Canad. Math. Bull. Vol. 63 (2), 2020 pp. 318–327
http://dx.doi.org/10.4153/S0008439519000304

Non-spectral Problem for Some Self-similar Measures

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Abstract. Suppose that $0 < |\rho| < 1$ and $m \geqslant 2$ is an integer. Let $\mu_{\rho,m}$ be the self-similar measure defined by $\mu_{\rho,m}(\cdot) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_{\rho,m}(\rho^{-1}(\cdot) - j)$. Assume that $\rho = \pm(q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}^+$ with $(p,q) = 1$ and $(p,m) = 1$. We prove that if $(q,m) = 1$, then there are at most m mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$ and m is the best possible. If $(q,m) > 1$, then there are any number of orthogonal exponential functions in $L^2(\mu_{\rho,m})$.

1 Introduction

Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. We say that μ is a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^d$ such that $E_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis of $L^2(\mu)$. In this case, we call Λ a spectrum of μ and (μ,Λ) a spectral pair, respectively.

A non-spectral measure μ belongs to one of the following two cases. In the first case, there exists an infinite set of orthogonal exponential functions but no such set forms a basis of $L^2(\mu)$. In the second case, there are only finitely many orthogonal exponential functions in $L^2(\mu)$. For the second case, it is natural to investigate the maximal cardinality of the orthogonal exponential functions in $L^2(\mu)$. We study this problem for one-dimensional self-similar measures.

The study of the spectral properties of self-similar measures started from the work of Jorgensen and Pedersen [8]. They constructed the first example of a singular, non-atomic spectral measure that is a Cantor-type measure and proved that there are only finitely many orthogonal exponential functions with respect to the middle-third Cantor measure. This kind of problem has been studied extensively since then [3,6,11,13–15]. A well-known family of self-similar measures is the Bernoulli convolutions μ_ρ $(0 < \rho < 1)$, which includes the classical Cantor measure and its variants [16]. Hu and Lau [7] characterized when the ρ-Bernoulli convolutions admit an infinite orthogonal set of exponential functions, and Dai [1] proved that $\mu_{1/(2k)}$ is the only class of spectral measures for ρ-Bernoulli convolutions.

As a generalization of the Bernoulli convolutions, Dai, He, and Lau [2] studied the spectral property of self-similar measures with consecutive digits and uniform weights. We use $\mu_{\rho,m}$ to denote this type of self-similar measure, which is the invariant

Received by the editors May 26, 2019.
Published online on Cambridge Core August 28, 2019.
The research is supported in part by the NNSF of China (No. 11831007, No.11571099).
AMS subject classification: **28A80**, 42C05, 46C05.
Keywords: self-similar measure, spectral measure, orthogonal exponential function, Fourier transform.

probability measure defined as

$$\mu_{\rho,m}(\cdot) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_{\rho,m}\big(\rho^{-1}(\cdot)-j\big), \tag{1.1}$$

where $0<|\rho|<1$ and $m\geq 2$. Deng [4] studied when $L^2(\mu_{\rho,m})$ admits an infinite orthogonal set of exponential functions under the assumption that m is a prime number. In [17], the authors removed that assumption and obtained the following more general conclusion.

Theorem A ([17, Theorem 1.1]) *Let $0<|\rho|<1$ and $m\geq 2$ be an integer. Suppose that $\mu_{\rho,m}$ is defined by (1.1); then $L^2(\mu_{\rho,m})$ contains an infinite orthonormal set of exponential functions if and only if $\rho=\pm(q/p)^{1/r}$ for some p, q, $r\in\mathbb{N}^+$ with $(p,q)=1$ and $(p,m)>1$.*

We note that the above theorem indicates some connections between number theory and spectral theory. It follows from Theorem A that if $(p,m)=1$, then every orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ is finite. One can naturally ask: What is the maximal cardinality of the orthogonal exponential functions in $L^2(\mu_{\rho,m})$? In fact, the origin of this kind of problem can be traced back to the study of non-spectrality of the middle-third Cantor measure in [8]. There is a special interest in studying non-spectral measures with only finitely many mutually orthogonal exponential functions, like the existence of the Fourier frame. The first examples of frame-spectral fractal measures with only finitely many mutually orthogonal exponential functions were constructed by Lai and Wang [12]. More recently, Dutkay et al. studied the existence of Riesz sequences with respect to non-spectral measures [5].

