Proceedings of the Royal Society of Edinburgh, **142A**, 1309–1324, *2012*

Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor

Young Jin Suh
Department of Mathematics, Kyungpook National University,
Taegu 702 701, Republic of Korea (yjsuh@knu.ac.kr)

(MS received 14 September 2010; accepted 2 November 2011)

We introduce the full expression of the curvature tensor of a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ from the Gauss equation. We then derive a new formula for the Ricci tensor of M in $G_2(\mathbb{C}^{m+2})$. Finally, we prove that there does not exist any Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor.

1. Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms, it can be easily checked that there does not exist any real hypersurface with parallel shape operator A by virtue of the Codazzi equation.

From this point of a view, many differential geometers have considered a weaker notion than the parallel second fundamental form, i.e. $\nabla A = 0$. In particular, Kimura and Maeda [6] have proved that a real hypersurface M in a complex projective space $\mathbb{C}P^m$ satisfying $\nabla_\xi A = 0$ is locally congruent to a real hypersurface of type A_1, A_2, that is, a tube over a totally geodesic complex submanifold $\mathbb{C}P^k$ with radius $0 < r < \frac{1}{2}\pi$. The structure vector field ξ mentioned above is defined by $\xi = -JN$, where J denotes a Kähler structure of $\mathbb{C}P^m$ and N denotes a local unit normal field of M in $\mathbb{C}P^m$. Moreover, in a class of Hopf hypersurfaces, Kimura [5] asserted that there do not exist any real hypersurfaces with parallel Ricci tensor, i.e. $\nabla S = 0$, where S denotes the Ricci tensor of a real hypersurface M in $\mathbb{C}P^m$.

On the other hand, in a quaternionic projective space $\mathbb{H}P^m$, Pérez [8] considered the notion of $\nabla_{\xi_i} A = 0$, $i = 1, 2, 3$, for real hypersurfaces in $\mathbb{H}P^m$ and classified M as locally congruent to a real hypersurface of A_1, A_2 type, i.e. a tube over $\mathbb{H}P^k$ with radius $0 < r < \frac{1}{4}\pi$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i denotes a quaternionic Kähler structure of $\mathbb{H}P^m$ and N denotes a unit normal field of M in $\mathbb{H}P^m$. Moreover, Pérez and Suh [9] considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{H}P^m$, and proved that M is locally congruent to a tube of radius $\frac{1}{4}\pi$ over $\mathbb{H}P^k$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the above situation is not so simple if we consider a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ [3,4,11–14]. Suh [11] showed that there does not exist any hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator A, i.e. $\nabla A = 0$.

We study the problem related to the parallel Ricci tensor S for real hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, that is, $\nabla_X S = 0$ for any vector field X tangent to M. The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not including J [2].

In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold that is not a hyper-Kähler manifold. So, we have considered the two natural geometric conditions for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, that the one-dimensional distribution $[\xi] = \operatorname{span}\{\xi\}$ is invariant under the shape operator and that the three-dimensional distribution $\mathfrak{D}^\perp = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such two geometric conditions and the results of Alekseevskii [1], Berndt and Suh [3] proved the following.

THEOREM 1.1. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(i) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(ii) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the structure vector ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, M is said to be a *Hopf real hypersurface*. In such cases, the integral curves of the structure vector field ξ are geodesics [4]. Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow*.

In the proof of theorem 1.1 it was proved that the one-dimensional distribution $[\xi]$ is contained in either the three-dimensional distribution $\mathfrak{D}^\perp$ or in the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. Case (i) of theorem 1.1 is just the case that the one- dimensional distribution $[\xi]$ belongs to the distribution $\mathfrak{D}^\perp$. Of course, it is not difficult to check that the Ricci tensor of any real hypersurface mentioned in theorem 1.1 is not parallel. Then it is natural to ask if real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor can exist.

Accordingly, the main result of this paper is to prove the non-existence of all Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor, that is, $\nabla S = 0$, as follows.

THEOREM 1.2. *There does not exist any Hopf real hypersurface with parallel Ricci tensor in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$.*

On the other hand, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *Einstein* if the Ricci tensor S is given by $g(SX, Y) = ag(X, Y)$ for a smooth function a and any vector fields X and Y on M. Naturally the Ricci tensor is parallel on M. So we also add the following corollary.

COROLLARY 1.3. *There does not exist any Einstein Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$.*

In § 2 we recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, and we will show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ in § 3. The formula for the Ricci tensor S and its covariant derivative ∇S will be shown explicitly in this section. In §§ 4 and 5 we shall give a complete proof of the main theorem according to the geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$ or vanishing geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^\perp$.

2. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$ (for details we refer the interested reader to [2–4,14,15]). By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = \mathrm{SU}(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We set $o = eK$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}(K)$-invariance of B, this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is 8.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature 8.

When $m = 2$, we note that the isomorphism $\mathrm{spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of orientated two-dimensional linear subspaces in $\mathbb{R}^6$. In the remainder of this paper, we shall assume $m \geqslant 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the centre of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the centre $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$ part induces a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost-Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1 J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used in the next sections.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost-Hermitian structures J_ν in $\mathfrak{J}$ such that

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu,$$

where the index is taken modulo 3. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, for any canonical local basis J_1, J_2 and J_3 of $\mathfrak{J}$, there exist three local 1-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

https://doi.org/10.1017/S0308210510001472 Published online by Cambridge University Press

Let $p \in G_2(\mathbb{C}^{m+2})$ and W be a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. Furthermore, we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (respectively, totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (respectively, totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ \\ + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ denotes any canonical local basis of $\mathfrak{J}$.

3. Some fundamental formulae for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some fundamental formulae which will be used in the proof of our main theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a submanifold in $G_2(\mathbb{C}^{m+2})$ with real codimension 1. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g).

Now let us set

$$JX = \phi X + \eta(X)N, \qquad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$.

From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$, there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M so that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{3.2}$$

for any vector field X on M.

On the other hand, from the quaternionic Kähler structure $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ and (3.1) we have an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, on M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$, in § 2 and (3.1), the relation between these two contact metric structures (ϕ, ξ, η, g) and

$(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be expressed by

$$\left.\begin{aligned} \phi_{\nu+1}\xi_\nu &= -\xi_{\nu+2}, \\ \phi_\nu\xi_{\nu+1} &= \xi_{\nu+2}, \\ \phi\xi_\nu &= \phi_\nu\xi, \\ \eta_\nu(\phi X) &= \eta(\phi_\nu X), \\ \phi_\nu\phi_{\nu+1}X &= \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ \phi_{\nu+1}\phi_\nu X &= -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1} \end{aligned}\right\} \tag{3.3}$$

for any vector field X on M.

Using expressions (2.2) and (3.1) for the curvature tensor $\bar{R}$, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X &- g(X,Z)Y \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &+ \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu \\ &+ g(AY,Z)AX - g(AX,Z)AY \end{aligned}$$

and

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y &- \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu, \end{aligned}$$

where R denotes the curvature tensor and A denotes the shape operator of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

Then, from formulae (2.1) and (3.1), together with (3.2) and (3.3), the Kähler structure and the quaternionic Kähler structure of $G_2(\mathbb{C}^{m+2})$ give

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.4}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.5}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{3.6}$$

Summing up these formulae, we find the following:

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{3.7}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{3.8}$$

4. Proof of the main theorem

Now let us contract Y and Z in the Gauss equation in § 3. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}SX &= \sum_{i=1}^{4m-1} R(X,e_i)e_i\\ &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu\\ &\quad + \sum_{\nu=1}^{3}\{(\operatorname{Tr}\phi_\nu\phi)\phi_\nu\phi X - (\phi_\nu\phi)^2X\} - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(X)\phi_\nu\phi\xi_\nu\}\\ &\quad - \sum_{\nu=1}^{3}\{(\operatorname{Tr}\phi_\nu\phi)\eta(X) - \eta(\phi_\nu\phi X)\}\xi_\nu + hAX - A^2X,\end{aligned} \tag{4.1}$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, $\operatorname{Tr} JJ_\nu = 0$, $\nu = 1,2,3$, for any basis $\{e_1,\dots,e_{4m-1},N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$, we calculate

$$\begin{aligned}0 &= \operatorname{Tr} JJ_\nu\\ &= \sum_{k=1}^{4m-1} g(JJ_\nu e_k,e_k) + g(JJ_\nu N,N)\\ &= \operatorname{Tr}\phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN)\\ &= \operatorname{Tr}\phi\phi_\nu - 2\eta_\nu(\xi)\end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
(\phi_\nu\phi)^2X &= \phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu) \\
&= \phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X)\phi_\nu^2\xi \\
&= X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}. \qquad (4.3)
\end{aligned}
$$

Substituting (4.2) and (4.3) into (4.1), we have

$$
\begin{aligned}
SX &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X \\
&= (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X. \qquad (4.4)
\end{aligned}
$$

