of L, we say that $(\hat{L}, \hat{K})$ is a *thinning expansion* if the following holds

(1) $\hat{K}$ is obtained from K by adding 2-cells and $\hat{L} = L \cup \hat{K}$;
(2)   If there exists a vertex reduced spherical diagram $f: C \to L$, $f(C) \nsubseteq K$, then there also exists a $\hat{K}$-thin vertex reduced spherical diagram $f': C' \to \hat{L}$, $f'(C') \nsubseteq \hat{K}$.

Thinning expansions always exist which can be seen in the following example.

Example 1 Let (L, K) be a 2-complex pair and consider a spherical diagram $f: C \to L$. We can remove open discs from C to obtain a planar diagram $g: F \to L - \{$open 2-cells of $K\}$, where F is a connected planar region. Each boundary component S of F maps to K and presents a trivial element in $\pi_1(K)$. For each S we attach a 2-cell to K using $g: S \to K$ as attaching map. We do this for all spherical diagrams over L and arrive at a complex $\hat{K}$. Note that we can attach discs to F and produce a thin spherical diagram $f': C' \to \hat{L} = L \cup \hat{K}$. If C does not contain a folding pair (d_1, d_2), $d_i \in L - K$, then neither does $f': C' \to \hat{L} = L \cup \hat{K}$. This construction gives the *minimal thinning expansion* of (L, K). Note that $(\pi_1(\hat{L}), \pi_1(\hat{K})) = (\pi_1(L), \pi_1(K))$.

Example 2 The *maximal thinning expansion* is obtained by adding a 2-cell for every closed edge path in K. In this case $(\pi_1(\hat{L}), \pi_1(\hat{K})) \neq (\pi_1(L), \pi_1(K))$.

If K is a 2-complex with oriented edges, then an attaching map of a 2-cell $d \in K$ has *exponent sum 0*, when traveling along the boundary of d in clockwise direction, one encounters the same number of positive as of negative edges. Here is a setting we will be using for applications.

Example 3 Suppose $K = K_1 \vee \ldots \vee K_n \subseteq L$ and that the attaching maps of 2-cells of K have exponent sum zero. We construct $\hat{K}_i$ from K_i by attaching 2-cells to every closed-edge path in K_i of exponent sum zero. We let $\hat{K} = \hat{K}_1 \vee \ldots \vee \hat{K}_n$ and let $\hat{L} = L \cup \hat{K}$. Note that $(\hat{L}, \hat{K})$ contains the minimal thinning expansion given in Example 1 and hence is itself a thinning expansion.

Let Γ be a graph and $\hat{\Gamma} = \Gamma_1 \cup \ldots \cup \Gamma_n$ be a union of disjoint subgraphs. We do not assume that Γ, or the Γ_i, are connected. We write $\Gamma / \hat{\Gamma}$ for the graph obtained from Γ by collapsing each Γ_i to a point.

Definition 3.2 Let Γ be a graph and $\hat{\Gamma} = \Gamma_1 \cup \ldots \cup \Gamma_n$ be a disjoint union as above. We say Γ is a *forest relative to* $\hat{\Gamma}$ if $\Gamma / \hat{\Gamma}$ has no cycles. Γ is called a *tree relative to* $\hat{\Gamma}$ if in addition Γ is connected.

Let C be a cell decomposition of the 2-sphere with oriented 1-cells. A *source* in C is a vertex with all its adjacent edges pointing away from it, and a *sink* is a vertex with all its adjacent edges pointing towards it. A 2-cell $d \in C$ is said to have *exponent sum 0* if, when traveling along the boundary of d in clockwise direction, one encounters the same number of positive as of negative edges. The following theorem is due to Gersten (see [8]):

Theorem 3.3 *Let C be a cell decomposition of the 2-sphere with oriented edges, such that all 2-cells have exponent sum 0. Then C contains a sink and a source.*

Proof Fix a vertex $v \in C$. If w is a vertex in C, define $h(w)$ to be the exponent sum of an edge path in C that connects v to w. The height $h(w)$ is well defined because of the exponent sum zero condition of C. A vertex of maximal height is a sink, and a vertex of minimal height is a source. ∎

A subcomplex K of a 2-complex L is called *full*, if for every 2-cell $d \in L$ where all boundary cells are in K we have $d \in K$. If $K = K_1 \vee \ldots \vee K_n \subseteq L$ then $\mathrm{lk}^+(K) = \mathrm{lk}^+(K_1) \cup \ldots \cup \mathrm{lk}^+(K_n)$ is a disjoint union, and $\mathrm{lk}^-(K) = \mathrm{lk}^-(K_1) \cup \ldots \cup \mathrm{lk}^-(K_n)$ is a disjoint union as well.