In this paper, we study the non-spectral property of $\mu_{\rho,m}$ for the case $\rho=\pm(\frac{q}{p})^{1/r}$ with $(p,m)=1$, where $\mu_{\rho,m}$ is defined as in (1.1). We first introduce a necessary definition.

Definition 1.1 ([14]) Let μ be a Borel probability measure with compact support on $\mathbb{R}$. Let Λ be a finite or countable subset of $\mathbb{R}$, and let $E_\Lambda=\{e^{2\pi i\lambda x}:\lambda\in\Lambda\}$. We denote E_Λ by E_Λ^* if E_Λ is a maximal orthogonal set of exponential functions in $L^2(\mu)$. Let

$$n^*(\mu) := \sup\{\#\Lambda : E_\Lambda^* \text{ is a maximal orthogonal set}\},$$

and call $n^*(\mu)$ the maximal cardinality of the orthogonal exponential functions in $L^2(\mu)$.

Our two main results are Theorems 1.2 and 1.3. We obtain the following theorems depending on whether $(q,m)=1$ or $(q,m)>1$. In Theorem 1.2, we derive the conclusion that if $(q,m)=1$, then the maximal cardinality of orthogonal exponential functions in $L^2(\mu_{\rho,m})$ is m; in Theorem 1.3, we show that if $(q,m)>1$, then there are any number of orthogonal exponential functions in $L^2(\mu_{\rho,m})$.

Theorem 1.2 *Let $\rho=\pm(\frac{q}{p})^{1/r}$ for some p, q, $r\in\mathbb{N}^+$ with $(p,q)=1$, $q<p$. Let $m\geq 2$ be an integer with $(p,m)=1$, $(q,m)=1$ and $\mu_{\rho,m}$ be defined as in (1.1). Then there are*

https://doi.org/10.4153/S0008439519000304 Published online by Cambridge University Press

at most m mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$, and m is the best possible.

Theorem 1.3 *Let $\rho = \pm(\frac{q}{p})^{1/r}$ for some p, q, $r \in \mathbb{N}^+$ with $(p,q) = 1$, $q < p$. Let $m \geq 2$ be an integer with $(p,m) = 1$, $(q,m) = d > 1$, and let $\mu_{\rho,m}$ be defined as in (1.1). Then there are any number of orthogonal exponential functions in $L^2(\mu_{\rho,m})$. Hence, $n^*(\mu_{\rho,m}) = \infty$.*

This paper is organized as follows. In Section 2, we set up some important lemmas to characterize the orthogonality of the set of exponential functions and give the proof of Theorem 1.2. In Section 3, we prove Theorem 1.3 by constructing an orthogonal set of exponential functions with cardinality equal to N, where N is any given positive integer.

2 The Proof of Theorem 1.2

Let $\hat{\mu}(\xi) = \int e^{-2\pi i\xi x}d\mu(x)$ be the Fourier transform of a measure μ and let $\mathcal{Z}(\hat{\mu}) = \{\xi : \hat{\mu}(\xi) = 0\}$ be the zero set of $\hat{\mu}$. Then

$$\mathcal{Z}(\hat{\mu}_{\rho,m}) = \Big\{\frac{l}{m\rho^k} : k \in \mathbb{N}^+, l \in \mathbb{Z} \setminus m\mathbb{Z}\Big\},$$

where $\mathbb{N}^+$ is the set of positive integers. Let $\#\Lambda$ be the cardinality of a set Λ and $\mathbb{N}_0 = \mathbb{N}^+ \cup \{0\}$. In this paper, we use (a,b) to denote the greatest common divisor between two positive integers a and b. Assume that $p,q,r \in \mathbb{N}^+$ with $(p,q) = 1$. We call $px^r - q = 0$ a minimal polynomial of $(\frac{q}{p})^{1/r}$ if $x^r - \frac{q}{p} = 0$ is the minimal polynomial of $(\frac{q}{p})^{1/r}$ over $\mathbb{Q}$.

The following lemma about the minimal polynomial will play a crucial role in the proof of our main theorems. We believe that the lemma is of independent interest.

