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PERCOLATION AND BEST-CHOICE PROBLEM FOR POWERS OF PATHS

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Abstract

The vertices of the kth power of a directed path with n vertices are exposed one by one to a selector in some random order. At any time the selector can see the graph induced by the vertices that have already appeared. The selector's aim is to choose online the maximal vertex (i.e. the vertex with no outgoing edges). We give upper and lower bounds for the asymptotic behaviour of $p_{n,k}n^{1/(k+1)}$, where $p_{n,k}$ is the probability of success under the optimal algorithm. In order to derive the upper bound, we consider a model in which the selector obtains some extra information about the edges that have already appeared. We give the exact asymptotics of the probability of success under the optimal algorithm in this case. In order to derive the lower bound, we analyse a site percolation process on a sequence of the kth powers of a directed path with n vertices.

Keywords: Site percolation; secretary problem; sequence of graphs; graph power; path

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1. Introduction

We consider the following online decision problem. The vertices of an acyclic directed graph G of known structure appear one by one in some random order. They are observed by a selector. At time t the selector can see the structure induced by the vertices that have already appeared. The selector can accept only one vertex and this choice can occur only at the time that the vertex appears. The aim is to maximize the probability of choosing a vertex from some previously defined set (e.g. the set of vertices with out-degree equal to 0).

The above formulation is a generalization of the so-called secretary problem, which is a classical problem in the field of optimal stopping. In the secretary problem, the selector sequentially observes n candidates for a job, who appear in a random order. There exists a linear ordering (i.e. candidates can be ranked from 1 to n) and the goal of the selector is to choose the absolutely best candidate (there is only one here). The selector observes the ranks of candidates relative to those examined so far, he/she knows the value of n and nothing else about the future candidates, and can hire only the presently examined candidate. The solution to this problem (published by Lindley in 1961, see [19]) is to reject a certain proportion of candidates

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(asymptotically n/e), regardless of their ranks, and after this, hire the first candidate who is the best seen so far (if such a candidate appears). Its asymptotic probability of success equals $1/\mathrm{e}$.

Many variants of the secretary problem have been considered (see Ferguson's survey [4]). The work of Stadje [24] was followed by a series of papers in which such a linear ordering was replaced by a partial one, including papers by several Russian authors who considered threshold strategies. This research was reviewed in Gnedin [9]. Optimal strategies for regular or simple posets can be found in [7], [13]–[15], [20], and [26]. The case where the selector knows in advance the total number of candidates but not the form of the ordering was considered by Preater [21]. Surprisingly, it turned out that there exists a stopping rule whose probability of success is bounded away from 0 by a constant for any poset. Preater's bound ($\frac{1}{8}$) was later improved by Georgiou *et al.* ($\frac{1}{4}$, see [8]), Kozik ($\frac{1}{4}+\varepsilon$, $\varepsilon>0$, see [16]), as well as Freij and Wästlund ($1/\mathrm{e}$, see [5]). Problems with richer, but still partial, information were considered by Garrod and Morris [6], as well as Kumar *et al.* [18].

A graph-theoretic generalization of the secretary problem was considered by Kubicki and Morayne in [17]. This generalization was based on the realization that orderings correspond to very rich directed graphs. The first and the most natural approach was to investigate a directed path instead of the linear ordering with the goal of choosing the top element, i.e. the sink. Some further graph-theoretic versions of the secretary problem were considered by Przykucki and Sulkowska [23]. The graph-theoretical analogue of Preater's problem was investigated by Goddard *et al.* [10], as well as Sulkowska [25]. Some further generalization to random graphs was considered by Przykucki in [22].

Throughout this paper we concentrate on a special family of graphs, the family of kth powers of a directed path with n vertices, denoted by $\{P_n^k\}_{1\le k\le n-1}$ (see Figure 1 or Section 2 for a strict definition of the power of a directed path). One can interpret one end ($k=1$) as a directed path and the other end ($k=n-1$) as a linear order that corresponds to the $(n-1)$th, i.e. full, power of a directed path. As mentioned above, the best-choice problems for these two cases have been solved (see [17] and [19]). One natural question is what happens in the intermediate cases. Grzesik *et al.* showed in [12] that the probability of success, $p_{n,k}$, under the optimal algorithm for choosing the sink from P_n^k is of the order $n^{-1/(k+1)}$. In this paper we give quite tight upper and lower bounds for the asymptotic behaviour of $p_{n,k}n^{1/(k+1)}$, even when k is a function of n as n goes to ∞. Nevertheless, the optimal algorithm itself is still not known.

In order to derive these upper bounds, we consider a model in which the selector obtains some extra information: while the vertices are being revealed, each edge of the graph induced by the observed vertices is labelled with the distance in P_n^k between its endpoints, and the selector sees those labels. In [12], the optimal stopping algorithm for choosing the sink from P_n^k in this case was derived and it was shown that its probability of success, $\tilde{p}_{n,k}$, is also of order $n^{-1/(k+1)}$. In this paper we also derive the exact values of $\lim_{n\to\infty}\tilde{p}_{n,k}n^{-1/(k+1)}$ for the whole range of k, $k=1,2,\ldots,n-1$.

In order to obtain lower bounds for the asymptotic behaviour of $p_{n,k}n^{1/(k+1)}$, we analyse the process of site (i.e. vertex) percolation for a sequence of kth powers of a directed path. An intensive study of percolation processes followed the work of Broadbent and Hammersley [2], who gave a probabilistic model for the flow of a liquid through some porous material. In this paper we will be interested in the probability of whether there exists an open passageway (flow) through $P_{n,k}$ when one declares each site (i.e. vertex) to be open with some probability p and closed otherwise, independently of all other sites. For some general results on percolations, we refer the reader to Grimmett [11].

This paper is organized as follows. In Section 2 we introduce some basic definitions and notation. In Section 3 we present two formal models: one for an optimal stopping problem on the kth power of a directed path P_n^k and the other for the site percolation on P_n^k. In Section 4 we consider the process of site percolation on the kth power of a directed path. In Section 5 we use the results from Section 4 to find the exact asymptotic behaviour of the probability of success, $\tilde{p}_{n,k}$, under the optimal algorithm for choosing the sink from P_n^k when the selector knows the distance between vertices in P_n^k that are connected by an edge in the induced graph. We use this result in Section 6 to derive an upper bound for the asymptotics of $p_{n,k}n^{1/(k+1)}$. In Section 6 we also analyse a special randomized algorithm to obtain lower bounds for the asymptotic behaviour of $p_{n,k}n^{1/(k+1)}$. In Section 7 we state a few open questions.