Theorem 3.4   *Let $K = K_1 \vee \ldots \vee K_n \subseteq L$. We assume the attaching maps of 2-cells in L have exponent sum 0, and the K_i are full. If $\mathrm{lk}^+(L)$ is a forest relative to $\mathrm{lk}^+(K)$ or $\mathrm{lk}^-(L)$ is a forest relative to $\mathrm{lk}^-(K)$ then L is VA relative to K. Furthermore, the inclusion induced homomorphism $\pi_1(K_i) \to \pi_1(L)$ is injective for every $i = 1, \ldots, n$.*

Proof   Let us assume that $\mathrm{lk}^+(L)$ is a forest relative to $\mathrm{lk}^+(K)$. Consider a thinning expansion $(\hat{L}, \hat{K})$ as in Example 3. Note that $\mathrm{lk}(L) - \mathrm{lk}(K) = \mathrm{lk}(\hat{L}) - \mathrm{lk}(\hat{K})$, and hence $\mathrm{lk}^+(\hat{L})$ is a forest relative to $\mathrm{lk}^+(\hat{K})$. We will first show that there is no $\hat{K}$-thin vertex reduced spherical diagram $f\colon C \to \hat{L}$, $f(C) \nsubseteq \hat{K}$. Suppose that there is such a diagram. Since we assumed the exponent sums of attaching maps of 2-cells in L are 0, the attaching maps of 2-cells in $\hat{L}$ have this quality as well. Thus C has a source and a sink by Theorem 3.3. Let $v \in C$ be a source. Then $z(v) = c_1 \ldots c_q \subseteq \mathrm{lk}^+(\hat{L})$, the image of the link of v in C, is homology reduced, and hence contained in $\mathrm{lk}^+(\hat{K})$ because $\mathrm{lk}^+(\hat{L})$ is a forest relative to $\mathrm{lk}^+(\hat{K})$. But this contradicts thinness of $f\colon C \to \hat{L}$.

Assume L is not VA relative to K. Then by Definition 3.1 there exists a $\hat{K}$-thin vertex reduced spherical diagram $f\colon C \to \hat{L}$, $f(C)$ not contained in $\hat{K}$. But we just proved that such a spherical diagram does not exist.

Suppose the map $\pi_1(K_i) \to \pi_1(L)$ is not injective for some i. Then there exists a vertex-reduced Van Kampen diagram $g\colon D \to L$ such that $g(\partial D)$ is a nontrivial element of $\pi_1(K_i)$. Note that D has to contain 2-cells that are not mapped to K because the map $\pi_1(K_i) \to \pi_1(K) = \pi_1(K_1) * \ldots * \pi_1(K_n)$ is injective. The boundary of D has exponent sum zero because the 2-cells in L are attached by maps of exponent sum 0. We can cap off D with a disc d_1 and obtain a spherical diagram $f\colon C \to \hat{L}$, $f(C) \nsubseteq \hat{K}$. Note that this spherical diagram is vertex reduced. If it were not, then there would have to be a folding vertex v on the boundary of D with folding pair (d_1, d_2), where d_2 is a 2-cell in D. But that would mean that $f(d_1) = f(d_2)$ is a 2-cell in L. Since we assumed K is full, this would imply that $f(d_1)$ is a 2-cell in K_i, which contradicts the fact that $g(\partial D)$ is a nontrivial element of $\pi_1(K_i)$. By Definition 3.1 there exists a $\hat{K}$-thin vertex-reduced spherical diagram $f'\colon C' \to \hat{L}$, $f(C) \nsubseteq \hat{K}$. But we know already from the beginning of this proof that no such spherical diagram exists.   ■

Let L be a 2-complex. We assign weights (or angles) $\omega(c) \in \mathbb{R}$ to the corners of the 2-cells and obtain an *angled* 2-complex. If $L = S$ is a closed orientable surface we define the curvature of a 2-cell $d \in S$ to be $\kappa(d) = \sum_{i=1}^q \omega(c_i) - (q - 2)$, where $c_1, \ldots, c_q$ are the corners in d. The curvature at a vertex is defined to be $\kappa(v) = 2 - \sum \omega(c_i)$ where the c_i are the corners at the vertex v. The combinatorial Gauss–Bonet theorem says

$$\kappa(S) = \sum_{v \in S} \kappa(v) + \sum_{d \in S} \kappa(d) = 2\chi(S).$$

Note that if $g: S \to L$ is a surface diagram and L is an angled 2-complex, we can pull back the weights and give S an induced angle structure. The idea behind a weight test is to give conditions on the link of an angled 2-complex that imply $\kappa(S) \le 0$ for every vertex-reduced surface diagram $g: S \to L$. This in turn implies that there can not exist vertex-reduced spherical diagrams $f: C \to L$, and hence L is aspherical.

A cycle in a graph is called *reduced* if no oriented edge in the cycle sequence is followed immediately by its inverse. Here is the weight test as defined by Gersten [8] which implies the asphericity of L:

Definition 3.5 Let L be a 2-complex. L satisfies the *weight test*, if there are weights $\omega(c) \in \mathbb{R}$ assigned to the corners of the 2-cells of L such that

(1) $\sum_i \omega(c_i) \le q - 2$ if $c_1, \ldots, c_q$ are the corners of a 2-cell of L and
(2) if $z = e_1 \ldots e_n$ is a reduced cycle in $lk(L)$ then $\sum_{i=1}^{n} \omega(e_i) \ge 2$.