Lemma 2.1 *Let $p,q,r \in \mathbb{N}^+$ with $(p,q) = 1$. If $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$, then there exist $p_1, q_1, r_1 \in \mathbb{N}^+$ with $(p_1,q_1) = 1$, $r_1|r$ and $r_1 < r$ such that $(\frac{q}{p})^{1/r} = (\frac{q_1}{p_1})^{1/r_1}$ and $p_1x^{r_1} - q_1 = 0$ is a minimal polynomial of $(\frac{q_1}{p_1})^{1/r_1}$.*

Proof Let $r = \alpha_1^{n_1}\alpha_2^{n_2}\cdots\alpha_t^{n_t}$ be the prime decomposition of r with all $n_j > 0$ and $\alpha_{j_1} \neq \alpha_{j_2}$ for different j_1, j_2. Let $N = \sum_{j=1}^t n_j$. Since $\frac{q}{p} \notin -4\mathbb{Q}^4$, by [9, Theorem 8.1.6], we obtain that there exists some $j_1 \in \{1,2,\dots t\}$, such that $\frac{q}{p} \in \mathbb{Q}^{\alpha_{j_1}}$. Then $\frac{q}{p} = (\frac{q_1}{p_1})^{\alpha_{j_1}}$ for some $p_1, q_1 \in \mathbb{N}^+$ with $(p_1,q_1) = 1$. Let $r_1 = \frac{r}{\alpha_{j_1}}$. Hence, $(\frac{q}{p})^{1/r} = (\frac{q_1}{p_1})^{1/r_1}$. We consider whether $p_1x^{r_1} - q_1 = 0$ is a minimal polynomial of $(\frac{q_1}{p_1})^{1/r_1}$.

Note that if $p_1x^{r_1} - q_1 = 0$ is a minimal polynomial of $(\frac{q_1}{p_1})^{1/r_1}$, then the conclusion follows. In the other case, arguing similarly as above, there exists some $j_2 \in \{1,2,\dots t\}$, such that $\frac{q_1}{p_1} \in \mathbb{Q}^{\alpha_{j_2}}$. Therefore, $\frac{q_1}{p_1} = (\frac{q_2}{p_2})^{\alpha_{j_2}}$ for some $p_2, q_2 \in \mathbb{N}^+$ with

$(p_2, q_2) = 1$. Let $r_2 = \frac{r_1}{\alpha_{j_2}} = \frac{r}{\prod_{k=1}^{2}\alpha_{j_k}}$. Thus,

$$\Big(\frac{q}{p}\Big)^{1/r} = \Big(\frac{q_1}{p_1}\Big)^{1/r_1} = \Big(\frac{q_2}{p_2}\Big)^{\alpha_{j_2}/r_1} = \Big(\frac{q_2}{p_2}\Big)^{1/r_2}.$$

We consider whether $p_2x^{r_2} - q_2 = 0$ is a minimal polynomial of $(\frac{q_2}{p_2})^{1/r_2}$. If it is, then we derive the conclusion. If not, we continue the discussion as above. Without loss of generality, we assume that through $N-1$ steps, we obtain $p_{N-1}, q_{N-1} \in \mathbb{N}^+$ with $(p_{N-1}, q_{N-1}) = 1$ such that $\frac{q_{N-2}}{p_{N-2}} = (\frac{q_{N-1}}{p_{N-1}})^{\alpha_{j_{N-1}}}$. Let $r_{N-1} = \frac{r_{N-2}}{\alpha_{N-1}} = \frac{r}{\prod_{k=1}^{N-1}\alpha_{j_k}}$. Thus, $(\frac{q}{p})^{1/r} = (\frac{q_{N-1}}{p_{N-1}})^{1/r_{N-1}}$.

If $p_{N-1}x^{r_{N-1}} - q_{N-1} = 0$ is a minimal polynomial of $(\frac{q_{N-1}}{p_{N-1}})^{1/r_{N-1}}$, then the conclusion follows. If not, then there exists some $j_N \in \{1, 2, \dots t\}$ such that $\frac{q_{N-1}}{p_{N-1}} \in \mathbb{Q}^{\alpha_{j_N}}$; *i.e.*, there exist $p_N, q_N \in \mathbb{N}^+$ with $(p_N, q_N) = 1$, such that $\frac{q_{N-1}}{p_{N-1}} = (\frac{q_N}{p_N})^{\alpha_{j_N}}$. Let $r_N = \frac{r_{N-1}}{\alpha_{j_N}} = \frac{r}{\prod_{k=1}^{N}\alpha_{j_k}} = 1$; thus $(\frac{q}{p})^{1/r} = (\frac{q_N}{p_N})^{1/r_N} = \frac{q_N}{p_N}$. Now we prove that $p_Nx^{r_N} - q_N = 0$ is a minimal polynomial of $(\frac{q_N}{p_N})^{1/r_N}$, *i.e.*, $p_Nx - q_N = 0$ is a minimal polynomial of $\frac{q_N}{p_N}$. Since $x - \frac{q_N}{p_N} = 0$ is an irreducible polynomial over $\mathbb{Q}$, it is the minimal polynomial of $\frac{q_N}{p_N}$. Hence, the lemma follows. ∎

In the following, we characterize the orthogonality of the set of exponential functions in $L^2(\mu_{\rho,m})$. We divide it into two cases. The conclusions are given in Lemmas 2.2 and 2.3 depending on whether $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r}$.