Now the covariant derivative of (4.4) becomes

$$
\begin{aligned}
(\nabla_Y S)X &= -3((\nabla_Y\eta)X)\xi - 3\eta(X)\nabla_Y\xi \\
&\quad - 3\sum_{\nu=1}^{3}(\nabla_Y\eta_\nu)(X)\xi_\nu - 3\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_Y\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{Y(\eta_\nu(\xi))\phi_\nu\phi X + \eta_\nu(\xi)(\nabla_Y\phi_\nu)\phi X \\
&\qquad + \eta_\nu(\xi)\phi_\nu(\nabla_Y\phi)X - (\nabla_Y\eta)(\phi_\nu X)\phi_\nu\xi \\
&\qquad - \eta((\nabla_Y\phi_\nu)X)\phi_\nu\xi - \eta(\phi_\nu X)\nabla_Y(\phi_\nu\xi) \\
&\qquad - (\nabla_Y\eta)(X)\eta_\nu(\xi)\xi_\nu - \eta(X)\nabla_Y(\eta_\nu(\xi))\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\} \\
&\quad + (Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X = 0 \qquad (4.5)
\end{aligned}
$$

for any vector fields X and Y tangent to M in $G_2(\mathbb{C}^{m+2})$. Then, from (4.5), together with the formulae in § 3, we have

$$
\begin{aligned}
(\nabla_Y S)X &= -3g(\phi AY, X)\xi - 3\eta(X)\phi AY \\
&\quad - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, X)\}\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu AY\}
\end{aligned}
$$

https://doi.org/10.1017/S0308210510001472 Published online by Cambridge University Press

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}[Y(\eta_\nu(\xi))\phi_\nu\phi X+\eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi X+q_{\nu+2}(Y)\phi_{\nu+1}\phi X\\
&\qquad\qquad +\eta_\nu(\phi X)AY-g(AY,\phi X)\xi_\nu\}\\
&\quad +\eta_\nu(\xi)\{\eta(X)\phi_\nu AY-g(AY,X)\phi_\nu\xi\}-g(\phi AY,\phi_\nu X)\phi_\nu\xi\\
&\quad +\{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X)-q_{\nu+2}(Y)\eta(\phi_{\nu+1}X)-\eta_\nu(X)\eta(AY)\\
&\qquad\qquad +\eta(\xi_\nu)g(AY,X)\}\phi_\nu\xi\\
&\quad -\eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi-q_{\nu+1}(Y)\phi_{\nu+2}\xi\\
&\qquad\qquad +\phi_\nu\phi AY-\eta(AY)\xi_\nu+\eta(\xi_\nu)AY\}\\
&\quad -g(\phi AY,X)\eta_\nu(\xi)\xi_\nu-\eta(X)Y(\eta_\nu(\xi))\xi_\nu-\eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu]\\
&+(Yh)AX+h(\nabla_Y A)X-(\nabla_Y A^2)X\\
&=0.
\end{aligned}
\tag{4.6}
$$

Setting $X=\xi$ in (4.6), we have

$$
\begin{aligned}
0&=-3\phi AY-3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi)-q_{\nu+1}(Y)\eta_{\nu+2}(\xi)+g(\phi_\nu AY,\xi)\}\xi_\nu\\
&\quad -3\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(Y)\xi_{\nu+1}-q_{\nu+1}(Y)\xi_{\nu+2}+\phi_\nu AY\}\\
&\quad +\sum_{\nu=1}^{3}[\eta_\nu(\xi)\{\phi_\nu AY-\eta(AY)\phi_\nu\xi\}-g(\phi AY,\phi_\nu\xi)\phi_\nu\xi\\
&\qquad -Y(\eta_\nu(\xi))\xi_\nu-\eta_\nu(\xi)\{q_{\nu+2}(Y)\xi_{\nu+1}-q_{\nu+1}(Y)\xi_{\nu+2}+\phi_\nu AY\}]\\
&\quad +(Yh)A\xi+h(\nabla_Y A)\xi-(\nabla_Y A^2)\xi
\end{aligned}
\tag{4.7}
$$

for any vector field Y tangent to M in $G_2(\mathbb{C}^{m+2})$.