We next define a relative weight test. It is coarse but will be sufficient for the applications we have in mind. Assume $K = K_1 \vee \ldots \vee K_n \subseteq L$. We assume L contains a single vertex v. We define $lk(L, K)$ in the following way: If $y_1, \ldots, y_l$ are the edges of K_i then we denote by $\Delta(K_i)$ the full graph on the vertices $y_j^{\pm 1}$ of $lk(K_i)$ together with an edge attached at each y_j^+ (a loop at that vertex) and at each y_j^-. Every pair of vertices in $\Delta(K_i)$ is connected by an edge, and at every vertex we have a loop. For each i we remove $lk(K_i)$ from $lk(L)$ and insert $\Delta(K_i)$ instead. The resulting graph is $lk(L, K)$. Note that if $\hat{K}$ is any 2-complex obtained from K by attaching 2-cells and $f: C \to \hat{L} = L \cup \hat{K}$ is a spherical diagram, then $f(lk(C, v))$ yields a cycle in $lk(L, K)$. Let $\Delta(K) = \Delta(K_1) \cup \ldots \cup \Delta(K_n)$, a disjoint union. Further note that $lk^+(L, K)$ is a forest relative to $\Delta^+(K)$ if and only if $lk^+(L)$ is a forest relative to $lk^+(K)$. And similarly for $-$ in place of $+$.

Definition 3.6 Assume $K = K_1 \vee \ldots \vee K_n \subseteq L$ and we are in the setting of Example 3, that is the attaching maps of 2-cells of K have exponent sum zero. We say L *satisfies the weight test* relative to K if there is a weight function ω on the set of edges of $lk(L, K)$, such that

(1) $\sum_i \omega(c_i) \le q - 2$ if $c_1, \ldots, c_q$ are the corners of a 2-cell of L not contained in K,
(2) if $z = e_1 \ldots e_n$ is a homology reduced cycle in $lk(L, K)$ containing at least one corner from $lk(L, K) - \Delta(K)$, then $\sum_{i=1}^{n} \omega(e_i) \ge 2$ and
(3) $\omega(c) = 0$ if c is an edge of $\Delta^+(K_i)$ or $\Delta^-(K_i)$, $\omega(c) = 1$ if c connects a vertex of $\Delta^+(K_i)$ with one of $\Delta^-(K_i)$, $i = 1, \ldots, n$.

Lemma 3.7 *Assume we are in the setting of Definition 3.6. If z is a cycle in $lk(L, K)$ containing at least one corner from $lk(L, K) - \Delta(K)$ and z is homology reduced relative to $\Delta(K)$, then $\omega(z) \ge 2$.*

Proof Let $z = c_1 \ldots c_q$ be a cycle as in the statement of the lemma. If z is homology reduced then $\omega(z) \ge 2$ since we assume Condition (2) holds. If z is not homology reduced then there exists a pair c_k, c_l satisfying $c_l = \bar{c}_k$. Since we assume z is homology reduced relative to $\Delta(K)$, $c_k, c_l \in lk(\Delta(K_i))$ for some i. Let $z_1 = c_1 \ldots c_{k-1} c_{l+1} \ldots c_q$ and $z_2 = c_{k+1} \ldots c_{l-1}$. Both z_1, z_2 are cycles and are homology reduced relative to $\Delta(K)$. Assume first that both z_1, z_2 contain at least one corner

from $\mathrm{lk}(L, K) - \Delta(K)$. Then by induction of the cycle length we have $\omega(z_i) \geq 2$. Since $c_k, c_l \in \Delta(K)$ both carry weights ≥ 0 and we have

$$\omega(z) = \omega(z_1) + \omega(z_2) + \omega(c_k) + \omega(c_l) \geq \omega(z_1) + \omega(z_2) \geq 2 + 2 = 4.$$

For the remaining case we assume z_1 contains a corner from $\mathrm{lk}(L, K) - \Delta(K)$ but z_2 does not. But in that case $z_2 \subseteq \Delta(K_i)$. Then $\omega(z_1) \geq 2$ and $\omega(z_2) \geq 0$. We have

$$\omega(z) = \omega(z_1) + \omega(z_2) + \omega(c_k) + \omega(c_l) \geq \omega(z_1) + \omega(z_2) \geq 2. \qquad \blacksquare$$

Theorem 3.8   *Assume $K = K_1 \vee \ldots \vee K_n \subseteq L$, each K_i is full and we are in the setting of Example 3, that is the attaching maps of 2-cells of K have exponent sum zero. Assume further that L satisfies the weight test relative to K. Then L is VA relative to K. If in addition the attaching maps of the 2-cells of L have exponent sum zero, then all the inclusion induced homomorphisms $\pi_1(K_i) \to \pi_1(L)$ are injective.*

