Generic point equivalence and Pisot numbers

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Abstract. Let $\beta > 1$ be an integer or, generally, a Pisot number. Put $T(x) = \{\beta x\}$ on [0, 1] and let $S : [0, 1] \to [0, 1]$ be a piecewise linear transformation whose slopes have the form $\pm \beta^m$ with positive integers m. We give a sufficient condition for T and S to have the same generic points. We also give an uncountable family of maps which share the same set of generic points.

1. *Introduction*

Let $b \geq 2$ be an integer and $T : [0, 1] \to [0, 1]$ be the map given by $T(x) = \{bx\}$, where $\{x\}$ denotes the fractional part of x. A real number $x \in [0, 1]$ is said to be normal in base b if in the base-b expansion of x any pattern of length L appears with relative frequency tending to b^{-L}. Wall [27] showed that x is normal in base b if and only if x is a T-generic point, that is, its orbital points $x, T(x), T^2(x), \ldots$ distribute uniformly. We recall that non-zero integers m and n are multiplicatively dependent if there exists $(i, j) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$ satisfying $m^i n^j = 1$. Maxfield [16] proved that if two positive integers b_1, b_2 are multiplicatively dependent, then base-b_1 normality is equivalent to base-b_2 normality. Schweiger [23] and Vandehey [25] showed that if two transformations T and S satisfy some mild conditions and $T^m = S^n$ for some positive integers n, m, then T-normality is equivalent to S-normality. Kraaikamp and Nakada [14] gave counterexamples to show that the other direction does not hold. They used the jump transformation to show the equivalence of normality: *normality equivalence*, in short.

   In this paper we relax a sufficient condition for normality equivalence and obtain infinite families of examples (see Examples 4.1 and 4.3). Moreover, we generalize the concept of

normality equivalence to include systems whose invariant measures may be different. Let $(X, \mathbb{B}, \mu, T)$ and $(X, \mathbb{B}, \nu, S)$ be two ergodic measure-preserving systems with a common underlying space X. We assume that X is a compact metric space, $\mathbb{B}$ is the sigma-algebra of Borel sets in X, and μ, ν are probability measures. A point $x \in X$ is called T-generic if $\lim_{N \to \infty} (1/N) \sum_{n=0}^{N-1} f(T^n x) = \int_X f \, d\mu$ for any continuous function f on X. We say that S and T are *generic point equivalent* if the set of S-generic points coincides with the set of T-generic points. The main purpose of this paper is to give sufficient conditions for generic point equivalence for $X = [0, 1]$, using the Pyatetskii–Shapiro criterion.

Let β be a *Pisot number*: a real algebraic integer greater than 1 whose Galois conjugates (except itself) have modulus less than 1. Note that any integer greater than 1 is a Pisot number. Put $T(x) = \{\beta x\}$ on $[0, 1]$. Let $S : [0, 1] \to [0, 1]$ be a piecewise linear transformation. In §3 we give a sufficient condition for generic point equivalence of S and T in the case where the slopes of S have the form $\pm \beta^m$ with positive integers m. More precisely, we show that if S admits an absolutely continuous invariant measure and the invariant density is bounded above and away from 0 and all intercepts are in $\mathbb{Q}(\beta)$, then T and S are generic point equivalent. In §2 we give Proposition 2.2, which can be used to prove generic point equivalence. Using this proposition, we shall prove our main result.

The Pisot slope condition is essential: our proof depends on the structure of the point set generated by Pisot numbers. The proof becomes simpler than those in the literature and applicable to a wide class of piecewise linear maps. In fact, we require no condition on the position of discontinuities. In particular, we provide a one-parameter family of maps (the cardinality of the maps is uncountable) by continuously shifting the discontinuity so that all the maps in the family are generic point equivalent (see Example 4.4). This appears to be the first result on generic point equivalence among generically non-Markov piecewise linear maps.

2. *Criteria for generic point equivalence*
We now review the Pyatetskii–Shapiro criterion. Let $(X, \mathbb{B}, \mu, T)$ be an ergodic measure-preserving system. Denote the characteristic function of $V \in \mathbb{B}$ by χ_V and the set of continuous functions on X by $C(X)$. Let $\mathcal{C} \subset \mathbb{B}$ be a semi-algebra generating $\mathbb{B}$ in the sense that the minimal sigma-algebra including $\mathcal{C}$ is $\mathbb{B}$. Then the Pyatetskii–Shapiro criterion reads as follows (see also [17]).