Lemma 2.2 *Let $\rho = \pm(\frac{q}{p})^{1/r}$ for some p, q, $r \in \mathbb{N}^+$ with $(p, q) = 1$, $q < p$. Let $m \geq 2$ be an integer with $(p, m) = 1$. Assume that Λ is a finite or countable subset of $\mathbb{R}$. If $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r}$, then E_Λ is an orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ if and only if there exists an $i \in \{0, 1, \dots, r-1\}$ such that $(\Lambda - \Lambda) \smallsetminus \{0\} \subseteq \mathcal{Z}_i$, where*

$$\mathcal{Z}_0 = \Big\{\frac{l}{m}\Big(\frac{p}{q}\Big)^n : n \in \mathbb{N}^+, l \in \mathbb{Z} \smallsetminus m\mathbb{Z}\Big\},$$

and

$$\mathcal{Z}_i = \Big\{\frac{l}{m}\Big(\frac{p}{q}\Big)^n\Big(\frac{p}{q}\Big)^{i/r} : n \in \mathbb{N}_0, l \in \mathbb{Z} \smallsetminus m\mathbb{Z}\Big\} \quad \textit{for } i \in \{1, 2, \dots r-1\}.$$

Proof It is easy to see that E_Λ is an orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ if and only if $(\Lambda - \Lambda) \smallsetminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{\rho,m})$. Assume that $(\Lambda - \Lambda) \smallsetminus \{0\} \subseteq \mathcal{Z}_i$ for some $i \in \{0, 1, \dots, r-1\}$, where $\mathcal{Z}_i$ is defined as in the lemma. Then the sufficiency comes from the fact that $\mathcal{Z}(\hat{\mu}_{\rho,m}) = \{\frac{l}{m\rho^k} : k \in \mathbb{N}^+, l \in \mathbb{Z} \smallsetminus m\mathbb{Z}\} = \{\frac{l}{m}(\frac{p}{q})^{\frac{k}{r}} : k \in \mathbb{N}^+, l \in \mathbb{Z} \smallsetminus m\mathbb{Z}\} = \bigcup_{i=0}^{r-1}\mathcal{Z}_i$.

Now we prove the necessity. Assume that E_Λ is an orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$. Then $(\Lambda - \Lambda) \smallsetminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{\rho,m})$. Without loss of generality, we assume that $0 \in \Lambda$. So we have

$$\Lambda \smallsetminus \{0\} \subset (\Lambda - \Lambda) \smallsetminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{\rho,m}) = \Big\{\frac{l}{m\rho^k} : k \in \mathbb{N}^+, l \in \mathbb{Z} \smallsetminus m\mathbb{Z}\Big\}.$$

Assume that λ_1 and λ_2 are any two distinct elements in $\Lambda \setminus \{0\}$ with

$$\lambda_1 = \frac{l_1}{m\rho^{k_1}} \quad \text{and} \quad \lambda_2 = \frac{l_2}{m\rho^{k_2}},$$

where $k_1, k_2 \in \mathbb{N}^+$ and $l_1, l_2 \in \mathbb{Z} \setminus m\mathbb{Z}$. Then there exists $\lambda_3 = \frac{l_3}{m\rho^{k_3}} \in \mathcal{Z}(\hat{\mu}_{\rho,m})$ where $l_3 \in \mathbb{Z} \setminus m\mathbb{Z}$, such that $\lambda_1 - \lambda_2 = \lambda_3$, *i.e.*,

$$\lambda_1 - \lambda_2 = \frac{l_1}{m\rho^{k_1}} - \frac{l_2}{m\rho^{k_2}} = \frac{l_3}{m\rho^{k_3}}.$$