Proof   Let $(\hat{L}, \hat{K})$ be the thinning expansion constructed in Example 3. We first make $\hat{L}$ into an angled 2-complex. Since $\hat{L} - \hat{K} = L - K$, we have already weights on the corners of 2-cells in $\hat{L} - \hat{K}$. If $\hat{d}$ is a 2-cell of $\hat{K}$ we assign to corners in $\mathrm{lk}^+(\hat{K})$ and in $\mathrm{lk}^-(\hat{K})$ weight 0, and weight 1 to all the other corners in $\hat{d}$. Suppose L is not VA relative to K. Then there exists a vertex-reduced spherical diagram $f: C \to L$ that is not already a diagram over K. By Definition 3.1 there also exists a $\hat{K}$-thin vertex-reduced spherical diagram $f': C' \to \hat{L}$ that is not already a diagram over $\hat{K}$. We pull back the weights of $\hat{L}$ and thus turn C' into an angled 2-complex. Condition (1) in the weight test implies that the curvature of a 2-cell not mapped to $\hat{K}$ is ≤ 0. If $d \in C'$ is a 2-cell which is mapped to $\hat{K}$ it has exponent sum 0, so there are at least 2 corners with weight 0 (the other corners of d have weight 1). So the curvature of d will also be ≤ 0.

Since $f': C' \to \hat{L}$ is $\hat{K}$-thin and vertex reduced, for every $v \in C'$ the image $f'(\mathrm{lk}(C', v))$ yields a cycle $z \in \mathrm{lk}(\hat{L}, \hat{K})$ that is homology reduced relative to $\Delta(K)$ and contains a corner from a 2-cell in $L - K$. Thus by Lemma 3.7 $\omega(z) \geq 2$ which implies that the curvature at v in C' is ≤ 0. So the curvature of C' is ≤ 0. This is a contradiction because C' is a 2-sphere and so the curvature is 2.

Injectivity of the homomorphisms $\pi_1(K_i) \to \pi_1(L)$ follows by the arguments already provided in the proof of Theorem 3.4. $\blacksquare$

An (ε, δ)-*corner* of $\mathrm{lk}(L, K)$ for $\varepsilon, \delta \in \{+, -\}$ is a corner between vertices x_i^ε and x_j^δ for some i, j.

Theorem 3.9   *Let $K = K_1 \vee \ldots \vee K_n \subseteq L$. Assume*

(1)   *the attaching maps of 2-cells in L have exponent sum zero;*
(2)   *$\mathrm{lk}^+(L)$ is a forest relative to $\mathrm{lk}^+(K) = \mathrm{lk}^+(K_1) \cup \ldots \cup \mathrm{lk}^+(K_n)$ and $\mathrm{lk}^-(L)$ is a forest relative to $\mathrm{lk}^-(K) = \mathrm{lk}^-(K_1) \cup \ldots \cup \mathrm{lk}^-(K_n)$.*

With the assignment ω: Edges of $\mathrm{lk}(L, K) \to \{0, 1\}$

$$\omega(c) = \begin{cases} 0 & \text{if } c \text{ is a } (++) - \text{corner or a } (--) - \text{corner} \\ 1 & \text{if } c \text{ is a } (+-) - \text{corner} \end{cases}$$

L satisfies the weight test relative to K.

Proof Note that $\mathrm{lk}^+(L)$ is a forest relative to $\mathrm{lk}^+(K)$ if and only if $\mathrm{lk}^+(L, K)$ is a forest relative to $\Delta^+(K)$ (the same for $-$ in place of $+$). Let $(\hat{L}, \hat{K})$ be a thinning expansion as in Example 3. Since we assumed that the attaching maps for the 2-cells of L have exponent sum zero, the same is true for the 2-cells of $\hat{L}$. Thus if $c_1, \ldots, c_q$ are the corners in a 2-cell of $\hat{L}$, then there is at least one $(++)$ and one $(--)$-corner among them. So $\sum \omega(c_i) \leq q - 2$ and the first condition of the weight test holds.

Let z be a homology reduced cycle in $\mathrm{lk}(L, K)$ containing at least one corner from a 2-cell of $L - K$. If z contains one $(+-)$-corner it has to contain at least two $(+-)$-corners and then $\omega(z) \geq 2$. So assume z contains only $(++)$-corners (or only $(--)$-corners). But since $\mathrm{lk}^+(L, K)$ is a forest relative to $\Delta^+(K)$ and z contains a corner of $L - K$, z is homology reducible. This is a contradiction. ∎

4 Applications to Labelled Oriented Trees

A standard reference for labeled oriented graphs, LOGs for short, is [13]. Here are the basic definitions. A LOG is an oriented finite graph Γ on vertices **x** and edges **e**, where each oriented edge is labeled by a vertex. Associated with a LOG Γ is the *LOG-complex* $K(\Gamma)$, a 2-complex with a single vertex, edges in correspondence with the vertices of Γ and 2-cells in correspondence with the edges of Γ. The attaching map of a 2-cell d_e is the word $xz(zy)^{-1}$, where e is an edge of Γ starting at x, ending at y, and labeled with z.

A labelled oriented graph is called *compressed* if no edge is labelled with one of its vertices. A LOG Γ is called *boundary reducible* if there is a boundary vertex $x \in \Gamma$ which does not occur as an edge label and *boundary reduced* otherwise. A LOG is *injective* if each vertex occurs as an edge label at most once. An injective LOG is called *reduced* if it is compressed and boundary reduced. A *labeled oriented tree*, LOT, is a labeled oriented graph where the underlying graph is a tree.

