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Scattering of a scalar time-harmonic wave by a penetrable obstacle with a thin layer

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This work looks at the asymptotic behaviour of the solution to the Helmholtz equation in a penetrable domain of $\mathbb{R}^3$ with a thin layer of thickness δ which tends to 0. We use the method of multi-scale expansion to derive and justify an asymptotic expansion of the solution with respect to the thickness δ up to any order. We then provide approximate transmission conditions of order two defined on an interface located inside the thin layer, with accuracy up to $O(\delta^2)$, which allow one to take into account the influence of the thin layer.

Key words: Asymptotic analysis, Asymptotic expansions, Wave scattering, Helmholtz equation, Thin structures.

1 Introduction

In this work, we study the asymptotic behaviour of the solution to the Helmholtz equation

$$\begin{cases} \operatorname{div}(\sigma_\delta \nabla u_\delta) + k_\delta^2 u_\delta = 0 & \text{in } \mathbb{R}^3, \\ \lim\limits_{|x|\to+\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right)(u_\delta - u_{inc}) = 0, \end{cases} \tag{1.1}$$

where σ_δ and k_δ^2 are piecewise constant functions defined by

$$\sigma_\delta(x) = \begin{cases} \sigma_{\text{ext}} & \text{if } x \in \Omega_{\text{ext},\delta}, \\ \widetilde{\sigma}_\delta & \text{if } x \in \Omega_\delta, \\ \sigma_{\text{int}} & \text{if } x \in \Omega_{\text{int},\delta}, \end{cases} \; ; \; k_\delta^2(x) = \begin{cases} k_{\text{ext}}^2 & \text{if } x \in \Omega_{\text{ext},\delta}, \\ \widetilde{k}_\delta^2 & \text{if } x \in \Omega_\delta, \\ k_{\text{int}}^2 & \text{if } x \in \Omega_{\text{int},\delta}, \end{cases}$$

where $\widetilde{\sigma}_\delta$ and σ_{int} are two strictly positive constants describing the contrast properties of Ω_δ and $\Omega_{\text{int},\delta}$ relative to the exterior propagation domain $\Omega_{\text{ext},\delta}$. The refractive properties of the media are defined by k_{int}^2 and k_δ^2 which are two complex numbers with strictly positive real parts and positive imaginary parts. We also assume that σ_{ext} and k_{ext} are strictly positive constants and that σ_{ext}, σ_{int}, k_{ext}, k_{int} are independent of δ. The domain $\Omega_{\text{int},\delta}$ is a three-dimensional bounded domain with regular boundary $\Gamma_{\delta,1}$, surrounded by

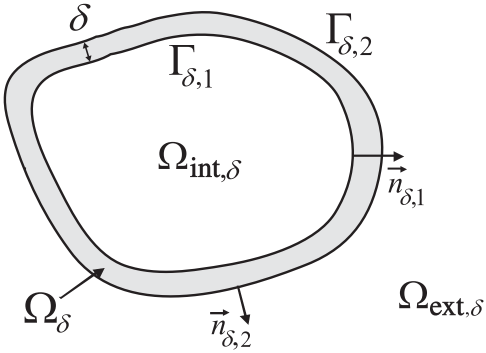

FIGURE 1. Geometric data.

a thin layer Ω_δ of thickness δ (which tends to 0) and $\Omega_{\mathrm{ext},\delta}$ is the exterior domain defined by $\Omega_{\mathrm{ext},\delta} = \mathbb{R}^3 \backslash \left(\overline{\Omega_{\mathrm{int},\delta} \cup \Gamma_{\delta,1} \cup \Omega_\delta \cup \Gamma_{\delta,2}}\right)$ (see Figure 1). This work looks at the scattering of an incident wave $u_{inc}(x) = e^{ik_{\mathrm{ext}}(x.d)/\sigma_{\mathrm{ext}}}$ by the penetrable domain $\left(\overline{\Omega_{\mathrm{int},\delta} \cup \Gamma_{\delta,1} \cup \Omega_\delta}\right)$, where d is a unit vector of $\mathbb{R}^3$ giving the direction of the plane wave u_{inc}.

Numerical simulations of scattering problems as the one considered here need to mesh the thin layer. Since this can be a very costly task [39], it is of great interest to take into account the effect of such thin layer thanks to suitable approximate boundary conditions. The latter can be derived by studying the asymptotic behaviour, as $\delta \to 0$, of the total field u_δ. The asymptotic behaviour of solutions to problems with thin layers has been addressed by many authors in the last decade (cf., e.g., [5, 7, 8, 12, 15, 17, 20, 33]. . .). Many different techniques have been used in these papers and a variety of results have been obtained. More precisely, approximate transmission conditions have been derived for the electro-quasistatic equations in [29] and time-harmonic Maxwell equations in [28] for thin layer and in [11, 12] for the Laplace equation in the case of thin periodic coating. Higher order approximation were derived in smooth geometries of conductive thin sheets for the Helmholtz equation in [37] and for the eddy current problem in [38]. The case of a thin ring with regularly spaced inhomogeneities has been treated in [15, 17] for the 2D Helmholtz equation.

Here, we derive transmission conditions to approximate the solution u_δ to Problem (1.1) by a solution u_δ^{ap} to a problem $\left(\mathscr{P}_\delta^{ap}\right)$ with the Helmholtz equation defined in a domain without a thin layer with Ventcel-type transmission conditions, involving tangential differential operators of order two, with accuracy up to $O(\delta^2)$. We propose a technique (see [8]) that consists of dividing Ω_δ into two thin layers separated by a surface Γ parallel to $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$ (see Figures 2 and 3) and choosing it in such a way that transmission conditions ensure existence and uniqueness of the solution u_δ^{ap} to the approximate problem. The main difficulty here, compared to the studies performed in [7, 8], is that the Helmholtz equation is not elliptic implying, for example, that we do not readily have a stability result which is uniform with respect to δ. Another difficulty comes from the unbounded setting of the study.

In order to accomplish our goal, we derive an asymptotic expansion of the solution u_δ to Problem (1.1). Two different approaches are often used: the matched asymptotic expansions method (cf., e.g., [2, 3, 15, 21, 24]) and the method of multi-scale expansion (cf.,

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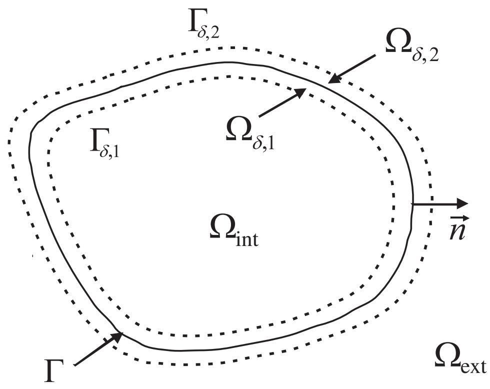

FIGURE 2. Geometry of the studied problem.

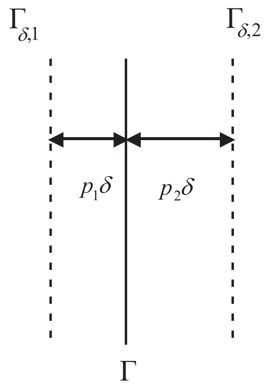

FIGURE 3. The thin layer Ω_δ.

e.g., [4, 7, 37]). However, the problem is not defined on the exterior of the thin layer and thus the multi-scale expansion method is more suitable since we are dealing with the transmission problem [7, 8].

The paper is organized as follows. In Section 2, we give the statement of the problem considered, the existence and uniqueness theorem for the solution to Problem (1.1) together with a uniform stability estimate for u_δ. Section 3 recalls some basic definitions and notation from the differential geometry of surfaces.

In Section 4, we construct a formal asymptotic expansion for the solution to Problem (1.1), while Section 5 focuses on the justification of the asymptotics and the convergence of this ansatz. In Section 6, we model the effect of the thin layer by a problem with Ventcel-type transmission conditions. The well-posedness of Problem $\left(\mathscr{P}_\delta^{ap}\right)$ will also be proved.

Finally, in Section 7, we extend our results to the case of materials having high magnetic permittivity in the domain Ω_δ.

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2 Problem setting

We consider a parallel surface Γ to $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$ dividing Ω_δ into two thin layers $\Omega_{\delta,1}$ and $\Omega_{\delta,2}$ of respective thickness $p_1\delta$ and $p_2\delta$, where p_1 and p_2 are strictly positive real numbers satisfying $p_1 + p_2 = 1$ and such that p_1 and p_2 belong to a small neighbourhood of 1/2 (see Figures 2 and 3). The term *small neighbourhood* means that the constants p_1 and p_2 are not too close to 1 or 0, in order to avoid having a layer too thin compared to the other because the following analysis does not lend itself to this case. Let us denote by $u_{\text{ext},\delta}$, $u_{d_2,\delta}$, $u_{d_1,\delta}$ and $u_{\text{int},\delta}$ the restrictions of u_δ respectively to the domains $\Omega_{\text{ext},\delta}$, $\Omega_{\delta,2}$, $\Omega_{\delta,1}$ and $\Omega_{\text{int},\delta}$. Under the aforementioned assumptions, we investigate in $H^1_{loc}(\mathbb{R}^3)$ the solution u_δ to the following equivalent problem:

$$\begin{cases} \operatorname{div}\left(\sigma_{\text{ext}}\nabla u_{\text{ext},\delta}\right) + k_{\text{ext}}^2 u_{\text{ext},\delta} = 0 & \text{in } \Omega_{\text{ext},\delta}, & (2.1a)\\ \operatorname{div}\left(\widetilde{\sigma}_\delta\nabla u_{d_2,\delta}\right) + \widetilde{k}_\delta^2 u_{d_2,\delta} = 0 & \text{in } \Omega_{\delta,2}, & (2.1b)\\ \operatorname{div}\left(\widetilde{\sigma}_\delta\nabla u_{d_1,\delta}\right) + \widetilde{k}_\delta^2 u_{d_1,\delta} = 0 & \text{in } \Omega_{\delta,1}, & (2.1c)\\ \operatorname{div}\left(\sigma_{\text{int}}\nabla u_{\text{int},\delta}\right) + k_{\text{int}}^2 u_{\text{int},\delta} = 0 & \text{in } \Omega_{\text{int},\delta}, & (2.1d) \end{cases}$$

with transmission conditions

$$\begin{cases} u_{d_2,\delta|\Gamma_{\delta,2}} = u_{\text{ext},\delta|\Gamma_{\delta,2}} & \text{on } \Gamma_{\delta,2}, & (2.1e)\\ \widetilde{\sigma}_\delta \partial_{\mathbf{n}_{\delta,2}} u_{d_2,\delta|\Gamma_{\delta,2}} = \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}} & \text{on } \Gamma_{\delta,2}, & (2.1f)\\ u_{d_1,\delta|\Gamma} = u_{d_2,\delta|\Gamma} & \text{on } \Gamma, & (2.1g)\\ \partial_{\mathbf{n}} u_{d_1,\delta|\Gamma} = \partial_{\mathbf{n}} u_{d_2,\delta|\Gamma} & \text{on } \Gamma, & (2.1h)\\ u_{\text{int},\delta|\Gamma_{\delta,1}} = u_{d_1,\delta|\Gamma_{\delta,1}} & \text{on } \Gamma_{\delta,1}, & (2.1i)\\ \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}} = \widetilde{\sigma}_\delta \partial_{\mathbf{n}_{\delta,1}} u_{d_1,\delta|\Gamma_{\delta,1}} & \text{on } \Gamma_{\delta,1}, & (2.1j) \end{cases}$$

and radiation condition

$$\lim_{|x|\to+\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right)\left(u_{\text{ext},\delta} - u_{inc}\right) = 0, \qquad (2.1k)$$

where $\partial_{\mathbf{n}_{\delta,1}}, \partial_{\mathbf{n}}, \partial_{\mathbf{n}_{\delta,2}}$ and $\partial_{\mathbf{n}_e}$ denote the derivatives in the direction of the unit normal vectors $\mathbf{n}, \mathbf{n}_{\delta,1}, \mathbf{n}_{\delta,2}$ and $\mathbf{n}_e$ to $\Gamma_{\delta,1}, \Gamma, \Gamma_{\delta,2}$ and $\partial\Omega$ respectively (see Figure 1). The following theorem gives the well-posedness of (2.1).

Theorem 2.1 *Problem (2.1) has one and only one solution u_δ in $H^1_{loc}(\mathbb{R}^3)$.*

Proof Uniqueness follows by Rellich's lemma (cf. [13, 35]). Existence of a solution is obtained by standard arguments involving the limiting absorption principle (cf., e.g., [30]).

We now rewrite the problem in a truncated domain (see [3–5] for a similar reduction) in order to get a uniform stability result with respect to δ. The latter is actually going to be useful for proving error estimates between u_δ and the asymptotic expansion that is going to be built in the next sections.

Let Ω be a bounded domain of class $\mathscr{C}^\infty$ which contains the thin layer Ω_δ as depicted in Figure 4.

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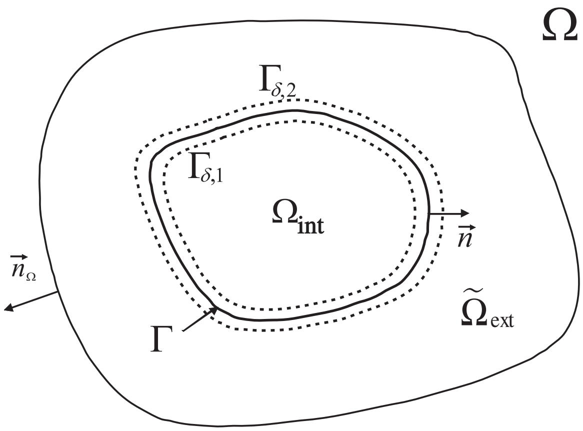

FIGURE 4. The truncated domain.

We denote by T the DtN operator (Dirichlet-to-Neumann) defined on $H^{1/2}(\partial\Omega)$ by $T\varphi := -\partial_{\mathbf{n}_\Omega}\omega$, where $\mathbf{n}_\Omega$ is the unit normal to $\partial\Omega$ directed out of Ω and ω is the unique solution to the following problem:

$$\begin{cases} \text{Find } \omega \in H^1_{loc}(\mathbb{R}^3\backslash\overline{\Omega}) & \\ \operatorname{div}(\sigma_{\text{ext}}\nabla\omega) + k^2_{\text{ext}}\omega = 0 & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \\ \omega_{|\partial\Omega} = \varphi & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to+\infty} |x|\left(\partial_{|x|} - ik_{\text{ext}}\right)\omega = 0. & \end{cases} \tag{2.2}$$

The DtN operator T is a pseudo-differential operator of order one [4] and is linearly continuous from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. The next lemma, whose proof uses standard elliptic regularity (cf., e.g., [4]), gives a useful decomposition of the DtN operator.

Lemma 2.1 *Let $\phi \in H^{1/2}(\partial\Omega)$ and $\varphi_0 \in H^1(\mathbb{R}^3 \setminus \overline{\Omega})$ be the unique solution to the following coercive scattering problem*:

$$\begin{cases} \Delta\varphi_0 - \varphi_0 = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\ \varphi_0 = \phi & \text{on } \partial\Omega. \end{cases}$$

Now let us consider $T_0\phi = -\partial_{\mathbf{n}_\Omega}\varphi_0$. Then, T_0 is bounded and coercive from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. In addition, there exists a compact operator K acting from $H^{1/2}(\partial\Omega)$ into $H^{3/2}(\partial\Omega)$ such that

$$T = T_0 + K. \tag{2.3}$$

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Using the DtN operator T, Problem (1.1) can be written as

$$\begin{cases} \text{Find } u_\delta \in H^1(\Omega) \text{ such that} \\ \operatorname{div}(\sigma_\delta \nabla u_\delta) + k_\delta^2 u_\delta = 0 & \text{in } \Omega, \\ \left(\partial_{\mathbf{n}_\Omega} + T\right) u_\delta = \left(\partial_{\mathbf{n}_\Omega} + T\right) u_{inc} & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

A variational formulation of (2.4) is then given by

$$\begin{cases} \text{Find } u_\delta \in H^1(\Omega),\ \forall v \in H^1(\Omega) \\ a_\delta(u_\delta, v) := \displaystyle\int_\Omega \sigma_\delta \nabla u_\delta . \nabla \overline{v} - k_\delta^2 u_\delta \overline{v}\, d\Omega + \sigma_{\text{ext}} \langle T u_\delta, \overline{v} \rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \\ \qquad\qquad = l_\delta(v), \end{cases} \tag{2.5}$$

where $\langle .,. \rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ and l_δ is an arbitrary linear form on $H^1(\Omega)$. For Problem (2.4), l_δ is defined by

$$l_\delta(v) := \sigma_{\text{ext}} \int_{\partial\Omega} \left(\partial_{\mathbf{n}_\Omega} + T\right) u_{inc} \overline{v}\, d\sigma.$$

For our purpose, one needs to know about the dependence of u_δ as δ goes to zero.

Theorem 2.2 (*Uniform stability*) *Suppose that*

$$\exists \varepsilon > 0,\ \widetilde{\sigma}_\delta = O(\delta^{-\frac{1}{2}+\varepsilon}),\ \widetilde{k}_\delta^2 = O(\delta^{-\frac{1}{2}+\varepsilon}); \tag{2.6}$$

$$\exists \alpha > 0,\ \forall \delta > 0,\ \Re(\sigma_\delta) > \alpha; \tag{2.7}$$

then, for all l_δ in $\left(H^1(\Omega)\right)'$, Problem (2.5) admits a unique solution in $H^1(\Omega)$. Furthermore, there exists a positive constant c independent of δ such that

$$\|u_\delta\|_{H^1(\Omega)} \leqslant c \|l_\delta\|_{\left(H^1(\Omega)\right)'}.$$

Proof We need to prove that

$$\|u_\delta\|_{H^1(\Omega)} \leqslant C \sup_{v \in H^1(\Omega)} \frac{|a_\delta(u_\delta, v)|}{\|v\|_{H^1(\Omega)}}.$$

To do so, we use a standard proof (cf. e.g., [16, 23]), and proceed by contradiction by assuming that there exists sequences $(\delta_n)_{n\geqslant 0}$ and $\left(u_{\delta_n}\right)_{n\geqslant 0}$ (denoted by $(u_n)_{n\geqslant 0}$) such that

$$\lim_{n\to+\infty} \delta_n \to 0,\ \|u_n\|_{H^1(\Omega)} = 1,\ \forall n \in \mathbb{N},\ \lim_{n\to+\infty} \sup_{\|\varphi\|_{H^1(\Omega)}=1} |a_{\delta_n}(u_n, \varphi)| = 0.$$

From Rellich's embedding theorem, we can extract a subsequence, still denoted by $(u_n)_{n\geqslant 0}$, such that

$$\begin{cases} u_n \to u_0 \text{ in } L^2(\Omega), \\ u_n \rightharpoonup u_0 \text{ in } H^1(\Omega). \end{cases}$$

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Since σ_δ and k_δ^2 satisfy (2.6), one gets

$$\begin{cases} \sigma_\delta \to \sigma_0 := \sigma_{\text{int}}\chi_{\Omega_{\text{int}}}(x) + \sigma_{\text{ext}}\chi_{\widetilde{\Omega}_{\text{ext}}}(x) \text{ in } L^2(\Omega), \\ k_\delta^2 \to k_0^2 := k_{\text{int}}^2\chi_{\Omega_{\text{int}}}(x) + k_{\text{ext}}^2\chi_{\widetilde{\Omega}_{\text{ext}}}(x) \text{ in } L^2(\Omega), \end{cases} \tag{2.8}$$

where $\Omega_{\text{int}} =$ "$\lim_{\delta\to 0}$"$\Omega_{\text{int},\delta}$ and $\widetilde{\Omega}_{\text{ext}} = \Omega\backslash\overline{\Omega}_{\text{int}}$ (see Figure 2) and $\chi_{\mathcal{O}}$ denotes the indicator function of the open set $\mathcal{O}$. Indeed,

$$\|\sigma_\delta - \sigma_0\|^2_{L^2(\Omega)} = \int_{\Omega_\delta} |\sigma_\delta - \sigma_0|^2 \, d\Omega_\delta \leqslant \int_{\Omega_{\delta,1}} |\sigma_{\text{int}}|^2 \, d\Omega_{\delta,1} + \int_{\Omega_\delta} |\widetilde{\sigma}_\delta|^2 \, d\Omega_\delta + \int_{\Omega_{\delta,2}} |\sigma_{\text{ext}}|^2 \, d\Omega_{\delta,2}.$$

Using Lebesgue dominated convergence theorem, we obtain

$$\lim_{\delta\to 0} \int_{\Omega_{\delta,1}} |\sigma_{\text{int}}|^2 \, d\Omega_{\delta,1} = \lim_{\delta\to 0} \int_{\Omega_{\delta,2}} |\sigma_{\text{ext}}|^2 \, d\Omega_{\delta,2} = 0.$$

Moreover, we have

$$\int_{\Omega_\delta} |\widetilde{\sigma}_\delta|^2 \, d\Omega_\delta \leqslant C \left(\delta^{-\frac{1}{2}+\varepsilon}\right)^2 meas(\Omega_\delta) = C\delta^{2\varepsilon} meas(\Gamma) \to_{\delta\to 0} 0.$$

Now, upon using (2.8), we get

$$\lim_{n\to+\infty} a_{\delta_n}(u_n, \varphi) = \int_\Omega \sigma_0 \nabla u_0 . \overline{\nabla\varphi} - k_0^2 u_0 \overline{\varphi} \, d\Omega \text{ for all } \varphi \text{ in } H^1(\Omega),$$

so $u_0 \in H^1(\Omega)$ satisfies

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla u_0) + k_0^2 u_0 = 0 & \text{in } \Omega, \\ \left(\partial_{\mathbf{n}_\Omega} + T\right) u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

As a result, well-known properties of uniqueness of the solution to this type of problem based on Rellich's lemma and the operator T imply that $u_0 = 0$. It only remains to show that $\lim_{n\to+\infty} \|u_n\|_{H^1(\Omega)} = 0$. Note that, since u_0 is uniquely determined, the whole sequence $(u_n)_{n\geqslant 0}$ that converges to u_0. To obtain the contradiction, we now show that $\lim_{n\to+\infty} \|\nabla u_n\|_{L^2(\Omega)} = 0$. From (2.7), we have

$$\begin{aligned} \|\nabla u_n\|^2_{L^2(\Omega)} &\leqslant C \int_\Omega \sigma_\delta |\nabla u_n|^2 \, d\Omega \\ &= C\Re\left(a_n(u_n, u_n) + \int_\Omega k_{\delta_n}^2 |u_n|^2 \, d\Omega - \sigma_{\text{ext}} \langle T u_n, \overline{u_n}\rangle_{H^{-1/2}\times H^{1/2}}\right). \end{aligned}$$

Using Lemma 2.1, we infer

$$\|\nabla u_n\|^2_{L^2(\Omega)} \leqslant C\left\{\Re\left[a_n(u_n, u_n) + \int_\Omega k_{\delta_n}^2 |u_n|^2 \, d\Omega - \sigma_{\text{ext}} \langle K u_n, \overline{u_n}\rangle_{H^{-1/2}\times H^{1/2}}\right]\right\}.$$

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As $u_n \underset{n\to+\infty}{\longrightarrow} u_0 = 0$ in $L^2(\Omega)$ and $k_{\delta_n}^2 \underset{n\to+\infty}{\longrightarrow} k_0^2$ in $L^2(\Omega)$, it follows that

$$\int_\Omega k_{\delta_n}^2 |u_n|^2 \, d\Omega \underset{n\to+\infty}{\longrightarrow} 0.$$

Since K is compact and $u_n \rightharpoonup 0$ in $H^1(\Omega)$, $\langle K u_n, \overline{u_n}\rangle_{H^{-1/2}\times H^{1/2}} \underset{n\to+\infty}{\longrightarrow} 0$. Finally, the hypothesis $\lim\limits_{n\to+\infty} \Re\left[a_n(u_n,u_n)\right] = 0$ yields $\lim\limits_{n\to+\infty} \|\nabla u_n\|_{L^2(\Omega)} = 0$ contradicting $\|u_n\|_{H^1(\Omega)} = 1$.

Remark 2.1 *In the proof of Theorem 2.2, we require the convergence in L^2 of σ_δ and k_δ. This justifies assumption (2.6) used to prove that (2.8) holds.*

3 Tools of differential geometry

The goal of this section is to define and to collect the main features of differential geometry [19] (see also [27]) in order to formulate our problem in a fixed domain (independent of δ). This technique is a key tool to determine the asymptotic expansion of the solution u_δ.

3.1 Parametrization of the surface Γ

Let $(\mathscr{U},\varphi)$ be a local coordinate patch for the surface Γ, with $\mathscr{U}$ being an open domain of $\mathbb{R}^2$ and

$$\begin{array}{rccl} \varphi: & \mathscr{U} & \to & \Gamma \\ & (\xi^1,\xi^2) & \to & m = \varphi\left(\xi^1,\xi^2\right). \end{array}$$

A basis of the tangent plane $T_m(\Gamma)$ to Γ at the point $m\in\Gamma$ is given by

$$\tau_\alpha(m) := \frac{\partial\varphi\left(\xi^1,\xi^2\right)}{\partial\xi^\alpha};\ \alpha = 1,2.$$

We assume that the coordinate patch $\{\tau_\alpha\}_{\alpha=1,2.}$ is compatible with the orientation, namely, the unit normal $\mathbf{n}(m)$ to Γ at point m is given by

$$\mathbf{n}(m) := \frac{\tau_1\times\tau_2}{|\tau_1\times\tau_2|},$$

where $\times$ and $|.|$ are respectively the usual cross product and norm in $\mathbb{R}^3$.
We denote by $\mathscr{R}$ the symmetric linear operator of the tangent plane $T_m(\Gamma)$ that characterizes the curvature of Γ at point m, and defined by

$$\frac{\partial\mathbf{n}(m)}{\partial\xi^\alpha} := \mathscr{R}(m)\tau_\alpha;\ \alpha = 1,2.$$

Let Π_m be the orthogonal projector from $\mathbb{R}^3$ into $T_m(\Gamma)$ and $\mathbf{w}$ a vector of $\mathbb{R}^3$, we have

$$\mathbf{w} = \mathbf{w}_T + w_n\mathbf{n} = \Pi_m\mathbf{w} + w_n\mathbf{n},$$

where $\mathbf{w}_T = \Pi_m\mathbf{w}$ is the tangential component and w_n is the normal component of $\mathbf{w}$.