THEOREM 2.1. [21, Theorem 6] *Let* $(X, \mathbb{B}, \mu, T)$ *be an ergodic measure-preserving system. Let* $x_0 \in X$ *and* $\mathcal{C} \subset \mathbb{B}$ *be a semi-algebra generating* $\mathbb{B}$. *We assume that any function* $f \in C(X)$ *is a limit point of the set of the (finite) linear combinations of the characteristic functions of* $V \in \mathcal{C}$ *with respect to the sup norm. Suppose that there exists a positive constant* C *satisfying*

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_I (T^n x_0) \leq C \mu(I) \tag{2.1}$$

for any $I \in \mathcal{C}$. *Then* x_0 *is a* T-*generic point.*

We now introduce a criterion for generic point equivalence deduced from Theorem 2.1.

PROPOSITION 2.2. *Let* $([0, 1], \mathbb{B}, \mu, T)$ *and* $([0, 1], \mathbb{B}, \nu, S)$ *be two ergodic measure-preserving systems. Let* $\mathcal{C}$ *be a semi-algebra generating* $\mathbb{B}$.

Let $x_0 \in [0, 1]$ *be a* T-*generic point. Suppose that there exist a positive integer* M, *a positive real number* C, *and a sequence* $(k(n))_{n=0}^{\infty}$ *of non-negative integers satisfying the following assertions.*

(1) *Let* $I \in \mathcal{C}$. *Then there exists* $\widetilde{I} = \bigcup_{i=1}^{r} \widetilde{I}_i$, *where* $\widetilde{I}_1, \ldots, \widetilde{I}_r$ *are subintervals of* $[0, 1]$, *such that*

$$\mu(\widetilde{I}) \leq C\nu(I)$$

and that, for any $n \geq 0$,

$$\text{if } S^n x_0 \in I \text{ then } T^{k(n)} x_0 \in \widetilde{I}.$$

(2) *For any non-negative integer* m, *we have*

$$\text{Card}\{n \geq 0 \mid k(n) = m\} \leq M,$$

where Card denotes the cardinality.

(3) *For any* $n \geq 0$, *we have*

$$k(n) \leq M \cdot \max\{1, n\}.$$

Then x_0 *is an* S-*generic point.*

Proof. Let $I \in \mathcal{C}$ and N be an integer greater than 1. Put

$$\rho(N) := \max\{k(n) \mid 0 \leq n \leq N - 1\} \leq M(N - 1) \leq MN - 1.$$

Then we see that

$$\frac{1}{N} \sum_{n=0}^{N-1} \chi_I(S^n x_0) = \frac{1}{N} \sum_{m=0}^{\rho(N)} \sum_{\substack{k(n)=m \\ 0 \leq n \leq N-1}} \chi_I(S^n x_0)$$

$$\leq \frac{1}{N} \sum_{m=0}^{\rho(N)} \sum_{\substack{k(n)=m \\ 0 \leq n \leq N-1}} \chi_{\widetilde{I}}(T^m x_0)$$

$$\leq \frac{M}{N} \sum_{m=0}^{\rho(N)} \chi_{\widetilde{I}}(T^m x_0) \leq M^2 \cdot \frac{1}{MN} \sum_{m=0}^{MN-1} \chi_{\widetilde{I}}(T^m x_0).$$

Since x_0 is T-generic, we get

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_I(S^n x_0) \leq M^2 \mu(\widetilde{I}) \leq M^2 C \nu(I),$$

which implies by Theorem 2.1 that x_0 is S-generic. □

Remark 2.3. Proposition 2.2 can be generalized for two ergodic measure-preserving systems $([0, 1]^d, \mathbb{B}, \mu, T)$ and $([0, 1]^d, \mathbb{B}, \nu, S)$.

3. *Pisot slope condition*

Let $\mathbb{N}$ be the set of positive integers. Given $\beta > 1$, let $T(x) = \{\beta x\}$ be a map on $[0, 1]$. Then T is ergodic with respect to a unique absolutely continuous invariant measure μ_β whose density is bounded and away from 0 (see [**18**]). Let $[0, 1] = \bigcup_{i=1}^{\ell} J_i$ be a finite partition of $[0, 1]$ into subintervals†. Let $S : [0, 1] \to [0, 1]$ be a transformation given by

$$S(x) = \epsilon_i \beta^{m_i} x + b_i \quad \text{for } x \in J_i,$$

where $\epsilon_i \in \{-1, 1\}$, $m_i \in \mathbb{N}$ and $b_i \in \mathbb{Q}(\beta)$ for $1 \le i \le \ell$.