It can be shown that an injective LOT can be transformed into a reduced injective LOT without altering the homotopy-type of the LOT-complex. If Γ is a LOT and Γ_1 is a sub-tree of Γ with at least one edge, such that each edge label of Γ_1 is a vertex of Γ_1, then we call Γ_1 a *sub-LOT* of Γ. A sub-LOT Γ_1 of Γ is *proper*, if $\Gamma_1 \neq \Gamma$. A sub-LOT Γ_1 of Γ is *maximal* if it is not contained in a bigger proper sub-LOT of Γ. A LOG is called *prime* if it does not contain proper sub-LOTs.

The LOT consisting of a single edge labeled by one of its vertices is prime but not boundary reduced. All other prime LOTs are boundary reduced, since a boundary reducible LOT containing more than one edge contains a proper sub-LOT. It follows that compressed prime LOTs containing at least two edges are boundary reduced and hence reduced.

In [9] the authors have shown that injective LOTs are aspherical. In this section we prove

Theorem 4.1 *Let Γ be a compressed injective LOT. Then $K(\Gamma)$ is VA.*
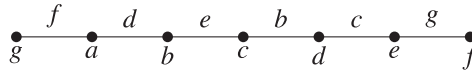
Figure 1: A reduced injective nonprime LOT which does not satisfy the weight test (with any orientation of its edges). See Huck–Rosebrock [12].

This generalizes a result obtained by Huck and Rosebrock.

Theorem 4.2 (Huck–Rosebrock [12])   *Let Γ be a reduced injective prime LOT. Then $K(\Gamma)$ satisfies the weight test with weights from $\{0,1\}$. In particular $K(\Gamma)$ is DR (and therefore also VA).*

A reduced non-prime injective LOT may not satisfy the weight test. An example is shown in Figure 1.

If Γ_1 is a proper sub-LOT of an injective LOT Γ, we can *collapse* Γ_1 in Γ and obtain a quotient LOT $\bar{\Gamma}_1$ in the following specific way: There exists a unique vertex x in Γ_1 that does not occur as an edge label in Γ_1, since Γ_1 is injective. x may or may not occur as an edge label in Γ. Collapse Γ_1 in Γ to x to obtain a quotient tree $\bar{\Gamma}_1$ of Γ with edge set the edges of $\Gamma - \Gamma_1$ and vertex set the vertices that come with the edges in $\Gamma - \Gamma_1$ together with x. If y is the label of an edge e in $\Gamma - \Gamma_1$, then that edge carries the same label y in $\bar{\Gamma}$. We denote the quotient map by $\Gamma \to \bar{\Gamma}_1$ and define $\bar{z}$ to be the image of a vertex z under that map.

Let Γ be a reduced injective LOT that is not prime. Choose a maximal proper sub-LOT Γ_1 and collapse it to obtain the quotient $\bar{\Gamma}_1$ of Γ. This quotient is injective. If it is not prime choose a maximal proper sub-LOT $\bar{\Gamma}_2$ in $\bar{\Gamma}_1$ and collapse it to obtain $\bar{\Gamma}_{12}$. We continue this process until we arrive at an injective prime quotient $\bar{\Gamma}_{12\ldots n}$:

$$
\begin{array}{ccccccccc}
\Gamma & \to & \bar{\Gamma}_1 & \to & \bar{\Gamma}_{12} & \ldots & \bar{\Gamma}_{12\ldots(n-1)} & \to & \bar{\Gamma}_{12\ldots n} \\
\cup & & \cup & & \cup & & \cup & & \\
\Gamma_1 & & \bar{\Gamma}_2 & & \bar{\Gamma}_3 & & \bar{\Gamma}_n & &
\end{array}
$$

Note that $\bar{\Gamma}_2$ does not contain the vertex x that Γ_1 is collapsed to, because that would contradict maximality of Γ_1. Thus the sub-LOT Γ_2 that maps to $\bar{\Gamma}_2$ under the first collapse is disjoint from Γ_1. This procedure gives a disjoint set of sub-LOTs $\Gamma_1, \ldots, \Gamma_n$ of Γ so that

$$\Gamma_i \to \bar{\Gamma}_i \text{ is a maximal sub-LOT of } \bar{\Gamma}_{12\ldots(i-1)}.$$

We call a set of sub-LOTs $\Gamma_1, \ldots, \Gamma_n$ of Γ obtained in this fashion *complete*, if $\bar{\Gamma}_{12\ldots n}$ is a compressed injective prime LOT that is not just a vertex. Such a set may or may not exist. It could happen that all injective prime quotients $\bar{\Gamma}_{12\ldots n}$ that we obtain are not compressed.

A labelled oriented graph Γ' is a *reorientation* of a labelled oriented graph Γ if Γ' is obtained from Γ by changing the orientation of some edges. The following lemma was used in the proof of Theorem 4.2. It is not explicitly stated as a lemma in [12], but

the first three lines of Section 3, page 288, of [12] state the result, and the proof is given on pages 289 and 290.