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3.2 Differential operators on Γ

Let v be a smooth function defined on Γ. The surfacic gradient $\nabla_\Gamma v(m)$ of v at $m \in \Gamma$ is defined by

$$\nabla_\Gamma v(m) := \sum_{\lambda=1}^{2}\left[\sum_{\alpha=1}^{2} G^{\lambda\alpha}(m)\frac{\partial}{\partial \xi^\alpha}(v\circ\varphi)\left(\xi^1,\xi^2\right)\right]\tau_\lambda(m),$$

where $(G^{\lambda\alpha}(m))$ is the inverse of the metric tensor $(\tau_\lambda(m).\tau_\alpha(m))_{\lambda,\alpha=1,2}$.

If $\widehat{v}$ is a function defined in a neighbourhood of Γ, we have

$$\nabla_\Gamma(\widehat{v})(m) := \Pi_m(\nabla\widehat{v}(m))\,;\quad m\in\Gamma.$$

Let $\mathbf{w}_T$ be a smooth tangent vector field defined on Γ. The surfacic divergence of $\mathbf{w}_T$ is the scalar function defined on Γ through Stokes formula

$$\int_\Gamma \phi\,\mathrm{div}_\Gamma \mathbf{w}_T\, d\Gamma := -\int_\Gamma \nabla_\Gamma\phi.\mathbf{w}_T\, d\Gamma,$$

where ϕ is any regular function on Γ and $d\Gamma = |\tau_1\times\tau_2|\,d\xi^1 d\xi^2$ denotes the surfacic measure on Γ. The scalar Laplace–Beltrami operator on Γ is finally given by

$$\Delta_\Gamma := \mathrm{div}_\Gamma(\nabla_\Gamma).$$

3.3 Parametrization of $\Omega_{\delta,\beta}$

In what follows, the Greek index β takes the values 1 and 2. Let $I_{\delta,1} = (-\delta,0)$ and $I_{\delta,2} = (0,\delta)$. We parameterize the thin shell $\Omega_{\delta,\beta}$ by the manifold $\Gamma\times I_{\delta,\beta}$ through the mapping ψ_β defined by

$$\begin{cases}\Gamma\times I_{\delta,\beta} & \xrightarrow{\psi_\beta} & \Omega_{\delta,\beta}\\ (m,\eta_\beta) & \to & x := m + p_\beta\eta_\beta\mathbf{n}(m).\end{cases}$$

As is well known [19], if the thickness of $\Omega_{\delta,\beta}$ is small enough, ψ_β is a C^∞-diffeomorphism of manifolds and it is also known [32, Remark 2.1] that the normal vector $\mathbf{n}_{\delta,\beta}$ to $\Gamma_{\delta,\beta}$ can be identified with $\mathbf{n}$.

With each function v_β defined on $\Omega_{\delta,\beta}$, we associate the function $\widetilde{v}_\beta$ defined on $\Gamma\times I_{\delta,\beta}$ by

$$\begin{cases}\widetilde{v}_\beta(m,\eta_\beta) & := v_\beta(x),\\ x & = \psi_\beta\left(m,\eta_\beta\right).\end{cases}$$

One then has

$$\frac{\partial\widetilde{v}_\beta}{\partial\xi^\alpha} = \sum_{i=1}^{3}\frac{\partial v_\beta}{\partial x^i}\frac{\partial x^i}{\partial \xi^\alpha} = \nabla v_\beta.\left(I + p_\beta\eta_\beta\mathscr{R}\right)\tau_\alpha;\ \alpha = 1,2$$

and

$$\frac{\partial\widetilde{v}_\beta}{\partial\eta_\beta} = \sum_{i=1}^{3}\frac{\partial v_\beta}{\partial x^i}\frac{\partial x^i}{\partial \eta_\beta} = p_\beta\nabla v_\beta.\mathbf{n},$$

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where I is the identity operator on the tangent plane $T_m(\Gamma)$.
Since the vector $(I + p_\beta \eta_\beta \mathscr{R})\tau_\alpha$ is in $T_m(\Gamma)$ and $(I + p_\beta \eta_\beta \mathscr{R})$ is a symmetric operator, we can write

$$\frac{\partial \widetilde{v}_\beta}{\partial \xi^\alpha} = \left(I + p_\beta \eta_\beta \mathscr{R}\right) \Pi_m \nabla v_\beta . \tau_\alpha,$$

or equivalently [18]

$$\Pi_m \nabla v_\beta = \left(I + p_\beta \eta_\beta \mathscr{R}\right)^{-1} \nabla_\Gamma \widetilde{v}_\beta.$$

One gets

$$\nabla v_\beta = \left(I + p_\beta \eta_\beta \mathscr{R}\right)^{-1} \nabla_\Gamma \widetilde{v}_\beta + p_\beta^{-1} \frac{\partial \widetilde{v}_\beta}{\partial \eta_\beta} \mathbf{n}.$$

The volume element on the thin shell $\Omega_{\delta,\beta}$ is given by

$$d\Omega_{\delta,\beta} = \frac{\partial x}{\partial \xi^1} \times \frac{\partial x}{\partial \xi^2} . \frac{\partial x}{\partial \eta_\beta} \, d\xi^1 d\xi^2 d\eta_\beta.$$

As

$$\frac{\partial x}{\partial \xi^1} \times \frac{\partial x}{\partial \xi^2} = \left(I + p_\beta \eta_\beta \mathscr{R}\right) \tau_1 \times \left(I + p_\beta \eta_\beta \mathscr{R}\right) \tau_2 = \det\left(I + p_\beta \eta_\beta \mathscr{R}\right) (\tau_1 \times \tau_2),$$

and

$$|\tau_1 \times \tau_2| \, d\xi^1 d\xi^2 = d\Gamma,$$

we obtain

$$d\Omega_{\delta,\beta} = p_\beta \det\left(I + p_\beta \eta_\beta \mathscr{R}\right) \; d\Gamma \, d\eta_\beta.$$

Now, we introduce the scaling $s_\beta = \eta_\beta / \delta$, and the intervals $I_1 = (-1, 0)$ and $I_2 = (0, 1)$ such that the C^∞-diffeomorphism Φ_β, defined by

$$\begin{cases} \Omega^\beta := \Gamma \times I_\beta & \overset{\Phi_\beta}{\rightarrow} \quad \Omega_{\delta,\beta} \\ \qquad (m, s_\beta) & \rightarrow \quad x := m + \delta p_\beta s_\beta \mathbf{n}(m), \end{cases}$$

parameterizes the thin shell $\Omega_{\delta,\beta}$.
To any function v_β defined on $\Omega_{\delta,\beta}$, the function $v^{[\beta]}$ defined on Ω^β is associated through

$$\begin{cases} v^{[\beta]}(m, s_\beta) & := v_\beta(x), \\ \qquad x & = \Phi_\beta(m, s_\beta). \end{cases}$$

Then, in local coordinates (ξ^1, ξ^2, s_β), the gradient takes the form

$$\nabla v_\beta = \left(I + \delta p_\beta s_\beta \mathscr{R}\right)^{-1} \nabla_\Gamma v^{[\beta]} + p_\beta^{-1} \delta^{-1} \frac{\partial v^{[\beta]}}{\partial s_\beta} \mathbf{n}. \tag{3.1}$$

The volume element on the thin shell $\Omega_{\delta,\beta}$ becomes

$$d\Omega_{\delta,\beta} = p_\beta \delta \det J_{\delta,\beta} \; d\Gamma \, ds_\beta, \tag{3.2}$$

where

$$J_{\delta,\beta} := I + p_\beta \delta s_\beta \mathscr{R},$$

and the surfacic measure on $\Gamma_{\delta,\beta}$ is

$$d\Gamma_{\delta,\beta} = \det\left(I + (-1)^\beta p_\beta \delta \mathscr{R}\right)\, d\Gamma.$$

Let u_β and v_β be two regular functions defined on $\Omega_{\delta,\beta}$. From (3.1) and (3.2), we get the change of variables formula

$$\int_{\Omega_{\delta,\beta}} \nabla u_\beta . \nabla v_\beta \, d\Omega_{\delta,\beta} = p_\beta \delta \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \det J_{\delta,\beta}\, d\Gamma\, ds_\beta$$
$$+ p_\beta^{-1} \delta^{-1} \int_{\Omega^\beta} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \det J_{\delta,\beta}\, d\Gamma\, ds_\beta, \tag{3.3}$$

$$\int_{\Omega_{\delta,\beta}} u_\beta v_\beta \, d\Omega_{\delta,\beta} = p_\beta \delta \int_{\Omega^\beta} u^{[\beta]} v^{[\beta]} \;\det J_{\delta,\beta}\, d\Gamma\, ds_\beta. \tag{3.4}$$

Remark 3.1 *For any function u defined in a neighbourhood of Γ, we denote, for convenience, by $u_{|\Gamma}$ the trace of u on Γ indifferently in local coordinates or in Cartesian coordinates.*

4 The asymptotic analysis

This section is devoted to the asymptotic analysis of the solution to Problem (2.1). From now on, we assume that $\widetilde{\sigma}_\delta$ and $\widetilde{k}_\delta$ are independent of δ (denoted by $\widetilde{\sigma}$ and $\widetilde{k}$ respectively) to simplify the overall presentation. We give a hierarchy of variational equations defined in a domain that does not depend on δ suited to the construction of a formal asymptotic expansion up to any order. We then calculate the first two terms and we conclude with a convergence theorem ensuring the validity of the ansatz.

Let v_d be in $H^1(\Omega_\delta)$. We denote by v_{d_β} its restriction to $\Omega_{\delta,\beta}$. Multiplying equation

$$\operatorname{div}\left(\widetilde{\sigma} \nabla u_{d,\delta}\right) + \widetilde{k}^2 u_{d,\delta} = 0 \text{ in } \Omega_\delta,$$

by test functions v_d, using (2.1*f*)–(2.1*h*), (2.1*j*) and Green's formula, we get

$$\begin{aligned}&\left\langle \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}}, v_{d_1|\Gamma_{\delta,1}} \right\rangle_{H^{-1/2}(\Gamma_{\delta,1})\times H^{1/2}(\Gamma_{\delta,1})}\\ &-\left\langle \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}}, v_{d_2|\Gamma_{\delta,2}} \right\rangle_{H^{-1/2}(\Gamma_{\delta,2})\times H^{1/2}(\Gamma_{\delta,2})}\\ &+\widetilde{\sigma} \int_{\Omega_\delta} \nabla u_{d,\delta} . \nabla v_d \, d\Omega_\delta - \widetilde{k}^2 \int_{\Omega_\delta} u_{d,\delta} v_d \, d\Omega_\delta = 0.\end{aligned}$$

Using the dilation of the thin layers, (3.3) and (3.4), one obtains

$$\begin{aligned}&\langle \sigma_{\text{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}} \circ \Phi_1(m,-1), v_d^{[1]}(m,-1) \rangle_{H^{-1/2}(\Gamma\times\{-1\})\times H^{1/2}(\Gamma\times\{-1\})}\\ &-\langle \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}} \circ \Phi_2(m,1), v_d^{[2]}(m,1) \rangle_{H^{-1/2}(\Gamma\times\{1\})\times H^{1/2}(\Gamma\times\{1\})}\\ &+\sum_{\beta=1}^{2} \left[\delta a_\delta^{[\beta]} \left(u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) + \delta b_\delta^{[\beta]} \left(u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) \right] = 0,\end{aligned} \tag{4.1}$$

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where the bilinear forms $a_\delta^{[\beta]}(.,.)$ and $b_\delta^{[\beta]}(.,.)$ ($\beta = 1, 2$) are defined by

$$a_\delta^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \widetilde{\sigma} p_\beta \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \det J_{\delta,\beta}\, d\Gamma\, ds_\beta \\ + \widetilde{\sigma} p_\beta^{-1} \delta^{-2} \int_{\Omega^\beta} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \det J_{\delta,\beta}\, d\Gamma\, ds_\beta, \tag{4.2}$$

and

$$b_\delta^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := -\widetilde{k}^2 p_\beta \int_{\Omega^\beta} u^{[\beta]} v^{[\beta]} \det J_{\delta,\beta}\, d\Gamma\, ds_\beta, \tag{4.3}$$

for every $u^{[\beta]}$ and $v^{[\beta]}$ in $H^1(\Omega^\beta)$. Standard regularity results for elliptic problems (see e.g. [1]) ensure that the trace of u_δ on $\Gamma_{\delta,1}$ or $\Gamma_{\delta,2}$ is $\mathscr{C}^\infty$. This fact allows us to write Problem (4.1) as

$$\int_\Gamma \sigma_{\text{int}} \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma_{\delta,1}} \circ \Phi_1(m, -1) v_d^{[1]}(m, -1) \det(I - p_1 \delta \mathscr{R})\, d\Gamma \\ - \int_\Gamma \sigma_{\text{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}} \circ \Phi_2(m, 1) v_d^{[2]}(m, 1)\ \det(I + p_2 \delta \mathscr{R})\, d\Gamma \\ + \sum_{\beta=1}^{2} \left[\delta a_\delta^{[\beta]}\left(u_{d,\delta}^{[\beta]}, v_d^{[\beta]}\right) + \delta b_\delta^{[\beta]}\left(u_{d,\delta}^{[\beta]}, v_d^{[\beta]}\right)\right] = 0, \tag{4.4}$$

which is the starting point of the asymptotic analysis.

4.1 Hierarchy of the variational equations

To carry out an asymptotic expansion of the solution u_δ of (2.1) in powers of δ, we consider two asymptotic expansions. Exterior expansions corresponding to the expansion of u_δ restricted to $\Omega_{\text{ext},\delta}$ and to $\Omega_{\text{int},\delta}$ are characterized by the ansatz

$$u_{\text{ext},\delta} = u_{\text{ext},0} + \delta u_{\text{ext},1} + \cdots, \tag{4.5}$$

$$u_{\text{int},\delta} = u_{\text{int},0} + \delta u_{\text{int},1} + \cdots, \tag{4.6}$$

where the terms $u_{\text{ext},n}$ and $u_{\text{int},n}$ ($n \in \mathbb{N}$) are independent of δ and respectively defined on $\Omega_{\text{ext}} := \Omega_{\text{ext},\delta} \cup \Gamma_{\delta,2} \cup \Omega_{\delta,2}$, and on $\Omega_{\text{int}} := \Omega_{\text{int},\delta} \cup \Gamma_{\delta,1} \cup \Omega_{\delta,1}$. The latter are the limits of $\Omega_{\text{ext},\delta}$ and $\Omega_{\text{int},\delta}$ when $\delta \to 0$. They fulfil

$$\begin{cases} \operatorname{div}\left(\sigma_{\text{int}} \nabla u_{\text{int},n}\right) + k_{\text{int}}^2 u_{\text{int},n} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}\left(\sigma_{\text{ext}} \nabla u_{\text{ext},n}\right) + k_{\text{ext}}^2 u_{\text{ext},n} = 0 & \text{in } \Omega_{\text{ext}}, \\ \lim\limits_{|x| \to +\infty} |x| \left(\partial_{|x|} - i k_{\text{ext}}\right) \left(u_{\text{ext},n} - \delta_{0,n} u_{inc}\right) = 0, \end{cases} \tag{4.7}$$

where $\delta_{0,n}$ indicates the Kronecker symbol. An interior expansion corresponding to the asymptotic expansion of $u_{d_\beta,\delta}$ written in a fixed domain is now defined by the ansatz

$$u_{d,\delta}^{[\beta]} = u_0^{[\beta]} + \delta u_1^{[\beta]} + \cdots, \text{ in } \Omega^\beta, \tag{4.8}$$

where the terms $u_n^{[\beta]}$, $n \in \mathbb{N}$, are independent of δ. Using a Taylor expansion in the normal variable, we formally infer

$$\begin{aligned} u_{\mathrm{int},\delta|\Gamma_{\delta,1}} &= u_{\mathrm{int},0|\Gamma} + \delta(u_{\mathrm{int},1|\Gamma} - p_1 \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}) + \cdots, \\ \partial_{\mathbf{n}_{\delta,1}} u_{\mathrm{int},\delta|\Gamma_{\delta,1}} &= \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} + \delta(\partial_{\mathbf{n}} u_{\mathrm{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\mathrm{int},0|\Gamma}) + \cdots, \end{aligned}$$

and

$$\begin{aligned} u_{\mathrm{ext},\delta|\Gamma_{\delta,2}} &= u_{\mathrm{ext},0|\Gamma} + \delta(u_{\mathrm{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}) + \cdots, \\ \partial_{\mathbf{n}_{\delta,2}} u_{\mathrm{ext},\delta|\Gamma_{\delta,2}} &= \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} + \delta(\partial_{\mathbf{n}} u_{\mathrm{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\mathrm{ext},0|\Gamma}) + \cdots. \end{aligned}$$

Transmission conditions (2.1e), (2.1g) and (2.1i) become

$$u_{\mathrm{ext},0|\Gamma} + \delta(u_{\mathrm{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}) + \cdots = u^{[2]}_{0|s_2=1} + \delta u^{[2]}_{1|s_2=1} + \cdots, \tag{4.9}$$

$$u^{[1]}_{0|s_1=0} + \delta u^{[1]}_{1|s_1=0} + \cdots = u^{[2]}_{0|s_2=0} + \delta u^{[2]}_{1|s_2=0} + \cdots, \tag{4.10}$$

$$u_{\mathrm{int},0|\Gamma} + \delta(u_{\mathrm{int},1|\Gamma} - p_1 \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}) + \cdots = u^{[1]}_{0|s_1=-1} + \delta u^{[1]}_{1|s_1=-1} + \cdots. \tag{4.11}$$

We now use the identity

$$\det J_{\delta,\beta} = 1 + 2p_\beta s_\beta \delta \mathscr{H} + \left(p_\beta s_\beta \delta\right)^2 \mathscr{K},$$

where $2\mathscr{H} := tr\mathscr{R}$ and $\mathscr{K} := \det \mathscr{R}$ are respectively the mean and the Gaussian curvatures of the surface Γ. We obtain

$$\begin{aligned} &\int_\Gamma \sigma_{\mathrm{int}} \partial_{\mathbf{n}_{\delta,1}} u_{\mathrm{int},\delta|\Gamma_{\delta,1}} \circ \Phi_1(m,-1) v_d^{[1]}(m,-1) \det(I - p_1 \delta \mathscr{R})\, d\Gamma \\ &= \int_\Gamma \sigma_{\mathrm{int}} \left[\partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} + \delta\left(\partial_{\mathbf{n}} u_{\mathrm{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\mathrm{int},0|\Gamma} - 2p_1 \mathscr{H} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}\right)\right. \\ &+ \delta^2 \left(\partial_{\mathbf{n}} u_{\mathrm{int},2|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\mathrm{int},1|\Gamma} + \frac{1}{2} p_1^2 \partial_{\mathbf{n}}^3 u_{\mathrm{int},0|\Gamma} - 2p_1 \mathscr{H} \partial_{\mathbf{n}} u_{\mathrm{int},1|\Gamma}\right. \\ &+ \left.\left. 2p_1^2 \mathscr{H} \partial_{\mathbf{n}}^2 u_{\mathrm{int},0|\Gamma} + p_1^2 \mathscr{K} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}\right) + \cdots\right] v_d^{[1]}(m,-1)\, d\Gamma, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\int_\Gamma \sigma_{\mathrm{ext}} \partial_{\mathbf{n}_{\delta,2}} u_{\mathrm{ext},\delta|\Gamma_{\delta,2}} \circ \Phi_2(m,1) v_d^{[2]}(m,1) \det(I + p_2 \delta \mathscr{R})\, d\Gamma \\ &= \int_\Gamma \sigma_{\mathrm{ext}} \left[\partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} + \delta\left(\partial_{\mathbf{n}} u_{\mathrm{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\mathrm{ext},0|\Gamma} + 2p_2 \mathscr{H} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}\right)\right. \\ &+ \delta^2 \left(\partial_{\mathbf{n}} u_{\mathrm{ext},2|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\mathrm{ext},1|\Gamma} + \frac{1}{2} p_2^2 \partial_{\mathbf{n}}^3 u_{\mathrm{ext},0|\Gamma} + 2p_2 \mathscr{H} \partial_{\mathbf{n}} u_{\mathrm{ext},1|\Gamma}\right. \\ &+ \left.\left. 2p_2^2 \mathscr{H} \partial_{\mathbf{n}}^2 u_{\mathrm{ext},0|\Gamma} + p_2^2 \mathscr{K} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}\right) + \cdots\right] v_d^{[2]}(m,1)\, d\Gamma. \end{aligned} \tag{4.13}$$

It remains to give the expansion of $a_\delta^{[\beta]}(.,.)$ defined in (4.2) in powers of δ. We use the identity (see [4, p. 1680])

$$J_{\delta,\beta}^{-2} := I - 2s_\beta p_\beta \delta \mathscr{R} + 3\left(p_\beta s_\beta \delta \mathscr{R}\right)^2 + \cdots + n\left(-p_\beta s_\beta \delta \mathscr{R}\right)^{n-1} + \left(-s_\beta p_\beta \delta \mathscr{R}\right)^n \left[n J_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}\right].$$

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The bilinear form $a_\delta^{[\beta]}(.,.)$ then admits the expansion

$$a_\delta^{[\beta]}(.,.) = \delta^{-2} a_{0,2}^{[\beta]} + \delta^{-1} a_{1,2}^{[\beta]} + \left(a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]}\right) + \delta a_{1,1}^{[\beta]} + \cdots + \delta^{n-1} a_{n-1,1}^{[\beta]} + \delta^n r_n^{[\beta]}(.,.), \quad (4.14)$$

where the forms $a_{k,l}^{[\beta]}$ are independent of δ and are given by

$$\begin{aligned}
a_{0,2}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} p_\beta^{-1} \widetilde{\sigma} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \, d\Gamma \, ds_\beta, \\
a_{1,2}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} 2\widetilde{\sigma} \mathscr{H} s_\beta \, \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \, d\Gamma \, ds_\beta, \\
a_{2,2}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} p_\beta \widetilde{\sigma} \mathscr{K} s_\beta^2 \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \, d\Gamma \, ds_\beta, \\
a_{0,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} p_\beta \widetilde{\sigma} \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \, d\Gamma \, ds_\beta, \\
a_{1,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} 2 p_\beta^2 \widetilde{\sigma} s_\beta \left(\mathscr{H} I - \mathscr{R}\right) \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \, d\Gamma \, ds_\beta, \\
a_{2,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} p_\beta^3 \widetilde{\sigma} \left(\mathscr{K} I - 4\mathscr{H}\mathscr{R} + 3\mathscr{R}^2\right) s_\beta^2 \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \, d\Gamma \, ds_\beta, \\
a_{n-1,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} p_\beta^n \widetilde{\sigma} \left[(n-2) \mathscr{K} \mathscr{R}^{n-3} - (n-1) 2\mathscr{H}\mathscr{R}^{n-2}\right. \\
&\qquad \left. + n\mathscr{R}^{n-1}\right] \left(-s_\beta\right)^{n-1} \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \, d\Gamma \, ds_\beta, \ n > 3,
\end{aligned}$$

in which the index 1 in the bilinear forms $a_{k,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right)$ corresponds to the derivatives of $u^{[\beta]}$ and $v^{[\beta]}$ with respect to the tangential variables and the index 2 refers to the derivatives of $u^{[\beta]}$ and $v^{[\beta]}$ with respect to the normal variable s_β. The remainder of Expansion (4.14) is expressed as follows:

$$r_n^{[\beta]}(u^{[\beta]}, v^{[\beta]}) := \int_{\Omega^\beta} \widetilde{\sigma} \left(B_{n,\delta} + 2\mathscr{H} B_{n-1,\delta} + \mathscr{K} B_{n-2,\delta}\right) s_\beta^n \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} d\Gamma \, ds_\beta,$$

with

$$B_{n,\delta} := \begin{cases} (-\mathscr{R})^n \left(n J_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}\right) & \text{if } n \geqslant 0, \\ J_{\delta,\beta}^{-2} & \text{otherwise.} \end{cases}$$

The form $b_\delta^{[\beta]}(.,.)$ has the finite expansion

$$b_\delta^{[\beta]}(.,.) = b_0^{[\beta]}(.,.) + \delta b_1^{[\beta]}(.,.) + \delta^2 b_2^{[\beta]}(.,.), \quad (4.15)$$

where

$$\begin{aligned}
b_0^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} -p_\beta \widetilde{k}^2 u^{[\beta]} v^{[\beta]} \, d\Gamma \, ds_\beta, \\
b_1^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} -p_\beta^2 \widetilde{k}^2 2 s_\beta \mathscr{H} u^{[\beta]} v^{[\beta]} \, d\Gamma \, ds_\beta, \\
b_2^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) &:= \int_{\Omega^\beta} -p_\beta^3 \widetilde{k}^2 s_\beta^2 \mathscr{K} u^{[\beta]} v^{[\beta]} \, d\Gamma \, ds_\beta.
\end{aligned}$$