For any $x \in [0, 1]$ and $h \ge 0$, let $i(h) = i(x; h)$ be defined by $S^h(x) \in J_{i(h)}$. Then we have, for any $n \ge 0$,

$$S^n(x) = \left(\prod_{h=0}^{n-1} \epsilon_{i(h)}\right) \beta^{\sum_{h=0}^{n-1} m_{i(h)}} x + \sum_{j=0}^{n-1} \left(\prod_{h>j}^{n-1} \epsilon_{i(h)}\right) \beta^{\sum_{h>j}^{n-1} m_{i(h)}} b_{i(j)}. \tag{3.1}$$

Put

$$\theta_n(x) := \sum_{h=0}^{n-1} m_{i(h)}, \tag{3.2}$$

where $\beta^{\theta_n(x)}$ gives the absolute value of the slope of S^n at x. Henceforth, unless explicitly stated otherwise, we assume that β is a Pisot number.

A subset Y of $\mathbb{R}$ is *uniformly discrete* if there exists a positive constant R such that for any two distinct points $y, y' \in Y$, we have $|y - y'| > R$.

LEMMA 3.1. *Let E be a finite subset of $\mathbb{Q}(\beta)$ and put*

$$F_E := \left\{ \sum_{j=0}^{r} d_j \beta^j \ \middle| \ d_j \in E, r = 0, 1, 2, \ldots \right\}.$$

Then F_E is uniformly discrete.

This follows from a standard discussion (e.g. Garsia [**3**]), but we show it for completeness.

Proof. Without loss of generality, we may assume that $0 \in E$. We claim that 0 is not an accumulation point of F_E. In fact, let $\beta^{(j)}$ be the Galois conjugates of β for $j = 1, \ldots, d$ with $\beta^{(1)} = \beta$. Take a positive integer L such that $E \subset (1/L)\mathbb{Z}[\beta]$. Suppose that $0 \ne \sum_{j=0}^{r} d_j \beta^j \in F_E$. Considering the image of the Galois conjugate map ϕ_i which sends β to $\beta^{(i)}$, we obtain

$$\left| L^d \prod_{i=1}^{d} \sum_{j=0}^{r} \phi_i(d_j)(\beta^{(i)})^j \right| \ge 1$$

because the product must be an integer. Since β is a Pisot number, we obtain

$$\left| \sum_{j=0}^{r} d_j \beta^j \right| \ge \frac{1}{L^d} \prod_{i=2}^{d} \left(\frac{A_i}{1 - |\beta^{(i)}|} \right)^{-1},$$

† Subinterval J_i can be closed, open or semi-open, even a singleton.

where $A_i = \max\{|\phi_i(d)| \mid d \in E\}$ is a positive constant because E is a finite set. This shows the claim. Note that the condition $0 \in E$ implies $F_E - F_E = F_{E-E}$. By the same proof replacing E by $E - E$, we obtain the assertion. □

If S and T are generic point equivalent, then the set of non-generic points of T and that of S are identical. Thus we may expect that eventually periodic orbits of T and those of S coincide. The next theorem confirms this expectation that T and S share the same set of eventually periodic orbits.

THEOREM 3.2. *The orbit $S^n(x)$ for $n = 0, 1, \ldots$ is eventually periodic if and only if $x \in \mathbb{Q}(\beta)$.*

Proof. Because $b_i \in \mathbb{Q}(\beta)$, every eventually periodic point of S belongs to $\mathbb{Q}(\beta)$. Assume that $x \in \mathbb{Q}(\beta)$. Take a positive integer L such that Lx and Lb_j are in $\mathbb{Z}[\beta]$. Then for all $n \geq 0$ we have $LS^n(x) \in \mathbb{Z}[\beta]$ by (3.1). Since β is a Pisot number, for each $i = 2, \ldots, d$ there is a constant $C_i > 0$ such that $|\phi_i(LS^n(x))| \leq C_i$ for all $n \geq 0$. We also have $|\phi_1(LS^n(x))| \leq L$. Since the image of the Minkowski embedding of $\mathbb{Z}[\beta]$ forms a lattice in $\mathbb{R}^d$, the orbit is eventually periodic. □

We are now in a position to state our main theorem.

THEOREM 3.3. *Let T, S be the maps defined above. Suppose that S preserves a probability measure ν, which is ergodic and absolutely continuous with respect to the Lebesgue measure λ. Moreover, assume that there exists a positive constant c satisfying*

$$c^{-1}\lambda(E) \leq \nu(E) \leq c\lambda(E) \tag{3.3}$$

for any Borel set $E \subset [0, 1]$. Then T and S are generic point equivalent.