Lemma 4.3 (Huck–Rosebrock [12]) *If Γ is a reduced injective prime LOT, then there is a reorientation Γ' of Γ such that $lk^+(K(\Gamma'))$ and $lk^-(K(\Gamma'))$ are trees. In particular $K(\Gamma')$ satisfies the weight test by assigning weight 0 to all corners in $lk^+(K(\Gamma'))$ and $lk^-(K(\Gamma'))$, and weight 1 to all other corners.*

We can adapt this to our more general setting:

Lemma 4.4 *If Γ is a reduced injective LOT with a complete set of sub-LOTs $\Gamma_1, \ldots, \Gamma_n$, then there is a reorientation Γ' of Γ such that $lk^+(K(\Gamma'))$ and $lk^-(K(\Gamma'))$ are trees relative to $lk^+(K(\Gamma'_1 \cup \ldots \cup \Gamma'_n))$ and $lk^-(K(\Gamma'_1 \cup \ldots \cup \Gamma'_n))$, respectively. In particular $K(\Gamma')$ satisfies the relative weight test by assigning weight 0 to all corners in $lk^+(K(\Gamma'))$ and in $lk^-(K(\Gamma'))$, and weight 1 to all other corners.*

Proof Collapsing each Γ_i in Γ results in an injective compressed prime LOT $\bar{\Gamma} = \bar{\Gamma}_{1\ldots n}$. By Lemma 4.3 we can reorient $\bar{\Gamma}$ to $\bar{\Gamma}'$ so that both $lk^+(K(\bar{\Gamma}'))$ and $lk^-(K(\bar{\Gamma}'))$ are trees. We pull back the edge-orientations of $\bar{\Gamma}'$ to edge-orientations of Γ to achieve a reorientation Γ' of Γ. Note that this reorientation does not affect the Γ_i (so $\Gamma'_i = \Gamma_i$). Since both $lk^+(K(\bar{\Gamma}'))$ and $lk^-(K(\bar{\Gamma}'))$ are trees, $lk^+(K(\Gamma'))$ is a tree relative to $lk^+(K(\Gamma'_1 \cup \ldots \Gamma'_n))$ and $lk^-(K(\Gamma'))$ is a tree relative to $lk^-(K(\Gamma'_1 \cup \ldots \Gamma'_n))$. Then Theorem 3.9 implies that $K(\Gamma')$ satisfies the weight test. ∎

The next lemma was also used in the proof of Theorem 4.2:

Lemma 4.5 (Huck–Rosebrock [11]) *Let Γ be a reduced injective LOT that satisfies the weight test with weights 0 and 1. Then any reorientation of Γ satisfies the weight test with weights 0 and 1.*

The analogous result for the general situation is

Lemma 4.6 *Let Γ be a reduced injective LOT with a complete set of sub-LOTs $\Gamma_1, \ldots, \Gamma_n$. If $K(\Gamma)$ satisfies the weight test relative to $K(\Gamma_1 \cup \ldots \cup \Gamma_n)$ with weights 0 and 1, then for any reorientation Γ' the 2-complex $K(\Gamma')$ satisfies the weight test relative to $K(\Gamma'_1 \cup \ldots \cup \Gamma'_n)$ with weights 0 and 1.*

Before we give a proof we introduce some useful notation. A *signed LOT* Γ is a labeled oriented tree where we allow vertices to carry signs. An oriented edge from x^{ε_1} to y^{ε_2}, $\varepsilon_j = \pm 1$, labeled by z gives a 2-cell in $K(\Gamma)$ with attaching map $x^{\varepsilon_1} z (z y^{\varepsilon_2})^{-1}$. Given a labeled oriented tree Γ and a subset X of the vertices of Γ, we define Γ_X to be the signed LOT obtained from Γ by replacing each vertex $x \in X$ with x^{-1}. It is important to note that edge labels and edge orientations in Γ and Γ_X are the same. One
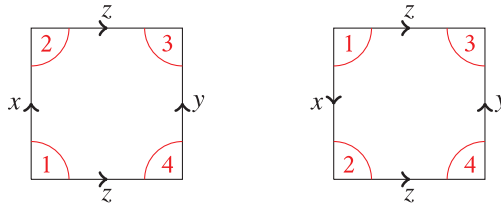
Figure 2: $lk(K(\Gamma)) = lk(K(\Gamma_x))$: The corners of the original 2-cell also appear in the 2-cell with the edge x reversed, only the order in which the corners appear changes.

of the key observations is that the link does not change under this vertex sign change: $lk(K(\Gamma)) = lk(K(\Gamma_X))$. See Figure 2. In particular if $K(\Gamma)$ satisfies the weight test, then so does $K(\Gamma_X)$.

Proof of Lemma 4.6.   Assume Γ' is obtained from Γ by reversing a single edge labeled x. Suppose first that x is not contained in any of the Γ_i. Then

$$lk(K(\Gamma), K(\Gamma_1 \cup \ldots \cup \Gamma_n)) = lk(K(\Gamma_x), K(\Gamma_1 \cup \ldots \cup \Gamma_n)).$$

Since $(K(\Gamma), K(\Gamma_1 \cup \ldots \cup \Gamma_n))$ satisfies the weight test with weights $0, 1$, so does $(K(\Gamma_x), K(\Gamma_1 \cup \ldots \cup \Gamma_n))$. Let $\phi_x : K(\Gamma_x) \to K(\Gamma')$ be the homeomorphism that changes the orientation of all x-edges in the attaching maps of 2-cells. It induces a homeomorphism of the corresponding expansions and preserves the weight test. Thus $K(\Gamma')$ satisfies the weight test relative $K(\Gamma_1 \cup \ldots \cup \Gamma_n)$ with weights $0, 1$.