Now inserting expansions (4.8) and (4.12)–(4.14) in (4.4) and matching the same powers of δ, we obtain the following variational equations, which hold for all $v^{[\beta]}$ in $H^1\left(\Gamma \times I_\beta\right)$ such that $v^{[1]}(.,0) = v^{[2]}(.,0)$,

$$a_{0,2}^{[1]}\left(u_0^{[1]}, v^{[1]}\right) + a_{0,2}^{[2]}\left(u_0^{[2]}, v^{[2]}\right) = 0, \tag{4.16}$$

$$\int_\Gamma \sigma_{\mathrm{int}} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} v^{[1]}(m,-1)\ d\Gamma + \sum_{\beta=1}^{2}\left[a_{1,2}^{[\beta]}\left(u_0^{[\beta]}, v^{[\beta]}\right) + a_{0,2}^{[\beta]}\left(u_1^{[\beta]}, v^{[\beta]}\right)\right]$$
$$- \int_\Gamma \sigma_{\mathrm{ext}} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} v^{[2]}(m,1)\ d\Gamma = 0, \tag{4.17}$$

$$\int_\Gamma \sigma_{\mathrm{int}}\left[-2p_1 \mathscr{H} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} + \partial_{\mathbf{n}} u_{\mathrm{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\mathrm{int},0|\Gamma}\right] v^{[1]}(m,-1)\ d\Gamma$$
$$+ \sum_{\beta=1}^{2}\left[a_{1,2}^{[\beta]}\left(u_1^{[\beta]}, v^{[\beta]}\right) + \left(a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} + b_0^{[\beta]}\right)\left(u_0^{[\beta]}, v^{[\beta]}\right) + a_{0,2}^{[\beta]}\left(u_2^{[\beta]}, v^{[\beta]}\right)\right]$$
$$- \int_\Gamma \sigma_{\mathrm{ext}}\left[2p_2 \mathscr{H} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} + \partial_{\mathbf{n}} u_{\mathrm{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\mathrm{ext},0|\Gamma}\right] v^{[2]}(m,1)\ d\Gamma = 0, \tag{4.18}$$

$$\int_\Gamma \sigma_{\mathrm{int}}\left[p_1^2 \mathscr{K} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} - 2p_1 \mathscr{H}\left(\partial_{\mathbf{n}} u_{\mathrm{int},1|\Gamma} - p_1 \partial_{\mathbf{n}}^2 u_{\mathrm{int},0|\Gamma}\right) + \partial_{\mathbf{n}} u_{\mathrm{int},2|\Gamma}\right.$$
$$\left. - p_1 \partial_{\mathbf{n}}^2 u_{\mathrm{int},1|\Gamma} + \frac{p_1^2}{2} \partial_{\mathbf{n}}^3 u_{\mathrm{int},0|\Gamma}\right] v^{[1]}(m,-1)\ d\Gamma + \sum_{\beta=1}^{2}\left[a_{0,2}^{[\beta]}\left(u_3^{[\beta]}, v^{[\beta]}\right)\right.$$
$$\left. + a_{1,2}^{[\beta]}\left(u_2^{[\beta]}, v^{[\beta]}\right)\left(a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} + b_0^{[\beta]}\right)\left(u_1^{[\beta]}, v^{[\beta]}\right) + \left(a_{1,1}^{[\beta]} + b_1^{[\beta]}\right)\left(u_0^{[\beta]}, v^{[\beta]}\right)\right]$$
$$- \int_\Gamma \sigma_{\mathrm{ext}}\left[p_2^2 \mathscr{K} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} + 2p_2 \mathscr{H}\left(\partial_{\mathbf{n}} u_{\mathrm{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}}^2 u_{\mathrm{ext},0|\Gamma}\right) + \partial_{\mathbf{n}} u_{\mathrm{ext},2|\Gamma}\right.$$
$$\left. + p_2 \partial_{\mathbf{n}}^2 u_{\mathrm{ext},1|\Gamma} + \frac{p_2^2}{2} \partial_{\mathbf{n}}^3 u_{\mathrm{ext},0|\Gamma}\right] v^{[2]}(m,1)\ d\Gamma = 0, \tag{4.19}$$

$$\int_\Gamma \sigma_{\mathrm{int}}\left[\partial_{\mathbf{n}} u_{\mathrm{int},n|\Gamma} + \cdots\right] v^{[1]}(m,-1)\ d\Gamma + \sum_{\beta=1}^{2}\left[a_{0,2}^{[\beta]}\left(u_{n+1}^{[\beta]}, v^{[\beta]}\right) + a_{1,2}^{[\beta]}\left(u_n^{[\beta]}, v^{[\beta]}\right)\right.$$
$$+ \left(a_{0,1}^{[\beta]} + b_0^{[\beta]}\right)\left(u_{n-1}^{[\beta]}, v^{[\beta]}\right) + \left(a_{1,1}^{[\beta]} + b_1^{[\beta]}\right)\left(u_{n-2}^{[\beta]}, v^{[\beta]}\right) + \left(a_{2,1}^{[\beta]} + b_2^{[\beta]}\right)\left(u_{n-3}^{[\beta]}, v^{[\beta]}\right)$$
$$\left. + \sum_{l=4}^{n} a_{l-1,1}^{[\beta]}\left(u_{n-l}^{[\beta]}, v^{[\beta]}\right)\right] - \int_\Gamma \sigma_{\mathrm{ext}}\left[\partial_{\mathbf{n}} u_{\mathrm{ext},n|\Gamma} + \cdots\right] v^{[2]}(m,1)\ d\Gamma = 0,\ n \geqslant 4. \tag{4.20}$$

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4.2 Computation of the first two terms

In this paragraph, we compute explicitly the first two terms in order to present a recursive method to define successively the asymptotic expansions. We need the following theorem whose proof can be found in [36].

Theorem 4.1 *Let $h \in H^{1/2}(\Gamma)$ and $\zeta \in H^{-1/2}(\Gamma)$. Then, the following problem*

$$\begin{cases} \operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ U_{\text{int}|\Gamma} - U_{\text{ext}|\Gamma} = h & \text{on } \Gamma, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}|\Gamma} = \zeta & \text{on } \Gamma, \\ \lim_{|x| \to +\infty} |x| \left(\partial_{|x|} - i k_{\text{ext}} \right) U_{\text{ext}} = 0, \end{cases} \tag{4.21}$$

admits a unique solution $(U_{\text{int}}, U_{\text{ext}})$ in $H^1(\Omega_{\text{int}}) \times H^1_{loc}\left(\overline{\Omega}_{\text{ext}}\right)$. Moreover, for $k_0 \in \mathbb{N}$, $h \in H^{k_0-1/2}(\Gamma)$, $\zeta \in H^{k_0-3/2}(\Gamma)$ and $\Gamma \cup \partial\Omega$ $\mathscr{C}^{k_0}$-continuous, let $(U_{\text{int}}, U_{\text{ext}}) \in H^1(\Omega_{\text{int}}) \times H^1_{loc}\left(\overline{\Omega}_{\text{ext}}\right)$ be the solution of (4.21). For any positive integer $k \leqslant k_0$, there exists a constant c_k such that

$$\|U_{\text{int}}\|_{H^k(\Omega_{\text{int}})} + \|U_{\text{ext}}\|_{H^k\left(\widetilde{\Omega}_{\text{ext}}\right)} \leqslant c_k \left(\|h\|_{H^{k-1/2}(\Gamma)} + \|\zeta\|_{H^{k-3/2}(\Gamma)} \right).$$

We also need the following technical result to determine terms of asymptotic expansions whose proof is obtained in a straightforward way.

Lemma 4.1 *For $\beta = 1, 2$, let $q^{[\beta]}$ be a given function in $L^2(\Gamma)$ and let $k^{[\beta]}$ be a vectorial function in $L^2(\Omega^\beta, \mathbb{C}^3)$ such that the partial application $s_\beta \to k^{[\beta]}(., s_\beta)$ is valued in the space of vectorial fields tangent to Γ and also $\operatorname{div}_\Gamma k^{[\beta]} \in L^2(\Omega^\beta)$. Then, the solution $h^{[\beta]}$ of the variational equation*

$$\begin{aligned} \mathscr{L}^{[\beta]} v^{[\beta]} := & \int_{\Omega^\beta} h^{[\beta]}\left(m, s_\beta\right) \partial_{s_\beta} v^{[\beta]}\left(m, s_\beta\right) \, d\Gamma \, ds_\beta \\ & + \int_{\Omega^\beta} k^{[\beta]}\left(m, s_\beta\right) . \nabla_\Gamma v^{[\beta]}\left(m, s_\beta\right) + \theta^{[\beta]}\left(m, s_\beta\right) v^{[\beta]}\left(m, s_\beta\right) \, d\Gamma \, ds_\beta \\ & + \int_\Gamma q^{[\beta]}(m) v^{[\beta]}\left(m, (-1)^\beta\right) \, d\Gamma = 0 \end{aligned}$$

for all $v^{[\beta]} \in H^1\left(\Omega^\beta\right)$; $v^{[\beta]}(., 0) = 0$ is explicitly given by

$$h^{[\beta]}\left(m, s_\beta\right) = (-1)^{\beta+1} q^{[\beta]}(m) + \int_{s_\beta}^{(-1)^\beta} \left(\operatorname{div}_\Gamma k^{[\beta]} - \theta^{[\beta]} \right)(m, \lambda) \, d\lambda.$$

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Moreover, for all $v^{[\beta]} \in H^1\left(\Omega^\beta\right)$, we have

$$\mathscr{L}^{[\beta]} v^{[\beta]} = (-1)^{\beta+1} \int_\Gamma h^{[\beta]}(m,0) v^{[\beta]}(m,0)\, d\Gamma$$
$$= \int_\Gamma \left[q^{[\beta]}(m) - (-1)^\beta \int_0^{(-1)^\beta} \left(div_\Gamma k^{[\beta]} - \theta^{[\beta]} \right) \left(m, s_\beta\right)\ ds_\beta \right] v^{[\beta]}\left(m,0\right)\ d\Gamma\,.$$

4.2.1 *Term of order 0*

Equation (4.16) implies that $\partial_{s_\beta} u_0^{[\beta]} = 0$. Using (4.9)–(4.11), we obtain

$$u_{\mathrm{int},0|\Gamma} = u_0^{[1]}(m, s_1) = u_0^{[2]}(m, s_2) = u_{\mathrm{ext},0|\Gamma}\,,\ \ m \in \Gamma\,. \tag{4.22}$$

The choice of $v^{[1]} = 0$ in (4.17) gives

$$a_{1,2}^{[2]}(u_0^{[2]}, v^{[2]}) + a_{0,2}^{[2]}(u_1^{[2]}, v^{[2]}) - \sigma_{\mathrm{ext}} \int_\Gamma \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}\, v^{[2]}(m,1) d\Gamma = 0.$$

We apply Lemma 4.1 with $h^{[2]} = p_2^{-1} \widetilde{\sigma} \partial_{s_2} u_1^{[2]}$, $q^{[2]}(m) = -\sigma_{\mathrm{ext}} \partial_{\mathbf{n}} u_{e,0|\Gamma}$, $k^{[2]} = 0$ and $\theta^{[2]} = 0$, to obtain

$$p_2^{-1} \widetilde{\sigma}\ \partial_{s_2} u_1^{[2]} = \sigma_{\mathrm{ext}} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}\,. \tag{4.23}$$

Similarly, choosing $v^{[2]} = 0$ in (4.17) gives

$$a_{1,2}^{[1]}(u_0^{[1]}, v^{[1]}) + a_{0,2}^{[1]}(u_1^{[1]}, v^{[1]}) + \sigma_{\mathrm{int}} \int_\Gamma \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}\, v^{[1]}\,(m,-1)\ d\Gamma = 0.$$

We apply Lemma 4.1 with $h^{[1]} = p_1^{-1} \widetilde{\sigma} \partial_{s_1} u_1^{[1]}, q^{[1]}(m) = \sigma_{\mathrm{int}} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}$, $k^{[1]} = 0$ and $\theta^{[1]} = 0$, to find

$$p_1^{-1} \widetilde{\sigma} \partial_{s_1} u_1^{[1]} = \sigma_{\mathrm{int}} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}\,. \tag{4.24}$$

From the second part of Lemma 4.1, one gets

$$\int_\Gamma \sigma_{\mathrm{int}} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma}\, v^{[1]}(m,0) d\Gamma = \int_\Gamma \sigma_{\mathrm{ext}} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}\, v^{[1]}(m,0) d\Gamma\,,$$

then

$$\sigma_{\mathrm{int}} \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} = \sigma_{\mathrm{ext}} \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}\,. \tag{4.25}$$

Let us define σ_0, u_n and k_0 by

$$\sigma_0(x) = \begin{cases} \sigma_{\mathrm{ext}} & \text{if } x \in \Omega_{\mathrm{ext}}, \\ \sigma_{\mathrm{int}} & \text{if } x \in \Omega_{\mathrm{int}}, \end{cases} \ ;\ k_0^2(x) = \begin{cases} k_{\mathrm{ext}}^2 & \text{if } x \in \Omega_{\mathrm{ext}}, \\ k_{\mathrm{int}}^2 & \text{if } x \in \Omega_{\mathrm{int}}, \end{cases}$$

and

$$u_n = \begin{cases} u_{\mathrm{ext},n} & \text{in } \Omega_{\mathrm{ext}}, \\ u_{\mathrm{int},n} & \text{in } \Omega_{\mathrm{int}}. \end{cases}$$

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Therefore, with (4.7), (4.22), (4.25) and Theorem 4.1, u_0 satisfies the following problem:

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla u_0) + k_0^2 u_0 = 0 & \text{in } \mathbb{R}^3, \\ \lim\limits_{|x| \to +\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right)(u_0 - u_{inc}) = 0. \end{cases}$$

The zeroth-order term is then determined. Note that u_0 is nothing but the solution to the scattering problem where there is no thin layer.

4.2.2 *Term of order 1*

Integrating Relations (4.23) and (4.24) in s_β and identifying terms of order 1 in (4.9) and (4.11), yields

$$u_1^{[1]}(m, s_1) = u_{\text{int},1|\Gamma} + p_1 \left[(s_1 + 1)\sigma_{\text{int}}\widetilde{\sigma}^{-1} - 1\right] \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}, \ \forall (m, s_1) \in \Omega^1,$$

and

$$u_1^{[2]}(m, s_2) = u_{\text{ext},1|\Gamma} + p_2 \left[(s_2 - 1)\sigma_{\text{ext}}\widetilde{\sigma}^{-1} + 1\right] \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}, \ \forall (m, s_2) \in \Omega^2.$$

The identification of first-order terms of (4.10) gives a first transmission condition on Γ

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1(1 - \sigma_{\text{int}}\widetilde{\sigma}^{-1})\partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + p_2(1 - \sigma_{\text{ext}}\widetilde{\sigma}^{-1})\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}. \tag{4.26}$$

The second one follows the same lines as for order 0. Indeed, we apply Lemma 4.1 to equation (4.18) once for $\beta = 2$ and another for $\beta = 1$, and using the identity [27, p. 75]

$$\Delta u = \Delta_\Gamma u + 2\mathscr{H}\partial_{\mathbf{n}} u + \partial_{\mathbf{n}}^2 u,$$

we obtain

$$\begin{aligned} \sigma_{\text{int}}\partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= p_1(\widetilde{\sigma} - \sigma_{\text{int}})\Delta_\Gamma u_{\text{int},0|\Gamma} + p_2(\widetilde{\sigma} - \sigma_{\text{ext}})\Delta_\Gamma u_{\text{ext},0|\Gamma} \\ &\quad + p_1 \left(\widetilde{k}^2 - k_{\text{int}}^2\right) u_{\text{int},0|\Gamma} + p_2 \left(\widetilde{k}^2 - k_{\text{ext}}^2\right) u_{\text{ext},0|\Gamma}. \end{aligned} \tag{4.27}$$

It follows from (4.7), (4.26), (4.27) and Theorem 4.1 that u_1 is the unique solution to the following problem:

$$\begin{cases} \operatorname{div}\left(\sigma_{\text{int}} \nabla u_{\text{int},1}\right) + k_{\text{int}}^2 u_{\text{int},1} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}\left(\sigma_{\text{ext}} \nabla u_{\text{ext},1}\right) + k_{\text{ext}}^2 u_{\text{ext},1} = 0 & \text{in } \Omega_{\text{ext}}, \\ \lim\limits_{|x| \to +\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right) u_{\text{ext},1} = 0, \end{cases}$$

with the following transmission conditions on Γ

$$\begin{aligned} u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} &= p_1(1 - \sigma_{\text{int}}\widetilde{\sigma}^{-1})\partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + p_2(1 - \sigma_{\text{ext}}\widetilde{\sigma}^{-1})\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}, \\ \sigma_{\text{int}}\partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} &= p_1(\widetilde{\sigma} - \sigma_{\text{int}})\Delta_\Gamma u_{\text{int},0|\Gamma} + p_2(\widetilde{\sigma} - \sigma_{\text{ext}})\Delta_\Gamma u_{\text{ext},0|\Gamma} \\ &\quad + p_1 \left(\widetilde{k}^2 - k_{\text{int}}^2\right) u_{\text{int},0|\Gamma} + p_2 \left(\widetilde{k}^2 - k_{\text{ext}}^2\right) u_{\text{ext},0|\Gamma}, \end{aligned}$$

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or

$$
\begin{aligned}
u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} &= \frac{p_1\sigma_{\text{ext}}\widetilde{\sigma} + p_2\sigma_{\text{int}}\widetilde{\sigma} - \sigma_{\text{int}}\sigma_{\text{ext}}}{2\sigma_{\text{int}}\sigma_{\text{ext}}\widetilde{\sigma}} \left(\sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},0|\Gamma} + \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}\right), \\
\sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},1|\Gamma} &= \frac{1}{2}\left(\widetilde{\sigma} - p_1\sigma_{\text{int}} - p_2\sigma_{\text{ext}}\right)\left(\Delta_\Gamma u_{\text{int},0|\Gamma} + \Delta_\Gamma u_{\text{ext},0|\Gamma}\right) \\
&\quad + \frac{1}{2}\left(\widetilde{k}^2 - p_1k_{\text{int}}^2 - p_2k_{\text{ext}}^2\right)\left(u_{\text{int},0|\Gamma} + u_{\text{ext},0|\Gamma}\right).
\end{aligned}
$$

5 Optimal error estimates

The process described in the previous section can be continued up to any order provided that the data are smooth enough. Here, the source term is given by a plane wave which is $\mathscr{C}^\infty$. We can also estimate the error made by truncating the series after a finite number of terms. Let n be in $\mathbb{N}$, we set $\widetilde{\Omega}_{\text{ext},\delta} = \widetilde{\Omega}_{\text{ext}} \backslash \overline{\Omega}_{\delta,2}$,

$$
u_{\text{int},\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{\text{int},j}, \quad u_{\text{ext},\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{\text{ext},j} \quad \text{and} \quad u_{d,\delta}^{(n)} := \begin{cases} u_{d_1,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{d_1,j} \text{ in } \Omega_{\delta,1}, \\ u_{d_2,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{d_2,j} \text{ in } \Omega_{\delta,2}, \end{cases}
$$

where $u_{d_\beta,j}(x) := \widetilde{u}_{d_\beta,j}(m, \delta s_\beta) := u_j^{[\beta]}(m, s_\beta)$; $\forall x = \Phi_\beta(m, s_\beta) \in \Omega_{\delta,\beta}$.

Theorem 5.1 (Convergence of the asymptotic expansion) *For all integers n, there exists a constant c independent of δ such that*

$$
\left\| u_{\text{int},\delta} - u_{\text{int},\delta}^{(n)} \right\|_{H^1(\Omega_{\text{int},\delta})} + \delta^{1/2} \left\| u_{d,\delta} - u_{d,\delta}^{(n)} \right\|_{H^1(\Omega_\delta)} + \left\| u_{\text{ext},\delta} - u_{\text{ext},\delta}^{(n)} \right\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} \leqslant c\delta^{n+1}.
$$

Proof Let us define the remainders $R_{D_1,n}, R_{D_2,n}, R_{N_1,n}$ and $R_{N_2,n}$ of Taylor expansions in the normal variable with respect to δ up to order n of $u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)}, u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)}, \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)}$ and $\partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)}$ respectively by

$$
\begin{aligned}
R_{D_1,n} &:= u_{\text{int},\delta|\Gamma_{\delta,1}}^{(n)} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} p_1^l \partial_{\mathbf{n}}^l u_{\text{int},j|\Gamma} \\
&= \sum_{j=0}^{n} \delta^j \widetilde{u}_{\text{int},j}(m, -\delta) - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} \partial_{s_1}^l \widetilde{u}_{\text{int},j}(m, 0),
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
R_{D_2,n} &:= u_{\text{ext},\delta|\Gamma_{\delta,2}}^{(n)} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} p_2^l \partial_{\mathbf{n}}^l u_{\text{ext},j|\Gamma} \\
&= \sum_{j=0}^{n} \delta^j \widetilde{u}_{\text{ext},j}(m, \delta) - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} \partial_{s_2}^l \widetilde{u}_{\text{ext},j}(m, 0),
\end{aligned} \tag{5.2}
$$

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$$R_{N_1,n} := \partial_{\mathbf{n}_{\delta,1}} u^{(n)}_{\text{int},\delta|\Gamma_{\delta,1}} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} p_1^l \partial_{\mathbf{n}}^{l+1} u_{\text{int},j|\Gamma}$$
$$= \sum_{j=0}^{n} \delta^j p_1^{-1} \partial_{s_1} \widetilde{u}_{\text{int},j}(m,-\delta) - \sum_{j=0}^{n}\sum_{l=0}^{n-j} p_1^{-1} \frac{(-1)^l \delta^{j+l}}{l!} \partial_{s_1}^{l+1} \widetilde{u}_{\text{int},j}(m,0), \tag{5.3}$$

and

$$R_{N_2,n} := \partial_{\mathbf{n}_{\delta,2}} u^{(n)}_{\text{ext},\delta|\Gamma_{\delta,2}} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} p_2^l \partial_{\mathbf{n}}^{l+1} u_{\text{ext},j|\Gamma}$$
$$= \sum_{j=0}^{n} \delta^j p_2^{-1} \partial_{s_2} \widetilde{u}_{\text{ext},j}(m,\delta) - \sum_{j=0}^{n}\sum_{l=0}^{n-j} p_2^{-1} \frac{\delta^{j+l}}{l!} \partial_{s_2}^{l+1} \widetilde{u}_{\text{ext},j}(m,0). \tag{5.4}$$

We shall rely on the following proposition to show the estimates of the remainders $R_{D_\beta,n}$ and $R_{N_\beta,n}$. The steps of its proof are very similar to those given in [37, Section 5].