Condition (3.3) implies that ν and λ are equivalent. Kowalski [13] showed under ergodicity of S that the converse holds as well in this setting.

Proof. If necessary, changing the constant c, we may assume that

$$c^{-1}\lambda(E) \leq \mu_\beta(E) \leq c\lambda(E)$$

for any Borel set $E \subset [0, 1]$ (Parry [18], Ito and Takahashi [7]). Let

$$E = \{\pm b_1, \pm b_2, \ldots, \pm b_k\} \cup \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}.$$

Putting $F := F_{E-E} = F_E - F_E$, we get by Lemma 3.1 that F is uniformly discrete.

First we assume that $x_0 \in [0, 1]$ is a T-generic point. For each $n \geq 0$, let $k(n) := \theta_n(x_0)$ be defined by (3.2). Then we see that

$$S^n(x_0) = \epsilon(\beta^{k(n)}x_0 - b), \quad T^{k(n)}(x_0) = \beta^{k(n)}x_0 - b',$$

for some $\epsilon \in \{1, -1\}$ and $b, b' \in F_E$. We now verify that $(k(n))_{n=0}^\infty$ and $M := \max\{m_1, \ldots, m_\ell\}$ satisfy the assumptions of Proposition 2.2, where $\mathcal{C}$ is the set of subintervals of $[0, 1]$. The first and the second assumptions are clear by $1 \leq k(n+1) - k(n) \leq M$ for any $n \geq 0$. For any interval I, put

$$\tilde{I} = \left(\bigcup_{t \in F \cap [-1, 2]} ((I + t) \cup (-I + t)) \right) \cap [0, 1].$$

Then we have

$$\mu_\beta(\tilde{I}) \le 2c \, \mathrm{Card}(F \cap [-1, 2])\lambda(I) \le 2c^2 \, \mathrm{Card}(F \cap [-1, 2])\nu(I).$$

We now assume for $n \ge 0$ that $S^n(x_0) \in I$. Noting that

$$b - b' = T^{k(n)}(x_0) - \epsilon S^n(x_0) \in [-1, 2] \cap F,$$

we obtain

$$T^{k(n)}(x_0) = \epsilon S^n(x_0) + (b - b') \in \tilde{I}.$$

Hence, x_0 is S-generic by Proposition 2.2.

We prove the other direction. Let $x_0 \in [0, 1]$ be an S-generic point. For each $n \ge 0$, we define $k(n)$ by

$$k(n) := \max\{k \mid \theta_k(x_0) \le \beta^n\}.$$

For any $h \ge 0$, we see that $k(n) = h$ if and only if

$$\theta_h(x_0) \le n < \theta_{h+1}(x_0) = \theta_h(x_0) + m_{i(h)}. \tag{3.4}$$

Moreover, we see for any $n \ge 0$ that

$$\theta_{k(n)}(x_0) = \beta^{n-j}, \tag{3.5}$$

for some $0 \le j < M = \max\{m_1, \ldots, m_\ell\}$. In what follows, we show that $(k(n))_{n=0}^\infty$ and M satisfy the assumptions of Proposition 2.2. The first and second assumptions are clear by (3.4) and $0 \le k(n+1) - k(n) \le 1$ for any $n \ge 0$. For any interval $I \subset [0, 1]$, put

$$\tilde{I} = \left(\bigcup_{j=0}^{M-1} \bigcup_{t \in F \cap [-1, 2]} ((T^{-j}(I) + t) \cup (-T^{-j}(I) + t)) \right) \cap [0, 1].$$

Then we get

$$\nu(\tilde{I}) \le 2c \, \mathrm{Card}(F \cap [-1, 2])M \max_{0 \le j \le M-1} \lambda(T^{-j}(I))$$

$$\le 2c^2 \, \mathrm{Card}(F \cap [-1, 2])M\mu_\beta(I).$$

Suppose for $n \ge 0$ that $T^n(x_0) \in I$. Let j be defined by (3.5). In the same way as the former part of the proof of Theorem 3.3, we get

$$\epsilon T^{n-j}(x_0) + S^{k(n)}(x_0) \in F \cap [-1, 2],$$

for some $\epsilon \in \{1, -1\}$. Therefore, we deduce that

$$S^{k(n)}(x_0) = -\epsilon T^{n-j}(x_0) + (\epsilon T^{n-j}(x_0) + S^{k(n)}(x_0)) \in \tilde{I}.$$