Let $K = \max\{k_j : 1 \le j \le 3\}$. Then

$$l_1\rho^{K-k_1} - l_2\rho^{K-k_2} = l_3\rho^{K-k_3};$$

i.e.,

$$l_1'|\rho|^{K-k_1} - l_2'|\rho|^{K-k_2} = l_3'|\rho|^{K-k_3},$$

where $l_j' = (\pm 1)^{K-K_j} l_j$ $(j = 1,2,3)$. Since $l_j' \in \mathbb{Z} \setminus \{0\}$, $K - k_j \ge 0$ $(j = 1,2,3)$ and $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r} (= |\rho|)$, by [4, Lemma 2.5], we have

$$K - k_1 = K - k_2 = K - k_3 \pmod r,$$

i.e., $k_1 = k_2 = k_3 \pmod r$. Denote $k_1 \pmod r$ by i. Thus, we have $\lambda_1, \lambda_2, \lambda_1 - \lambda_2 \in \mathcal{Z}_i$. Since λ_1, λ_2 are any two distinct elements in $\Lambda \setminus \{0\}$, the lemma follows. ∎

In fact, the conclusion of Lemma 2.2 also holds even if $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$. The only modification we need is to substitute p, q, r by p_1, q_1, r_1, where p_1, q_1, r_1 are defined in Lemma 2.1. We provide a short proof in the next lemma.

Lemma 2.3 *Under the assumptions of Lemma 2.2, if $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$, then let p_1, q_1, r_1 be defined as in Lemma 2.1. Then E_Λ is an orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ if and only if there exists a $j \in \{0, 1, \dots, r_1 - 1\}$ such that $(\Lambda - \Lambda) \setminus \{0\} \subseteq \widetilde{\mathcal{Z}}_j$, where*

$$\widetilde{\mathcal{Z}}_0 = \Big\{\frac{l}{m}\Big(\frac{p_1}{q_1}\Big)^n : n \in \mathbb{N}^+, l \in \mathbb{Z} \setminus m\mathbb{Z}\Big\}$$

and

$$\widetilde{\mathcal{Z}}_j = \Big\{\frac{l}{m}\Big(\frac{p_1}{q_1}\Big)^n \Big(\frac{p_1}{q_1}\Big)^{j/r_1} : n \in \mathbb{N}_0, l \in \mathbb{Z} \setminus m\mathbb{Z}\Big\} \quad \textit{for } j \in \{1, 2, \dots r_1 - 1\}.$$

Proof From Lemma 2.1, we obtain $p_1, q_1, r_1 \in \mathbb{N}^+$ with $(p_1, q_1) = 1$, $r_1 | r$ and $r_1 < r$. Moreover, $(\frac{q}{p})^{1/r} = (\frac{q_1}{p_1})^{1/r_1}$. Since $q < p$, we have $q_1 < p_1$. Let $\alpha = \frac{r}{r_1} \in \mathbb{N}^+$. Now $(\frac{q}{p})^{1/r} = (\frac{q_1}{p_1})^{1/r_1}$ implies $\frac{q}{p} = (\frac{q_1}{p_1})^\alpha$. Thus, $p = \frac{q}{q_1^\alpha} \cdot p_1^\alpha$. Since $p \in \mathbb{N}^+$ and $(p_1, q_1) = 1$, we get $q_1^\alpha | q$. Let $\beta = \frac{q}{q_1^\alpha} \in \mathbb{N}^+$. Hence, $p = \beta p_1^\alpha$. Since $(p, m) = 1$, we have $(p_1, m) = 1$.

Since $\rho = \pm(\frac{q}{p})^{1/r} = \pm(\frac{q_1}{p_1})^{1/r_1}$ and $p_1 x^{r_1} - q_1 = 0$ is a minimal polynomial of $(\frac{q_1}{p_1})^{1/r_1}$, the conclusion follows from Lemma 2.2. ∎

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Now we are ready to prove Theorem 1.2. We divide the proof into two cases depending on whether $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r}$. In the first case, we first prove that there are at most m mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$ based on Lemma 2.2. Then prove that m is the best possible by constructing an orthogonal set of exponential functions E_{Λ_0} such that $\#E_{\Lambda_0} = m$. We prove that the conclusion holds in the second case by applying Lemma 2.3 and using the result in the first case.

Proof of Theorem 1.2 We divide the proof into two cases.

Case 1: $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r}$.