2. Definitions and notation

A *directed graph*, or simply digraph, is a pair (V, E), where V is a set whose elements are called *vertices* (or, in percolation theory language, *sites*) and E is a set of ordered pairs of vertices, which are called *edges* (or, in percolation theory language, *bonds*). We call a vertex $v \in V$ a *maximal element* or a *sink* if it has no outgoing edges. Let $G = (V, E)$ be a digraph. The set of maximal elements of G will be denoted by $\max(G)$. We say that G is (weakly) connected if it is possible to reach any vertex starting from any other by traversing edges in some direction (i.e. not necessarily in the direction they point). An induced subdigraph $G' = (W, E \cap W^2)$ of G, where $W \subseteq V$, is called a *component* if it is a maximal (weakly) connected induced subdigraph. A *directed path* is a graph $P_n = (V_n, E_n)$ such that $V_n = \{v_1, v_2, \dots, v_n\}$ and $E_n = \{(v_{i+1}, v_i) : i \in \{1, 2, \dots, n-1\}\}$. The *length* of P_n is $n - 1$. The only maximal vertex of P_n will be denoted by $\mathbf{1}$ (i.e. $v_1 = \mathbf{1}$).

The kth *power* of a graph $G = (V, E)$ is the graph with the set of vertices V and an edge between two vertices if and only if there is a path of length at most k between them in G. Throughout this paper we consider the structures of kth powers of a directed path P_n. We denote them by P_n^k. All the edges of P_n^k will be always drawn in an 'upward directed'. We call P_n^{n-1} a *full power* of a directed path. The kth powers of P_4, $k = 1, 2, 3$, are presented in Figure 1. Note that, for $k \geq n$, we have $P_n^k = P_n^{n-1}$.

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$. For a power of a directed path we define a gap function $d : E \to \mathbb{N}$ as follows: $d((v_i, v_j)) = i - j - 1$, e.g. $d((v_n, v_1)) = n-2$ in P_n^{n-1}, and always $d((v_{i+1}, v_i)) = 0$; note that $d(e)$ stands for the number of vertices between the endpoints of e and these are the values that we will use as labels to the edges (in one of our models).

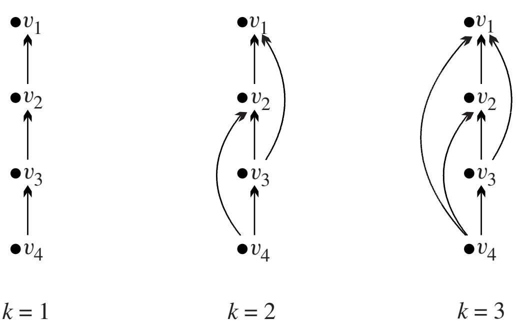

FIGURE 1: The kth powers of a directed path with four vertices.

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3. Formal models

3.1. Optimal stopping model

Here, we describe two (very similar) models. The first is the *unlabelled* one, where the selector sees only the (unlabelled) subdigraph induced by the vertices that he/she has observed so far. Let $P_n^k = (V, E)$ be a kth power of a directed path P_n (where $|V| = n$) and let S_n denote the family of all permutations of the set V. Let $\pi = (\pi_1, \pi_2, \dots, \pi_n) \in S_n$. By $P_{(m)} = P_{(m)}(\pi) = (V_{(m)}, E_{(m)}), m \leq n$, we denote the subdigraph of P_n^k induced by $\{\pi_1, \dots, \pi_m\}$, i.e.

$$V_{(m)} = \{\pi_1, \pi_2, \dots, \pi_m\},$$
$$E_{(m)} = \{(v_i, v_j) \colon \{v_i, v_j\} \subseteq \{\pi_1, \pi_2, \dots, \pi_m\} \wedge (v_i, v_j) \in E\}.$$

By $c(P_{(m)})$ we denote the number of components in $P_{(m)}$. Let us define the probability space $(\Omega, \mathcal{F}, \mathbb{P})\colon \Omega = S_n, \mathcal{F} = \mathcal{P}(\Omega)$, the probability measure $\mathbb{P}\colon \mathcal{F} \to [0, 1]$ is defined by $\mathbb{P}[\{\pi\}] = 1/n!$ for each $\pi \in S_n$. Let $R \subseteq \mathbb{N}^2$. We write $(\pi_1, \pi_2, \dots, \pi_m) \cong R$ if, for all i, $j \leq m, i \neq j, (\pi_i, \pi_j) \in E$ if and only if $(i, j) \in R$. Let

$$\mathcal{F}_t = \sigma\{\{\pi \in \Omega \colon (\pi_1, \pi_2, \dots, \pi_t) \cong R\} \colon R \subseteq \mathbb{N}^2\}, \qquad 1 \leq t \leq n,$$

be our *filtration* (a sequence of σ-algebras such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \mathcal{F}_n \subseteq \mathcal{F}$). Here, $\mathcal{F}_t$ contains all the events that can happen in our model till time t. A random variable $\tau \colon \Omega \to \{1, 2, \dots, n\}$ is a *stopping time* with respect to the filtration $(\mathcal{F}_t)_{t=1}^n$ if $\tau^{-1}(\{t\}) \in \mathcal{F}_t$ for each $t \leq n$. This means that the decision to stop is based only on past and present events. Let D be a subset of vertices of P_n^k (i.e. $D \subseteq V$). An *optimal stopping time* for choosing an element from D is any stopping time τ^* for which

$$\mathbb{P}[\pi_{\tau^*} \in D] = \max_{\tau \in \mathcal{S}} \mathbb{P}[\pi_\tau \in D],$$

where $\mathcal{S}$ denotes the set of all stopping times and $[\pi_\tau \in D]$ is the set $\{\pi \in \Omega \colon \pi_{\tau(\pi)} \in D\}$. Throughout this paper $D = \{\mathbf{1}\}$.