Now we apply Proposition 2.2 to complete the proof.   □

Remark 3.4. It is natural to assume that all slopes in modulus are certain powers of a fixed number, since we cannot expect generic point equivalence for multiplicatively independent slopes. Indeed, if a and b are multiplicatively independent positive integers, then Schmidt [22] showed that there are uncountably many a-normal numbers which are not b-normal. Moreover, Pollington [20] calculated the Hausdorff dimension of

such numbers. Consider a partition of the set $\{2, 3, \ldots\}$ into A and B so that all multiplicatively dependent integers fall into the same class. Then the set of real numbers normal in any base from A and in no base from B has Hausdorff dimension 1. Explicit construction of numbers which are a-normal but not b-normal is exploited when a divides b (e.g. [**9**, **10**, **26**]). However, we do not yet know a concrete example of a 2-normal number which is not 3-normal.

Remark 3.5. Theorem 3.3 does not extend to an infinite partition, due to an example by Jäger [**8**] for the case of $\beta = 10$. Let $T = \{10x\}$ on $[0, 1]$ and $x = (0.x_1x_2 \ldots)$ be the coding of x by T, that is, the decimal expansion of x. Let m be the first occurrence of a fixed digit $r \in \{0, 1, \ldots, 9\}$ where $x_m = r$; then we define a jump transform $S_r(x) := (0.x_{m+1}x_{m+2} \ldots)$. If there is no occurrence of r, put $S_r(x) := 0$. Then every T-generic point is S_r-generic, but the converse does not hold.

Remark 3.6. We show that condition (3.3) is not preserved after taking flips. Let $\beta > 1$ be a real number, and $0 = t_0 < t_1 < \cdots < t_k = 1$ is a finite partition of $[0, 1]$. Suppose that T is a map on $[0, 1]$ which has slope $\pm\beta^{m_i}$ on $[t_{i-1}, t_i)$ and has an invariant measure which is equivalent to the Lebesgue measure. If S is a locally flipped map of T on $[0, 1]$, that is, on one interval $[t_{i-1}, t_i)$, S has the opposite slope $\mp\beta^{m_i}$ and $T((t_{i-1} + t_i)/2) = S((t_{i-1} + t_i)/2)$, then one might expect that S also has an invariant measure equivalent to the Lebesgue measure. Unfortunately, this is not true. Here is a counterexample. Let $1 < \beta < \sqrt{2}$ and put

$$S(x) = \begin{cases} -\beta x + 1, & x \in [0, 1/\beta), \\ \beta x - 1, & x \in [1/\beta, 1]. \end{cases}$$

The map S is a locally flipped map of the beta transformation T having density away from zero. Since the dynamics of S on $[\beta - 1, -\beta^2 + \beta + 1]$ is dissipative, the density of S on $[\beta - 1, -\beta^2 + \beta + 1]$ is zero. The explicit densities of flipped beta expansions are given in Gora [**4**].

4. *Examples*

We apply Theorem 3.3 to certain families of piecewise linear maps on $[0, 1]$.

Example 4.1. Let r be an integer greater than 1. For $\mathbf{s} = (s_0, s_1, \ldots, s_{r-1}) \in \{0, 1\}^r$, let $T(r, \mathbf{s}; x) : [0, 1] \to [0, 1]$ be a map defined by

$$T(r, \mathbf{s}; x) := \begin{cases} s_i + (-1)^{s_i}\{rx\} & \text{if } x \in [i/r, (i+1)/r), \\ 0 & \text{if } x = 1. \end{cases}$$

Then $([0, 1], \mathbb{B}, \lambda, T(r, s; x))$ is an ergodic measure-preserving system, where λ is the Lebesgue measure. Let q be integers greater than 1 and $\mathbf{t} \in \{0, 1\}^q$. Assume that q and r are multiplicatively dependent. Then there exist positive integers b, k, and l such that $q = b^k$ and $r = b^l$. Thus, $T(q, \mathbf{t}; x)$ and $T(r, \mathbf{s}; x)$ are both generic point equivalent to $T(x) = \{bx\}$. As a special case, the tent map