Proposition 5.1 *There exists a constant $c > 0$, independent of δ, such as*

$$\left\| R_{N_\beta,n} \right\|_{L^2(\Gamma_{\delta,\beta})} \leqslant c\delta^{n+1/2},$$
$$\left\| \nabla_\Gamma^{(j)} R_{D_\beta,n} \right\|_{L^2(\Gamma)} \leqslant c\delta^{n+1/2}, \quad \text{for } j = 0, 1.$$

Moreover, there exists an extension $\mathscr{P}R$ of $R_{D_\beta,n}$ into Ω_δ with

$$\partial_{\eta_\beta} \widetilde{\mathscr{P}R}\left(m,\eta_\beta\right)_{|\eta_\beta=(-1)^\beta p_\beta \delta} = 0 \text{ and } \|\mathscr{P}R\|_{H^1(\Omega_\delta)} \leqslant c\delta^n.$$

Continuation of the proof of Theorem 5.1 Let $r^n_{\text{int},\delta}$, $r^n_{d,\delta}$ and $r^n_{\text{ext},\delta}$ be the remainders got by truncating Series (4.5), (4.6) and (4.8)

$$r^n_{\text{int},\delta} := u_{\text{int},\delta} - u^{(n)}_{\text{int},\delta},\ r^n_{\text{ext},\delta} := u_{\text{ext},\delta} - u^{(n)}_{\text{ext},\delta},\ r^n_{d,\delta} := u_{d,\delta} - u^{(n)}_{d,\delta},$$

and $\mathscr{L}_\delta$ be the linear form defined on $H^1(\Omega)$

$$\mathscr{L}_\delta v := \int_{\Omega_{\text{int},\delta}} \sigma_{\text{int}} \nabla r^n_{\text{int},\delta}.\nabla \overline{v}_{\text{int}} - k^2_{\text{int}} r^n_{\text{int},\delta} \overline{v}_{\text{int}}\ d\Omega_{\text{int},\delta}$$
$$+ \int_{\Omega_\delta} \widetilde{\sigma} \nabla (r^n_{d,\delta} - \mathscr{P}R).\nabla \overline{v}_d - \widetilde{k}^2 (r^n_{d,\delta} - \mathscr{P}R) \overline{v}_d\ d\Omega_\delta$$
$$+ \int_{\widetilde{\Omega}_{\text{ext},\delta}} \sigma_{\text{ext}} \nabla r^n_{\text{ext},\delta}.\nabla \overline{v}_{\text{ext}} - k^2_{\text{ext}} r^n_{\text{ext},\delta} \overline{v}_{\text{ext}}\ d\widetilde{\Omega}_{\text{ext},\delta}$$
$$+ \sigma_{\text{ext}} \left\langle T r^n_{\text{ext},\delta|\partial\Omega}, \overline{v}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}, \tag{5.5}$$

in which $\mathscr{P}R$ is the extension function of $R_{D_\beta,n}$ into Ω_δ and v_{int}, v_d and $v_{\text{ext},}$ are the restrictions of v respectively to the domains $\Omega_{\text{int},\delta}$, Ω_δ and $\widetilde{\Omega}_{\text{ext},\delta}$. Using Green's formula in $\Omega_{\text{int},\delta}$ and in $\widetilde{\Omega}_{\text{ext},\delta}$ with the help of (4.7), we obtain

$$\begin{aligned}\mathscr{L}_\delta v = &-\int_{\Gamma_{\delta,1}} \sigma_{\text{int}} \left(\partial_{\mathbf{n}_{\delta,1}} u_{\text{int},0|\Gamma_{\delta,1}} + \cdots + \delta^n \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},n|\Gamma_{\delta,1}}\right) \overline{v}_{\text{int}|\Gamma_{\delta,1}}\, d\Gamma_{\delta,1} \\ &-\sum_{\beta=1}^{2} \left[\delta a^{[\beta]}(u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \overline{v^{[\beta]}}) + \delta b^{[\beta]}\left(u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \overline{v^{[\beta]}}\right)\right] \\ &+\int_{\Gamma_{\delta,2}} \sigma_{\text{ext}} \left(\partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},0|\Gamma_{\delta,2}} + \cdots + \delta^n \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},n|\Gamma_{\delta,2}}\right) \overline{v}_{\text{ext}|\Gamma_{\delta,2}}\, d\Gamma_{\delta,2} \\ &-\int_{\Omega_\delta} \widetilde{\sigma}\nabla\mathscr{P}R.\nabla\overline{v}_d - \widetilde{k}^2\mathscr{P}R\overline{v}_d\, d\Omega_\delta.\end{aligned}$$

It follows, from (5.1)–(5.4), that

$$\begin{aligned}\mathscr{L}_\delta v = &-\int_{\Gamma_{\delta,1}} \sigma_{\text{int}} R_{N_1,n}\overline{v}_{\text{int}|\Gamma_{\delta,1}}\, d\Gamma_{\delta,1} + \int_{\Gamma_{\delta,2}} \sigma_{\text{ext}} R_{N_2,n}\overline{v}_{\text{ext}|\Gamma_{\delta,2}}\, d\Gamma_{\delta,2} \\ &-\sum_{\beta=1}^{2} \left[\delta a_\delta^{[\beta]}(u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \overline{v^{[\beta]}}) + \delta b_\delta^{[\beta]}(u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \overline{v^{[\beta]}})\right] \\ &-\int_{\Gamma} \sigma_{\text{int}} \left(\sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{(-1)^l\delta^{k+l}}{l!} p_1^l \partial_{\mathbf{n}}^{l+1} u_{\text{int},k|\Gamma}\right) \overline{v^{[1]}}(m,-1) \det\left(1 - 2\delta p_1\mathscr{H} + \delta^2 p_1^2\mathscr{K}\right) d\Gamma \\ &+\int_{\Gamma} \sigma_{\text{ext}} \left(\sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{\delta^{k+l}}{l!} p_1^l \partial_{\mathbf{n}}^{l+1} u_{\text{ext},k|\Gamma}\right) \overline{v^{[2]}}(m,1)\ \det\left(1 + 2\delta p_2\mathscr{H} + \delta^2 p_2^2\mathscr{K}\right) d\Gamma \\ &-\int_{\Omega_\delta} \widetilde{\sigma}\nabla\mathscr{P}R.\nabla\overline{v}_d - \widetilde{k}^2\mathscr{P}R\overline{v}_d\, d\Omega_\delta,\end{aligned}$$

where $R_{N_1,n}$ and $R_{N_2,n}$ are respectively the remainders of Taylor expansions in the normal variable with respect to δ up to order n of $\partial_{\mathbf{n}_{\delta,1}} u^{(n)}_{\text{int},\delta|\Gamma_{\delta,1}}$ and $\partial_{\mathbf{n}_{\delta,2}} u^{(n)}_{\text{ext},\delta|\Gamma_{\delta,2}}$; $2\mathscr{H}$ and $\mathscr{K}$ are respectively the mean and the Gaussian curvatures of the surface Γ. Now, we use that $u_0^{[\beta]}, \ldots, u_{n+1}^{[\beta]}$, $(\beta = 1, 2)$ are solutions of equations (4.16)–(4.20), and obtain

$$\begin{aligned}\mathscr{L}_\delta v = \delta^{n+1} \sum_{\beta=1}^{2} \Big\{ &\delta^{-1} a_{0,2}^{[\beta]}\left(u_{n+1}^{[\beta]}, \overline{v^{[\beta]}}\right) - \left(a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]} + b_0^{[\beta]}\right)\left(u_n^{[\beta]}, \overline{v^{[\beta]}}\right) \\ &- \left(a_{1,1}^{[\beta]} + b_1^{[\beta]}\right)\left(u_{n-1}^{[\beta]} + \delta u_n^{[\beta]}, \overline{v^{[\beta]}}\right) - \left(a_{2,1}^{[\beta]} + b_2^{[\beta]}\right)\left(u_{n-2}^{[\beta]} + \delta u_{n-1}^{[\beta]} + \delta^2 u_n^{[\beta]}, \overline{v^{[\beta]}}\right) \\ &- \cdots - a_{n-1,1}^{[\beta]}\left(u_1^{[\beta]} + \cdots + \delta^{n-1} u_{n-1}^{[\beta]}, \overline{v^{[\beta]}}\right) - r_n^{[\beta]}\left(u_1^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, \overline{v^{[\beta]}}\right)\Big\} \\ &+ \int_{\Gamma_{\delta,1}} \sigma_{\text{int}} R_{N_1,n}\overline{v}_{\text{int}|\Gamma_{\delta,1}}\, d\Gamma_{\delta,1} - \int_{\Gamma_{\delta,2}} \sigma_{\text{ext}} R_{N_2,n}\overline{v}_{\text{ext}|\Gamma_{\delta,2}}\, d\Gamma_{\delta,2} \\ &- \int_{\Omega_\delta} \widetilde{\sigma}\nabla\mathscr{P}R.\nabla\overline{v}_d - \widetilde{k}^2\mathscr{P}R\overline{v}_d\, d\Omega_\delta.\end{aligned}$$

By the estimates based on the explicit expressions of the bilinear forms $a_{k,l}^{[\beta]}(.,.)$ and those from Propositions 5.1, we have

$$|\mathscr{L}_\delta v| \leqslant c\delta^{n+1} \sum_{\beta=1}^{2} \left(\left\| \nabla_\Gamma v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{-1} \left\| \partial_{s_\beta} v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \left\| v^{[\beta]} \right\|_{L^2(\Omega^\beta)} \right)$$
$$+ c\delta^n \left(\|v_{\text{int}}\|_{H^1(\Omega_{\text{int},\delta})} + \sum_{\beta=1}^{2} \|v_\beta\|_{H^1(\Omega_{\delta,\beta})} + \|v_{\text{ext}}\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} \right).$$

This implies,

$$|\mathscr{L}_\delta v| \leqslant c\delta^{n+\frac{1}{2}} \sum_{\beta=1}^{2} \left(\delta^{\frac{1}{2}} \left\| \nabla_\Gamma v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{-\frac{1}{2}} \left\| \partial_{s_\beta} v^{[\beta]} \right\|_{L^2(\Omega^\beta)} + \delta^{\frac{1}{2}} \left\| v^{[\beta]} \right\|_{L^2(\Omega^\beta)} \right)$$
$$+ c\delta^n \left(\|v_{\text{int}}\|_{H^1(\Omega_{\text{int},\delta})} + \sum_{\beta=1}^{2} \|v_\beta\|_{H^1(\Omega_{\delta,\beta})} + \|v_{\text{ext}}\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} \right).$$

Therefore,

$$|\mathscr{L}_\delta v| \leqslant c\delta^n \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega). \tag{5.6}$$

We set in (5.5) $v_{\text{int}} = r^n_{\text{int},\delta}$, $v_d = r^n_{d,\delta} - \mathscr{P}R$ and $v_{\text{ext}} = r^n_{\text{ext},\delta}$. Then, v is continuous over the interfaces $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$. Hence, $v \in H^1(\Omega)$. Using (5.6) and the stability theorem 2.2, we obtain

$$\left\| r^n_{\text{int},\delta} \right\|_{H^1(\Omega_{\text{int},\delta})} + \left\| r^n_{d,\delta} - \mathscr{P}R \right\|_{H^1(\Omega_\delta)} + \left\| r^n_{\text{ext},\delta} \right\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} \leqslant c\delta^n.$$

Thanks to Proposition 5.1, we find

$$\left\| r^n_{\text{int},\delta} \right\|_{H^1(\Omega_{\text{int},\delta})} + \left\| r^n_{d,\delta} \right\|_{H^1(\Omega_\delta)} + \left\| r^n_{\text{ext},\delta} \right\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} \leqslant c\delta^n. \tag{5.7}$$

Finally, since $\|u_{\text{ext},j}\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} = O(1)$, $\|u_{\text{int},j}\|_{H^1(\Omega_{\text{int},\delta})} = O(1)$ and $\left\|u_{d_\beta,j}\right\|_{H^1(\Omega_{\delta,\beta})} = O(\delta^{-1/2})$, one gets

$$\begin{aligned}
\left\| r^n_{\text{int},\delta} \right\|_{H^1(\Omega_{\text{int},\delta})} &= \left\| \delta^{n+1} u_{\text{int},n+1} + r^{n+1}_{\text{int},\delta} \right\|_{H^1(\Omega_{\text{int},\delta})} \\
&\overset{(5.7)}{\leqslant} c\delta^{n+1} + c\delta^{n+1} \leqslant c\delta^{n+1}, \\
\left\| r^n_{\text{ext},\delta} \right\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} &= \left\| \delta^{n+1} u_{\text{ext},n+1} + r^{n+1}_{\text{ext},\delta} \right\|_{H^1(\widetilde{\Omega}_{\text{ext},\delta})} \\
&\overset{(5.7)}{\leqslant} c\delta^{n+1} + c\delta^{n+1} \leqslant c\delta^{n+1}, \\
\left\| r^n_{d,\delta} \right\|_{H^1(\Omega_\delta)} &= \left\| r^{n+1}_{d,\delta} + \delta^{n+1} u_{d,n+1} \right\|_{H^1(\Omega_\delta)} \\
&\overset{(5.7)}{\leqslant} c\delta^{n+1} + c\delta^{n+1/2} \leqslant c\delta^{n+1/2},
\end{aligned}$$

which completes the proof. □

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6 The first-order approximate transmission conditions

In this section, we model the effect of the thin layer by a problem with appropriate transmission conditions and prove that the modelling error is of order two in δ. We begin to truncate the series defining the asymptotic expansions, keeping only the first two terms. This yields

$$
\begin{aligned}
u_{\mathrm{int},\delta} &\simeq u^{(1)}_{\mathrm{int},\delta} := u_{\mathrm{int},0} + \delta u_{\mathrm{int},1} \ \text{ in } \Omega_{\mathrm{ext}},\\
u_{\mathrm{ext},\delta} &\simeq u^{(1)}_{\mathrm{ext},\delta} := u_{\mathrm{ext},0} + \delta u_{\mathrm{ext},1} \ \text{ in } \Omega_{\mathrm{ext}},\\
u_{d_1,\delta}(x) &\simeq u^{(1)}_{d_1,\delta}(m,s_1) := u^{[1]}_0(m,s_1) + \delta u^{[1]}_1(m,s_1),\ \forall x = \Phi_1(m,s_1) \in \Omega_{\delta,1},\\
u_{d_2,\delta}(x) &\simeq u^{(1)}_{d_2,\delta}(m,s_2) := u^{[2]}_0(m,s_2) + \delta u^{[2]}_1(m,s_2),\ \forall x = \Phi_2(m,s_2) \in \Omega_{\delta,2},
\end{aligned}
$$

where

$$
U^{(1)}_\delta := \begin{cases} u^{(1)}_{\mathrm{ext},\delta} \ \text{ in } \Omega_{\mathrm{ext}},\\ u^{(1)}_{\mathrm{int},\delta} \ \text{ in } \Omega_{\mathrm{int}},\end{cases}
$$

is the solution to

$$
\begin{cases}
\operatorname{div}\left(\sigma_{\mathrm{int}}\nabla u^{(1)}_{\mathrm{int},\delta}\right) + k^2_{\mathrm{int}} u^{(1)}_{\mathrm{int},\delta} = 0 & \text{in } \Omega_{\mathrm{int}},\\
\operatorname{div}\left(\sigma_{\mathrm{ext}}\nabla u^{(1)}_{\mathrm{ext},\delta}\right) + k^2_{\mathrm{ext}} u^{(1)}_{\mathrm{ext},\delta} = 0 & \text{in } \Omega_{\mathrm{ext}},\\
u^{(1)}_{\mathrm{int},\delta|\Gamma} - u^{(1)}_{\mathrm{ext},\delta|\Gamma} = \delta\mathscr{A}\left(u^{(1)}_{\mathrm{int},\delta}, u^{(1)}_{\mathrm{ext},\delta}\right) - \delta^2\xi_\delta & \text{on } \Gamma,\\
\sigma_{\mathrm{int}}\partial_{\mathbf{n}} u^{(1)}_{\mathrm{int},\delta|\Gamma} - \sigma_{\mathrm{ext}}\partial_{\mathbf{n}} u^{(1)}_{\mathrm{ext},\delta|\Gamma} = \delta\mathscr{B}\left(u^{(1)}_{\mathrm{int},\delta}, u^{(1)}_{\mathrm{ext},\delta}\right) - \delta^2\rho_\delta & \text{on } \Gamma,\\
\lim\limits_{|x|\to+\infty} |x|\left(\partial_{|x|} - ik_{\mathrm{ext}}\right)\left(u^{(1)}_{\mathrm{ext},\delta} - u_{inc}\right) = 0,
\end{cases}
\tag{6.1}
$$

with

$$
\begin{aligned}
\mathscr{A}(u,v) &:= \frac{p_1\sigma_{\mathrm{ext}}\widetilde{\sigma} + p_2\sigma_{\mathrm{int}}\widetilde{\sigma} - \sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}}{2\sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}\widetilde{\sigma}}\left(\sigma_{\mathrm{int}}\partial_{\mathbf{n}} u_{|\Gamma} + \sigma_{\mathrm{ext}}\partial_{\mathbf{n}} v_{|\Gamma}\right),\\
\mathscr{B}(u,v) &:= \frac{1}{2}\left(\widetilde{\sigma} - p_1\sigma_{\mathrm{int}} - p_2\sigma_{\mathrm{ext}}\right)\left(\Delta_\Gamma u_{|\Gamma} + \Delta_\Gamma v_{|\Gamma}\right)\\
&\quad + \frac{1}{2}\left(\widetilde{k}^2 - p_1 k^2_{\mathrm{int}} - p_2 k^2_{\mathrm{ext}}\right)\left(u_{|\Gamma} + v_{|\Gamma}\right),\\
\xi_\delta &:= \frac{p_1\sigma_{\mathrm{ext}}\widetilde{\sigma} + p_2\sigma_{\mathrm{int}}\widetilde{\sigma} - \sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}}{2\sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}\widetilde{\sigma}}\left(\sigma_{\mathrm{int}}\partial_{\mathbf{n}} u_{\mathrm{int},1|\Gamma} + \sigma_{\mathrm{ext}}\partial_{\mathbf{n}} u_{\mathrm{ext},1|\Gamma}\right),\\
\rho_\delta &:= \frac{1}{2}\left(\widetilde{\sigma} - p_1\sigma_{\mathrm{int}} - p_2\sigma_{\mathrm{ext}}\right)\left(\Delta_\Gamma u_{\mathrm{int},1|\Gamma} + \Delta_\Gamma u_{\mathrm{ext},1|\Gamma}\right)\\
&\quad + \frac{1}{2}\left(\widetilde{k}^2 - p_1 k^2_{\mathrm{int}} - p_2 k^2_{\mathrm{ext}}\right)\left(u_{\mathrm{int},1|\Gamma} + u_{\mathrm{ext},1|\Gamma}\right).
\end{aligned}
$$

The first-order approximation is defined by

$$
U^{ap}_\delta := \begin{cases} u^{ap}_{\mathrm{ext},\delta} \ \text{ in } \Omega_{\mathrm{ext}},\\ u^{ap}_{\mathrm{int},\delta} \ \text{ in } \Omega_{\mathrm{int}},\end{cases}
$$

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where U_δ^{ap} is the solution of (6.1) with $\rho_\delta = 0$ and $\xi_\delta = 0$. The approximate problem $\left(\mathscr{P}_\delta^{ap}\right)$ is then defined by equation (6.1) with the following transmission conditions:

$$\begin{cases} u^{ap}_{\mathrm{int},\delta|\Gamma} - u^{ap}_{\mathrm{ext},\delta|\Gamma} = \delta \frac{p_1\sigma_{\mathrm{ext}}\widetilde{\sigma}+p_2\sigma_{\mathrm{int}}\widetilde{\sigma}-\sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}}{2\sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}\widetilde{\sigma}} \left(\sigma_{\mathrm{int}}\partial_{\mathbf{n}} u^{ap}_{\mathrm{int},\delta|\Gamma} + \sigma_{\mathrm{ext}}\partial_{\mathbf{n}} u^{ap}_{\mathrm{ext},\delta|\Gamma}\right), \\ \sigma_{\mathrm{int}}\partial_{\mathbf{n}} u^{ap}_{\mathrm{int},\delta|\Gamma} - \sigma_{\mathrm{ext}}\partial_{\mathbf{n}} u^{ap}_{\mathrm{ext},\delta|\Gamma} = \delta\frac{1}{2}\left(\widetilde{\sigma} - p_1\sigma_{\mathrm{int}} - p_2\sigma_{\mathrm{ext}}\right)\left(\Delta_\Gamma u^{ap}_{\mathrm{int},\delta|\Gamma} + \Delta_\Gamma u^{ap}_{\mathrm{ext},\delta|\Gamma}\right) \\ \quad +\delta\frac{1}{2}\left(\widetilde{k}^2 - p_1 k^2_{\mathrm{int}} - p_2 k^2_{\mathrm{ext}}\right)\left(u^{ap}_{\mathrm{int},\delta|\Gamma} + u^{ap}_{\mathrm{ext},\delta|\Gamma}\right). \end{cases} \tag{6.2}$$

Before proving that U_δ^{ap} is indeed an approximation of the field u_δ far from the thin layer with error $O(\delta^2)$, we study the well-posedness of $\left(\mathscr{P}_\delta^{ap}\right)$. However, the bilinear form associated to $\left(\mathscr{P}_\delta^{ap}\right)$ is neither positive nor negative. To show the existence and uniqueness of the solution U_δ^{ap}, we reformulate Problem $\left(\mathscr{P}_\delta^{ap}\right)$ into a non-local equation on the interface Γ (cf. e.g., [6,9] for different problems). We introduce the DtN operators (Dirichlet-to-Neumann) S_{int} and S_{ext} defined from $H^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$ by $S_{\mathrm{int}}\varphi := \sigma_{\mathrm{int}}\partial_{\mathbf{n}} u_{\mathrm{int}|\Gamma}$, where u_{int} is the solution to the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma_{\mathrm{int}}\nabla u_{\mathrm{int}}) + k^2_{\mathrm{int}} u_{\mathrm{int}} = 0 & \text{in } \Omega_{\mathrm{int}}, \\ u_{\mathrm{int}|\Gamma} = \varphi & \text{on } \Gamma, \end{cases}$$

and by $S_{\mathrm{ext}}\psi := \sigma_{\mathrm{ext}}\partial_{-\mathbf{n}} u_{\mathrm{ext}|\Gamma}$, where u_{ext} is the solution to the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma_{\mathrm{ext}}\nabla u_{\mathrm{ext}}) + k^2_{\mathrm{ext}} u_{\mathrm{ext}} = 0 & \text{in } \Omega_{\mathrm{ext}}, \\ u_{\mathrm{ext}|\Gamma} = \psi & \text{on } \Gamma, \\ \lim\limits_{|x|\to+\infty} |x|\left(\partial_{|x|} - ik_{\mathrm{ext}}\right) u_{\mathrm{ext}} = 0. \end{cases}$$

Remark 6.1 *The function u_{int} is defined only in the case where the constant $k^2_{\mathrm{int}}/\sigma_{\mathrm{int}}$ does not belong to the spectrum of the closed operator $\left(-\Delta, H^1_0(\Omega_{\mathrm{int}})\right)$. Fortunately, its spectrum is discrete since this operator has a compact resolvent and is composed only of real numbers so we can always assume that u_{int} is well defined.*

The following theorem gives the uniqueness of the solution U_δ^{ap} to Problem $\left(\mathscr{P}_\delta^{ap}\right)$.

Theorem 6.1 *Assume that the following hypotheses hold*

$$\Im\left(\widetilde{k}^2 - p_1 k^2_{\mathrm{int}}\right) \geqslant 0, \tag{6.3}$$

$$\lambda_{\delta,1} := \frac{1}{\delta}\frac{2\sigma_{\mathrm{ext}}\widetilde{\sigma}\sigma_{\mathrm{int}}}{p_1\sigma_{\mathrm{ext}}\widetilde{\sigma} + p_2\sigma_{\mathrm{int}}\widetilde{\sigma} - \sigma_{\mathrm{int}}\sigma_{\mathrm{ext}}} \notin \sigma\left(S_{\mathrm{int}}\right), \tag{6.4}$$

problem $\left(\mathscr{P}_\delta^{ap}\right)$ admits at most one solution.

Remark 6.2 *Note that we can always choose p_1 and p_2 in such a manner that the condition on $\lambda_{\delta,1}$ is fulfilled.*

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Proof Let us consider the homogeneous problem associated to $\left(\mathscr{P}_{\delta}^{ap}\right)$:

$$\begin{cases} \operatorname{div}\left(\sigma_{\text{int}} \nabla u_{\text{int},\delta}^{ap}\right) + k_{\text{int}}^2 u_{\text{int},\delta}^{ap} = 0 & \text{in } \Omega_{\text{int}}, \quad (6.5) \\ \operatorname{div}\left(\sigma_{\text{ext}} \nabla u_{\text{ext},\delta}^{ap}\right) + k_{\text{ext}}^2 u_{\text{ext},\delta}^{ap} = 0 & \text{in } \Omega_{\text{ext}}, \quad (6.6) \\ \lim_{|x|\to+\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right) u_{\text{ext},\delta}^{ap} = 0, \end{cases}$$

with transmission conditions on the interface Γ

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = \lambda_{\delta,1}^{-1}\left(\sigma_{\text{int}}\partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} + \sigma_{\text{ext}}\partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap}\right), \\ \sigma_{\text{int}}\partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{ap} - \sigma_{\text{ext}}\partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{ap} = \delta\frac{1}{2}\left(\widetilde{\sigma} - p_1\sigma_{\text{int}} - p_2\sigma_{\text{ext}}\right)\left(\Delta_\Gamma u_{\text{int},\delta|\Gamma}^{ap} + \Delta_\Gamma u_{\text{ext},\delta|\Gamma}^{ap}\right) \\ \qquad\qquad + \delta\frac{1}{2}\left(\widetilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2\right)\left(u_{\text{int},\delta|\Gamma}^{ap} + u_{\text{ext},\delta|\Gamma}^{ap}\right). \end{cases} \tag{6.7}$$

Standard regularity results for elliptic problems (see e.g. [1]) show that $\left(u_{\text{int},\delta}^{ap}, u_{\text{ext},\delta}^{ap}\right) \in \mathscr{C}^\infty\left(\overline{\Omega_{\text{int}}}\right) \times \mathscr{C}^\infty\left(\overline{\Omega_{\text{ext}}}\right)$. Let B_R denote the ball with centre O and radius R large enough to contain Ω_{int} and Ω_R be the domain of $\mathbb{R}^3$ defined by $\Omega_R := B_R \cap \Omega_{\text{ext}}$. Multiplying equations (6.5) and (6.6) respectively by $\overline{u_{\text{int},\delta}^{ap}}$ and $\overline{u_{\text{ext},\delta}^{ap}}$, integrating in B_R and using Green formula, we obtain

$$\begin{aligned} &\sigma_{\text{int}} \int_{\Omega_{\text{int}}} \left|\nabla u_{\text{int},\delta}^{ap}\right|^2 \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \left|u_{\text{int},\delta}^{ap}\right|^2 \, d\Omega_{\text{int}} + \sigma_{\text{ext}} \int_{\Omega_R} \left|\nabla u_{\text{ext},\delta}^{ap}\right|^2 \, d\Omega_R \\ &- k_{\text{ext}}^2 \int_{\Omega_R} \left|u_{\text{ext},\delta}^{ap}\right|^2 \, d\Omega_R + \gamma_1 \int_\Gamma \left(\left|u_{\text{int},\delta|\Gamma}^{ap}\right|^2 + \left|u_{\text{ext},\delta|\Gamma}^{ap}\right|^2\right) d\Gamma \\ &+ \delta\frac{1}{4}\left(\widetilde{\sigma} - p_1\sigma_{\text{int}} - p_2\sigma_{\text{ext}}\right) \int_\Gamma \left|\nabla_\Gamma u_{\text{int},\delta|\Gamma}^{ap} + \nabla_\Gamma u_{\text{ext},\delta|\Gamma}^{ap}\right|^2 \, d\Gamma \\ &+ 2\gamma_2 \int_\Gamma \Re\left(u_{\text{int},\delta|\Gamma}^{ap} \overline{u_{\text{ext},\delta|\Gamma}^{ap}}\right) d\Gamma = \sigma_{\text{ext}} \int_{S_R} \partial_{\mathbf{R}} u_{\text{ext},\delta|S_R}^{ap} \overline{u_{\text{ext},\delta|S_R}^{ap}} \, dS_R, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \gamma_1 &:= -\frac{\sigma_{\text{int}}\sigma_{\text{ext}}\widetilde{\sigma}}{\delta\left(p_1\sigma_{\text{ext}}\widetilde{\sigma} + p_2\sigma_{\text{int}}\widetilde{\sigma} - \sigma_{\text{int}}\sigma_{\text{ext}}\right)} - \delta\frac{\widetilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2}{4}, \\ \gamma_2 &:= \frac{\sigma_{\text{int}}\sigma_{\text{ext}}\widetilde{\sigma}}{\delta\left(p_1\sigma_{\text{ext}}\widetilde{\sigma} + p_2\sigma_{\text{int}}\widetilde{\sigma} - \sigma_{\text{int}}\sigma_{\text{ext}}\right)} - \delta\frac{\widetilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2}{4}, \end{aligned}$$

and S_R denotes the sphere with centre O and radius R. Hence, taking the imaginary part of (6.8) and using (6.3), we have

$$\Im\left(\int_{S_R} \partial_{\mathbf{R}} u_{\text{ext},\delta|S_R}^{ap} \overline{u_{\text{ext},\delta|S_R}^{ap}} \, dS_R\right) \leqslant 0. \tag{6.9}$$

It follows from Rellich's lemma and radiation condition (2.1k) that

$$u_{\text{ext},\delta}^{ap} = 0 \quad \text{on } \Omega_{\text{ext}}. \tag{6.10}$$

Problems (6.6) and (6.7) are reduced to

$$\begin{cases} \operatorname{div}\left(\sigma_{\text{int}}\nabla u^{ap}_{\text{int},\delta}\right)+k^2_{\text{int}}u^{ap}_{\text{int},\delta}=0 & \text{in } \Omega_{\text{int}}, \\ u^{ap}_{\text{int},\delta|\Gamma}=\lambda^{-1}_{\delta,1}\partial_{\mathbf{n}}u^{ap}_{\text{int},\delta|\Gamma} & \text{on } \Gamma, \\ \sigma_{\text{int}}\partial_{\mathbf{n}}u^{ap}_{\text{int},\delta|\Gamma}=\delta\frac{1}{2}\left(\widetilde{\sigma}-p_1\sigma_{\text{int}}-p_2\sigma_{\text{ext}}\right)\Delta_\Gamma u^{ap}_{\text{int},\delta|\Gamma} & \\ \quad+\delta\frac{1}{2}\left(\widetilde{k}^2-p_1k^2_{\text{int}}-p_2k^2_{\text{ext}}\right)u^{ap}_{\text{int},\delta|\Gamma} & \text{on } \Gamma. \end{cases} \tag{6.11}$$

The equation

$$u^{ap}_{\text{int},\delta|\Gamma}=\lambda^{-1}_{\delta,1}\partial_{\mathbf{n}}u^{ap}_{\text{int},\delta|\Gamma}, \tag{6.12}$$

implies

$$\left(S_{\text{int}}-\lambda_{\delta,1}I\right)\varphi_{\text{int}}=0, \tag{6.13}$$

where φ_{int} is the trace of $u^{ap}_{\text{int},\delta}$ on the surface Γ. By virtue of (6.4), we get $u^{ap}_{\text{int},\delta}=0$ on Γ and therefore $u^{ap}_{\text{int},\delta}=0$ in Ω_{int}. □

The existence of U^{ap}_δ is based on properties of Laplace–Beltrami and Dtn operators. The latter are given in the next lemma whose proof can be found, for example, in [40].