Finally, for the *labelled model*, i.e. when the selector also knows the value $d(e)$ of each edge e that appears in the induced subdigraph, only the filtration changes. In this case, for $\varphi\colon R \to \mathbb{N}$ and $R_\varphi = \{(i, j, \varphi(i, j)) \colon (i, j) \in R\}$, we write $(\pi_1, \pi_2, \dots, \pi_m) \cong R_\varphi$ if, for all $i, j \leq m, i \neq j, (\pi_i, \pi_j) \in E$ if and only if $(i, j) \in R$ and $\varphi(i, j) = d((\pi_i, \pi_j))$, and the filtration is

$$\tilde{\mathcal{F}}_t = \sigma\{\{\pi \in \Omega \colon (\pi_1, \pi_2, \dots, \pi_t) \cong R_\varphi\} \colon R \subseteq \mathbb{N}^2, \varphi\colon R \to \mathbb{N}\}, \qquad 1 \leq t \leq n.$$

The optimal strategy $\tilde{\tau}_{n,k}$ for choosing a sink $\mathbf{1}$ from P_n^k in the labelled model is known [12]. It can be stated as follows. *Stop when there is a positive conditional (given history) probability that the presently examined candidate is the sink and the probability that the sink can be among the future candidates is equal to 0.* In other words, $\tilde{\tau}_{n,k}$ tells the selector to play till the last moment where there is still a chance of success. It tells him/her to stop at time m if π_m is a sink in $P_{(m)}$ (recall that $P_{(m)} = (V_{(m)}, E_{(m)})$ is the graph induced by $\{\pi_1, \pi_2, \dots, \pi_m\}$) and all the remaining vertices are necessary either to connect the components of $P_{(m)}$ or to join the components of $P_{(m)}$ as their 'inner' vertices. Thus, the strategy $\tilde{\tau}_n$ may be also stated as follows:

$$\tilde{\tau}_{n,k}(\pi) = \min\{t \leq n \colon n - t = k(c(P_{(t)}) - 1) + b_t, \pi_t \in \max(P_{(t)})\},$$

where $b_t = \sum_{e \in E_{(m)}} d(e)$. (An example of a situation when $\tilde{\tau}_{9,2}$ stops the search at π_6 playing on P_9^2 is presented in Figure 2.) The optimality of this strategy for $k = 1$ was proved by Kubicki and Morayne [17]. They proved also that its probability of success satisfies $\lim_{n\to\infty} \sqrt{n}\mathbb{P}[\pi_{\tilde{\tau}_{n,1}} = \mathbf{1}] = \sqrt{\pi}/2$. The optimality of $\tilde{\tau}_{n,k}$ for the whole range of k $(1 \le k \le n-1)$ was proved by Grzesik *et al.* [12]. The authors proved also that its probability of success is of the order $n^{-1/(k+1)}$. In this paper, among other things, we refine this result giving the exact value of the limit $\lim_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$ for the whole range of k.

3.2. Percolation model

Let $p \in [0, 1]$. Given a graph G, each of its sites (i.e. vertices), independently of the others, is declared *open* with probability p and *closed* otherwise. A path in G is called open if all its sites are open. When $G = P_n^k$ we say that it percolates if there exists an open path joining v_1 and v_n. More generally, we write $v_i \overset{p}{\longleftrightarrow} v_j$ if there exists an open path joining v_i and v_j (in particular, v_i and v_j must be open). By $\psi_{n,k}(p)$ we denote the probability that P_n^k percolates, i.e. $\psi_{n,k}(p) = \mathbb{P}[v_1 \overset{p}{\longleftrightarrow} v_n]$. For any k and n, we give general lower and upper bounds for $\psi_{n,k}(p)$. Then, for a sequence $(P_n^{k(n)})_{n=1}^{\infty}$, where $k = k(n)$ can be any function of n, we use the previous bounds to find the asymptotic behaviour of the probability of success of the optimal stopping algorithm $\tilde{\tau}_n$ for choosing a sink from P_n^k in the labelled model.

4. Site percolation probability for P_n^k

In this section we consider the site percolation process on $(P_n^k)_n$. Recall that $\psi_{n,k}(p)$ denotes the probability that P_n^k percolates. Whenever the context is clear we write $\psi_{n,k}$ instead of $\psi_{n,k}(p)$.

Lemma 4.1. *For any positive integers k and n, the probability that P_n^k percolates, $\psi_{n,k} = \psi_{n,k}(p)$, satisfies*

$$\psi_{1,k} = p, \quad \psi_{n,k} = p^2 \quad \text{for } 2 \le n \le k+1,$$
$$\psi_{n,k} = p\psi_{n-1,k} + p(1-p)\psi_{n-2,k} + \cdots + p(1-p)^{k-1}\psi_{n-k,k} \quad \text{for } n \ge k+1 \tag{4.1}$$

(note that the $n = k+1$ case is covered twice).

Proof. It is obvious that $\psi_{1,k} = p$. Whenever $2 \le n \le k+1$ then there exists an edge joining v_1 and v_n. Thus, P_n^k percolates if and only if v_1 and v_n are open (which happens with probability p^2). From this, we can also easily check that (4.1) holds when $n = k+1$.

In the remaining case $(n > k+1)$, $\psi_{n,k}$ is expressed as a sum of terms that are the probabilities of some disjoint events. For $0 \le j \le n-2$, let

$$A_j = [(v_1 \text{ is open}) \cap (v_2, \ldots, v_{j+1} \text{ are not open}) \cap (v_{j+2} \overset{p}{\longleftrightarrow} v_n)].$$

Note that

$$\mathbb{P}[A_j] = p(1-p)^j \psi_{n-(j+1),k}.$$

Note also that there is an edge joining v_1 and v_{j+2} if and only if $j \le k-1$. Moreover, the events A_j are disjoint. Taking the union of the A_j over $j = 0, 1, \ldots, k-1$, we obtain the event that P_n^k percolates. □

Lemma 4.2. *For $n > k+1$, we have*

$$\psi_{n,k} = \psi_{n-1,k} - p(1-p)^k\psi_{n-k-1,k}.$$

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Proof. By Lemma 4.1, for $n > k+1$, we have

$$\begin{aligned}\psi_{n,k} &= p\psi_{n-1,k} + p(1-p)\psi_{n-2,k} + \cdots + p(1-p)^{k-1}\psi_{n-k,k}\\ &= p\psi_{n-1,k} + (1-p)(p\,\psi_{n-2,k} + \cdots + p(1-p)^{k-2}\psi_{n-k,k})\\ &= p\psi_{n-1,k} + (1-p)(\psi_{n-1,k} - p(1-p)^{k-1}\psi_{n-k-1,k})\\ &= \psi_{n-1,k} - p(1-p)^{k}\psi_{n-k-1,k}.\end{aligned}$$