We first prove that there are at most m mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$. We prove this by contradiction. Assume that there are $m+1$ mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$ with exponent set Λ. Without loss of generality, assume that $0 \in \Lambda$. From Lemma 2.2, we see that

$$\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}_i,$$

for some fixed $i \in \{0, 1, \dots r-1\}$ with

$$\Lambda \setminus \{0\} = \{\lambda_1, \dots, \lambda_m\},$$

where $\lambda_j = \frac{l_j}{m}(\frac{p}{q})^{n_j}(\frac{p}{q})^{i/r}$. Let $n = \max\{n_j : 1 \le j \le m\}$; then

$$\Lambda \setminus \{0\} = \frac{1}{m}\frac{1}{q^n}\Big(\frac{p}{q}\Big)^{i/r}\big\{p^{n_j}q^{n-n_j}l_j : 1 \le j \le m\big\}.$$

Let $\Lambda' = \{p^{n_j}q^{n-n_j}l_j : 1 \le j \le m\}$. Since $(p,m) = 1$, $(q,m) = 1$, and $m \nmid l$, we have

$$m \nmid p^{n_j}q^{n-n_j}l_j, \quad \forall j \in \{1, 2, \dots m\},$$

i.e.,

$$p^{n_j}q^{n-n_j}l_j \ne 0 \pmod{m}.$$

Together with $\#\Lambda' = m$, we see that there exist at least two different j_1, $j_2 \in \{1, 2, \dots m\}$, such that

$$p^{n_{j_1}}q^{n-n_{j_1}}l_{j_1} = p^{n_{j_2}}q^{n-n_{j_2}}l_{j_2} \pmod{m}.$$

So,

$$\lambda_{j_1} - \lambda_{j_2} = \frac{1}{m}\Big(\frac{p}{q}\Big)^{i/r}\frac{1}{q^n}(p^{n_{j_1}}q^{n-n_{j_1}}l_{j_1} - p^{n_{j_2}}q^{n-n_{j_2}}l_{j_2}) := \frac{1}{m}\Big(\frac{p}{q}\Big)^{i/r}\frac{1}{q^n}km \notin \mathcal{Z}_i,$$

where $k \in \mathbb{Z} \setminus \{0\}$. The last relation comes from $(q,m) = 1$ and the definition of $\mathcal{Z}_i$. Since if there exists some $\lambda = \frac{l}{m}(\frac{p}{q})^{i/r}(\frac{p}{q})^{n'} \in \mathcal{Z}_i$, such that $\lambda_{j_1} - \lambda_{j_2} = \lambda$, then we have

$$\frac{1}{m}\Big(\frac{p}{q}\Big)^{i/r}\frac{1}{q^n}km = \frac{l}{m}\Big(\frac{p}{q}\Big)^{i/r}\Big(\frac{p}{q}\Big)^{n'},$$

i.e., $\frac{1}{q^n}km = (\frac{p}{q})^{n'}l$, hence $q^{n'}km = q^n p^{n'}l$. Since $(p,q) = 1$ and $(p,m) = 1$, we have $p^{n'} \mid k$ and thus $q^{n'}k'm = q^n l$, where $k' \in \mathbb{Z} \setminus \{0\}$.

If $n' \ge n$, we have $q^{n'-n}k'm = l$, which contradicts $m \nmid l$; If $n' < n$, we have $k'm = q^{n-n'}l$. Since $(q,m) = 1$, we have $k''m = l$ for some $k'' \in \mathbb{Z} \setminus \{0\}$, which also contradicts the condition on l. Hence, there are at most m mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$.

In the following, we prove that m is the best possible by constructing an orthogonal set of exponential functions with cardinality equal to m. Let

$$\Lambda_0 = \{0\} \cup \left\{ \frac{j}{m}\left(\frac{p}{q}\right)^t \left(\frac{p}{q}\right)^{i/r} : 1 \le j \le m-1 \right\},$$

for some $i \in \{0, 1, \dots r-1\}$ and some $t \in \mathbb{N}^+$. It is easy to check that $(\Lambda_0 - \Lambda_0) \setminus \{0\} \subset \mathcal{Z}_i$. Thus, E_{Λ_0} is an orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ by Lemma 2.2. Hence m is the best possible.

Case 2: $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$.

Let p_1, q_1, r_1 be as in Lemma 2.1. Then $p_1, q_1, r_1 \in \mathbb{N}^+$, $(p_1, q_1) = 1$ and $r_1 | r$. Recall from the proof of Lemma 2.3 that $\alpha = \frac{r}{r_1} \in \mathbb{N}^+$, $\beta = \frac{q}{q_1^\alpha} \in \mathbb{N}^+$. From $q = \beta q_1^\alpha$ and $(q, m) = 1$, we see that $(q_1, m) = 1$. Also recall from the proof of Lemma 2.3 that $(p_1, m) = 1$ and $q_1 < p_1$.