On the other hand, we know that

$$
\begin{aligned}
Y(\eta_\nu(\xi))&=(\nabla_Y\eta_\nu)\xi+\eta_\nu(\nabla_Y\xi)\\
&=q_{\nu+2}(Y)\eta_{\nu+1}(\xi)-q_{\nu+1}(Y)\eta_{\nu+2}(\xi)+g(\phi_\nu AY,\xi)+\eta_\nu(\phi AY)\\
&=q_{\nu+2}(Y)\eta_{\nu+1}(\xi)-q_{\nu+1}(Y)\eta_{\nu+2}(\xi)+2\eta_\nu(\phi AY).
\end{aligned}
$$

Now, if we suppose that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, then (4.7) together with $A\xi=\alpha\xi$ implies

$$
\begin{aligned}
0&=(h\alpha-\alpha^2-3)\phi AY+Y(\alpha h)\xi-hA\phi AY-Y(\alpha^2)\xi+A^2\phi AY\\
&\quad -4\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi)-q_{\nu+1}(Y)\eta_{\nu+2}(\xi)\}_\nu\\
&\quad -5\sum_{\nu=1}^{3}g(\phi_\nu AY,\xi)\xi_\nu-4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+2}(Y)\xi_{\nu+1}
\end{aligned}
$$

https://doi.org/10.1017/S0308210510001472 Published online by Cambridge University Press

$$+ 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+1}(Y)\xi_{\nu+2} - 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu AY$$
$$- \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\eta(AY) + g(\phi AY, \phi_\nu\xi)\}\phi_\nu\xi, \tag{4.8}$$

where we have used

$$\sum_{\nu=1}^{3} Y(\eta_\nu(\xi))\xi_\nu = \sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi) - g(\phi_\nu AY, \xi)\}\xi_\nu.$$

On the other hand, by differentiating $A\xi = \alpha\xi$ and using the Codazzi equation in § 3, we have the following:

$$\begin{aligned} - 2g(\phi X, Y) &+ 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned} \tag{4.9}$$

Setting $X = \xi$ gives

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y).$$

From this, substituting into the above equation, we have the following:

$$\begin{aligned} hA\phi AY = {}& \tfrac{1}{2}\alpha h(A\phi + \phi A)Y + h\phi Y \\ & + h\sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y \\ & \qquad - 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\}. \end{aligned} \tag{4.10}$$

Then, substituting (4.10) into (4.9), we have

$$\begin{aligned} 0 = {}& \{h\alpha - \alpha^2 - 3\}\phi AY + Y(h\alpha)\xi - hA\phi AY \\ & - (Y\alpha^2)\xi + \tfrac{1}{2}\alpha A^2\phi Y + A\phi Y \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(Y)A\phi\xi_\nu + \eta_\nu(\phi Y)A\xi_\nu + \eta_\nu(\xi)A\phi_\nu Y \\ & \qquad - 2\eta(Y)\eta_\nu(\xi)A\phi\xi_\nu - 2\alpha\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} \\ & + \tfrac{1}{4}\alpha^2(A\phi + \phi A)Y + \tfrac{1}{2}\alpha\phi Y \\ & - \tfrac{1}{2}\alpha\sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu \\ & \qquad + \eta_\nu(\xi)\phi_\nu Y - 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} \end{aligned}$$

$$
\begin{aligned}
&- 4\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi)\}\xi_\nu \\
&- 5\sum_{\nu=1}^{3} g(\phi_\nu AY, \xi)\xi_\nu - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+2}(Y)\xi_{\nu+1} \\
&+ 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+1}(Y)\xi_{\nu+2} - 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu AY \\
&- \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\eta(AY) + g(\phi AY, \phi_\nu\xi)\}\phi_\nu\xi. \qquad (4.11)
\end{aligned}
$$

From this, let us verify that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. In order to do this we suppose that $\xi = X_1 + X_2$ for some $X_1 \in \mathfrak{D}$ and $X_2 \in \mathfrak{D}^\perp$. Now, setting $Y = \xi$ in (4.11), we have

$$
\begin{aligned}
&- (\xi\alpha^2)\xi + \xi(h\alpha)\xi - 4\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(X_2) - q_{\nu+1}(\xi)\eta_{\nu+2}(X_2)\}\xi_\nu \\
&\quad - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+2}(\xi)\xi_{\nu+1} + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+1}(\xi)\xi_{\nu+2} - 4\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = 0. \qquad (4.12)
\end{aligned}
$$