□

Lemma 4.3. *For $n > k+1$, we have*

$$p^2(1-(1-p)^k)^{n-k} \le \psi_{n,k} \le p^2(1-p(1-p)^k)^{n-k-1}.$$

Proof. Since $\psi_{n,k}$ is weakly decreasing in n, by Lemma 4.1, for $n > k$, we have

$$\begin{aligned}\psi_{n,k} &= p\psi_{n-1,k} + p(1-p)\psi_{n-2,k} + \cdots + p(1-p)^{k-1}\psi_{n-k,k}\\ &\ge p\psi_{n-1,k} + p(1-p)\psi_{n-1,k} + \cdots + p(1-p)^{k-1}\psi_{n-1,k}\\ &= p\psi_{n-1,k}(1 + (1-p) + \cdots + (1-p)^{k-1})\\ &= \psi_{n-1,k}(1-(1-p)^k).\end{aligned}$$

Thus,

$$\begin{aligned}\psi_{n,k} &\ge \psi_{n-1,k}(1-(1-p)^k)\\ &\ge \psi_{n-2,k}(1-(1-p)^k)^2\\ &\;\;\vdots\\ &\ge \psi_{k,k}(1-(1-p)^k)^{n-k}\\ &= p^2(1-(1-p)^k)^{n-k}.\end{aligned}$$

On the other hand, since $\psi_{n,k}$ is weakly decreasing in n, we have, for $n > k+1$,

$$\begin{aligned}\psi_{n,k} &= \psi_{n-1,k} - p(1-p)^k\psi_{n-k-1,k}\\ &\le \psi_{n-1,k} - p(1-p)^k\psi_{n-1,k}\\ &= \psi_{n-1,k}(1-p(1-p)^k),\end{aligned}$$

where the first equality follows from Lemma 4.2. Thus,

$$\begin{aligned}\psi_{n,k} &\le \psi_{n-1,k}(1-p(1-p)^k)\\ &\le \psi_{n-2,k}(1-p(1-p)^k)^2\\ &\;\;\vdots\\ &\le \psi_{k+1,k}(1-p(1-p)^k)^{n-k-1}\\ &= p^2(1-p(1-p)^k)^{n-k-1}.\end{aligned}$$

□

5. The probability of success of $\tilde{\tau}_{n,k}$ (labelled model)

Throughout this section we always assume that π is a random permutation of vertices of P_n^k. We give the exact values of the limit $\lim_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$ for the whole range

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of $k = k(n)$, with $1 \leq k \leq n-1$. Recall that $\tilde{\tau}_{n,k}$ is the optimal algorithm for choosing the sink from P_n^k in the labelled model.

Instead of selecting directly π uniformly among all permutations of $V(P_n^k)$, we will do this with the following technique which was introduced by Freij and Wästlund [5] and also used in [12]. Let us associate with each vertex v_i a random variable A_i from the uniform distribution on the interval [0, 1]. We may think of A_i as of the arrival time of v_i. By ordering $V(P_n^k)$ according to the values of the A_i (in a nondecreasing way), we obtain a uniform random order of $V(P_n^k)$. Though we consider now a different probability space, our problem of optimal stopping and its probability of success are equivalent to those of our original model (as all the permutations of vertices are still equiprobable). We omit formal details. The arrival time of the sink will be denoted by p, i.e. $p = A_1$. Note that, since all the A_i are independent, given $A_1 = p$, the probability that a particular vertex appears before the sink is equal to p. Also, the vertices appear before the sink (with probability p) independently. In an analogy to the percolation model, given p, for $2 \leq i \leq n$, we will say that v_i is open if $A_i \leq p$, and it is closed otherwise.

Since p is uniformly chosen from [0, 1], the following equality holds (see [3, Equation (10.1)]):

$$\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1} \mid A_1 = p]\,\mathrm{d}p. \tag{5.1}$$

(Here, we define $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1} \mid A_1 = p] = \lim_{h\to 0^+} \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1} \mid A_1 \in (p, p+h)]$.)

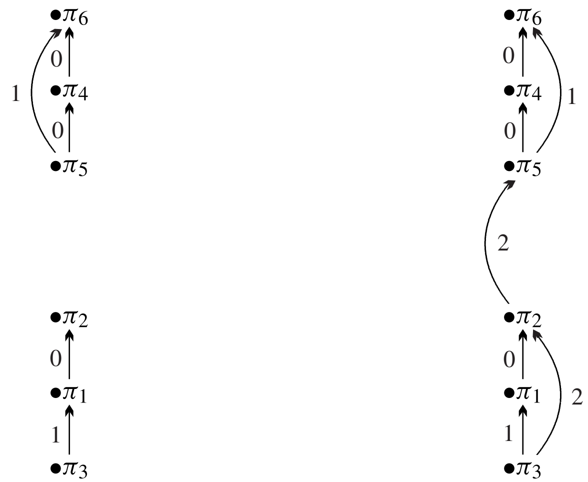
In the following two lemmas we give two different formulas for $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$. We will use the first one to obtain the lower bound and the second one to obtain the upper bound for the desired limit $\lim_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$.

Remark 5.1. (*The $k = n-1$ case.*) When we deal with the $(n-1)$th (i.e. full) power of P_n^k, the vertices $\mathbf{1}$ and v_n are easily identified when they both show up in the induced graph, because it is the only pair of vertices connected by an edge whose label is equal to $n-2$. Therefore, if only $\mathbf{1}$ appears after v_n in a random permutation π, $\tilde{\tau}_{n,n-1}$ stops at $\mathbf{1}$. Note also that $\tilde{\tau}_{n,n-1}$ loses if $\mathbf{1}$ precedes v_n in π. Thus, $\mathbb{P}[\pi_{\tilde{\tau}_{n,n-1}} = \mathbf{1}] = \frac{1}{2}$ (and also $\lim_{n\to\infty} n^{1/n}\, \mathbb{P}[\pi_{\tilde{\tau}_{n,n-1}} = \mathbf{1}] = \frac{1}{2}$). Throughout the rest of this section we always assume that $k < n-1$.