Since $\rho = \pm(\frac{q}{p})^{1/r} = \pm(\frac{q_1}{p_1})^{1/r_1}$ and $p_1 x^{r_1} - q_1 = 0$ is a minimal polynomial of $(\frac{q_1}{p_1})^{1/r_1}$, by the result of Case 1, we see that there are at most m mutually orthogonal exponential functions in $L^2(\mu_{\rho,m})$, and m is the best possible.

This completes the proof. ∎

3 Proof of Theorem 1.3

It follows from Theorem A that if $(p, m) = 1$, then every orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ is finite. But to our surprise, there are any number of orthogonal exponential functions in $L^2(\mu_{\rho,m})$ under the assumption that $(q, m) > 1$. We prove the conclusion by constructing an orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ with cardinality equal to N, where N is any given positive integer. The construction of such sets is divided into two cases depending on whether $(\frac{q}{p})^{1/r}$ admits $px^r - q = 0$ as a minimal polynomial.

We first introduce a definition that plays an important part in the construction of such sets.

Definition 3.1 ([10]) Let $n > 0$, $(a, n) = 1$ and let s be the smallest positive integer such that $a^s = 1 \pmod{n}$; then s is called the *multiplicative order of* $a \pmod{n}$.

Lemma 3.2 *Under the assumptions of Theorem 1.3, assume that $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r}$. For any given positive integer N, and for any given integer i with $i \in \{0, 1, \dots, r-1\}$, let*

$$\Lambda^* = \left\{ \lambda_n = \frac{p^{(N+n)s}}{m}\left(\frac{p}{q}\right)^{ns}\left(\frac{p}{q}\right)^{i/r} : 1 \le n \le N \right\} \subset \mathcal{Z}_i,$$

where s is the multiplicative order of $p \pmod{m}$, and $\mathcal{Z}_i$ is defined as in Lemma 2.2. Then $(\Lambda^* - \Lambda^*) \setminus \{0\} \subset \mathcal{Z}_i$.

https://doi.org/10.4153/S0008439519000304 Published online by Cambridge University Press

Proof Without loss of generality, we assume that $i = 0$. For any integers n_1, n_2 with $1 \le n_2 < n_1 \le N$, we have

$$\begin{aligned}\lambda_{n_1} - \lambda_{n_2} &= \frac{p^{(N+n_1)s}}{m}\Big(\frac{p}{q}\Big)^{n_1 s} - \frac{p^{(N+n_2)s}}{m}\Big(\frac{p}{q}\Big)^{n_2 s}\\ &= \frac{1}{m}\Big(\frac{p}{q}\Big)^{n_1 s}\Big(p^{(N+n_1)s} - p^{(N+n_2)s}\cdot\Big(\frac{q}{p}\Big)^{(n_1-n_2)s}\Big)\\ &= \frac{1}{m}\Big(\frac{p}{q}\Big)^{n_1 s}\Big(p^{(N+n_1)s} - p^{(N+n_2)s-(n_1-n_2)s}\cdot q^{(n_1-n_2)s}\Big)\\ &= \frac{1}{m}\Big(\frac{p}{q}\Big)^{n_1 s}\Big(p^{(N+n_1)s} - p^{(N-n_1+2n_2)s}\cdot q^{(n_1-n_2)s}\Big).\end{aligned}$$

In order to prove that $\lambda_{n_1} - \lambda_{n_2} \in \mathcal{Z}_0$, we only need to prove that

(3.1) $$p^{(N+n_1)s} - p^{(N-n_1+2n_2)s}\cdot q^{(n_1-n_2)s} \ne 0 \pmod{m}.$$

Since $p^s \pmod{m} = 1$, we have

(3.2) $$p^{ks} \pmod{m} = 1$$

holds for any positive integer k.

We claim that for any two positive integers a, b,

(3.3) $$p^{as}q^b = q^b \ne 1 \pmod{m}.$$

We prove this by contradiction. Assume that $q^b \pmod{m} = 1$; then there exists some integer t, such that $q^b = tm + 1$, *i.e.*, $q^b - tm = 1$. Since $d \mid q$ and $d \mid m$, we have $d \mid 1$, which contradicts the fact that $d > 1$. Hence the claim follows.