By comparing the $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components in the above equation we have, respectively,

$$
\{-\xi(\alpha^2) + \xi(h\alpha)\}X_1 - 4\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_1 = 0 \qquad (4.13)
$$

and

$$
\begin{aligned}
&\{-\xi(\alpha^2) + \xi(h\alpha)\}X_2 - 4\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(X_2) - q_{\nu+1}(\xi)\eta_{\nu+2}(X_2)\}\xi_\nu \\
&\quad - 4\sum_{\nu=1}^{3}\eta_\nu(X_2)q_{\nu+2}(\xi)\xi_{\nu+1} + 4\sum_{\nu=1}^{3}\eta_\nu(X_2)q_{\nu+1}(\xi)\xi_{\nu+2} \\
&\qquad - 4\alpha\sum_{\nu=1}^{3}\eta_\nu(X_2)\phi_\nu X_2 = 0. \qquad (4.14)
\end{aligned}
$$

Now, taking the inner product of (4.13) with X_1, we have

$$
-\xi(\alpha^2) + \xi(h\alpha) = 0,
$$

which gives

$$
4\alpha\sum_{\nu=1}^{3}\eta_\nu(X_2)\phi_\nu X_1 = 0.
$$

Then $\alpha = 0$ or $\eta_\nu(X_2) = 0$, $\nu = 1, 2, 3$. This gives $X_2 = 0$. From this we conclude that if a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has non-vanishing geodesic Reeb flow,

https://doi.org/10.1017/S0308210510001472 Published online by Cambridge University Press

i.e. $\alpha \neq 0$, then $\xi \in \mathfrak{D}$. In the case of vanishing geodesic Reeb flow, i.e. $\alpha = 0$, we shall show the following.

LEMMA 4.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor. If M has non-vanishing geodesic Reeb flow, then $\xi \in \mathfrak{D}$. If M has vanishing geodesic Reeb flow, then $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$.*

Proof. For the case $\alpha \neq 0$ we have already proved the above result. Now let us consider the case $\alpha = 0$. By differentiating $A\xi = 0$ and using the same method as in [3], for any tangent vector field Y on M, we have

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y).$$

This gives

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y) = 0.$$

From this, replacing Y by ϕY for any $Y \in \mathfrak{D}$, we have

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)^2\eta(Y) = 0.$$

Using a similar method as in [10], we have two cases. First, in the case where $\eta(Y) \neq 0$ for some $Y \in \mathfrak{D}$, we have $\eta_\nu(\xi) = 0$ for $\nu = 1, 2, 3$. This means $\xi \in \mathfrak{D}$. Next, in the case where $\eta(Y) = 0$ for any $Y \in \mathfrak{D}$, we have $\xi \in \mathfrak{D}^{\perp}$. This completes the proof of our lemma. □

By virtue of lemma 4.1, in order to give the proof of theorem 1.2, in § 5 we consider the case where M has a geodesic Reeb flow, including both vanishing and non-vanishing Reeb flow, with $\xi \in \mathfrak{D}$. In § 6, completing the proof of theorem 1.2, we shall discuss the remaining case when M has vanishing geodesic Reeb flow with $\xi \in \mathfrak{D}^{\perp}$.

5. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$

In this section, let us show that the distribution $\mathfrak{D}$ of a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

The Reeb vector ξ is said to be a *Hopf* vector if it is a principal vector for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$, that is, the Reeb vector ξ is invariant under the shape operator A.

On the other hand, it was proved in [3] that the Reeb vector ξ of M belongs to the distribution $\mathfrak{D}$ when M is a hypersurface of type (ii) in theorem 1.1. Naturally we are able to consider a converse problem. It should be an interesting problem to check whether a real hypersurface of type (ii), that is, a tube over a totally real totally geodesic $\mathbb{H}P^n$, $m = 2n$, is always a hypersurface with its Reeb vector ξ belonging to the distribution $\mathfrak{D}$.

From such a viewpoint, we affirmatively answer this question in [7] as follows.

Theorem 5.1. *Let M be a connected orientatable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Now it remains to check whether or not the Ricci tensor of a real hypersurface M of type (ii) is parallel. So let us suppose that the Ricci tensor S is parallel. That is, $(\nabla_Y S)X = 0$ for any vector fields X and Y tangent to M. In this case $\xi \in \mathfrak{D}$. If we set $X = \xi$ in (4.5), the parallel Ricci tensor implies

$$\begin{aligned} 0 &= (\nabla_Y S)\xi \\ &= -3\nabla_Y\xi - 3\sum_{\nu=1}^{3}(\nabla_Y\eta_\nu)(\xi)\xi_\nu \\ &\quad + \sum_{\nu=1}^{3}\{-(\nabla_Y\eta)(\phi_\nu\xi)\phi_\nu\xi - \eta((\nabla_Y\phi_\nu)\xi)\phi_\nu\xi\} + h(\nabla_Y A)\xi - (\nabla_Y A^2)\xi. \end{aligned}$$

Since we have assumed that M is a Hopf hypersurface, it follows that

$$\begin{aligned} 0 = -3\phi AY + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AY)\phi_\nu\xi \\ + \alpha h\phi AY - hA\phi AY - \alpha^2\phi AY + A^2\phi AY. \end{aligned} \tag{5.1}$$

Now let us apply a proposition from [3] as follows.