Lemma 5.1. *For $\psi_{n,k+1}(p)$ being the probability that P_n^{k+1} percolates, π being a random permutation of vertices of P_n^k, and $\tilde{\tau}_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the labelled model, we have*

$$\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 \frac{\psi_{n,k+1}(p)}{p}\,\mathrm{d}p.$$

Proof. For a given $t \in [0, 1]$, assume that $\tilde{\tau}_{n,k}$ stops at time t, i.e. the algorithm stops while observing some vertex whose arrival time is t. Consider the subdigraph G_t of P_n^k induced on the vertices v_i for which $A_i < t$. Compare it to the subdigraph G_t^* of P_n^{k+1} induced by the same set of vertices of G_t. Recall that $\tilde{\tau}_{n,k}$ stops at the vertex that is maximal so far only if there is no chance that the sink is among the vertices that are still to come. Note that this means that there is a path from v_1 to v_n in G_t^*. Conversely, for any value of t for which there is a vertex v_i whose arrival time is t such that G_t^* has a path joining v_1 and v_n, and v_i is a sink in G_t, the algorithm $\tilde{\tau}_{n,k}$ will stop at time t and select v_i (see Figure 2). Now, we conclude that: if we are given $A_1 = p$, then the probability that $\tilde{\tau}_{n,k}$ stops at the sink, i.e. it stops at time p, is equal to the probability that P_n^{k+1} percolates, given that v_1 is assumed to be open and each other site is



(a) $\tilde{\tau}_{9,2} = 6$ on P_9^2 with input π (b) P_9^3 percolates

FIGURE 2: (a) We have the observed graph at step 6 for the optimal stopping algorithm on labelled P_9^2, using the permutation $\pi = \{v_7, v_6, v_9, v_2, v_3, v_1, v_5, v_4, v_8\}$ as input. Note that $\tilde{\tau}_{9,2}(\pi) = 6$. The label of an edge e represents $d(e)$. (b) The graph induced by the same vertices in P_9^3: even if we did not know π, since there is a labelled path from π_3 to π_6, it follows that $\pi_6 = \mathbf{1}$. In fact, the remaining three vertices (that were not observed until step 6) have to be used to 'close the gaps' in this path.

open with probability p, independently. Hence, we obtain

$$\begin{aligned}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1} \mid A_1 = p] &= \mathbb{P}[v_1 \overset{p}{\longleftrightarrow} v_n \mid v_1 \text{is open}] \\ &= \frac{\mathbb{P}[v_1 \overset{p}{\longleftrightarrow} v_n \text{ and } v_1 \text{ is open}]}{\mathbb{P}[v_1 \text{ is open}]} \\ &= \frac{\psi_{n,k+1}(p)}{p}, \end{aligned} \tag{5.2}$$

which together with (5.1) gives $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 (\psi_{n,k+1}(p)/p)\,\mathrm{d}p$. Note that although we consider the optimal stopping on P_n^k, the notation $v_1 \overset{p}{\longleftrightarrow} v_n$ refers here to the percolation on P_n^{k+1}. □

Now, let us recall the formula for $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$ introduced in [12]. Consider the following sequence of indicator random variables that depend only on $v_2, \ldots, v_{n-1}$:

$$X_i^{(p)} = \begin{cases} 1 & \text{if each of } v_{i+1}, v_{i+2}, \ldots, v_{i+k+1} \text{ is closed,} \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \le i \le n-k-2. \tag{5.3}$$

Let also $X^{(p)} = \sum_{i=1}^{n-k-2} X_i^{(p)}$.

Lemma 5.2. (See [12, Lemma 4.5].) *For $X^{(p)}$ defined as above, π being a random permutation of vertices of P_n^k and $\tilde{\tau}_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the labelled model, we have*

$$\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 p\mathbb{P}[X^{(p)} = 0]\,\mathrm{d}p.$$

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Proof. Given p, consider an induced graph, say G_p, at the time p, i.e. the subdigraph of P_n^k induced by the vertices v_i for which $A_i < p$. Note that we may order the components of G_p in a natural way: a component C will appear before a component C' in such order if and only if, for every $v_i \in C$ and $v_j \in C'$, we have $i < j$ (note that this is possible due to the structure of P_n^k. Then the equality $X^{(p)} = 0$ means that there are no two consecutive components for which their distance in the original graph (P_n^k) is greater than k (by the distance between two components we mean the length of the shortest path in P_n that joins vertices from the different components). Thus, whenever $[X^{(p)} = 0]$, v_1 is open and v_n is open then P_n^{k+1} percolates (i.e. in the subdigraph of P_n^{k+1} induced by the same vertices of G_p there is a path from v_1 to v_n). Conversely, if v_1 is closed, or v_n is closed, or $[X^{(p)} > 0]$, then P_n^{k+1} does not percolate. Therefore,

$$\mathbb{P}[v_1 \overset{p}{\longleftrightarrow} v_n \mid v_1 \text{is open}] = \mathbb{P}[v_n \text{ is open} \wedge X^{(p)} = 0].$$

Now, this lemma follows from (5.1) and (5.2). □

Recall the definitions of the gamma and beta functions.

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \qquad x > 0,\ a > 0,\ b > 0.$$

The following three lemmas will be helpful later on.

Lemma 5.3. (See [1].) *For every real $a > 0$, $b > 0$ we have $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.*

Lemma 5.4. (See [1].) *Let $\alpha \in \mathbb{R}$. We have $\lim_{n\to\infty} \Gamma(n)\, n^\alpha / \Gamma(n+\alpha) = 1$.*

Lemma 5.5. *Let $\alpha(n) \to 0$ as $n \to \infty$. We have $\lim_{n\to\infty} \Gamma(n) n^{\alpha(n)}/\Gamma(n+\alpha(n)) = 1$.*

Proof. To prove this lemma it is enough to apply Stirling's formula for the gamma function $\Gamma(x+1) = \sqrt{2\pi x}(x/\mathrm{e})^x(1 + O(1/x))$. □

Theorem 5.1. *For π being a random permutation of vertices of P_n^k and $\tilde{\tau}_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the labelled model, we have*

$$\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \ge \begin{cases} \Gamma\Big(1 + \dfrac{1}{k+1}\Big) & \text{if } k \text{ is a constant,} \\ 1 & \text{if } k = o(\log n) \text{ and } k(n) \to \infty \text{ as } n \to \infty, \\ 1 - \dfrac{1}{2\mathrm{e}^{1/c}} & \text{if } k(n) = c\log n, \\ \frac{1}{2} & \text{if } k(n) = \omega(\log n). \end{cases}$$