Next assume that x is contained in one of the Γ_i, say Γ_1. If we proceed as above we run into a technical difficulty: the attaching maps of the 2-cells in the subcomplex $K(\Gamma_{1_x} \cup \Gamma_2 \cup \ldots \cup \Gamma_n)$ do not all have exponent sum zero, so we are not in the setting of the weight test as given in Definition 3.6 anymore. Here is how we fix this. Let X be the set of vertices of Γ_1. Note that the attaching maps in $K(\Gamma_{1_x})$ do have exponent sum zero. Now we argue exactly as above with X in place of x. The homeomorphism $\phi_X : K(\Gamma_X) \to K(\Gamma')$ now changes the orientation of all x-edges, $x \in X$, in the attaching maps of 2-cells.

Proposition 4.7   *Let Γ be a reduced injective LOT which is not prime. Assume there is a complete set of sub-LOTs $\Gamma_1, \ldots, \Gamma_n$. Then $K(\Gamma)$ is VA relative to $K(\Gamma_1) \cup \ldots \cup K(\Gamma_n)$.*

Proof   By Lemma 4.4 there exists a reorientation Γ' so that $K(\Gamma')$ satisfies the weight test relative $K(\Gamma'_1 \cup \ldots \cup \Gamma'_n)$ with weights 0 and 1. By Lemma 4.6 $K(\Gamma)$ itself satisfies the weight test relative $K(\Gamma_1 \cup \ldots \cup \Gamma_n)$ with weights 0 and 1. It follows from Theorem 3.8 that $K(\Gamma)$ is VA relative to $K(\Gamma_1) \cup \ldots \cup K(\Gamma_n)$.   ∎

This proposition would quickly lead to a proof of our main Theorem 4.1 if all nonprime reduced injective LOTs had a complete set of sub-LOTs. This, however, is not true. See Figure 3. We thank the referee for catching this problem.
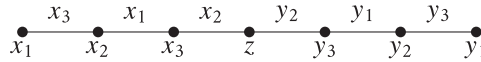
$$
\begin{array}{cccccccc}
x_3 & x_1 & x_2 & y_2 & y_1 & y_3 \\
\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \\
x_1 & x_2 & x_3 & z & y_3 & y_2 & y_1
\end{array}
$$

Figure 3: The figure shows a reduced injective LOT that does not contain a complete set of sub-LOTs (for any orientation of its edges). Note that it freely decomposes.

We say a LOT Γ *freely decomposes* if $\Gamma = \Gamma_L \cup \Gamma_R$, where the Γ_L and Γ_R are proper sub-LOTs of Γ, intersecting in a single vertex.

Proposition 4.8   *Suppose Γ is a reduced and injective LOT that is not prime. Then either*

(1)   *Γ contains a complete set of sub-LOTs, or*
(2)   *Γ freely decomposes.*

Proof   We make two observations:

(1)   *Let Γ be an injective compressed LOT and let Γ_1 be a maximal proper sub-LOT. If the quotient $\bar{\Gamma}_1$ is not compressed, then $\Gamma = \Gamma_1 \cup e$, where e is an edge attached to Γ_1, whose vertex y not in Γ_1 does not occur as an edge label in Γ_1. In particular, Γ is not boundary reduced.*

   There exists an edge $\bar{e}$ in $\bar{\Gamma}_1$ with vertices $\bar{x}$ and $\bar{y}$ and edge label $\bar{x}$. Since Γ is compressed it follows that $\bar{x}$ is the vertex the sub-LOT Γ_1 is collapsed to, and it follows that we have a sub-LOT $\Gamma_1 \cup e$ in Γ, where the edge label of e is a vertex in Γ_1. So $\Gamma_1 \cup e$ is a sub-LOT of Γ containing Γ_1. Maximality of Γ_1 in Γ implies that $\Gamma = \Gamma_1 \cup e$. All vertices in Γ_1 except x occur in Γ_1 as edge labels (by the way we collapse), so y is not an edge label of Γ_1.

(2)   *Let Γ be an injective compressed LOT and suppose*

$$
\begin{array}{ccccc}
\Gamma & \to & \bar{\Gamma}_1 & \to & \bar{\Gamma}_{12} \\
\cup & & \cup & & \\
\Gamma_1 & & \bar{\Gamma}_2 & &
\end{array}
$$

   is a 2-stage quotient of Γ so that $\bar{\Gamma}_1$ is compressed but $\bar{\Gamma}_{12}$ is not. Then Γ freely decomposes.