Proposition 5.2. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and that ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \qquad \beta = 2\cot(2r), \qquad \gamma = 0, \qquad \lambda = \cot(r), \qquad \mu = -\tan(r)$$

with some $r \in (0, \frac{1}{4}\pi)$. The corresponding multiplicities are

$$m(\alpha) = 1, \qquad m(\beta) = 3 = m(\gamma), \qquad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \qquad T_\beta = \mathfrak{J}J\xi, \qquad T_\gamma = \mathfrak{J}\xi, \qquad T_\lambda, \qquad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \qquad \mathfrak{J}T_\lambda = T_\lambda, \qquad \mathfrak{J}T_\mu = T_\mu, \qquad JT_\lambda = T_\mu.$$

Setting $Y = \xi_1 \in T_\beta$ in (5.1), by proposition 5.2, we have

$$(\alpha h - \alpha^2 - 4)\beta = 0. \tag{5.2}$$

On the other hand, the trace h of type (ii) is given by

$$\begin{aligned} h &= \alpha + 6\cot 2r + (4n-4)(\cot r - \tan r) \\ &= \alpha + (4n-1)(\cot r - \tan r). \end{aligned}$$

Substituting this into (5.2), we have $0 = -16n$, which gives a contradiction.

6. Real hypersurfaces with vanishing geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^{\perp}$

Now let us consider a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor and vanishing geodesic Reeb flow, that is, $\alpha = 0$, satisfying $\xi \in \mathfrak{D}^{\perp}$. Since we assume $\xi \in \mathfrak{D}^{\perp} = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$, there exists a Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1 N$, that is, $\xi = \xi_1$.

Now setting $\xi = \xi_1$ in (4.8), we have

$$\begin{aligned} 0 &= (h\alpha - \alpha^2 - 3)\phi AY + Y(\alpha h)\xi_1 - hA\phi AY - (Y\alpha^2)\xi_1 + A^2\phi AY \\ &\quad - 4[\{q_3(Y)\eta_2(\xi) - q_2(Y)\eta_3(\xi)\}\xi_1 + \{q_1(Y)\eta_3(\xi) - q_3(Y)\}\xi_2 \\ &\qquad + \{q_2(Y) - q_1(Y)\eta_2(\xi)\}\xi_3] \\ &\quad - 5g(\phi_2 AY, \xi)\xi_2 - 5g(\phi_3 AY, \xi)\xi_3 - 4q_3(Y)\xi_2 + 4q_2(Y)\xi_3 - 3\phi_1 AY \\ &\quad - \{\eta_2(\xi)\eta(AY) + g(\phi AY, \phi_2\xi)\}\phi_2\xi - \{\eta_3(\xi)\eta(AY) + g(\phi AY, \phi_3\xi)\}\phi_3\xi \\ &= (h\alpha - \alpha^2 - 3)\phi AY + Y(\alpha h)\xi_1 - hA\phi AY - (Y\alpha^2)\xi \\ &= A^2\phi AY - 3\phi_1 AY + 6\eta_2(A\xi)\xi_3 - 6\eta_3(AY)\xi_2 - 3\phi_1 AY. \end{aligned} \tag{6.1}$$

Substituting (4.10), this can be rearranged as follows:

$$\begin{aligned} 0 &= \{h\alpha - \alpha^2 - 3 - \tfrac{1}{2}\alpha h\}\phi AY \\ &\quad - \tfrac{1}{2}\alpha h A\phi Y + Y(\alpha h)\xi_1 - (Y\alpha^2)\xi + A^2\phi AY - h\phi Y \\ &\quad - h\sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y \\ &\qquad - 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} \\ &\quad + 6\eta_2(AY)\xi_3 - \eta_3(AY)\xi_2 - 3\phi_1 AY \\ &= \{h\alpha - \alpha^2 - 3 - \tfrac{1}{2}\alpha h\}\phi AY \\ &\quad - \tfrac{1}{2}\alpha h A\phi Y + Y(\alpha h)\xi_1 - (Y\alpha^2)\xi + A^2\phi AY - h\phi Y \\ &\quad - h\{\phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2\} \\ &\quad + 6\eta_2(AY) - \eta_3(AY)\xi_2 - 3\phi_1 AY, \end{aligned} \tag{6.2}$$

where we have used the formulae $\eta_2(\phi Y) = \eta_3(Y)$ and $\eta_3(\phi Y) = -\eta_2(Y)$.