Proof. By Lemma 5.1, we know that $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 (\psi_{n,k+1}(p)/p)\,\mathrm{d}p$. By Lemma 4.3,

$$\psi_{n,k+1}(p) \ge p^2(1-(1-p)^{k+1})^{n-k-1} \quad \text{for } n > k+2;$$

hence, we can write $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \ge \int_0^1 p(1-(1-p)^{k+1})^{n-k-1}\,\mathrm{d}p$. Letting $q = 1-p$, we obtain

$$\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}] \ge \int_0^1 (1-q)(1-q^{k+1})^{n-k-1}\,\mathrm{d}q$$

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which yields

$$\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \geq \int_0^1 (1 - q^{k+1})^n \, \mathrm{d}q - \int_0^1 q(1 - q^{k+1})^n \, \mathrm{d}q. \tag{5.4}$$

Substituting $x = q^{k+1}$ in the first integral, integrating by parts, and using Lemma 5.3, we obtain

$$\begin{aligned}
\int_0^1 (1 - q^{k+1})^n \, \mathrm{d}q &= \int_0^1 \frac{1}{k+1} x^{-k/(k+1)} (1 - x)^n \, \mathrm{d}x \\
&= [x^{1/(k+1)}(1 - x)^n]_0^1 + \int_0^1 x^{1/(k+1)} n(1 - x)^{n-1} \, \mathrm{d}x \\
&= n \int_0^1 x^{1/(k+1)} (1 - x)^{n-1} \, \mathrm{d}x \\
&= nB\biggl(1 + \frac{1}{k+1}, n\biggr) \\
&= n \frac{\Gamma(1 + 1/(k+1))\Gamma(n)}{\Gamma(n + 1 + 1/(k+1))} \\
&= \Gamma\biggl(1 + \frac{1}{k+1}\biggr) \frac{\Gamma(n+1)}{\Gamma(n + 1 + 1/(k+1))}.
\end{aligned}$$

Thus, by Lemmas 5.4 and 5.5, we obtain

$$\begin{aligned}
\lim_{n\to\infty} n^{1/(k+1)} \int_0^1 (1 - q^{k+1})^n \, \mathrm{d}q &= \lim_{n\to\infty} \Gamma\biggl(1 + \frac{1}{k+1}\biggr) \frac{\Gamma(n+1) n^{1/(k+1)}}{\Gamma(n + 1 + 1/(k+1))} \\
&= \begin{cases} \Gamma\biggl(1 + \dfrac{1}{k+1}\biggr) & \text{if } k \text{ is a constant,} \\ 1 & \text{if } k(n) \to \infty \text{ as } n \to \infty. \end{cases}
\end{aligned} \tag{5.5}$$

Substituting $x = q^{k+1}$ in the second integral of (5.4), integrating by parts and using Lemma 5.3 in a similar way, we obtain

$$\begin{aligned}
\int_0^1 q(1 - q^{k+1})^n \, \mathrm{d}q &= \int_0^1 \frac{1}{k+1} x^{-k/(k+1)} x^{1/(k+1)} (1 - x)^n \, \mathrm{d}x \\
&= \frac{1}{2}\biggl(\Bigl[x^{2/(k+1)}(1 - x)^n\Bigr]_0^1 - \int_0^1 -x^{2/(k+1)} n(1 - x)^{n-1} \, \mathrm{d}x\biggr) \\
&= \frac{n}{2} \int_0^1 x^{2/(k+1)} (1 - x)^{n-1} \, \mathrm{d}x \\
&= \frac{n}{2} B\biggl(1 + \frac{2}{k+1}, n\biggr) \\
&= \frac{n}{2} \frac{\Gamma(1 + 2/(k+1))\Gamma(n)}{\Gamma(n + 1 + 2/(k+1))} \\
&= \frac{1}{2}\Gamma\biggl(1 + \frac{2}{k+1}\biggr) \frac{\Gamma(n+1)}{\Gamma(n + 1 + 2/(k+1))}.
\end{aligned}$$

Again, by Lemmas 5.4 and 5.5, we have

$$\lim_{n\to\infty} n^{1/(k+1)} \int_0^1 q(1-q^{k+1})^n \mathrm{d}q = \lim_{n\to\infty} \frac{1}{2} \frac{\Gamma(1+2/(k+1))}{n^{1/(k+1)}} \frac{\Gamma(n+1)n^{2/(k+1)}}{\Gamma(n+1+2/(k+1))}$$
$$= \begin{cases} 0 & \text{if } k(n) = o(\log n), \\ \dfrac{1}{2\mathrm{e}^{1/c}} & \text{if } k(n) = c\log n, \\ \frac{1}{2} & \text{if } k(n) = \omega(\log n). \end{cases} \tag{5.6}$$

Thus, by (5.4)–(5.6), we obtain

$$\liminf_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$$
$$\geq \lim_{n\to\infty} \biggl(n^{1/(k+1)} \int_0^1 (1-q^{k+1})^n \,\mathrm{d}q - n^{1/(k+1)} \int_0^1 q(1-q^{k+1})^n \biggr)$$
$$= \begin{cases} \Gamma\Bigl(1+\dfrac{1}{k+1}\Bigr) & \text{if } k \text{ is a constant}, \\ 1 & \text{if } k(n) = o(\log n) \text{ and } k(n) \to \infty \text{ as } n \to \infty, \\ 1 - \dfrac{1}{2\mathrm{e}^{1/c}} & \text{if } k(n) = c\log n, \\ \frac{1}{2} & \text{if } k(n) = \omega(\log n). \end{cases}$$

This completes the proof. □

Theorem 5.2. *For π being a random permutation of vertices of P_n^k and $\tilde{\tau}_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the labelled model, we have*

$$\limsup_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \leq \begin{cases} 1 & \textit{if } k = o(\log n) \textit{ and } k(n) \to \infty \textit{ as } n \to \infty, \\ 1 - \dfrac{1}{2\mathrm{e}^{1/c}} & \textit{if } k(n) = c\log n, \\ \frac{1}{2} & \textit{if } k(n) = \omega(\log n). \end{cases}$$

Remark 5.2. The case for k being a constant will be considered separately later on.