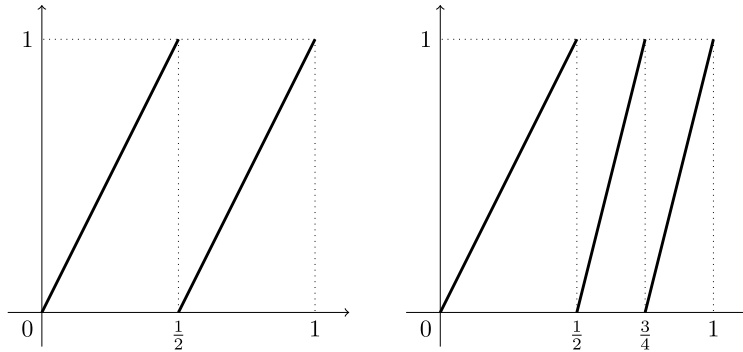$$f(x) = \begin{cases} 2x, & 0 \le x < 1/2, \\ 2(1-x), & 1/2 \le x \le 1, \end{cases}$$

FIGURE 1. The maps T_1 and S_1 in Example 4.2.

and the binary expansion map $T(x) = \{2x\}$ are generic point equivalent. This simple case already seems new. Indeed, this serves an alternative [**1**, proof of Corollary 19] which solves several conjectures posed in [**24**], as the set of 2-normal numbers lies exactly in the third Borel hierarchy by [**11**].

The following examples were shown by Kraaikamp and Nakada in [**14**].

Example 4.2. Consider the maps $T_1 : [0, 1] \to [0, 1]$ and $S_1 : [0, 1] \to [0, 1]$ defined by $T_1(x) = \{2x\}$ and

$$S_1(x) := \begin{cases} 2x, & x \in [0, 1/2), \\ \{4x\}, & x \in [1/2, 1]. \end{cases}$$

Let $\beta = (\sqrt{5} + 1)/2$. Define $T_2 : [0, 1] \to [0, 1]$ and $S_2 : [0, 1] \to [0, 1]$ by $T_2(x) = \{\beta x\}$ and

$$S_2(x) := \begin{cases} \beta x, & x \in [0, 1/\beta), \\ \beta^2 x - \beta, & x \in [1/\beta, 1]. \end{cases}$$

Let $i \in \{1, 2\}$ be fixed. Then Theorem 3.3 implies that $x \in [0, 1]$ is T_i-generic if and only if x is S_i-generic. The graphs of T_1, S_1 and graphs of T_2, S_2 are shown in Figures 1 and 2, respectively.

Examples 4.1 and 4.2 are generalized as follows.

Example 4.3. Let β be a Perron number: an algebraic integer greater than 1 whose conjugates have modulus less than β. Handelman [**6**] showed that β has no other positive conjugates if and only if there exist an $\ell \in \mathbb{N}$ and a non-negative integer vector $(a_1, \ldots, a_\ell)$ satisfying

$$1 = \sum_{i=1}^{\ell} \frac{a_i}{\beta^i}.$$

If there exists such a vector, then there are infinitely many different expressions for 1 of this form. Assume further that β is a Pisot number having no other positive conjugate. For such a vector $(a_1, \ldots, a_\ell)$ we can partition $[0, 1]$ into $a_1 + \cdots + a_\ell$ sub-intervals, a_i intervals
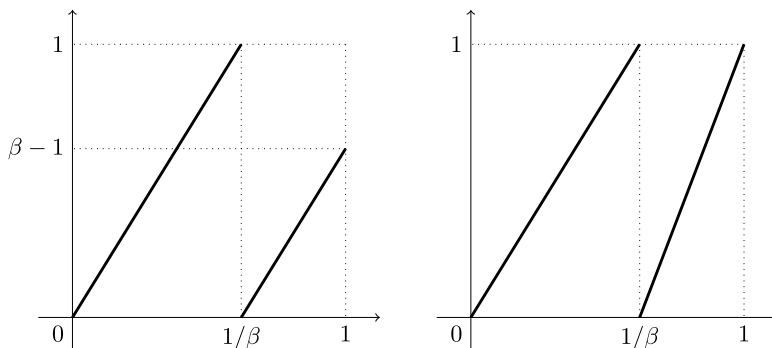
FIGURE 2. The maps T_2 and S_2 in Example 4.2.

of length β^{-i}, arranged in arbitrary order, and construct a piecewise linear transformation S of slopes $\pm\beta^i$ for $i = 1, \ldots, \ell$ all of whose discontinuities are mapped to $\{0, 1\}$. The invariant measure of S is the Lebesgue measure. All the maps S produced from a fixed Pisot number β in this manner are normality equivalent, because all of them are generic point equivalent to $T(x) = \{\beta x\}$ by Theorem 3.3 (cf. [**2**]).