By taking the inner product of (6.2) with ξ, we know $Y(\alpha h - \alpha^2) = 0$. This, together with (6.2), gives

$$\begin{aligned} A^2\phi AY &= \tfrac{1}{2}\alpha h A\phi Y - \{h\alpha - \alpha^2 - 3 - \tfrac{1}{2}\alpha h\}\phi AY \\ &\quad + h\{\phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2\} \\ &\quad - 6\eta_2(AY)\xi_3 + 6\eta_3(AY)\xi_2 + 3\phi_1 AY. \end{aligned} \tag{6.3}$$

On the other hand, we have assumed that the structure vector ξ is principal. Denote by $\mathfrak{H}$ the orthogonal complement of the real span $[\xi]$ of the structure vector ξ in TM. Then if we take the inner product of the Codazzi equation in § 3 with ξ and use $A\xi = \alpha\xi$, we again obtain formula (4.9).

Setting $X = \xi$ in (4.9), we have

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any tangent vector field Y on M. Substituting this formula into (4.9), it can be rewritten as follows:

$$\begin{aligned} & -2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &\quad = 4\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\eta_\nu(\xi) \\ &\qquad + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned} \tag{6.4}$$

From this formula, setting $\xi = \xi_1$, we are able to assert that

$$A\phi AX = \tfrac{1}{2}\alpha(A\phi + \phi A)X + \phi X + \{\phi_1 X - 2\eta_2(X)\xi_3 + 2\eta_3(X)\xi_2\}, \tag{6.5}$$

where we have used the formula $\eta_2(\phi X) = \eta_3(X)$ and $\eta_3(\phi X) = -\eta_2(X)$.

By applying the shape operator A to (6.5) from the left-hand side and using (6.5) once more, we have

$$\begin{aligned} 2A^2\phi Y = {} & 2\{A\phi_1 Y - 2\eta_2(Y)A\xi_3 + 2\eta_3(Y)A\xi_2\} + \alpha A^2\phi Y + 2A\phi Y \\ & + \tfrac{1}{2}\alpha^2(A\phi + \phi A)Y + \alpha\phi Y + \alpha\{\phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2\} \end{aligned} \tag{6.6}$$

for any vector field Y on M. Then, by setting $Y = \xi_2$ in both (6.3) and (6.6), the formula is as follows for $\alpha = 0$:

$$3\phi A\xi_2 - 2h\xi_3 - 6\eta_2(A\xi_2)\xi_3 + 6\eta_3(A\xi_2)\xi_2 + 3\phi_1 A\xi_2 = -2A\xi_3.$$

From this, by taking the inner product with ξ_2, we have $g(A\xi_3, \xi_2) = 0$, and the formula becomes

$$3\phi A\xi_2 - 2h\xi_3 - 6\eta_2(A\xi_2)\xi_3 + 3\phi_1 A\xi_2 = -2A\xi_3. \tag{6.7}$$

Similarly, we also have

$$3\phi A\xi_3 + 6\eta_3(A\xi_3)\xi_2 + 3\phi_1 A\xi_3 + 2h\xi_2 = 2A\xi_2. \tag{6.8}$$

Then, by applying ϕ and ϕ_1 to (6.3), we have, respectively,

$$\phi A\xi_3 = \tfrac{3}{2}A\xi_2 + h\xi_2 + 3\eta_2(A\xi_2)\xi_2 - \tfrac{3}{2}\phi\phi_1 A\xi_2$$

and

$$\phi_1 A\xi_3 = -\tfrac{3}{2}\phi_1\phi A\xi_2 - h\xi_2 - 3\eta_2(A\xi_2)\xi_2 + \tfrac{3}{2}A\xi_2.$$