Proof. By Lemma 5.2, we know that $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 p\mathbb{P}[X^{(p)} = 0]\,\mathrm{d}p$, where $X^{(p)} = \sum_{i=1}^{n-k-2} X_i^{(p)}$ and the $X_i^{(p)}$ are given by (5.3). Let $m = \lfloor (n-2)/(k+1) \rfloor$. Since

$$X_1^{(p)}, X_{(k+1)+1}^{(p)}, X_{2(k+1)+1}^{(p)}, \dots, X_{(m-1)(k+1)+1}^{(p)}$$

are independent and $\mathbb{P}[X_i^{(p)} = 0] = 1-(1-p)^{k+1}$ for $i = 1, 2, \dots, n-k-2$, we have

$$\mathbb{P}[X^{(p)} = 0] \leq \mathbb{P}[X_1^{(p)} = 0 \wedge X_{(k+1)+1}^{(p)} = 0 \wedge \dots \wedge X_{(m-1)(k+1)+1}^{(p)} = 0]$$
$$= (1-(1-p)^{k+1})^m.$$

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Thus, $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \le \int_0^1 p(1-(1-p)^{k+1})^m \,\mathrm{d}p$. Letting $q = 1-p$, integrating as in Theorem 5.1, and applying again Lemmas 5.4 and 5.5, we obtain

$$
\begin{aligned}
&n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]\\
&\quad\le n^{1/(k+1)}\int_0^1 (1-q^{k+1})^m \,\mathrm{d}q - n^{1/(k+1)}\int_0^1 q(1-q^{k+1})^m \,\mathrm{d}q\\
&\quad= \Gamma\Big(1+\frac{1}{k+1}\Big)\frac{\Gamma(m+1)n^{1/(k+1)}}{\Gamma(m+1+1/(k+1))} - \frac{1}{2}\frac{\Gamma(1+2/(k+1))}{n^{1/(k+1)}}\frac{\Gamma(m+1)n^{2/(k+1)}}{\Gamma(m+1+2/(k+1))}\\
&\quad\to \begin{cases} 1 & \text{if } k(n) = o(\log n) \text{ and } k(n)\to\infty \text{ as } n\to\infty,\\ 1-\dfrac{1}{2\mathrm{e}^{1/c}} & \text{if } k(n) = c\log n,\\ \frac{1}{2} & \text{if } k(n) = \omega(\log n), \end{cases} \qquad \text{as } n\to\infty.
\end{aligned}
$$

This completes the proof. □

Now, let us prove the two technical lemmas that will be helpful in finding the tight upper bound for $\limsup_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$ when k is a constant.

Lemma 5.6. *Let $k \ge 1$ be a constant and let $\delta_n = (1/n)^{(k+3/2)/(k+1)(k+2)}$. Then*

$$\lim_{n\to\infty} n^{1/(k+1)}\int_{\delta_n}^1 (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q = 0$$

(which also implies $\lim_{n\to\infty} n^{1/(k+1)}\int_{\delta_n}^1 (1-q^{k+1})^n\,\mathrm{d}q = 0$).

Proof. Note that if $n > ((k+2)/(k+1))^{(k+1)(k+2)/(k+3/2)}$ then $\delta_n < (k+1)/(k+2)$ and we can split the above integral into two parts

$$
\begin{aligned}
\int_{\delta_n}^1 (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q &= \int_{\delta_n}^{(k+1)/(k+2)} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q\\
&\quad+\int_{(k+1)/(k+2)}^1 (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q \qquad (5.7)
\end{aligned}
$$

and consider each of them separately. Therefore, throughout the rest of the proof we always assume that $n > ((k+2)/(k+1))^{(k+1)(k+2)/(k+3/2)}$.

The function $f(q) = 1-q^{k+1}+q^{k+2}$ is decreasing on $[\delta_n, (k+1)/(k+2)]$; thus,

$$
\begin{aligned}
&\int_{\delta_n}^{(k+1)/(k+2)} (1-q^{k+1}+q^{k+2})^{n-k-2}\mathrm{d}q\\
&\quad\le \int_{\delta_n}^{(k+1)/(k+2)} (1-\delta_n^{k+1}+\delta_n^{k+2})^{n-k-2}\,\mathrm{d}q\\
&\quad= \int_{\delta_n}^{(k+1)/(k+2)} (1-(1-\delta_n)\delta_n^{k+1})^{n-k-2}\,\mathrm{d}q\\
&\quad\le (1-\delta_n)\bigg(1-(1-\delta_n)\Big(\frac{1}{n}\Big)^{(k+3/2)/(k+2)}\bigg)^{n-k-2}.
\end{aligned}
$$

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Therefore, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n^{1/(k+1)} \int_{\delta_n}^{(k+1)/(k+2)} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q \\
&\quad \le \lim_{n\to\infty} n^{1/(k+1)}(1-\delta_n)\bigg(1-(1-\delta_n)\Big(\frac{1}{n}\Big)^{(k+3/2)/(k+2)}\bigg)^{n-k-2} \\
&\quad = \lim_{n\to\infty} n^{1/(k+1)}\bigg[\bigg(1-\frac{1-\delta_n}{n^{(k+3/2)/(k+2)}}\bigg)^{n^{(k+3/2)/(k+2)}/(1-\delta_n)}\bigg]^{(n-k-2)(1-\delta_n)/n^{(k+3/2)/(k+2)}} \\
&\quad = \lim_{n\to\infty} n^{1/(k+1)} \exp\bigg\{-\frac{(n-k-2)(1-\delta_n)}{n^{(k+3/2)/(k+2)}}\bigg\} \\
&\quad = 0. \qquad\qquad (5.8)
\end{aligned}$$

Now, let us consider the second part of (5.7). Note that the function $f(q)=1-q^{k+1}+q^{k+2}$ is increasing and convex on $[(k+1)/(k+2),1]$, thus, can be bounded on $[(k+1)/(k+2),1]$ from above by the linear function $h(q)$ going through the points $((k+1)/(k+2), f((k+1)/(k+2)))$ and $(1,1)$. Let $a_k=(k+2)(1-f((k+1)/(k+2)))$. We have $h(q)=a_kq+(1-a_k)$ and

$$\int_{(k+1)/(k+2)}^{1} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q \le \int_{(k+1)/(k+2)}^{1} (a_kq+(1-a_k))^{n-k-2}\,\mathrm{d}q.$$

Letting $x=a_kq+(1-a_k)$, we obtain

$$\begin{aligned}
\int_{(k+1)/(k+2)}^{1} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q &\le \frac{1}{a_k}\int_{f((k+1)/(k+2))}^{1} x^{n-k-2}\,\mathrm{d}x \\
&= \frac{1}{a_k(n-k-1)}\bigg(1-\bigg(f\Big(\frac{k+1}{k+2}\Big)\bigg)^{n-k-1}\bigg).
\end{aligned}$$