Example 4.4. Take a real number $\beta > 1$ and $t \in [0, \lceil\beta\rceil/\beta - 1]$. Define a map $S_t : [0, 1] \to [0, 1]$ by

$$S_t(x) := \begin{cases} \beta x - \lfloor\beta x\rfloor, & x \in [0, \lfloor\beta\rfloor/\beta - t), \\ \beta(x - 1) + 1 & x \in [\lfloor\beta\rfloor/\beta - t, 1]. \end{cases}$$

See Figure 3 for the graphs of S_t for some t. As the map S_t has only one non-trivial discontinuity at $r_0 = l_0 = \lfloor\beta\rfloor/\beta - t$, it is ergodic with respect to a unique absolutely continuous invariant measure (cf. [**15**]). Its invariant density is made explicit as

$$h(x) = C + \sum_{x \geq r_n} \frac{1}{\beta^n} + \sum_{x < l_n} \frac{1}{\beta^n},$$

where the sums are taken over positive integers n. Here $r_n = S_t^n(\lfloor\beta\rfloor/\beta - t + 0)$ and $l_n = S_t^n(\lfloor\beta\rfloor/\beta - t - 0)$. The constant C is computed as

$$C = \frac{\beta - 2}{\beta - 1} + \sum_{n=1}^{\infty} \frac{\iota^+(n) - \iota^-(n)}{\beta^n},$$

with

$$\iota^+(n) = \begin{cases} 1, & r_n \geq r_0, \\ 0, & r_n < r_0, \end{cases} \quad \text{and} \quad \iota^-(n) = \begin{cases} 1, & l_n \geq l_0, \\ 0, & l_n < l_0. \end{cases}$$

Though C can be negative, we claim for any pair (β, t) that:
(*)   There exists a positive c that $c^{-1} < h(x) < c$ if and only if $\beta \geq 2$.
Its proof is given in the Appendix A. Moreover, we shall show that c depends only on β.

Hence, we see that if β is a Pisot number not less than 2, then the map S satisfies the assumptions in Theorem 3.3. Therefore, if $\beta \notin \mathbb{Z}$, then all maps in the one-parameter
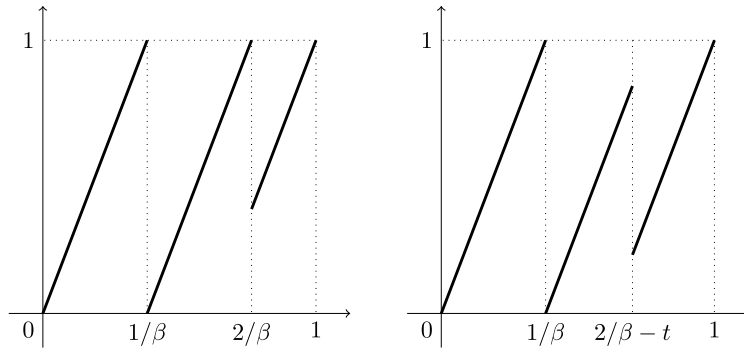
FIGURE 3. The maps S_0 and S_t in Example 4.4 for $\beta = (3 + \sqrt{5})/2$.

family with cardinality of continuum

$$\left\{ S_t \,\middle|\, t \in \mathbb{R},\, 0 \leq t \leq \frac{\lceil \beta \rceil}{\beta} - 1 \right\}$$

are generic point equivalent by Theorem 3.3.

A. Appendix. Positivity of invariant density

To study the invariant densities of a piecewise linear map, a general method is established by Kopf [12] and Gora [5]. It works well for a given map. To deal with the parametrized family of maps in Example 4.4, we follow an analogy of Parry [18, 19] to calculate the invariant density and deduce the claim (*). For simplicity, we write $S = S_t$. When $\beta < 2$, the map S is dissipative in $Y := [0, r_1) \cup [l_1, 1)$ and $h(x) = 0$ in Y. For an integer $\beta > 1$, the map S is the β-adic transformation and preserves the Lebesgue measure. Therefore we have to show that $h(x)$ is positive for $\beta > 2$ and $\beta \notin \mathbb{Z}$. Putting

$$d_n^+(x) = \begin{cases} 1, & x \geq r_n, \\ 0, & x < r_n, \end{cases} \quad \text{and} \quad d_n^-(x) = \begin{cases} 1, & x < l_n, \\ 0, & x \geq l_n, \end{cases}$$

for $n = 1, 2, \ldots$, we see that $h(x) = C + \sum_{n=1}^{\infty} d_n(x)/\beta^n$ with $d_n(x) := d_n^+(x) + d_n^-(x)$. Define the digit $\alpha(x) := \beta x - S(x)$ for $x \in [0, 1)$. Then

$$\mathcal{D} = \{\alpha(x) \,|\, x \in [0, 1)\} = \{0, 1, \ldots, \lfloor \beta \rfloor - 1\} \cup \{\beta - 1\}.$$