Lemma 6.1

(1) *The Laplace–Beltrami operator $-\Delta_\Gamma$ on Γ is a pseudo-differential operator of real symbol of order 2. It is Fredholm of index 0.*

(2) *The Dirichlet-to-Neumann operators S_{int} and S_{ext} are elliptic pseudo-differential operators of real symbol of order 1.*

Using the definition of S_{int} and S_{ext}, Problem $\left(\mathscr{P}^{ap}_\delta\right)$ is equivalent to the boundary equations

$$\begin{cases} \left(S_{\text{int}}-\lambda_{\delta,1}I\right)\omega-\left(S_{\text{ext}}-\lambda_{\delta,1}I\right)\varkappa=g & (6.14) \\ S_{\text{int}}\omega+S_{\text{ext}}\varkappa-\delta\dfrac{1}{2}\left(\widetilde{\sigma}-p_1\sigma_{\text{int}}-p_2\sigma_{\text{ext}}\right)\left(\Delta_\Gamma\omega+\Delta_\Gamma\varkappa\right) & \\ -\delta\dfrac{1}{2}\left(\widetilde{k}^2-p_1k^2_{\text{int}}-p_2k^2_{\text{ext}}\right)(\omega+\varkappa)=-g, & (6.15) \end{cases}$$

where

$$g:=-\sigma_{\text{ext}}\partial_{\mathbf{n}}u_{inc|\Gamma}-S_{\text{ext}}u_{inc|\Gamma}\in\mathscr{C}^\infty(\Gamma), \tag{6.16}$$

ω and $\varkappa$ are the traces of $u^{ap}_{\text{int},\delta}$ and $u^{ap}_{\text{ext},\delta}$ on the surface Γ respectively. From (6.4), $-\lambda_{\delta,1}\notin\sigma\left(S_{\text{int}}\right)$ thus the next pseudo-differential operator of order -1 is well defined

$$K_\delta:=\left(S_{\text{int}}-\lambda_{\delta,1}I\right)^{-1}. \tag{6.17}$$

Equation (6.14) then reduces to

$$\omega=K_\delta\left(S_{\text{ext}}-\lambda_{\delta,1}I\right)\varkappa+K_\delta g, \tag{6.18}$$

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and Problems (6.14) and (6.15) are equivalent to the boundary equation

$$A_\delta \varkappa := B_\delta \varkappa - \lambda_{\delta,2} \Delta_\Gamma K_\delta \left(S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1} I\right) \varkappa = \theta, \tag{6.19}$$

where

$$\lambda_{\delta,2} := \delta \frac{1}{2} \left(\widetilde{\sigma} - p_1 \sigma_{\text{int}} - p_2 \sigma_{\text{ext}}\right), \tag{6.20}$$

$$\begin{aligned}
\theta &:= -g - S_{\text{int}} K_\delta g + \lambda_{\delta,2} \Delta_\Gamma K_\delta g + \lambda_{\delta,3} K_\delta g, \\
B_\delta &:= S_{\text{int}} K_\delta S_{\text{ext}} - \lambda_{\delta,1} S_{\text{int}} K_\delta + S_{\text{ext}} - \lambda_{\delta,3} K_\delta S_{\text{ext}} + \lambda_{\delta,3} \lambda_{\delta,1} K_\delta - \lambda_{\delta,3} I, \\
\lambda_{\delta,3} &:= \delta \frac{1}{2} \left(\widetilde{k}^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2\right).
\end{aligned}$$

Some properties of the operator A_δ are given in the next proposition.

Proposition 6.1 *For all integers k in $\mathbb{N}$, the operator A_δ defined from $H^{k+1/2}(\Gamma)$ to $H^{k-3/2}(\Gamma)$ is Fredholm with index zero.*

Proof Let k be an integer in $\mathbb{N}$. Since S_{int} and S_{ext} are pseudo-differential operators of order 1, they map $H^k(\Gamma)$ to $H^{k-1}(\Gamma)$. K_δ being a pseudo-differential operator of order -1, it maps $H^k(\Gamma)$ to $H^{k+1}(\Gamma)$. As a consequence, B_δ maps $H^k(\Gamma)$ to $H^{k-1}(\Gamma)$. The injection $H^{k-1}(\Gamma) \hookrightarrow H^{k-2}(\Gamma)$ being compact, the operator A_δ defined from $H^{k+1/2}(\Gamma)$ to $H^{k-3/2}(\Gamma)$ is a compact perturbation of $\lambda_{\delta,2} \Delta_\Gamma K_\delta \left(S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1} I\right)$. Since Δ_Γ is Fredholm with index zero, to show that A_δ is Fredholm with index zero, it suffices to show that $S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1} I$ is invertible.
Let us consider the equation

$$\left(S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1} I\right) \varphi = \psi, \ \ \psi \in H^{k-1/2}(\Gamma), \ k \in \mathbb{N}. \tag{6.21}$$

Using the definition of S_{ext} and S_{int}, equation (6.21) is equivalent to the following problem:

$$\begin{cases}
\operatorname{div}(\sigma_{\text{int}} \nabla V_{\text{int}}) + k_{\text{int}}^2 V_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\
\operatorname{div}(\sigma_{\text{ext}} \nabla V_{\text{ext}}) + k_{\text{ext}}^2 V_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\
V_{\text{int}|\Gamma} - V_{\text{ext}|\Gamma} = 0 & \text{in} \Gamma \\
\sigma_{\text{int}} \partial_{\mathbf{n}} V_{\text{int}|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} V_{\text{ext}|\Gamma} = 2\lambda_{\delta,1} V_{\text{int}|\Gamma} + \psi & \text{in} \Gamma \\
\lim_{|x| \to +\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right) V_{\text{ext}} = 0,
\end{cases} \tag{6.22}$$

where $\varphi = V_{\text{int}|\Gamma} = V_{\text{ext}|\Gamma}$. Standard arguments involving Rellich's lemma and the Fredholm alternative show that, for all k in $\mathbb{N}$, if $\psi \in H^{k-1/2}(\Gamma)$, then Problem (6.22) admits a unique solution $(V_{\text{int}}, V_{\text{ext}})$ in $H^{k+1}(\Omega_{\text{int}}) \times H_{loc}^{k+1}\left(\overline{\Omega_{\text{ext}}}\right)$, and hence a unique trace $\varphi \in H^{k+1/2}(\Gamma)$. As a consequence, the operator $S_{\text{ext}} + S_{\text{int}} - 2\lambda_{\delta,1} I$, defined from $H^{k+1/2}(\Gamma)$ to $H^{k-1/2}(\Gamma)$, is invertible. □

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We are now in position to state the existence theorem.

Theorem 6.2 *Under the assumptions of Theorem 6.1, Problem* ($\mathscr{P}_\delta^{ap}$) *admits a unique solution* $(u^{ap}_{\text{int},\delta}, u^{ap}_{\text{ext},\delta})$ *in* $H^{k+1}(\Omega_{\text{int}}) \times H^{k+1}_{loc}(\overline{\Omega_{\text{ext}}})$, $\forall k \in \mathbb{N}$.

Proof It follows from Proposition 6.1 that the uniqueness of U^{ap}_δ implies the existence. From Theorem 6.1, we then infer that, for all k in $\mathbb{N}$, there exists a unique solution $(u^{ap}_{\text{int},\delta}, u^{ap}_{\text{ext},\delta})$ in $H^{k+1}(\Omega_{\text{int}}) \times H^{k+1}_{loc}(\overline{\Omega_{\text{ext}}})$. □

Let us denote by u^{ap}_δ the approximate solution defined on Ω by

$$u^{ap}_\delta := \begin{cases} u^{ap}_{\text{int},\delta} & \text{in } \Omega_{\text{int},\delta}, \\ u^{ap}_{d_\beta,\delta} & \text{in } \Omega_{\delta,\beta},\ (\beta = 1 \text{ or } 2), \\ u^{ap}_{\text{ext},\delta} & \text{in } \Omega_{\text{ext},\delta}, \end{cases}$$

such that $u^{ap}_{d_\beta,\delta}$ are defined on $\Omega_{\delta,\beta}$ by

$$u^{ap}_{d_1,\delta}(x) := u^{[1],ap}_{d_1,\delta}(m,s_1) := u^{ap}_{\text{int},\delta|\Gamma} + \delta p_1 \left[(s_1+1)\sigma_{\text{int}}\widetilde{\sigma}^{-1} - 1\right] \partial_{\mathbf{n}} u^{ap}_{\text{int},\delta|\Gamma},$$
$$\forall x = \Phi_1(m,s_1) \in \Omega_{\delta,1},$$

and

$$u^{ap}_{d_2,\delta}(x) := u^{[2],ap}_{d_2,\delta}(m,s_2) := u^{ap}_{\text{ext},\delta|\Gamma} + \delta p_2 \left[(s_2-1)\sigma_{\text{ext}}\widetilde{\sigma}^{-1} + 1\right] \partial_{\mathbf{n}} u^{ap}_{\text{ext},\delta|\Gamma},$$
$$\forall x = \Phi_2(m,s_2) \in \Omega_{\delta,2}.$$

Finally, we want to derive an error estimate between u_δ and the approximate solution u^{ap}_δ. To do so, we need once again a uniform stability result for the approximate problem. Let $\mathbb{H}^1(\Omega)$ be the Hilbert space defined by

$$\mathbb{H}^1(\Omega) := \left\{ v = (v_{\text{int}}, v_{\text{ext}}) \in H^1(\Omega_{\text{int}}) \times H^1\left(\widetilde{\Omega}_{\text{ext}}\right) \right\},$$

equipped with its natural norm and b_δ be a bilinear form defined on $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ by

$$\begin{aligned} b_\delta(u,v) := {} & \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \nabla u_{\text{int}}.\nabla v_{\text{int}}\ d\Omega_{\text{int}} - k^2_{\text{int}} \int_{\Omega_{\text{int}}} u_{\text{int}} v_{\text{int}}\ d\Omega_{\text{int}} \\ & + \sigma_{\text{ext}} \int_{\widetilde{\Omega}_{\text{ext}}} \nabla u_{\text{ext}}.\nabla v_{\text{ext}}\ d\widetilde{\Omega}_{\text{ext}} - k^2_{\text{ext}} \int_{\widetilde{\Omega}_{\text{ext}}} u_{\text{ext}} v_{\text{ext}}\ d\widetilde{\Omega}_{\text{ext}} \\ & - \frac{1}{2}\lambda_{\delta,1} \int_\Gamma \left(u_{\text{int}|\Gamma} - u_{\text{ext}|\Gamma}\right)\left(v_{\text{int}|\Gamma} - v_{\text{ext}|\Gamma}\right)\ d\Gamma \\ & + \sigma_{\text{ext}} \left\langle T u_{\text{ext}|\partial\Omega}, v_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}. \end{aligned}$$

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We have the following lemma:

Lemma 6.2

(1) *For all* h_δ *in* $\left(\mathbb{H}^1(\Omega)\right)'$, *there exists a positive constant* c *independent of* δ *such that the solution to the variational problem*

$$\begin{cases} \text{Find } u_\delta \in \mathbb{H}^1(\Omega), \ \forall v \in \mathbb{H}^1(\Omega), \\ b_\delta(u_\delta, v) = h_\delta(v), \end{cases}$$

satisfies

$$\|u_\delta\|_{\mathbb{H}^1(\Omega)} \leqslant c\delta^{-1/2} \|h_\delta\|_{\left(\mathbb{H}^1(\Omega)\right)'}. \tag{6.23}$$

(2) *Furthermore, if* $p_1\sigma_{\text{ext}}\widetilde{\sigma}_\delta + p_2\sigma_{\text{int}}\widetilde{\sigma}_\delta - \sigma_{\text{int}}\sigma_{\text{ext}} \leqslant 0$, *one has*

$$\|u_\delta\|_{\mathbb{H}^1(\Omega)} \leqslant c \|h_\delta\|_{\left(\mathbb{H}^1(\Omega)\right)'}. \tag{6.24}$$

Proof

(1) We need to prove that

$$\|u_\delta\|_{\mathbb{H}^1(\Omega)} \leqslant c\delta^{-1/2} \sup_{v \in \mathbb{H}^1(\Omega)} \frac{|b_\delta(u_\delta, v)|}{\|v\|_{\mathbb{H}^1(\Omega)}}.$$

We proceed by contradiction, assuming there exist sequences $(\delta_n)_{n\geqslant 0}$ and $\left(u_{\delta_n}\right)_{n\geqslant 0}$, denoted by $(u_n)_{n\geqslant 0}$, such that

$$\lim_{n\to+\infty} \delta_n = 0, \ \left\|\sqrt{\delta} u_n\right\|_{\mathbb{H}^1(\Omega)} = 1, \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \sup_{\|\varphi\|_{\mathbb{H}^1(\Omega)}=1} |b_{\delta_n}(u_n, \varphi)| = 0. \tag{6.25}$$

We can extract a subsequence of $\left(\sqrt{\delta} u_n\right)_{n\geqslant 0}$, still denoted by $\left(\sqrt{\delta} u_n\right)_{n\geqslant 0}$, such that

$$\begin{cases} \sqrt{\delta} u_n \to u_0 \text{ in } \mathbb{L}^2(\Omega), \\ \sqrt{\delta} u_n \rightharpoonup u_0 \text{ in } \mathbb{H}^1(\Omega). \end{cases} \tag{6.26}$$

Furthermore, for all v in $\mathscr{C}^\infty(\overline{\Omega_{\text{int}}}) \times \mathscr{C}^\infty(\overline{\widetilde{\Omega}_{\text{ext}}})$, we have

$$\begin{aligned} &\lim_{n\to+\infty} \frac{1}{2}\sqrt{\delta}\lambda_{\delta,1} \int_\Gamma \left(u_{\text{int},n|\Gamma} - u_{\text{ext},n|\Gamma}\right)\left(v_{\text{int}|\Gamma} - v_{\text{ext}|\Gamma}\right) \, d\Gamma \\ &= \sigma_{\text{int}} \int_{\Omega_{\text{int}}} \sqrt{\delta}\nabla u_{\text{int},n}.\nabla v_{\text{int}} \, d\Omega_{\text{int}} - k_{\text{int}}^2 \int_{\Omega_{\text{int}}} \sqrt{\delta} u_{\text{int},n} v_{\text{int}} \, d\Omega_{\text{int}} \\ &\quad + \sigma_{\text{ext}} \int_{\widetilde{\Omega}_{\text{ext}}} \sqrt{\delta}\nabla u_{\text{ext},n}.\nabla v_{\text{ext}} \, d\widetilde{\Omega}_{\text{ext}} - k_{\text{ext}}^2 \int_{\widetilde{\Omega}_{\text{ext}}} \sqrt{\delta} u_{\text{ext},n} v_{\text{ext}} \, d\widetilde{\Omega}_{\text{ext}} \end{aligned}$$

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$$
\begin{aligned}
&-\sqrt{\delta}b_\delta\left(u_n,v\right)+\sigma_{\rm ext}\left\langle \sqrt{\delta}Tu_{{\rm ext},n|\partial\Omega},v_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}\\
&=\sigma_{\rm int}\int_{\Omega_{\rm int}}\nabla u_0.\nabla v_{\rm int}\ d\Omega_{\rm int}-k_{\rm int}^2\int_{\Omega_{\rm int}}u_0v_{\rm int}\ d\Omega_{\rm int}\\
&+\sigma_{\rm ext}\int_{\widetilde{\Omega}_{\rm ext}}\nabla u_0.\nabla v_{\rm ext}\ d\widetilde{\Omega}_{\rm ext}-k_{\rm ext}^2\int_{\widetilde{\Omega}_{\rm ext}}u_0v_{\rm ext}\ d\widetilde{\Omega}_{\rm ext}\\
&+\sigma_{\rm ext}\left\langle Tu_{{\rm ext},0|\partial\Omega},v_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.
\end{aligned}
$$

As the right-hand side is independent of δ, we have

$$
\frac{1}{2}\sqrt{\delta}\lambda_{\delta,1}\int_\Gamma\left(u_{{\rm int},n|\Gamma}-u_{{\rm ext},n|\Gamma}\right)\left(v_{{\rm int}|\Gamma}-v_{{\rm ext}|\Gamma}\right)\ d\Gamma=O\left(1\right),\quad\forall v\in\mathscr{C}^\infty(\overline{\Omega_{\rm int}})\times\mathscr{C}^\infty(\overline{\widetilde{\Omega}_{\rm ext}}),\tag{6.27}
$$

and by density of $\mathscr{C}^\infty(\overline{\Omega_{\rm int}})\times\mathscr{C}^\infty(\overline{\widetilde{\Omega}_{\rm ext}})$ in $\mathbb{H}^1\left(\Omega\right)$, we conclude that the equality is true for all v in $\mathbb{H}^1\left(\Omega\right)$. Setting $v=\overline{u_n}$, we obtain

$$
\left\|u_{{\rm int},n|\Gamma}-u_{{\rm ext},n|\Gamma}\right\|_{L^2(\Gamma)}\leqslant c\delta^{1/4}.\tag{6.28}
$$

It follows that $u_{\rm int,0}=u_{\rm ext,0}$ on Γ and, for all v in $H^1\left(\Omega\right)$, one gets

$$
\begin{aligned}
\lim_{n\to+\infty}\sqrt{\delta}b_\delta\left(u,v\right)\ :=\ &\sigma_{\rm int}\int_{\Omega_{\rm int}}\nabla u_{\rm int,0}.\nabla v_{\rm int}\ d\Omega_{\rm int}-k_{\rm int}^2\int_{\Omega_{\rm int}}u_{\rm int,0}v_{\rm int}\ d\Omega_{\rm int}\\
&+\sigma_{\rm ext}\int_{\widetilde{\Omega}_{\rm ext}}\nabla u_{\rm ext,0}.\nabla v_{\rm ext}\ d\widetilde{\Omega}_{\rm ext}-k_{\rm ext}^2\int_{\widetilde{\Omega}_{\rm ext}}u_{\rm ext,0}v_{\rm ext}\ d\widetilde{\Omega}_{\rm ext}\\
&+\sigma_{\rm ext}\left\langle Tu_{{\rm ext},0|\partial\Omega},v_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}=0.
\end{aligned}\tag{6.29}
$$

Theorem 4.1 ensures that the problem: *Find* u_0 *in* $H^1\left(\Omega\right)$ *satisfying (6.29),* $\forall v\in H^1\left(\Omega\right)$, is well posed. We then infer that $u_0=0$ and it only remains to show that $\lim_{n\to+\infty}\left\|\sqrt{\delta}u_n\right\|_{\mathbb{H}^1(\Omega)}=0$. Note that, since u_0 is uniquely determined, the whole sequence converges to $u_0=0$ in $\mathbb{L}^2\left(\Omega\right)$. To obtain a contradiction, we have to show that $\lim_{n\to+\infty}\left\|\sqrt{\delta}\nabla u_n\right\|_{\mathbb{L}^2(\Omega)}=0$. One has

$$
\begin{aligned}
\left\|\sqrt{\delta}\nabla u_n\right\|_{\mathbb{L}^2(\Omega)}^2\leqslant c\delta&\left(\sigma_{\rm int}\int_{\Omega_{\rm int}}\left|\nabla u_{{\rm int},n}\right|^2\ d\Omega_{\rm int}+\sigma_{\rm ext}\int_{\Omega_{\rm ext}}\left|\nabla u_{{\rm ext},n}\right|^2\ d\Omega_{\rm ext}\right)\\
=c\Re&\left(\delta b_\delta(u_n,\overline{u_n})+k_{\rm int}^2\int_{\Omega_{\rm int}}\delta\left|u_{{\rm int},n}\right|^2\ d\Omega_{\rm int}+k_{\rm ext}^2\int_{\Omega_{\rm ext}}\delta\left|u_{{\rm ext},n}\right|^2\ d\Omega_{\rm ext}\right.\\
&+\frac{1}{2}\delta\lambda_{\delta,1}\int_\Gamma\left|u_{{\rm int},n|\Gamma}-u_{{\rm ext},n|\Gamma}\right|^2\ d\Gamma\\
&\left.-\ \delta\sigma_{\rm ext}\left\langle Tu_{{\rm ext},n|\partial\Omega},\overline{u_{{\rm ext},n}}_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}\right).
\end{aligned}
$$

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Using Lemma 2.1, we infer

$$\left\|\sqrt{\delta}\nabla u_n\right\|^2_{L^2(\Omega)} \leqslant c\Re\left[\delta b_n(u_n,\overline{u_n})+k^2_{\mathrm{int}}\int_{\Omega_{\mathrm{int}}}\delta\left|u_{\mathrm{int},n}\right|^2\, d\Omega_{\mathrm{int}}+k^2_{\mathrm{ext}}\int_{\Omega_{\mathrm{ext}}}\delta\left|u_{\mathrm{ext},n}\right|^2\, d\Omega_{\mathrm{ext}}\right.$$
$$+\frac{1}{2}\delta\lambda_{\delta,1}\int_{\Gamma}\left|u_{\mathrm{int},n|\Gamma}-u_{\mathrm{ext},n|\Gamma}\right|^2\, d\Gamma$$
$$\left.-\sigma_{\mathrm{ext}}\left\langle\sqrt{\delta}K u_{\mathrm{ext},n|\partial\Omega},\sqrt{\delta}\overline{u_{\mathrm{ext},n|\partial\Omega}}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}\right].$$

Since K is compact and $\sqrt{\delta}u_n \rightharpoonup 0$ in $\mathbb{H}^1(\Omega)$,

$$\left\langle\sqrt{\delta}K u_{\mathrm{ext},n|\partial\Omega},\sqrt{\delta}\overline{u_{\mathrm{ext},n|\partial\Omega}}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \to 0.$$

Finally, the assumption $\lim_{n\to+\infty}\Re\left[b_n(u_n,\overline{u_n})\right]=0$ and (6.28) yield $\lim_{n\to+\infty}\left\|\sqrt{\delta}\nabla u_n\right\|_{\mathbb{L}^2(\Omega)}=0$ contradicting $\left\|\sqrt{\delta}u_n\right\|_{\mathbb{H}^1(\Omega)}=1$.