From (3.2) and (3.3), we obtain that (3.1) holds. Hence,

$$\lambda_{n_1} - \lambda_{n_2} \in \mathcal{Z}_0.$$

Since n_1 and n_2 are arbitrary, together with the property $\mathcal{Z}_0 = -\mathcal{Z}_0$, we obtain that $(\Lambda^* - \Lambda^*) \setminus \{0\} \subset \mathcal{Z}_0$. ■

If $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$, then let p_1, q_1, r_1 be defined as in Lemma 2.1. Recall from the proof of Lemma 2.3 that $(p_1, m) = 1$. We prove that $(q_1, m) > 1$ and then construct the set $\widetilde{\Lambda}^*$ by using the conclusion of Lemma 3.2.

Lemma 3.3 *Under the assumptions of Theorem 1.3, if $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$, let p_1, q_1, r_1 be defined as in Lemma 2.1. For any given positive integer N, and for any given integer j with $j \in \{0, 1, \dots, r_1 - 1\}$, let*

$$\widetilde{\Lambda}^* = \Big\{\widetilde{\lambda}_n = \frac{p_1^{(N+n)\widetilde{s}}}{m}\Big(\frac{p_1}{q_1}\Big)^{n\widetilde{s}}\Big(\frac{p_1}{q_1}\Big)^{j/r_1} : 1 \le n \le N\Big\} \subset \widetilde{\mathcal{Z}}_j,$$

where $\widetilde{s}$ is the multiplicative order of $p_1 \pmod{m}$, and $\widetilde{\mathcal{Z}}_j$ is defined as in Lemma 2.3. Then $(\widetilde{\Lambda}^* - \widetilde{\Lambda}^*) \setminus \{0\} \subset \widetilde{\mathcal{Z}}_j$.

Proof From Lemma 2.1, we obtain that $p_1, q_1, r_1 \in \mathbb{N}^+$ with $(p_1, q_1) = 1$, $r_1 | r$ and $r_1 < r$. Moreover, $(\frac{q}{p})^{1/r} = (\frac{q_1}{p_1})^{1/r_1}$. Recall from the proof of Lemma 2.3 that $\alpha = \frac{r}{r_1} \in \mathbb{N}^+$, $\beta = \frac{q}{q_1^\alpha} \in \mathbb{N}^+$, and $p = \beta p_1^\alpha$. Since $(p, m) = 1$, we obtain that $(\beta, m) = 1$. From

$q = \beta q_1^{\alpha}$ and $(q, m) > 1$, we have $(q_1, m) > 1$. Also recall from the proof of Lemma 2.3 that $(p_1, m) = 1$ and $q_1 < p_1$.

Since $\rho = \pm(\frac{q}{p})^{1/r} = \pm(\frac{q_1}{p_1})^{1/r_1}$ and $p_1 x_1^r - q_1 = 0$ is a minimal polynomial of $(\frac{q_1}{p_1})^{1/r_1}$, from Lemma 3.2, we see that $(\widetilde{\Lambda}^* - \widetilde{\Lambda}^*) \setminus \{0\} \subset \widetilde{\mathcal{Z}}_j$. ■

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 If $px^r - q = 0$ is a minimal polynomial of $(\frac{q}{p})^{1/r}$, then for any given positive integer N and for any given $i \in \{0, 1, \ldots, r-1\}$, let Λ^* be defined as in Lemma 3.2. Since $(\Lambda^* - \Lambda^*) \setminus \{0\} \subset \mathcal{Z}_i$, from Lemma 2.2, we see that E_{Λ^*} is an orthogonal set in $L^2(\mu_{\rho,m})$ with $\# E_{\Lambda^*} = N$, *i.e.*, there are any number of orthogonal exponential functions in $L^2(\mu_{\rho,m})$. Hence, $n^*(\mu_{\rho,m}) = \infty$. The conclusion follows.

If $px^r - q = 0$ is not a minimal polynomial of $(\frac{q}{p})^{1/r}$, then we can derive the same conclusion from Lemmas 3.3 and 2.3. This completes the proof. ■

Acknowledgments The authors would like to thank Professor Palle Jorgensen for many valuable suggestions and Professor Jing-Cheng Liu for many helpful discussions. The authors also would like to thank Dr. Sayan Das for helpful comments on the presentation of this paper.

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https://doi.org/10.4153/S0008439519000304 Published online by Cambridge University Press