Put

$$\mathcal{D}(x) = \begin{cases} \mathcal{D} \backslash \{\beta - 1\}, & x \in [0, r_1), \\ \mathcal{D}, & x \in [r_1, l_1), \\ \mathcal{D} \backslash \{\lfloor \beta \rfloor - 1\}, & x \in [l_1, 1), \end{cases}$$

$$e_n^+(x) = \text{Card}\{d \in \mathcal{D}(x) \,|\, d > \alpha(r_n)\} - \begin{cases} 1 & \alpha(r_n) = \lfloor \beta \rfloor - 1 \text{ and } x \geq l_1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$e_n^-(x) = \text{Card}\{d \in \mathcal{D}(x) \,|\, d < \alpha(l_n)\} - \begin{cases} 1 & \alpha(l_n) = \beta - 1 \text{ and } x < r_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we observe the key equality:

$$\sum_{y \in S^{-1}(\{x\})} d_n(y) = e_n(x) + d_{n+1}(x)$$

with $e_n(x) := e_n^+(x) + e_n^-(x)$. Therefore

$$\frac{1}{\beta} \sum_{y \in S^{-1}(\{x\})} h(y) = \frac{1}{\beta} \sum_{y \in S^{-1}(\{x\})} C + \sum_{n=1}^{\infty} \frac{1}{\beta^{1+n}} \sum_{y \in S^{-1}(\{x\})} d_n(y)$$

$$= \frac{(\lfloor \beta \rfloor - 1 + d_1(x)) C}{\beta} + \sum_{n=1}^{\infty} \frac{e_n(x) + d_{n+1}(x)}{\beta^{n+1}}.$$

To be an invariant density, we have to show that this is nothing more than $h(x)$. It is sufficient to confirm that

$$C\left(1 - \frac{\lfloor \beta \rfloor - 1}{\beta} - \frac{d_1(x)}{\beta}\right) = \sum_{n=1}^{\infty} \frac{e_n(x)}{\beta^{n+1}} - \frac{d_1(x)}{\beta}.$$

We can check that the integration over $[0, 1]$ of both sides vanishes. Moreover, both sides take only two values, that is, they are constant in $[0, r_1) \cup [l_1, 1)$ and in $[r_1, l_1)$. This shows the existence of a constant C. Computation of C is therefore done at any point x in $[0, 1)$. Evaluating at $x = 0$, we have $e_n^+(0) = \lfloor \beta \rfloor - 1 - \lfloor \alpha(r_n) \rfloor$ and $e_n^-(0) = \lfloor \alpha(l_n) \rfloor$. Then we apply

$$r_0 = \sum_{n=1}^{\infty} \frac{\alpha(r_{n-1})}{\beta^n}, \quad l_0 = \sum_{n=1}^{\infty} \frac{\alpha(l_{n-1})}{\beta^n}$$

to obtain

$$C = 1 - \frac{1}{\beta - 1} + \sum_{n=1}^{\infty} \frac{\iota^+(n) - \iota^-(n)}{\beta^n}$$

and

$$h(x) = 1 + \sum_{n=1}^{\infty} \frac{\iota^+(n) - \iota^-(n)}{\beta^n} + \sum_{n=1}^{\infty} \frac{d_n(x) - 1}{\beta^n}.$$

We have $\iota^+(n) - \iota^-(n) \in \{-1, 0, 1\}$ and $d_n(x) - 1 \in \{-1, 0, 1\}$. Note that $\iota^+(n) - \iota^-(n) = -1$ if and only if $r_n < \lfloor \beta \rfloor / \beta - t \leq l_n$, and $r_n < l_n$ implies $d_n(x) \geq 1$. Moreover, $\iota^+(n) - \iota^-(n) = 1$ if and only if $l_n < \lfloor \beta \rfloor / \beta - t \leq r_n$, and $l_n < r_n$ implies $d_n(x) \leq 1$. Therefore we obtain

$$\iota^+(n) - \iota^-(n) + d_n(x) - 1 \in \{-1, 0, 1\}$$

and

$$\frac{\beta - 2}{\beta - 1} \leq h(x) \leq \frac{\beta}{\beta - 1}.$$

The condition $\beta > 2$ asserts that the lower bound is positive.

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