(2) Similar arguments to those used to prove (6.23) guarantee Inequality (6.24).

We can now prove optimal error estimates.

Theorem 6.3 *There exists a constant c independent of δ such that*

$$\left\|u_{\mathrm{int},\delta}-u^{ap}_{\mathrm{int},\delta}\right\|_{H^1\left(\Omega_{\mathrm{int},\delta}\right)}+\delta^{1/2}\sum_{\beta=1}^{2}\left\|u_{d_\beta,\delta}-u^{ap}_{d_\beta,\delta}\right\|_{H^1\left(\Omega_{\delta,\beta}\right)}$$
$$+\left\|u_{\mathrm{ext},\delta}-u^{ap}_{\mathrm{ext},\delta}\right\|_{H^1\left(\widetilde{\Omega}_{\mathrm{ext},\delta}\right)}\leqslant c\delta^2.$$

Proof According to the Convergence Theorem, it is enough to estimate the error $U^{ap}_\delta - U^{(1)}_\delta$. Therefore, as in [41], we perform an asymptotic expansion for U^{ap}_δ which amounts to postulating the ansatz

$$U^{ap}_\delta=\sum_{j\geqslant 0}\delta^j w_j, \tag{6.30}$$

where $w_{j|\Omega_{\mathrm{ext}}}:=w_{\mathrm{ext},j}$ and $w_{j|\Omega_{\mathrm{int}}}:=w_{\mathrm{int},j}$, satisfy the recurrence relations

$$\begin{cases}\operatorname{div}\left(\sigma_{\mathrm{int}}\nabla w_{\mathrm{int},j}\right)+k^2_{\mathrm{int}}w_{\mathrm{int},j}=0 & \text{in } \Omega_{\mathrm{int}},\\ \operatorname{div}\left(\sigma_{\mathrm{ext}}\nabla w_{\mathrm{ext},j}\right)+k^2_{\mathrm{ext}}w_{\mathrm{ext},j}=0 & \text{in } \Omega_{\mathrm{ext}},\\ w_{\mathrm{int},j|\Gamma}-w_{\mathrm{ext},j|\Gamma}=\mathscr{A}\left(w_{\mathrm{int},j-1},w_{\mathrm{ext},j-1}\right) & \text{on } \Gamma,\\ \sigma_{\mathrm{int}}\partial_{\mathbf{n}}w_{\mathrm{int},j|\Gamma}-\sigma_{\mathrm{ext}}\partial_{\mathbf{n}}w_{\mathrm{ext},j|\Gamma}=\mathscr{B}\left(w_{\mathrm{int},j-1},w_{\mathrm{ext},j-1}\right) & \text{on } \Gamma,\\ \lim\limits_{|x|\to+\infty}|x|\left(\partial_{|x|}-ik_{\mathrm{ext}}\right)\left(w_{\mathrm{ext},j}-\delta_{0,j}u_{inc}\right)=0, & \end{cases}$$

with the convention that $w_{-1}=0$. A simple computation shows that the two first terms $(w_{\mathrm{int},0},w_{\mathrm{ext},0})$ and $(w_{\mathrm{int},1},w_{\mathrm{ext},1})$ coincide with the two first terms of (4.5) and (4.6).

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Furthermore, each term of (6.30) is bounded in H^1. Let $\mathscr{R}_w$ be the remainder made by truncating Series (6.30):

$$\mathscr{R}_{w|\Omega_{\rm int}} := \mathscr{R}_{{\rm int},w} := u^{ap}_{{\rm int},\delta} - w_{{\rm int},0} - \delta w_{{\rm int},1} - \delta^2 w_{{\rm int},2} - \delta^3 w_{{\rm int},3}$$

and

$$\mathscr{R}_{w|\Omega_{\rm ext}} := \mathscr{R}_{{\rm ext},w} := u^{ap}_{{\rm ext},\delta} - w_{{\rm ext},0} - \delta w_{{\rm ext},1} - \delta^2 w_{{\rm ext},2} - \delta^3 w_{{\rm ext},3}.$$

Then $\mathscr{R}_w$ is a solution of the following problem:

$$\begin{cases} \operatorname{div}\left(\sigma_{\rm int}\nabla\mathscr{R}_{{\rm int},w}\right) + k^2_{\rm int}\mathscr{R}_{{\rm int},w} = 0 & \text{in } \Omega_{\rm int}, \\ \operatorname{div}\left(\sigma_{\rm ext}\nabla\mathscr{R}_{{\rm ext},w}\right) + k^2_{\rm ext}\mathscr{R}_{{\rm ext},w} = 0 & \text{in } \Omega_{\rm ext}, \\ \mathscr{R}_{{\rm int},w|\Gamma} - \mathscr{R}_{{\rm ext},w|\Gamma} = \delta\mathscr{A}\left(\mathscr{R}_{{\rm int},w}, \mathscr{R}_{{\rm ext},w}\right) + \delta^4\mathscr{A}\left(w_{{\rm int},3}, w_{{\rm ext},3}\right) & \text{on } \Gamma, \\ \sigma_{\rm int}\partial_{\mathbf{n}}\mathscr{R}_{{\rm int},w|\Gamma} - \sigma_{\rm ext}\partial_{\mathbf{n}} w_{{\rm ext},j|\Gamma} = \delta\mathscr{B}\left(\mathscr{R}_{{\rm int},w}, \mathscr{R}_{{\rm ext},w}\right) & \\ \qquad\qquad + \delta^4\mathscr{B}\left(w_{{\rm int},3}, w_{{\rm ext},3}\right) & \text{on } \Gamma, \\ \lim\limits_{|x|\to+\infty} |x|\left(\partial_{|x|} - ik_{\rm ext}\right)\left(\mathscr{R}_{{\rm ext},w}\right) = 0, & \end{cases}$$

which gives, for all $v = (v_{\rm int}, v_{\rm ext})$ in $\mathbb{H}^1(\Omega)$,

$$\begin{aligned} &\sigma_{\rm int}\int_{\Omega_{\rm int}} \nabla\mathscr{R}_{{\rm int},w}.\nabla v_{\rm int}\; d\Omega_{\rm int} - k^2_{\rm int}\int_{\Omega_{\rm int}} \mathscr{R}_{{\rm int},w} v_{\rm int}\; d\Omega_{\rm int} \\ &+\sigma_{\rm ext}\int_{\widetilde{\Omega}_{\rm ext}} \nabla\mathscr{R}_{{\rm ext},w}.\nabla v_{\rm ext}\; d\widetilde{\Omega}_{\rm ext} - k^2_{\rm ext}\int_{\widetilde{\Omega}_{\rm ext}} \mathscr{R}_{{\rm ext},w} v_{\rm ext}\; d\widetilde{\Omega}_{\rm ext} \\ &-\frac{1}{2}\lambda_{\delta,1}\int_\Gamma \left(\mathscr{R}_{{\rm int},w|\Gamma} - \mathscr{R}_{{\rm ext},w|\Gamma}\right)\left(v_{{\rm int}|\Gamma} - v_{{\rm ext}|\Gamma}\right)\; d\Gamma \\ &-\delta\frac{1}{4}\left(\widetilde{k}^2_\delta - p_1 k^2_{\rm int} - p_2 k^2_{\rm ext}\right)\int_\Gamma \left(\mathscr{R}_{{\rm int},w|\Gamma} + \mathscr{R}_{{\rm ext},w|\Gamma}\right)\left(v_{{\rm int}|\Gamma} + v_{{\rm ext}|\Gamma}\right)\; d\Gamma \\ &-\delta\frac{1}{4}\left(\widetilde{\sigma}_\delta - p_1\sigma_{\rm int} - p_2\sigma_{\rm ext}\right)\int_\Gamma \left(\Delta_\Gamma\mathscr{R}_{{\rm int},w|\Gamma} + \Delta_\Gamma\mathscr{R}_{{\rm ext},w|\Gamma}\right)\left(v_{{\rm int}|\Gamma} + v_{{\rm ext}|\Gamma}\right)\; d\Gamma \\ &+\sigma_{\rm ext}\left\langle T\mathscr{R}_{{\rm ext},w|\partial\Omega}, v_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \\ &= \frac{1}{2}\delta^4\int_\Gamma \mathscr{B}\left(w_{{\rm int},3}, w_{{\rm ext},3}\right)\left(v_{{\rm int}|\Gamma} + v_{{\rm ext}|\Gamma}\right)\; d\Gamma \\ &+\frac{1}{2}\delta^4\int_\Gamma \lambda_{\delta,1}\mathscr{A}\left(w_{{\rm int},3}, w_{{\rm ext},3}\right)\left(v_{{\rm int}|\Gamma} - v_{{\rm ext}|\Gamma}\right)\; d\Gamma. \end{aligned}$$

Putting all terms of order 1 in δ on the right-hand side, we get

$$b_\delta\left(\mathscr{R}_w, v\right) = h_\delta\left(v\right),$$

where

$$h_\delta(v) := \delta\frac{1}{4}\left(\widetilde{k}_\delta^2 - p_1 k_{\text{int}}^2 - p_2 k_{\text{ext}}^2\right)\int_\Gamma \left(\mathscr{R}_{\text{int},w|\Gamma} + \mathscr{R}_{\text{ext},w|\Gamma}\right)\left(v_{\text{int}|\Gamma} + v_{\text{ext}|\Gamma}\right)\, d\Gamma$$
$$+\delta\frac{1}{4}\left(\widetilde{\sigma}_\delta - p_1\sigma_{\text{int}} - p_2\sigma_{\text{ext}}\right)\int_\Gamma \left(\Delta_\Gamma \mathscr{R}_{\text{int},w|\Gamma} + \Delta_\Gamma \mathscr{R}_{\text{ext},w|\Gamma}\right)\left(v_{\text{int}|\Gamma} + v_{\text{ext}|\Gamma}\right)\, d\Gamma$$
$$+\frac{1}{2}\delta^4\int_\Gamma \mathscr{B}\left(w_{\text{int},3}, w_{\text{ext},3}\right)\left(v_{\text{int}|\Gamma} + v_{\text{ext}|\Gamma}\right)\, d\Gamma$$
$$+\delta^4\int_\Gamma \lambda_{\delta,1}\mathscr{A}\left(w_{\text{int},3}, w_{\text{ext},3}\right)\left(v_{\text{int}|\Gamma} - v_{\text{ext}|\Gamma}\right)\, d\Gamma.$$

From Lemma 6.2, there exists a constant c independent of δ such that

$$\|\mathscr{R}_w\|_{\mathbb{H}^1(\Omega)} \leqslant c\delta^{-1/2}\|h_\delta\|_{(\mathbb{H}^1(\Omega))'}.$$

Hence, we obtain

$$\|\mathscr{R}_w\|_{\mathbb{H}^1(\Omega)} \leqslant c\left(\delta^{1/2}\|\mathscr{R}_w\|_{\mathbb{H}^1(\Omega)} + \delta^{5/2}\|w_3\|_{\mathbb{H}^1(\Omega)}\right),$$

so

$$\|\mathscr{R}_w\|_{\mathbb{H}^1(\Omega)} \leqslant \frac{\delta^{5/2}c}{\left(1 - c\delta^{1/2}\right)}\|w_3\|_{\mathbb{H}^1(\Omega)}.$$

Since δ is small enough, we have

$$\|\mathscr{R}_w\|_{\mathbb{H}^1(\Omega)} \leqslant c\delta^2\|w_3\|_{\mathbb{H}^1(\Omega)},$$

which gives the desired result. □

Remark 6.3 *There is a particularly interesting case when σ_{ext}, $\widetilde{\sigma}_\delta$ and σ_{int} are strictly positive constants satisfying $\sigma_{\text{int}} < \widetilde{\sigma} < \sigma_{\text{ext}}$ or $\sigma_{\text{ext}} < \widetilde{\sigma} < \sigma_{\text{int}}$, it corresponds to the case where the solution U_δ^{ap} is continuous when crossing Γ. Indeed, if we set*

$$\sigma_{\text{int}}\sigma_{\text{ext}} - p_1\sigma_{\text{ext}}\widetilde{\sigma}_\delta - p_2\sigma_{\text{int}}\widetilde{\sigma}_\delta = 0,$$

we obtain

$$p_1 = \frac{\sigma_{\text{int}}\left(\sigma_{\text{ext}} - \widetilde{\sigma}\right)}{\widetilde{\sigma}\left(\sigma_{\text{ext}} - \sigma_{\text{int}}\right)} \text{ and } p_2 = \frac{\sigma_{\text{ext}}\left(\widetilde{\sigma} - \sigma_{\text{int}}\right)}{\widetilde{\sigma}\left(\sigma_{\text{ext}} - \sigma_{\text{int}}\right)}.$$

Then, the transmission conditions (6.2) become

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = 0, \\ \sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},\delta|\Gamma}^{ap} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},\delta|\Gamma}^{ap} = \delta\dfrac{\left(\sigma_{\text{ext}} - \widetilde{\sigma}\right)\left(\sigma_{\text{int}} - \widetilde{\sigma}\right)}{\widetilde{\sigma}}\Delta_\Gamma u_{\text{int},\delta}^{ap} \\ \quad +\delta\dfrac{\widetilde{\sigma}\widetilde{k}^2\left(\sigma_{\text{ext}} - \sigma_{\text{int}}\right) - \sigma_{\text{int}}k_{\text{int}}^2\left(\sigma_{\text{ext}} - \widetilde{\sigma}\right) - \sigma_{\text{ext}}k_{\text{ext}}^2\left(\widetilde{\sigma} - \sigma_{\text{int}}\right)}{\widetilde{\sigma}\left(\sigma_{\text{ext}} - \sigma_{\text{int}}\right)}u_{\text{ext},\delta|\Gamma}^{ap}. \end{cases}$$

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Problem $\left(\mathscr{P}_\delta^{ap}\right)$ *is equivalent to the boundary equation*

$$\begin{aligned} B_\delta\omega := & -\Delta_\Gamma\omega + \frac{\widetilde{\sigma}\sigma_{\text{int}}}{\delta\left(\sigma_{\text{ext}}-\widetilde{\sigma}\right)\left(\sigma_{\text{int}}-\widetilde{\sigma}\right)}S_{\text{int}}\omega + \frac{\widetilde{\sigma}\sigma_{\text{ext}}}{\delta\left(\sigma_{\text{ext}}-\widetilde{\sigma}\right)\left(\sigma_{\text{int}}-\widetilde{\sigma}\right)}S_{\text{ext}}\omega \\ & + \frac{\sigma_{\text{int}}k_{\text{int}}^2\left(\sigma_{\text{ext}}-\widetilde{\sigma}\right)+\sigma_{\text{ext}}k_{\text{ext}}^2\left(\widetilde{\sigma}-\sigma_{\text{int}}\right)-\widetilde{\sigma}\widetilde{k}^2\left(\sigma_{\text{ext}}-\sigma_{\text{int}}\right)}{\left(\sigma_{\text{ext}}-\widetilde{\sigma}\right)\left(\sigma_{\text{int}}-\widetilde{\sigma}\right)\left(\sigma_{\text{ext}}-\sigma_{\text{int}}\right)}\omega \\ = & \frac{\sigma_{\text{ext}}\widetilde{\sigma}}{\delta\left(\sigma_{\text{ext}}-\widetilde{\sigma}\right)\left(\sigma_{\text{int}}-\widetilde{\sigma}\right)}\partial_{\mathbf{n}}u_{inc|\Gamma} + \frac{\sigma_{\text{ext}}\widetilde{\sigma}}{\delta\left(\sigma_{\text{ext}}-\widetilde{\sigma}\right)\left(\sigma_{\text{int}}-\widetilde{\sigma}\right)}S_{\text{ext}}u_{inc|\Gamma} \text{ on } \Gamma, \end{aligned}$$

where ω *is the trace of* $u_{\text{int},\delta}^{ap}$ *on the surface* Γ*. As above, the existence and uniqueness are obtained with a Fredholm alternative, and similar error estimates can be shown.*

7 Extension to thin layer with high magnetic permittivity

In this section, we consider the case of a high value of magnetic permittivity of the domain Ω_δ (cf. e.g. [25, 34] for similar problems). More precisely, we consider the case where $\widetilde{\sigma}_\delta := \widetilde{\sigma}/\delta$ and $\widetilde{k}_\delta^2 := \widetilde{k}^2/\delta$ where $\widetilde{\sigma}$ is a strictly positive constant and $\widetilde{k}^2$ is a complex number with strictly positive real part and positive imaginary part.

The asymptotic analysis can be done in the same way and we are thus going to only give the approximate transmission conditions without doing all computations. Although the derivation of these new conditions can be done without additional difficulties, the uniform stability estimate, which is the basis for optimal error estimates, cannot be proved as Theorem 2.2. Actually, the singularity of both the contrast and the refractive index of the thin layer yield a limiting equation that involves Ventcel-like transmission condition. All the well-posedness and regularity results used below to get such a uniform stability estimate are postponed to the appendix. The non-standard nature of transmission conditions of the Ventcel problem lead us to introduce the Sobolev spaces $H^{1,1}\left(\Omega_{\text{int}}\right), H_{loc}^{1,1}\left(\overline{\Omega_{\text{ext}}}\right)$ and $H_\Gamma^1\left(\Omega\right)$ defined by

$$\begin{aligned} H^{1,1}\left(\Omega_{\text{int}}\right) & := \left\{v \in H^1\left(\Omega_{\text{int}}\right),\ v_{|\Gamma} \in H^1\left(\Gamma\right)\right\}, \\ H_\Gamma^1\left(\Omega\right) & := \left\{v \in H^1\left(\Omega\right),\ v_{|\Gamma} \in H^1\left(\Gamma\right)\right\}, \\ H_{loc}^{1,1}\left(\overline{\Omega_{\text{ext}}}\right) & := \left\{v \in H_{loc}^1\left(\overline{\Omega_{\text{ext}}}\right),\ v_{|\Gamma} \in H^1\left(\Gamma\right)\right\}, \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^3$ containing Ω_{int} (cf. Figure 4), equipped with their natural norms and semi-norm.

This section is then organized as follows, we first prove the uniform stability estimate and give next the first-order transmission conditions to take into account the effect of the thin layer.

7.1 Uniform stability estimate

We prove here the uniform stability result for the high-permittivity case.

Theorem 7.1 (Uniform stability) *If* $l_\delta \in \left(H^1(\Omega)\right)'$*, then Problem (2.5) admits a unique solution in* $H^1(\Omega)$*. Furthermore, there exists a positive constant* c *independent of* δ *such*

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that

$$\|u_\delta\|_{H^1(\Omega)} \leqslant c\,\|l_\delta\|_{(H^1(\Omega))'}. \tag{7.1}$$

Proof We recall below the definition of the bilinear form

$$a_\delta(u_\delta, v) := \int_\Omega \sigma_\delta \nabla u_\delta . \nabla \overline{v} - k_\delta^2 u_\delta \overline{v}\ d\Omega + \sigma_{\text{ext}} \left\langle T u_{\delta|\partial\Omega}, \overline{v}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}, \tag{7.2}$$

and we need to prove that

$$\|u_\delta\|_{H^1(\Omega)} \leqslant c \sup_{v\in H^1(\Omega)} \frac{|a_\delta(u_\delta, v)|}{\|v\|_{H^1(\Omega)}}. \tag{7.3}$$

To do so, we proceed by contradiction, assuming there exist sequences $(\delta_n)_{n\geqslant 0}$ and $\left(u_{\delta_n}\right)_{n\geqslant 0}$ such that

$$\lim_{n\to+\infty} \delta_n = 0,\ \|u_{\delta_n}\|_{H^1(\Omega)} = 1,\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \sup_{\|\varphi\|_{H^1(\Omega)}=1} |a_{\delta_n}(u_{\delta_n}, \varphi)| = 0. \tag{7.4}$$

In the following, we assume that there exist two positive constants ε et δ_1 such that

$$\delta_n < \varepsilon < \delta_1 \leqslant \delta_0. \tag{7.5}$$

Note that this can be done at least by extracting a subsequence of $(\delta_n)_{n\geqslant 0}$. We show in three steps that there exists a subsequence of $\left(u_{\delta_n}\right)$ such that $\lim_{n\to+\infty} \|u_{\delta_n}\|_{H^1(\Omega)} = 0$, which will lead to a contradiction.

Step 1: *There exists a subsequence* $\left(u_{\delta_n}\right)_{n\geqslant 0}$ *such that* $\left\|u_{d_\beta,\delta_n}\right\|_{H^1\left(\Omega_{\delta_n,\beta}\right)} \leqslant c\delta_n^{1/2}$.

From (7.4) and Rellich's theorem, we can extract a subsequence of $\left(u_{\delta_n}\right)_{n\geqslant 0}$, still denoted by $\left(u_{\delta_n}\right)_{n\geqslant 0}$, such that

$$\begin{cases} u_{\delta_n} \to u_0 \text{ in } L^2(\Omega), \\ u_{\delta_n} \rightharpoonup u_0 \text{ in } H^1(\Omega). \end{cases} \tag{7.6}$$

Furthermore, for all v in $\mathscr{C}^\infty(\overline{\Omega})$, $a_\delta(.,.)$ can be written as

$$\begin{aligned} a_\delta(u_\delta, v) = & \int_\Omega \sigma_0 \nabla u_\delta . \nabla \overline{v} - k_0^2 u_\delta \overline{v}\ d\Omega + \sigma_{\text{ext}} \left\langle T u_{\delta|\partial\Omega}, \overline{v}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \\ & - \int_{\Omega_{\delta,1}} \sigma_{\text{int}} \nabla u_{d_1,\delta} . \nabla \overline{v_{d_1}}\ d\Omega_{\delta,1} + \int_{\Omega_{\delta,1}} k_{\text{int}}^2 u_{d_1,\delta} \overline{v_{d_1}}\ d\Omega_{\delta,1} - \int_{\Omega_{\delta,2}} \sigma_{\text{ext}} \nabla u_{d_2,\delta} . \nabla \overline{v_{d_2}}\ d\Omega_{\delta,2} \\ & + \int_{\Omega_{\delta,2}} k_{\text{ext}}^2 u_{d_2,\delta} \overline{v_{d_2}}\ d\Omega_{\delta,2} + \int_{\Omega_\delta} \frac{\widetilde{\sigma}}{\delta} \nabla u_{d,\delta} . \nabla \overline{v_d}\ d\Omega_\delta - \int_{\Omega_\delta} \frac{\widetilde{k}^2}{\delta} u_{d,\delta} \overline{v_d}\ d\Omega_\delta, \end{aligned} \tag{7.7}$$

with

$$\begin{cases} \sigma_0 := \sigma_{\text{int}} \chi_{\Omega_{\text{int}}}(x) + \sigma_{\text{ext}} \chi_{\Omega_{\text{ext}}}(x), \\ k_0^2 := k_{\text{int}}^2 \chi_{\Omega_{\text{int}}}(x) + k_{\text{ext}}^2 \chi_{\Omega_{\text{ext}}}(x), \end{cases} \tag{7.8}$$

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and $u_{d_\beta,\delta}$ and v_{d_β} are respectively the restriction of u_δ and v to the domain $\Omega_{\delta,\beta}$. Applying the Cauchy–Schwarz and triangular inequalities, we get

$$\left|\int_{\Omega_{\delta,1}} \sigma_{\text{int}}\nabla u_{d_1,\delta}.\nabla\overline{v_{d_1}} - k_{\text{int}}^2 u_{d_1,\delta}\overline{v_{d_1}}\ d\Omega_{\delta,1}\right| \leqslant c\,\|u_{d_1,\delta}\|_{H^1(\Omega_{\delta,1})}\,\|v_{d_1}\|_{W^{1,\infty}(\Omega_{\delta,1})}\sqrt{|\Omega_{\delta,1}|}$$
$$\leqslant c\,\|u_\delta\|_{H^1(\Omega)}\,\|v\|_{W^{1,\infty}(\Omega)}\sqrt{|\Omega_{\delta,1}|}.$$

Using $\|u_{\delta_n}\|_{H^1(\Omega)} = 1$ and that $|\Omega_{\delta_n,1}| = c\delta_n$, we infer

$$\lim_{n\to+\infty}\left|\int_{\Omega_{\delta_n,1}} \sigma_{\text{int}}\nabla u_{d_1,\delta}.\nabla\overline{v_{d_1}} - k_{\text{int}}^2 u_{d_1,\delta}\overline{v_{d_1}}\ d\Omega_{\delta_n,1}\right| = 0,\ \forall v\in\mathscr{C}^\infty(\overline{\Omega}). \tag{7.9}$$

Similarly, we show that

$$\lim_{n\to+\infty}\left|\int_{\Omega_{\delta_n,2}} \sigma_{\text{ext}}\nabla u_{d_2,\delta}.\nabla\overline{v_{d_2}} - k_{\text{ext}}^2 u_{d_2,\delta}\overline{v_{d_2}}\ d\Omega_{\delta_n,2}\right| = 0,\ \forall v\in\mathscr{C}^\infty(\overline{\Omega}). \tag{7.10}$$

As a consequence, we infer from (7.4)–(7.10)

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\Omega_{\delta_n}} \frac{\widetilde{\sigma}}{\delta_n}\nabla u_{d,\delta_n}.\nabla\overline{v_d} - \frac{\widetilde{k}^2}{\delta_n}u_{d,\delta_n}\overline{v_d}\ d\Omega_{\delta_n}\\
&= \lim_{n\to+\infty}\Bigg[a_{\delta_n}\left(u_{\delta_n},v\right) - \int_\Omega \sigma_0\nabla u_{\delta_n}.\nabla\overline{v} - k_0^2 u_{\delta_n}\overline{v}\ dx\\
&-\sigma_{\text{ext}}\left\langle Tu_{\delta_n|\partial\Omega},\overline{v}_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} + \int_{\Omega_{\delta,1}}\sigma_{\text{int}}\nabla u_{d_1,\delta}.\nabla\overline{v_{d_1}} - k_{\text{int}}^2 u_{d_1,\delta}\overline{v_{d_1}}\ d\Omega_{\delta,1}\\
&+\int_{\Omega_{\delta_n,2}}\sigma_{\text{ext}}\nabla u_{d_2,\delta}.\nabla\overline{v_{d_2}} - k_{\text{ext}}^2 u_{d_2,\delta}\overline{v_{d_2}}\ d\Omega_{\delta_n,2}\Bigg]\\
&= -\int_\Omega \sigma_0\nabla u_0.\nabla\overline{v} - k_0^2 u_0\overline{v}\ dx - \sigma_{\text{ext}}\left\langle Tu_{0|\partial\Omega},\overline{v}_{|\partial\Omega}\right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)},\quad \forall v\in\mathscr{C}^\infty(\overline{\Omega}).
\end{aligned}$$

As the right-hand side is independent of δ, one has

$$\int_{\Omega_{\delta_n}} \frac{\widetilde{\sigma}}{\delta_n}\nabla u_{d,\delta_n}.\nabla\overline{v_d} - \frac{\widetilde{k}^2}{\delta_n}u_{d,\delta_n}\overline{v_d}\ d\Omega_{\delta_n} = O\left(1\right),\quad \forall v\in\mathscr{C}^\infty(\overline{\Omega}), \tag{7.11}$$

and by density, we conclude that the equality is true for all v in $H^1(\Omega)$.