Substituting these two formulae into (6.8) gives

$$-7A\xi_2 = -9\phi\phi_1 A\xi_2 + 6\eta_3(A\xi_3)\xi_2 + 2h\xi_2. \tag{6.9}$$

Applying ϕ to (6.9), we have

$$7\phi A\xi_2 = -9\phi_1 A\xi_2 + 6\eta_3(A\xi_3)\xi_3 + 2h\xi_3. \tag{6.10}$$

Applying ϕ_1 to (6.9) and substituting (6.10), we have

$$\begin{aligned} -7\phi_1 A\xi_2 &= -9\phi_1^2\phi A\xi_2 + 6\eta_3(A\xi_3)\phi_1\xi_2 + 2h\phi_1\xi_2 \\ &= 9\phi A\xi_2 + 6\eta_3(A\xi_3)\xi_3 + 2h\xi_3 \\ &= 9(-\tfrac{9}{7}\phi_1 A\xi_2 + \tfrac{6}{7}\eta_3(A\xi_3)\xi_3 + \tfrac{2}{7}h\xi_3) + 6\eta_3(A\xi_3)\xi_3 + 2h\xi_3, \end{aligned}$$

which gives $\phi_1 A\xi_2 = 3\eta_3(A\xi_3)\xi_3 + h\xi_3$. Accordingly, we can write

$$A\xi_2 = 3\eta_3(A\xi_3)\xi_2 + h\xi_2.$$

Similarly, we have

$$A\xi_3 = 3\eta_2(A\xi_2)\xi_3 + h\xi_3.$$

These two formulae for a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ give the invariancy of the shape operator A of M, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. Then, by virtue of theorem 1.1, we deduce that M is locally congruent to a tube of certain radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with vanishing geodesic Reeb flow, $\alpha = 0$. However, in proposition 5.2 the principal curvature α never vanishes. Consequently, such a case is not possible. This completes the proof of theorem 1.2.

Remark 6.1. It was proved in [11] that there do not exist any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator, i.e. $\nabla A = 0$. Such a geometric condition is stronger than the parallelism of the Ricci tensor mentioned in this paper.

Remark 6.2. Suh [12] proved the non-existence property of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, $A\phi_i = \phi_i A$, $i = 1, 2, 3$, and in [16] gave a characterization of type (i) in theorem 1.1 in terms of the commuting Ricci tensor, i.e. $S\phi = \phi S$.

Remark 6.3. Corollary 1.3 was also proved in [10]. By giving a classification of pseudo-Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$ we have shown that there does not exist any Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Acknowledgements

This work was supported by Grant no. BSRP-2011-0025687 and Project no. NRF-2011-220-C00002 from the National Research Foundation of Korea.

References

1 D. V. Alekseevskii. Compact quaternion spaces. *Funct. Analysis Applic.* **2** (1967), 106–114.
2 J. Berndt. Riemannian geometry of complex two-plane Grassmannians. *Rend. Sem. Mat. Univ. Polit. Torino* **55** (1997), 19–83.
3 J. Berndt and Y. J. Suh. Real hypersurfaces in complex two-plane Grassmannians. *Monatsh. Math.* **127** (1999), 1–14.
4 J. Berndt and Y. J. Suh. Isometric flows on real hypersurfaces in complex two-plane Grassmannians. *Monatsh. Math.* **137** (2002), 87–98.

5 M. Kimura. Real hypersurfaces of a complex projective space. *Bull. Austral. Math. Soc.* **33** (1986), 383–387.

6 M. Kimura and S. Maeda. On real hypersurfaces of a complex projective space. II. *Tsukuba J. Math.* **15** (1991), 547–561.

7 H. Lee and Y. J. Suh. Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector. *Bull. Korean Math. Soc.* **47** (2009), 551–561.

8 J. D. Pérez. Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} A = 0$. *J. Geom.* **49** (1994), 166–177.

9 J. D. Pérez and Y. J. Suh. Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$. *Diff. Geom. Applic.* **7** (1997), 211–217.

10 J. D. Pérez, Y. J. Suh and Y. Watanabe. Generalized Einstein hypersurfaces in complex two-plane Grassmannians. *J. Geom. Phys.* **60** (2010), 1806–1818.

11 Y. J. Suh. Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator. *Bull. Austral. Math. Soc.* **67** (2003), 493–502.

12 Y. J. Suh. Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator. *Bull. Austral. Math. Soc.* **68** (2003), 379–393.

13 Y. J. Suh. Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator. II. *J. Korean Math. Soc.* **41** (2004), 535–565.

14 Y. J. Suh. Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives. *Can. Math. Bull.* **49** (2006), 134–143.

15 Y. J. Suh. Real hypersurfaces of type B in complex two-plane Grassmannians. *Monatsh. Math.* **141** (2006), 337–355.

16 Y. J. Suh. Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor. *J. Geom. Phys.* **60** (2010), 1792–1805.

(*Issued 7 December 2012*)