We know from Observation (1) that $\bar{\Gamma}_1 = \bar{\Gamma}_2 \cup \bar{e}$ where $\bar{e}$ is attached to $\bar{\Gamma}_2$ at a vertex $\bar{x}$, the other vertex $\bar{y}$ does not label an edge in $\bar{\Gamma}_2$, and $\bar{e}$ has as edge label a vertex from $\bar{\Gamma}_2$. Let $\bar{z}$ be the vertex Γ_1 is collapsed to. Note that $\bar{z}$ can not be in $\bar{\Gamma}_2$ because of maximality, so $\bar{z} = \bar{y}$. But then it follows that $\Gamma = \Gamma_2 \cup e \cup \Gamma_1$. If we take $\Gamma_L = \Gamma_2 \cup e$ and $\Gamma_R = \Gamma_1$ we have a free decomposition of Γ.

Using the two observations we complete the proof of Proposition 4.8. Suppose Γ is not prime and $\bar{\Gamma}_{12\ldots n}$ is an injective prime quotient that is not compressed. Note that $n \geq 2$ because of the first observation (1): if $n = 1$ then Γ is not boundary reduced, contradicting the assumption that Γ is reduced. There exists a j so that $\bar{\Gamma}_{12\ldots j} \to \bar{\Gamma}_{12\ldots j+1} \to \bar{\Gamma}_{12\ldots j+2}$, so that both $\bar{\Gamma}_{12\ldots j}$ and $\bar{\Gamma}_{12\ldots j+1}$ are injective and compressed, but $\bar{\Gamma}_{12\ldots j+2}$ is not. Then by observation (2) $\bar{\Gamma}_{12\ldots j}$ freely decomposes: $\bar{\Gamma}_{12\ldots j} = \bar{\Gamma}_L \cup \bar{\Gamma}_R$.

We claim $\Gamma = \bar{\Gamma}_{12\ldots j}$. If not, then let $\bar{x}$ be the vertex in $\bar{\Gamma}_{1\ldots j}$ that $\bar{\Gamma}_j \subseteq \bar{\Gamma}_{12\ldots(j-1)}$ is collapsed to. But then $\bar{x}$ is in $\bar{\Gamma}_L$ or in $\bar{\Gamma}_R$, contradicting maximality of $\bar{\Gamma}_j$ in $\bar{\Gamma}_{12\ldots(j-1)}$. So $\Gamma = \bar{\Gamma}_{12\ldots j}$, and we have that Γ freely decomposes.   ∎

We need two more lemmas before we can complete the proof of Theorem 4.1.

Lemma 4.9 *Suppose $\Gamma_0 \subseteq \Gamma$ and Γ_0 is obtained from Γ by a boundary reduction. Then $K(\Gamma)$ is VA if and only if $K(\Gamma_0)$ is VA.*

Proof   If $K(\Gamma)$ is VA, then so is $K(\Gamma_0)$ because the VA property is hereditary. Assume that $K(\Gamma_0)$ is VA. If Γ consists of one edge then $K(\Gamma)$ is VA. Let Γ contain at least two edges. Then $\Gamma = \Gamma_0 \cup e$ where e is attached to Γ_0 at a vertex x, and the other vertex y of e does not occur as an edge label in Γ. $K(\Gamma)$ contains a 2-cell with a free boundary edge y. So any vertex-reduced spherical diagram $f\colon C \to K(\Gamma)$ maps to $K(\Gamma_0)$.   ∎

Lemma 4.10   *If an injective LOT Γ freely decomposes $\Gamma = \Gamma_L \cup \Gamma_R$ and both $K(\Gamma_L)$ and $K(\Gamma_R)$ are VA, then $K(\Gamma)$ is VA.*

Proof   Consider a vertex-reduced spherical diagram $f\colon C \to K(\Gamma)$. $f(C)$ cannot be contained in one of $K(\Gamma_L)$ or $K(\Gamma_R)$ only, because $K(\Gamma_L)$ and $K(\Gamma_R)$ are VA. Let x be the common vertex of Γ_L and Γ_R. x is an edge label at most once in Γ since Γ is injective. So assume x is not an edge label in Γ_R. Let $\gamma \in C$ be a boundary component of a maximal region, which maps to $K(\Gamma_R)$. Each edge of γ maps to x, and so γ maps to a word in x and x^{-1} of exponent sum zero. So there is a vertex $v \in \gamma$ where two edges $e_1, e_2 \in \gamma$ end. Now v is a folding vertex with a folding pair of 2-cells d_1 and d_2 which both map to $K(\Gamma_R)$ containing e_1 and e_2, respectively.   ∎

Proof of Theorem 4.1   Let Γ be a smallest injective compressed LOT such that $K(\Gamma)$ is not VA. Minimality implies that Γ is boundary reduced because of Lemma 4.9, so Γ is reduced. Γ is not prime because of Theorem 4.2. Γ does not freely decompose by Lemma 4.10. So by Proposition 4.8 Γ contains a complete set of sub-LOTs $\Gamma_1, \ldots, \Gamma_n$. Proposition 4.7 implies that $K(\Gamma)$ is VA relative to $K(\Gamma_1) \cup \ldots \cup K(\Gamma_n)$. Since $K(\Gamma)$ is the smallest non-VA injective compressed LOT we have that all the $K(\Gamma_i)$ are VA. But then Theorem 2.3 implies that $K(\Gamma)$ is VA in contradiction to our assumption.

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