Taking now $v\in\mathscr{C}^\infty(\overline{\Omega})$ one has for all $x=\psi_\beta(m,\eta_\beta)$ in $\Omega_{\delta,\beta}$,

$$v_{d_\beta}(x) = \widetilde{v}_{d_\beta}\left(m,\eta_\beta\right) = \widetilde{v}_{d_\beta}(m,0) + \int_0^{\eta_\beta}\partial_{\eta_\beta}\widetilde{v}_{d_\beta}(m,\lambda)\ d\lambda, \tag{7.12}$$

so

$$\left|\widetilde{v}_{d_\beta}\left(m,\eta_\beta\right)\right|^2 \leqslant 2\left|\widetilde{v}_{d_\beta}(m,0)\right|^2 + 2\left|\int_0^{\eta_\beta}\partial_{\eta_\beta}\widetilde{v}_{d_\beta}(m,\lambda)\ d\lambda\right|^2. \tag{7.13}$$

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By integrating on Γ, we find

$$\int_{\Gamma}\left|\widetilde{v}_{d_\beta}\left(m,\eta_\beta\right)\right|^2 d\Gamma \leqslant 2\int_{\Gamma}\left|\widetilde{v}_{d_\beta}\left(m,0\right)\right|^2\ d\Gamma + 2\int_{\Gamma}\int_0^{p_\beta\delta}\left|\partial_{\eta_\beta}\widetilde{v}_{d_\beta}\left(m,\lambda\right)\right|^2\ d\lambda d\Gamma, \tag{7.14}$$

so

$$\int_{\Gamma}\left|\widetilde{v}_{d_\beta}\left(m,\eta_\beta\right)\right|^2 d\Gamma \leqslant 2\left\|v_{d_\beta|\Gamma}\right\|^2_{L^2(\Gamma)} + 2\left\|\partial_{\eta_\beta}\widetilde{v}_{d_\beta}\right\|^2_{L^2\left(\Omega_{\delta,\beta}\right)}. \tag{7.15}$$

Integrating a second time with respect with η_β, we obtain

$$\int_0^{p_\beta\delta}\int_{\Gamma}\left|\widetilde{v}_{d_\beta}\left(m,\eta_\beta\right)\right|^2 d\Gamma\ d\eta_\beta \leqslant c\delta\left(\left\|v_{d_\beta|\Gamma}\right\|^2_{L^2(\Gamma)} + \left\|\partial_{\eta_\beta}\widetilde{v}_{d_\beta}\right\|^2_{L^2\left(\Omega_{\delta,\beta}\right)}\right), \tag{7.16}$$

so

$$\left\|v_{d_\beta}\right\|_{L^2\left(\Omega_{\delta,\beta}\right)} \leqslant c\delta^{1/2}\left\|v\right\|_{H^1(\Omega)}. \tag{7.17}$$

Since $v \in \mathscr{C}^\infty(\overline{\Omega})$ is arbitrary, we conclude, by density, that the last estimate is true for all v in $H^1(\Omega)$. Hence, for $v = u_{\delta_n}$, we have

$$\left\|u_{d_\beta,\delta_n}\right\|_{L^2\left(\Omega_{\delta_n,\beta}\right)} \leqslant c\delta_n^{1/2}. \tag{7.18}$$

Using both (7.11) and (7.18), one gets

$$\left\|u_{d_\beta,\delta_n}\right\|_{H^1\left(\Omega_{\delta_n,\beta}\right)} \leqslant c\delta_n^{1/2}, \tag{7.19}$$

which proves the first claim.

Step 2: *We show that $u_0 = 0$ in Ω.*

In view of (7.19) and (3.4), one has

$$\begin{aligned}\left\|u_\delta\right\|^2_{H^1(\Omega_{\delta,\beta})} = p_\beta\delta\int_{\Omega^\beta} J_{\delta,\beta}^{-2}\left|\nabla_\Gamma u_\delta^{[\beta]}\right|^2 \det J_{\delta,\beta}\ d\Gamma\, ds_\beta \\ + p_\beta^{-1}\delta^{-1}\int_{\Omega^\beta}\left|\partial_{s_\beta}u_\delta^{[\beta]}\right|^2 \det J_{\delta,\beta}\ d\Gamma\, ds_\beta + \delta p_\beta\int_{\Omega^\beta}\left|u_\delta^{[\beta]}\right|^2\ \det J_{\delta,\beta}\ d\Gamma\, ds_\beta,\end{aligned}$$

we then infer the estimates

$$\left\|\nabla_\Gamma u_{\delta_n}^{[\beta]}\right\|_{\mathbb{L}^2\left(\Omega^\beta\right)} \leqslant c, \tag{7.20}$$

$$\left\|\partial_{s_\beta}u_{\delta_n}^{[\beta]}\right\|_{L^2\left(\Omega^\beta\right)} \leqslant c\delta_n, \tag{7.21}$$

$$\left\|u_{\delta_n}^{[\beta]}\right\|_{L^2\left(\Omega^\beta\right)} \leqslant c. \tag{7.22}$$

To compute the limiting equation, we introduce X as the Hilbert space defined by

$$\begin{aligned}X := \Big\{V := \left(v, v^{[1]}, v^{[2]}\right) \in H^1(\Omega)\times H^1\left(I_1, H^1\left(\Gamma\right)\right)\times H^1\left(I_2, H^1\left(\Gamma\right)\right)\ ; \\ v^{[\beta]}\left(m,0\right) = v_{|\Gamma},\ \beta = 1,2\Big\},\end{aligned}$$

It follows from (7.4) and (7.20)–(7.22) that the sequence $\left(U_{\delta_n}\right)_n$ defined by $U_{\delta_n} := \left(u_{\delta_n}, u_{\delta_n}^{[1]}, u_{\delta_n}^{[2]}\right)$ is bounded in X. Therefore, there exists a subsequence of $\left(U_{\delta_n}\right)_{n\geqslant 0}$, still

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denoted by $\left(U_{\delta_n}\right)_{n\geqslant 0}$, such that $U_{\delta_n} \rightharpoonup U_0 := \left(u_0, \omega_0^{[1]}, \omega_0^{[2]}\right)$ in X . Inequality (7.21) implies $\partial_{s_\beta}\omega_0^{[\beta]} = 0$ resulting in

$$\omega_0^{[\beta]}(m, s_\beta) = u_{0|\Gamma}, \ \forall (m, s_\beta) \in \Omega^\beta. \tag{7.23}$$

Let now v be a smooth function v in $H^1(\Omega)$, $a_\delta(.,.)$ becomes

$$\begin{aligned} a_\delta(u_\delta, v) = &\int_\Omega \sigma_0 \nabla u_\delta . \nabla \overline{v} - k_0^2 u_\delta \overline{v} \; d\Omega + \sigma_{\text{ext}} \left\langle T u_{\delta|\partial\Omega}, \overline{v}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \\ &- \int_{\Omega_{\delta,1}} \sigma_{\text{int}} \nabla u_{d_1,\delta}.\nabla \overline{v_{d_1}} \; d\Omega_{\delta,1} + \int_{\Omega_{\delta,1}} k_{\text{int}}^2 u_{d_1,\delta} \overline{v_{d_1}} \; d\Omega_{\delta,1} - \int_{\Omega_{\delta,2}} \sigma_{\text{ext}} \nabla u_{d_2,\delta}.\nabla \overline{v_{d_2}} \; d\Omega_{\delta,2} \\ &+ \int_{\Omega_{\delta,2}} k_{\text{ext}}^2 u_{d_2,\delta} \overline{v_{d_2}} \; d\Omega_{\delta,2} + \sum_{\beta=1}^{2} \left[p_\beta \widetilde{\sigma} \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u_\delta^{[\beta]} . \nabla_\Gamma \overline{v^{[\beta]}} \det J_{\delta,\beta} \; d\Gamma \, ds_\beta \right. \\ &\left. + p_\beta^{-1} \delta^{-2} \widetilde{\sigma} \int_{\Omega^\beta} \partial_{s_\beta} u_\delta^{[\beta]} \partial_{s_\beta} \overline{v^{[\beta]}} \det J_{\delta,\beta} \; d\Gamma \, ds_\beta - p_\beta \widetilde{k}^2 \int_{\Omega^\beta} u_\delta^{[\beta]} \overline{v^{[\beta]}} \; \det J_{\delta,\beta} \; d\Gamma \, ds_\beta \right]. \end{aligned} \tag{7.24}$$

Now choosing a smooth v in $H(\Omega)$ (see Lemma A.1) leads to

$$\begin{aligned} &p_\beta \widetilde{\sigma} \int_{\Omega^\beta} J_{\delta_n,\beta}^{-2} \nabla_\Gamma u_{\delta_n}^{[\beta]} . \nabla_\Gamma \overline{v^{[\beta]}} \det J_{\delta_n,\beta} \; d\Gamma \, ds_\beta + p_\beta^{-1} \delta_n^{-2} \widetilde{\sigma} \int_{\Omega^\beta} \partial_{s_\beta} u_{\delta_n}^{[\beta]} \partial_{s_\beta} \overline{v^{[\beta]}} \det J_{\delta_n,\beta} \; d\Gamma \, ds_\beta \\ &\quad - p_\beta \widetilde{k}^2 \int_{\Omega^\beta} u_{\delta_n}^{[\beta]} \overline{v^{[\beta]}} \; \det J_{\delta_n,\beta} \; d\Gamma \, ds_\beta \\ &\quad \underset{n\to+\infty}{\longrightarrow} p_\beta \widetilde{\sigma} \int_{\Omega^\beta} \nabla_\Gamma \omega_0^{[\beta]} . \nabla_\Gamma \overline{v^{[\beta]}} \; d\Gamma \, ds_\beta - p_\beta \widetilde{k}^2 \int_\Gamma \omega_0^{[\beta]} \overline{v^{[\beta]}} \; d\Gamma \, ds_\beta ; \end{aligned} \tag{7.25}$$

otherwise, as $\omega_0^{[\beta]}$ is independent of s_β, we would have

$$p_\beta \widetilde{\sigma} \int_{\Omega^\beta} \nabla_\Gamma \omega_0^{[\beta]} . \nabla_\Gamma \overline{v^{[\beta]}} \; d\Gamma \, ds_\beta = p_\beta \widetilde{\sigma} \int_\Gamma \nabla_\Gamma u_{0|\Gamma} . \nabla_\Gamma \overline{v_{|\Gamma}} \; d\Gamma \tag{7.26}$$

$$- p_\beta \widetilde{k}^2 \int_{\Omega^\beta} \omega_0^{[\beta]} \overline{v^{[\beta]}} \; d\Gamma \, ds_\beta = - p_\beta \widetilde{k}^2 \int_\Gamma u_{0|\Gamma} \overline{v_{|\Gamma}} \; d\Gamma . \tag{7.27}$$

Since $\omega_0^{[\beta]}$ is independent of s_β and $U_0 \in X$, $\omega_0^{[\beta]}(m, s_\beta) = \omega_0^{[\beta]}(m, 0) = u_{0|\Gamma} \in H^1(\Gamma)$, which gives meaning to the last two equalities. As a consequence, in view of (7.4) and (7.25)–(7.27), we obtain

$$\begin{aligned} 0 &= \lim_{n\to+\infty} a_{\delta_n}\left(u_{\delta_n}, v\right) \\ &= \int_\Omega \sigma_0 \nabla u_0 . \nabla \overline{v} - k_0^2 u_0 \overline{v} \; d\Omega + \sigma_{\text{ext}} \left\langle T u_{0|\partial\Omega}, \overline{v}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \\ &\quad + \widetilde{\sigma} \int_\Gamma \nabla_\Gamma u_{0|\Gamma} . \nabla_\Gamma \overline{v_{|\Gamma}} \; d\Gamma - \widetilde{k}^2 \int_\Gamma u_{0|\Gamma} \overline{v_{|\Gamma}} \; d\Gamma . \end{aligned} \tag{7.28}$$

By density (Lemma A.1), we deduce that (7.28) is true for all v in $H_\Gamma^1(\Omega)$. It follows from Theorem A.1 that the problem: Find u_0 in $H_\Gamma^1(\Omega)$ satisfying (7.28), $\forall v \in H_\Gamma^1(\Omega)$, is well posed, moreover $u_0 = 0$.

Step 3: *Getting the contradiction.*

To obtain the contradiction, we show that $\lim_{n\to+\infty} \|u_{\delta_n}\|_{H^1(\Omega)} = 0$. Since u_0 is uniquely determined, the whole sequence $(U_\delta)_\delta$ converges to $U_0 = 0$, then the whole sequence $(u_\delta)_\delta$ converges to $u_0 = 0$ in $L^2(\Omega)$. It only remains to show that $\lim_{n\to+\infty} \|\nabla u_{\delta_n}\|_{\mathbb{L}^2(\Omega)} = 0$. We have

$$
\begin{aligned}
\|\nabla u_{\delta_n}\|^2_{\mathbb{L}^2(\Omega)} &\leqslant c \int_\Omega \sigma_{\delta_n} |\nabla u_{\delta_n}|^2 \, d\Omega \\
&= c\Re\Big\{ a_{\delta_n}(u_{\delta_n}, u_{\delta_n}) + \int_\Omega k^2_{\delta_n} |u_{\delta_n}|^2 \, d\Omega \\
&\quad - \sigma_{\text{ext}} \left\langle T u_{\delta_n|\partial\Omega}, \overline{u_{\delta_n}}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \Big\}.
\end{aligned}
$$

Using Lemma 2.1, we infer

$$
\begin{aligned}
\|\nabla u_{\delta_n}\|^2_{\mathbb{L}^2(\Omega)} &\leqslant c\Re\Big\{ a_{\delta_n}(u_{\delta_n}, u_{\delta_n}) + \int_\Omega k^2_{\delta_n} |u_{\delta_n}|^2 \, d\Omega \\
&\quad - \sigma_{\text{ext}} \left\langle K u_{\delta_n|\partial\Omega}, \overline{u_{\delta_n}}_{|\partial\Omega} \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \Big\}.
\end{aligned}
$$

As

$$
\begin{aligned}
\int_\Omega k^2_{\delta_n} |u_{\delta_n}|^2 \, d\Omega &= \int_{\Omega_{\text{int},\delta_n}} k^2_{\text{int}} |u_{\text{int},\delta_n}|^2 \, d\Omega_{\text{int},\delta_n} + \int_{\Omega_{\text{ext},\delta_n}} k^2_{\text{ext}} |u_{\text{ext},\delta_n}|^2 \, d\Omega_{\text{ext},\delta_n} \\
&\quad + \delta_n^{-1} \sum_{\beta=1}^{2} \int_{\Omega_{\delta_n,\beta}} \widetilde{k}^2 \left|u_{d_\beta,\delta_n}\right|^2 \, d\Omega_{\delta_n,\beta} \\
&= \int_{\Omega_{\text{int},\delta_n}} k^2_{\text{int}} |u_{\text{int},\delta_n}|^2 \, d\Omega_{\text{int},\delta_n} + \int_{\Omega_{\text{ext},\delta_n}} k^2_{\text{ext}} |u_{\text{ext},\delta_n}|^2 \, d\Omega_{\text{ext},\delta_n} \\
&\quad + \sum_{\beta=1}^{2} \int_{\Omega^\beta} \widetilde{k}^2 \left|u^{[\beta]}_{\delta_n}\right|^2 \det J_{\delta_n,\beta} \, d\Gamma \, ds_\beta \\
&= \int_\Omega k_0^2 |u_{\delta_n}|^2 \, d\Omega + \sum_{\beta=1}^{2} \int_{\Omega^\beta} \widetilde{k}^2 \left|u^{[\beta]}_{\delta_n}\right|^2 \det J_{\delta_n,\beta} \, d\Gamma \, ds_\beta \\
&\quad - \delta_n \int_{\Omega^1} k^2_{\text{int}} \left|u^{[1]}_{\delta_n}\right|^2 \det J_{\delta,1} \, d\Gamma \, ds_1 - \int_{\Omega^2} k^2_{\text{ext}} \left|u^{[2]}_{\delta_n}\right|^2 \det J_{\delta_n,2} \, d\Gamma \, ds_2,
\end{aligned}
$$

$u_{\delta_n} \underset{n\to+\infty}{\longrightarrow} u_0 = 0$ in $L^2(\Omega)$ and $u^{[\beta]}_{\delta_n} \underset{n\to+\infty}{\longrightarrow} u^{[\beta]}_{\text{int},0} = u_{0|\Gamma} = 0$ in $L^2\left(I_\beta, H^1(\Gamma)\right)$, it follows $\int_\Omega k^2_{\delta_n} |u_{\delta_n}|^2 \, d\Omega \underset{n\to+\infty}{\longrightarrow} 0$. Since K is compact and $u_{\delta_n} \rightharpoonup 0$ in $H^1(\Omega)$, $\langle K u_{\delta_n|\partial\Omega}, \overline{u_{\delta_n}}_{|\partial\Omega}\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \underset{n\to+\infty}{\longrightarrow} 0$. Finally, the hypothesis $\lim_{n\to+\infty} \Re\left[a_{\delta_n}(u_{\delta_n}, u_{\delta_n})\right] = 0$ leads to $\lim_{n\to+\infty} \|\nabla u_{\delta_n}\|_{\mathbb{L}^2(\Omega)} = 0$ contradicting $\|u_{\delta_n}\|_{H^1(\Omega)} = 1$. □

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7.2 Approximate transmission conditions of order 2

Using the same techniques as in the previous section, we derive an asymptotic expansion of the total field u_δ and establish a convergence theorem. We now give the first two terms of the asymptotic expansion, denoted by, $\left(u_{\text{int},n}, u_{\text{ext},n}\right)_{0\leqslant n\leqslant 1}$ satisfy the following problem:

$$\begin{cases} \operatorname{div}\left(\sigma_{\text{ext}}\nabla u_{\text{ext},n}\right) + k_{\text{ext}}^2 u_{\text{ext},n} = 0 & \text{in } \Omega_{\text{ext}}, \\ \operatorname{div}\left(\sigma_{\text{int}}\nabla u_{\text{int},n}\right) + k_{\text{int}}^2 u_{\text{int},n} = 0 & \text{in } \Omega_{\text{int}}, \\ \lim\limits_{|x|\to+\infty} |x| \left(\partial_{|x|} - ik_{\text{ext}}\right)\left(u_{\text{ext},n} - \delta_{0,n} u_{inc}\right) = 0, & \end{cases}$$

where $\delta_{0,n}$ indicates the Kronecker symbol. The transmission conditions on Γ are described below:

At order 0.

$$u_{\text{int},0|\Gamma} = u_{\text{ext},0|\Gamma}, \tag{7.29}$$

$$\sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},0|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma} = \widetilde{\sigma}\Delta_\Gamma u_{\text{int},0|\Gamma} + \widetilde{k}^2 u_{\text{int},0|\Gamma}. \tag{7.30}$$

At order 1.

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1\partial_{\mathbf{n}}u_{\text{int},0|\Gamma} + p_2\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}, \tag{7.31}$$

$$\begin{aligned} \sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},1|\Gamma} &= p_1\widetilde{\sigma}\Delta_\Gamma u_{\text{int},1|\Gamma} + p_2\widetilde{\sigma}\Delta_\Gamma u_{\text{ext},1|\Gamma} + p_1\widetilde{k}^2 u_{\text{int},1|\Gamma} \\ &+ p_2\widetilde{k}^2 u_{\text{ext},1|\Gamma} - p_2\sigma_{\text{ext}}\Delta_\Gamma u_{\text{ext},0|\Gamma} - p_1\sigma_{\text{int}}\Delta_\Gamma u_{\text{int},0|\Gamma} \\ &- p_1^2\widetilde{k}^2\mathscr{H} u_{\text{int},0|\Gamma} - p_1 k_{\text{int}}^2 u_{\text{int},0|\Gamma} + p_2^2\widetilde{k}^2\mathscr{H} u_{\text{ext},0|\Gamma} \\ &- p_2 k_{\text{ext}}^2 u_{\text{ext},0|\Gamma} + p_2^2\widetilde{\sigma} div_\Gamma\left((\mathscr{H}I - \mathscr{R})\nabla_\Gamma u_{\text{ext},0|\Gamma}\right) \\ &- p_1^2\widetilde{\sigma} div_\Gamma\left((\mathscr{H}I - \mathscr{R})\nabla_\Gamma\right) u_{\text{int},0|\Gamma} + p_2^2\widetilde{\sigma}\Delta_\Gamma\left(\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}\right) \\ &+ p_2^2\widetilde{k}^2\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma} - p_1^2\widetilde{\sigma}\Delta_\Gamma\left(\partial_{\mathbf{n}}u_{\text{int},0|\Gamma}\right) - p_1^2\widetilde{k}^2\partial_{\mathbf{n}}u_{\text{int},0|\Gamma}. \end{aligned} \tag{7.32}$$

Moreover, $\left(u_n^{[\beta]}\right)_{0\leqslant n\leqslant 1}$ is determined by the following expressions:

$$\begin{aligned} u_0^{[1]}(m,s_1) &= u_0^{[2]}(m,s_2) = u_{\text{int},0|\Gamma} = u_{\text{ext},0|\Gamma}, \\ u_1^{[1]}(m,s_1) &= u_1^{[2]}(m,s_2) = u_{\text{ext},1|\Gamma} + p_2\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma} \\ &= u_{\text{int},1|\Gamma} - p_1\partial_{\mathbf{n}}u_{\text{int},0|\Gamma}, \quad \forall (m,s_\beta) \in \Omega^\beta. \end{aligned}$$

We follow the approach used in Section 6 to derive an approximate problem of order 1. The proof of the uniqueness of the solution is then going to encounter two difficulties. The first comes from terms $\Delta_\Gamma\left(\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}\right)$ and $\Delta_\Gamma\left(\partial_{\mathbf{n}}u_{\text{int},0|\Gamma}\right)$ in Condition (7.32). To bypass this difficulty, we determine constants p_1 and p_2 making these terms vanish. From (7.29)

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and (7.30), Condition (7.32) becomes

$$\begin{aligned}
\sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},1|\Gamma} &= p_1\widetilde{\sigma}\Delta_\Gamma u_{\text{int},1|\Gamma} + p_2\widetilde{\sigma}\Delta_\Gamma u_{\text{ext},1|\Gamma} + p_1\widetilde{k}^2u_{\text{int},1|\Gamma} \\
&+ p_2\widetilde{k}^2u_{\text{ext},1|\Gamma} - p_2\sigma_{\text{ext}}\Delta_\Gamma u_{\text{ext},0|\Gamma} - p_1\sigma_{\text{int}}\Delta_\Gamma u_{\text{int},0|\Gamma} \\
&+ p_2^2\widetilde{k}^2\mathscr{H}u_{\text{ext},0|\Gamma} - p_2k_{\text{ext}}^2u_{\text{ext},0|\Gamma} - p_1^2\widetilde{k}^2\mathscr{H}u_{\text{int},0|\Gamma} - p_1k_{\text{int}}^2u_{\text{int},0|\Gamma} \\
&+ p_2^2\widetilde{\sigma}div_\Gamma\left((\mathscr{H}I-\mathscr{R})\nabla_\Gamma u_{\text{ext},0|\Gamma}\right) - p_1^2\widetilde{\sigma}div_\Gamma\left((\mathscr{H}I-\mathscr{R})\nabla_\Gamma\right)u_{\text{int},0|\Gamma} \\
&+ \left(\sigma_{\text{int}}p_2^2 - p_1^2\sigma_{\text{ext}}\right)\frac{\widetilde{\sigma}}{\sigma_{\text{int}}}\Delta_\Gamma\left(\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}\right) + \left(\sigma_{\text{int}}p_2^2 - p_1^2\sigma_{\text{ext}}\right)\frac{\widetilde{k}^2}{\sigma_{\text{int}}}\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma} \\
&- 2p_1^2\widetilde{\sigma}\frac{\widetilde{k}^2}{\sigma_{\text{int}}}\Delta_\Gamma u_{\text{int},0|\Gamma} - p_1^2\frac{\widetilde{\sigma}}{\sigma_{\text{int}}}\left(\Delta_\Gamma^2 u_{\text{int},0|\Gamma}\right) - p_1^2\frac{\widetilde{k}^4}{\sigma_{\text{int}}}u_{\text{int},0|\Gamma}. \qquad (7.33)
\end{aligned}$$

Then, by setting $\sigma_{\text{int}}p_2^2 - p_1^2\sigma_{\text{ext}} = 0$, one obtains

$$p_1 = \frac{\sqrt{\sigma_{\text{int}}}\left(\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}}\right)}{\sigma_{\text{int}} - \sigma_{\text{ext}}} \quad \text{and} \quad p_2 = \frac{\sqrt{\sigma_{\text{ext}}}\left(\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}}\right)}{\sigma_{\text{int}} - \sigma_{\text{ext}}}.$$

As a consequence, (7.33) becomes

$$\begin{aligned}
\sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},1|\Gamma} &= p_1\widetilde{\sigma}\Delta_\Gamma u_{\text{int},1|\Gamma} + p_2\widetilde{\sigma}\Delta_\Gamma u_{\text{ext},1|\Gamma} \\
&+ p_1\widetilde{k}^2u_{\text{int},1|\Gamma} + p_2\widetilde{k}^2u_{\text{ext},1|\Gamma} - p_2\sigma_{\text{ext}}\Delta_\Gamma u_{\text{ext},0|\Gamma} - p_1\sigma_{\text{int}}\Delta_\Gamma u_{\text{int},0|\Gamma} \\
&+ p_2^2\widetilde{k}^2\mathscr{H}u_{\text{ext},0|\Gamma} - p_2k_{\text{ext}}^2u_{\text{ext},0|\Gamma} - p_1^2\widetilde{k}^2\mathscr{H}u_{\text{int},0|\Gamma} - p_1k_{\text{int}}^2u_{\text{int},0|\Gamma} \\
&+ p_2^2\widetilde{\sigma}div_\Gamma\left[(\mathscr{H}I-\mathscr{R})\nabla_\Gamma u_{\text{ext},0|\Gamma}\right] - p_1^2\widetilde{\sigma}div_\Gamma\left[(\mathscr{H}I-\mathscr{R})\nabla_\Gamma\right]u_{\text{int},0|\Gamma} \\
&- 2p_1^2\widetilde{\sigma}\frac{\widetilde{k}^2}{\sigma_{\text{int}}}\Delta_\Gamma u_{\text{int},0|\Gamma} - p_1^2\frac{\widetilde{\sigma}}{\sigma_{\text{int}}}\left(\Delta_\Gamma^2 u_{\text{int},0|\Gamma}\right) - p_1^2\frac{\widetilde{k}^4}{\sigma_{\text{int}}}u_{\text{int},0|\Gamma}. \qquad (7.34)
\end{aligned}$$

As a result, we assume in what follows that the following constraints hold

$$p_1 = \frac{\sqrt{\sigma_{\text{int}}}\left(\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}}\right)}{\sigma_{\text{int}} - \sigma_{\text{ext}}} \quad \text{and} \quad p_2 = \frac{\sqrt{\sigma_{ext}}\left(\sqrt{\sigma_{\text{int}}} - \sqrt{\sigma_{\text{ext}}}\right)}{\sigma_{\text{int}} - \sigma_{\text{ext}}}.$$

The second difficulty comes from complexity of Condition (7.34) which can be overcome by rewriting the whole transmission condition. Thus, from (7.29) and (7.34), we deduce a form, albeit longer but better adapted to treat, still, the uniqueness of the solution of the

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approximate problem.

$$\begin{aligned}
\sigma_{\text{int}}\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - \sigma_{\text{ext}}\partial_{\mathbf{n}}u_{\text{ext},1|\Gamma} &= p_1\widetilde{k}^2 u_{\text{int},1|\Gamma} + p_2\widetilde{k}^2 u_{\text{ext},1|\Gamma} \\
&+ p_1\widetilde{\sigma}\Delta_\Gamma u_{\text{int},1|\Gamma} + p_2\widetilde{\sigma}\Delta_\Gamma u_{\text{ext},1|\Gamma} \\
&+ \frac{\sigma_{\text{ext}}p_1}{\sigma_{\text{int}}p_2+\sigma_{\text{ext}}p_1}\left(p_2^2\widetilde{k}^2\mathscr{H} - p_1^2\widetilde{k}^2\mathscr{H} - p_2k_{\text{ext}}^2 - p_1k_{\text{int}}^2 - p_1^2\frac{\widetilde{k}^4}{\sigma_{\text{int}}}\right)u_{\text{ext},0|\Gamma} \\
&+ \frac{\sigma_{\text{int}}p_2}{\sigma_{\text{int}}p_2+\sigma_{\text{ext}}p_1}\left(p_2^2\widetilde{k}^2\mathscr{H} - p_1^2\widetilde{k}^2\mathscr{H} - p_2k_{\text{ext}}^2 - p_1k_{\text{int}}^2 - p_1^2\frac{\widetilde{k}^4}{\sigma_{\text{int}}}\right)u_{\text{int},0|\Gamma} \\
&+ \frac{\sigma_{\text{ext}}p_1}{\sigma_{\text{int}}p_2+\sigma_{\text{ext}}p_1}\left(-p_2\sigma_{\text{ext}} - p_1\sigma_{\text{int}} - 2p_1^2\widetilde{\sigma}\frac{\widetilde{k}^2}{\sigma_{\text{int}}}\right)\Delta_\Gamma u_{\text{ext},0|\Gamma} \\
&+ \frac{\sigma_{\text{int}}p_2}{\sigma_{\text{int}}p_2+\sigma_{\text{ext}}p_1}\left(-p_2\sigma_{\text{ext}} - p_1\sigma_{\text{int}} - 2p_1^2\widetilde{\sigma}\frac{\widetilde{k}^2}{\sigma_{\text{int}}}\right)\Delta_\Gamma u_{\text{int},0|\Gamma} \\
&+ \frac{\sigma_{\text{ext}}p_1}{\sigma_{\text{int}}p_2+\sigma_{\text{ext}}p_1}(p_2-p_1)\widetilde{\sigma}div_\Gamma\,[(\mathscr{H}I-\mathscr{R})\nabla_\Gamma]\,u_{\text{ext},0|\Gamma} \\
&+ \frac{\sigma_{\text{int}}p_2}{\sigma_{\text{int}}p_2+\sigma_{\text{ext}}p_1}(p_2-p_1)\widetilde{\sigma}div_\Gamma\,[(\mathscr{H}I-\mathscr{R})\nabla_\Gamma]\,u_{\text{int},0|\Gamma} \\
&+ \frac{\sigma_{\text{ext}}p_1}{\sigma_{\text{int}}p_2+\sigma_{ext}p_1}\left(-p_1^2\frac{\widetilde{\sigma}}{\sigma_{\text{int}}}\right)\left(\Delta_\Gamma^2 u_{ext,0|\Gamma}\right) \\
&+ \frac{\sigma_{\text{int}}p_2}{\sigma_{\text{int}}p_2+\sigma_{ext}p_1}\left(-p_1^2\frac{\widetilde{\sigma}}{\sigma_{\text{int}}}\right)\left(\Delta_\Gamma^2 u_{\text{int},0|\Gamma}\right).
\end{aligned}$$

We are now in position to give the first-order model. Once again (see Section 6) we define the approximate solution u_δ^{ap} on Ω by

$$u_\delta^{ap} := \begin{cases} u_{\text{int},\delta}^{ap} & \text{in } \Omega_{\text{int},\delta}, \\ u_{d_\beta,\delta}^{ap} & \text{in } \Omega_{\delta,\beta}, (\beta = 1\text{or } 2), \\ u_{\text{ext},\delta}^{ap} & \text{in } \Omega_{\text{ext},\delta}, \end{cases}$$

where $u_{d_\beta,\delta}^{ap}$ are defined on $\Omega_{\delta,\beta}$ by

$$\begin{aligned}
u_{d_\beta,\delta}^{ap}(x) :&= u_{d_\beta,\delta}^{[\beta],ap}(m,s_\beta) := u_{\text{int},\delta|\Gamma}^{ap} - \delta p_1\partial_{\mathbf{n}}u_{\text{int},\delta|\Gamma}^{ap} \\
:&= u_{\text{ext},\delta|\Gamma}^{ap} + \delta p_2\partial_{\mathbf{n}}u_{\text{ext},\delta|\Gamma}^{ap},\ \forall x = \Phi_\beta(m,s_\beta) \in \Omega_{\delta,\beta},
\end{aligned}$$

and $\left(u_{\text{int},\delta}^{ap}, u_{\text{ext},\delta}^{ap}\right)$ is the solution to the following problem:

$$\begin{cases} \text{div}\left(\sigma_{\text{ext}}\nabla u_{\text{ext},\delta}^{ap}\right) + k_{\text{ext}}^2 u_{\text{ext},\delta}^{ap} = 0 & \text{in } \Omega_{\text{ext}}, \\ \text{div}\left(\sigma_{\text{int}}\nabla u_{\text{int},\delta}^{ap}\right) + k_{\text{int}}^2 u_{\text{int},\delta}^{ap} = 0 & \text{in } \Omega_{\text{int}}, \\ \lim_{|x|\to+\infty} |x|\left(\partial_{|x|} - ik_{\text{ext}}\right)\left(u_{\text{ext},\delta}^{ap} - u_{inc}\right) = 0, & \end{cases} \tag{7.35}$$

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with Ventcel-type transmission conditions Γ

$$\begin{cases} u^{ap}_{\mathrm{int},\delta|\Gamma} - u^{ap}_{\mathrm{ext},\delta|\Gamma} = \delta p_1 \partial_{\mathbf{n}} u^{ap}_{\mathrm{int},\delta|\Gamma} + \delta p_2 \partial_{\mathbf{n}} u^{ap}_{\mathrm{ext},\delta|\Gamma}, \\ \sigma_{\mathrm{int}} \partial_{\mathbf{n}} u^{ap}_{\mathrm{int},\delta|\Gamma} - \sigma_{\mathrm{ext}} \partial_{\mathbf{n}} u^{ap}_{\mathrm{ext},\delta|\Gamma} = \mathscr{K}\left(u^{ap}_{\mathrm{int},\delta|\Gamma}, u^{ap}_{\mathrm{ext},\delta|\Gamma}\right), \end{cases} \tag{7.36}$$

where

$$\begin{aligned} \mathscr{K}\left(u_{|\Gamma}, v_{|\Gamma}\right) := {} & \alpha_{\delta,1} u_{|\Gamma} + \alpha_{\delta,2} v_{|\Gamma} + \alpha_{\delta,3} \Delta_\Gamma u_{|\Gamma} \\ & + \alpha_{\delta,4} \Delta_\Gamma v_{|\Gamma} - \alpha_{\delta,7} \Delta^2_\Gamma u_{|\Gamma} - \alpha_{\delta,8} \Delta^2_\Gamma v_{|\Gamma} \\ & + \alpha_{\delta,5} div_\Gamma \left[(\mathscr{H} I - \mathscr{R}) \nabla_\Gamma \right] u_{|\Gamma} \\ & + \alpha_{\delta,6} div_\Gamma \left[(\mathscr{H} I - \mathscr{R}) \nabla_\Gamma \right] v_{|\Gamma} \end{aligned}$$

and

$$\begin{aligned} \alpha_{\delta,1} &:= p_1 \widetilde{k}^2 + \delta \frac{\sigma_{\mathrm{int}} p_2}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} \left(p_2 \widetilde{k}^2 \mathscr{H} - p_1 \widetilde{k}^2 \mathscr{H} - p_2 k^2_{\mathrm{ext}} - p_1 k^2_{\mathrm{int}} - p_1^2 \frac{\widetilde{k}^4}{\sigma_{\mathrm{int}}} \right), \\ \alpha_{\delta,2} &:= p_2 \widetilde{k}^2 + \delta \frac{\sigma_{\mathrm{ext}} p_1}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} \left(p_2 \widetilde{k}^2 \mathscr{H} - p_1 \widetilde{k}^2 \mathscr{H} - p_2 k^2_{\mathrm{ext}} - p_1 k^2_{\mathrm{int}} - p_1^2 \frac{\widetilde{k}^4}{\sigma_{\mathrm{int}}} \right), \\ \alpha_{\delta,3} &:= p_1 \widetilde{\sigma} - \delta \frac{\sigma_{\mathrm{int}} p_2}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} \left(p_2 \sigma_{\mathrm{ext}} + p_1 \sigma_{\mathrm{int}} + 2 p_1^2 \widetilde{\sigma} \frac{\widetilde{k}^2}{\sigma_{\mathrm{int}}} \right), \\ \alpha_{\delta,4} &:= p_2 \widetilde{\sigma} - \delta \frac{\sigma_{\mathrm{ext}} p_1}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} \left(p_2 \sigma_{\mathrm{ext}} + p_1 \sigma_{\mathrm{int}} + 2 p_1^2 \widetilde{\sigma} \frac{\widetilde{k}^2}{\sigma_{\mathrm{int}}} \right), \\ \alpha_{\delta,5} &:= \delta \frac{\sigma_{\mathrm{int}} p_2}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} (p_2 - p_1) \widetilde{\sigma}, \quad \alpha_{\delta,6} := \delta \frac{\sigma_{\mathrm{ext}} p_1}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} (p_2 - p_1) \widetilde{\sigma}, \\ \alpha_{\delta,7} &:= \delta \frac{\sigma_{\mathrm{int}} p_2}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} \left(p_1^2 \frac{\widetilde{\sigma}^2}{\sigma_{\mathrm{int}}} \right), \quad \alpha_{\delta,8} := \delta \frac{\sigma_{\mathrm{ext}} p_1}{\sigma_{\mathrm{int}} p_2 + \sigma_{\mathrm{ext}} p_1} \left(p_1^2 \frac{\widetilde{\sigma}^2}{\sigma_{\mathrm{int}}} \right). \end{aligned}$$

Similar ideas to those used in Section 6 guarantee the existence and the uniqueness of the solution to Problems (7.35) and (7.36). Optimal error estimates can also be obtained. Nevertheless, note that we suppose here that σ_δ and k^2_δ are strictly positive constants.

8 Conclusion

In this work, we determined and justified an asymptotic expansion of the exact solution to Problem (1.1) for different values of contrast and wavenumber. For each case, we derive approximate transmission conditions, validated thanks to optimal error estimates (see Theorem 6.3 and the end of Section 7) to take into account the effect of the thin layer. Ventcel-type transmission conditions, involving tangential differential operators of order two, have also been obtained in the case of high values of magnetic permittivity and wavenumber.

An interesting continuation of this work could be to consider the full Maxwell's system describing the scattering of electromagnetic waves by an obstacle.

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Appendix A

This appendix contains some technical results needed in the proof of the uniform stability estimate in the high-permittivity case (see Section 7). We first give the well-posedness and regularity results for a Helmholtz equation with Ventcel-type transmission conditions and provide next a density result.

A.1 Well-posedness and regularity results for a solution to a Ventcel transmission problem

Recalling that $\widetilde{\Omega}_{\text{ext}} = \Omega \backslash \overline{\Omega_{\text{int}}}$, one has the following result.

Theorem A.1 *Let $h \in H^1(\Gamma)$ and $\zeta \in H^{-1}(\Gamma)$. Then, the following problem*

$$\begin{cases} \operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ \operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ U_{\text{int}|\Gamma} - U_{\text{ext}|\Gamma} = h & \text{on } \Gamma, \\ \sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}|\Gamma} = p_1 \widetilde{\sigma} \Delta_\Gamma U_{\text{int}|\Gamma} + p_1 \widetilde{k}^2 U_{\text{int}|\Gamma} & \\ + p_2 \widetilde{\sigma} \Delta_\Gamma U_{\text{ext}|\Gamma} + p_2 \widetilde{k}^2 U_{\text{ext}|\Gamma} + \zeta & \text{on } \Gamma, \\ \lim_{|x| \to +\infty} |x| \left(\partial_{|x|} - i k_{\text{ext}} \right) U_{\text{ext}} = 0, & \end{cases} \tag{A 1}$$

admits a unique solution $(U_{\text{int}}, U_{\text{ext}})$ in $H^{1,1}(\Omega_{\text{int}}) \times H^{1,1}_{loc}\left(\overline{\Omega_{\text{ext}}}\right)$ satisfying the inequality

$$\| U_{\text{int}} \|_{H^{1,1}(\Omega_{\text{int}})} + \| U_{\text{ext}} \|_{H^{1,1}\left(\widetilde{\Omega}_{\text{ext}}\right)} \leqslant c_k \left(\|h\|_{H^1(\Gamma)} + \|\zeta\|_{H^{-1}(\Gamma)} \right). \tag{A 2}$$

Moreover, for all $k \in \mathbb{N}^*$, if $h \in H^k(\Gamma)$, $\zeta \in H^{k-2}(\Gamma)$ and Γ $\mathscr{C}^{k+1}$-continuous, then $(U_{\text{int}}, U_{\text{ext}}) \in H^{k+1/2}(\Omega_{\text{int}}) \times H^{k+1/2}_{loc}\left(\overline{\Omega_{\text{ext}}}\right)$.

Proof Uniqueness follows by Rellich's lemma. Existence of a solution can be obtained by the Fredholm alternative. To show the regularity result, we proceed by induction in k. For $k = 1$, we showed above that if $h \in H^1(\Gamma)$ and $\zeta \in H^{-1}(\Gamma)$, then Problem (A 1) admits a unique solution $(U_{\text{int}}, U_{\text{ext}})$ in $H^{1,1}(\Omega_{\text{int}}) \times H^{1,1}_{loc}\left(\overline{\Omega_{\text{ext}}}\right)$; hence $U_{\text{ext}|\Gamma}, U_{\text{int}|\Gamma} \in H^1(\Gamma)$. Since

$$\operatorname{div}(\sigma_{\text{ext}} \nabla U_{\text{ext}}) + k_{\text{ext}}^2 U_{\text{ext}} = 0 \text{ in } \Omega_{\text{ext}},$$
$$\operatorname{div}(\sigma_{\text{int}} \nabla U_{\text{int}}) + k_{\text{int}}^2 U_{\text{int}} = 0 \text{ in } \Omega_{\text{int}}$$

and Γ is $\mathscr{C}^2$, we get (cf. [10]) $(U_{\text{int}}, U_{\text{ext}}) \in H^{3/2}(\Omega_{\text{int}}) \times H^{3/2}_{loc}\left(\overline{\Omega_{\text{ext}}}\right)$. Now assume that the assertion holds up to $k-1$. Let $h \in H^k(\Gamma)$ and $\zeta \in H^{k-2}(\Gamma)$. Since $H^k(\Gamma) \subset H^{k-1}(\Gamma)$ and $H^{k-2}(\Gamma) \subset H^{k-3}(\Gamma)$, $h \in H^{k-1}(\Gamma)$ and $\zeta \in H^{k-3}(\Gamma)$. Then, there exists a unique solution $(U_{\text{int}}, U_{\text{ext}})$ of (A 1) in $H^{k-1/2}(\Omega_{\text{int}}) \times H^{k-1/2}_{loc}\left(\overline{\Omega_{\text{ext}}}\right)$. Applying trace theorem of functions in $H^{k-1/2}$ (cf. [10]), we obtain $\partial_{\mathbf{n}} U^{\pm}_{|\Gamma} \in H^{k-2}(\Gamma)$.

Now from

$$\sigma_{\text{int}} \partial_{\mathbf{n}} U_{\text{int}|\Gamma} - \sigma_{\text{ext}} \partial_{\mathbf{n}} U_{\text{ext}|\Gamma} = p_1 \widetilde{\sigma} \Delta_\Gamma U_{\text{int}|\Gamma} + p_1 \widetilde{k}^2 U_{\text{int}|\Gamma} + p_2 \widetilde{\sigma} \Delta_\Gamma U_{\text{ext}|\Gamma} + p_2 \widetilde{k}^2 U_{\text{ext}|\Gamma} + \zeta, \tag{A 3}$$

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we can write

$$\widetilde{\sigma}\Delta_\Gamma\left(p_1 U_{\mathrm{int}|\Gamma} + p_2 U_{\mathrm{ext}|\Gamma}\right) = \sigma_{\mathrm{int}}\partial_{\mathbf{n}} U_{\mathrm{int}|\Gamma} - \sigma_{\mathrm{ext}}\partial_{\mathbf{n}} U_{\mathrm{ext}|\Gamma} - p_1\widetilde{k}^2 U_{\mathrm{int}|\Gamma} - p_2\widetilde{k}^2 U_{\mathrm{ext}|\Gamma} - \zeta. \quad \text{(A 4)}$$

Thus, $\Delta_\Gamma(p_1 U_{\mathrm{int}|\Gamma} + p_2 U_{\mathrm{ext}|\Gamma}) \in H^{k-2}(\Gamma)$. As $p_1 U_{\mathrm{int}|\Gamma} + p_2 U_{\mathrm{ext}|\Gamma} \in H^{k-1}(\Gamma)$, $\Delta_\Gamma(p_1 U_{\mathrm{int}|\Gamma} + p_2 U_{\mathrm{ext}|\Gamma}) \in H^{k-2}(\Gamma)$ and the operator Δ_Γ is elliptic of order two on a compact manifold without boundary Γ of class $\mathscr{C}^{k+2}$, $p_1 U_{\mathrm{int}|\Gamma} + p_2 U_{\mathrm{ext}|\Gamma} \in H^k(\Gamma)$ but $U_{\mathrm{int}|\Gamma} - U_{\mathrm{ext}|\Gamma} \in H^k(\Gamma)$, then $U_{\mathrm{ext}|\Gamma}, U_{\mathrm{int}|\Gamma} \in H^k(\Gamma)$. Summarizing, we have

$$\begin{aligned} \mathrm{div}\,(\sigma_{\mathrm{ext}}\nabla U_{\mathrm{ext}}) + k_{\mathrm{ext}}^2 U_{\mathrm{ext}} &= 0 \text{ in } \Omega_{\mathrm{ext}}, \\ \mathrm{div}\,(\sigma_{\mathrm{int}}\nabla U_{\mathrm{int}}) + k_{\mathrm{int}}^2 U_{\mathrm{int}} &= 0 \text{ in } \Omega_{\mathrm{int}} \end{aligned}$$

and $U_{\mathrm{ext}|\Gamma}, U_{\mathrm{int}|\Gamma} \in H^k(\Gamma)$, as a consequence $(U_{\mathrm{int}}, U_{\mathrm{ext}}) \in H^{k+1/2}(\Omega_{\mathrm{int}}) \times H^{k+1/2}_{loc}(\overline{\Omega_{\mathrm{ext}}})$.

A.2 The density lemma

Recall that, in view of the thin shells assumption (cf. [18]), there exists $\delta_0 > 0$ such that

$$\Omega_{\delta_0} = \left\{x \in \mathbb{R}^3 \ ; \ x := m + \eta\mathbf{n}(m) \text{ where } -\delta_0 \leqslant \eta \leqslant \delta_0 \text{ and } m \in \Gamma\right\} \quad \text{(A 5)}$$

defines a bijection between $\overline{\Omega}_{\delta_0}$ and $\Gamma \times [-\delta_0, \delta_0]$. Let now $\varepsilon > 0$, satisfying $\varepsilon < \delta_0$. We denote by $H(\Omega)$ the space of functions defined by

$$H(\Omega) := \left\{v \in H^1_\Gamma(\Omega) \ / \ \exists\varepsilon > 0 \text{ such as } \partial_\eta v(m,\eta) = 0, \ \forall |\eta| < \varepsilon\right\}. \quad \text{(A 6)}$$

We have the following density lemma.

Lemma A.1 *$H(\Omega)$ is dense in $H^1_\Gamma(\Omega)$.*

Proof Let $v \in H^1_\Gamma(\Omega)$. We construct a sequence $(v_\varepsilon)_\varepsilon \subset H(\Omega)$, $\varepsilon > 0$, such that $\lim\limits_{\varepsilon\to 0} v_\varepsilon \to v$ in $H^1_\Gamma(\Omega)$. Since $C^\infty\left(\overline{\Omega}\right)$ is dense in $H^1_\Gamma(\Omega)$, it is sufficient to construct such a sequence for $v \in C^\infty\left(\overline{\Omega}\right)$. Let $\varepsilon > 0$, we introduce the function φ_ε defined on $[-\delta_0, \delta_0]$ by

$$\varphi_\varepsilon(\eta) := \begin{cases} 0 & \text{if} \quad |\eta| < \varepsilon, \\ \delta_1 \frac{\eta-\varepsilon}{\delta_1-\varepsilon} & \text{if} \quad \varepsilon \leqslant \eta \leqslant \delta_1, \\ \delta_1 \frac{\eta+\varepsilon}{\delta_1-\varepsilon} & \text{if} \quad -\delta_1 \leqslant \eta \leqslant -\varepsilon, \\ \eta & \text{if} \quad |\eta| > \delta_1, \end{cases} \quad \text{(A 7)}$$

where δ_1 satisfies $\varepsilon < \delta_1 \leqslant \delta_0$. Then, we set

$$v_\varepsilon(x) := \begin{cases} v(m, \varphi_\varepsilon(\eta)) & \text{if} \quad x = (m,\eta) \in \overline{\Omega_{\delta_0}}, \\ v(x) & \text{if} \quad x \in \Omega_{\mathrm{int},\delta_0} \cup \widetilde{\Omega}_{\mathrm{ext},\delta_0}. \end{cases} \quad \text{(A 8)}$$

It is easy to show that $v_\varepsilon \in H(\Omega)$ and $v_\varepsilon \xrightarrow[\varepsilon\to 0]{} v$ in $H^1_\Gamma(\Omega)$. Hence the lemma. □

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