Since $k\ge 1$ and $f((k+1)/(k+2))<1$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n^{1/(k+1)} \int_{(k+1)/(k+2)}^{1} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q \\
&\quad \le \lim_{n\to\infty} \frac{n^{1/(k+1)}}{a_k(n-k-1)}\bigg(1-\bigg(f\Big(\frac{k+1}{k+2}\Big)\bigg)^{n-k-1}\bigg) \\
&\quad = 0,
\end{aligned}$$

which with (5.7) and (5.8) yields $\lim_{n\to\infty} n^{1/(k+1)}\int_{\delta_n}^{1}(1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q=0$. □

Lemma 5.7. *Let $k\ge 1$ be a constant and $\delta_n=(1/n)^{(k+3/2)/(k+1)(k+2)}$. Let $q\in[0,\delta_n]$. Then*

$$\lim_{n\to\infty}\frac{(1-q^{k+1}+q^{k+2})^{n-k-2}}{(1-q^{k+1})^n}=1.$$

Proof. Of course, we have

$$\frac{(1-q^{k+1}+q^{k+2})^{n-k-2}}{(1-q^{k+1})^n}\ge 1.$$

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The function $h(q) = q^{k+2}/(1-q^{k+1})$ is increasing on $[0, 1)$. Moreover, for $n \geq ((k+2)/(k+1))^{(k+1)(k+2)/(k+3/2)}$, the function $f(q) = (1-q^{k+1}+q^{k+2})^{k+1}$ is decreasing on $[0, \delta_n]$. Therefore, throughout the rest of the proof we always assume that $n \geq ((k+2)/(k+1))^{(k+1)(k+2)/(k+3/2)}$. We obtain

$$\begin{aligned}\frac{(1-q^{k+1}+q^{k+2})^{n-k-2}}{(1-q^{k+1})^n} &= \left(1+\frac{q^{k+2}}{1-q^{k+1}}\right)^n \frac{1}{(1-q^{k+1}+q^{k+2})^{k+2}} \\ &\leq \left(1+\frac{(1/n)^{(k+3/2)/(k+1)}}{1-\delta_n^{k+1}}\right)^n \frac{1}{(1-\delta_n^{k+1}+\delta_n^{k+2})^{k+2}} \\ &=: \alpha_n.\end{aligned}$$

Since $\lim_{n\to\infty} 1/(1-\delta_n^{k+1}+\delta_n^{k+2})^{k+2} = 1$ and

$$\begin{aligned}&\lim_{n\to\infty}\left(1+\frac{(1/n)^{(k+3/2)/(k+1)}}{1-\delta_n^{k+1}}\right)^n \\ &= \lim_{n\to\infty}\left[\left(1+\frac{1}{nn^{1/(2(k+1))}(1-\delta_n^{k+1})}\right)^{nn^{1/(2(k+1))}(1-\delta_n^{k+1})}\right]^{n/nn^{1/(2(k+1))}(1-\delta_n^{k+1})} \\ &= \mathrm{e}^0 \\ &= 1,\end{aligned}$$

we obtain $\lim_{n\to\infty} \alpha_n = 1$. □

Theorem 5.3. *For k being a constant, π being a random permutation of vertices of P_n^k, and $\tilde{\tau}_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the labelled model, we have*

$$\limsup_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \leq \Gamma\left(1+\frac{1}{k+1}\right).$$

Proof. By Lemma 5.1, we know that $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \int_0^1 \psi_{n,k+1}(p)/p\,\mathrm{d}p$. By Lemma 4.3, $\psi_{n,k+1}(p) \leq p^2(1-p(1-p)^{k+1})^{n-k-2}$ for $n > k+2$, hence, we can write $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \leq \int_0^1 p(1-p(1-p)^{k+1})^{n-k-2}\,\mathrm{d}p$. Letting $q = 1-p$, we obtain $\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \leq \int_0^1 (1-q)(1-(1-q)q^{k+1})^{n-k-2}\,\mathrm{d}q$, which yields

$$\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \leq \int_0^1 (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q. \tag{5.9}$$

By (5.9) and Lemma 5.6, we know that, for $\delta_n = (1/n)^{(k+3/2)/(k+1)(k+2)}$,

$$\begin{aligned}\limsup_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] &\leq \lim_{n\to\infty} n^{1/(k+1)} \int_0^1 (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q \\ &= \lim_{n\to\infty} n^{1/(k+1)} \int_0^{\delta_n} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q.\end{aligned} \tag{5.10}$$

By Lemma 5.6 and (5.5), we obtain also

$$\lim_{n\to\infty} n^{1/(k+1)} \int_0^1 (1-q^{k+1})^n\,\mathrm{d}q = \lim_{n\to\infty} n^{1/(k+1)} \int_0^{\delta_n} (1-q^{k+1})^n\,\mathrm{d}q = \Gamma\left(1+\frac{1}{k+1}\right). \tag{5.11}$$

From Lemma 5.7, there exists $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\begin{aligned}\lim_{n\to\infty} \frac{n^{1/(k+1)} \int_0^{\delta_n} (1-q^{k+1}+q^{k+2})^{n-k-2}\,\mathrm{d}q}{n^{1/(k+1)} \int_0^{\delta_n} (1-q^{k+1})^n\,\mathrm{d}q} &\le \lim_{n\to\infty} \frac{\int_0^{\delta_n} (1-q^{k+1})^n (1+\varepsilon_n)\,\mathrm{d}q}{\int_0^{\delta_n} (1-q^{k+1})^n\,\mathrm{d}q} \\ &= \lim_{n\to\infty} (1+\varepsilon_n) \\ &= 1,\end{aligned}$$

which with (5.10) and (5.11) implies

$$\limsup_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] \le \Gamma\biggl(1+\frac{1}{k+1}\biggr). \qquad \square$$

Corollary 5.1. *For π being a random permutation of vertices of P_n^k and $\tilde{\tau}_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the labelled model, we have*

$$\lim_{n\to\infty} n^{1/(k+1)}\, \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}] = \begin{cases} \Gamma\Bigl(1+\dfrac{1}{k+1}\Bigr) & \text{if } k \text{ is a constant,} \\ 1 & \text{if } k = o(\log n) \text{ and } k(n) \to \infty \text{ as } n \to \infty, \\ 1 - \dfrac{1}{2\mathrm{e}^{1/c}} & \text{if } k(n) = c\log n, \\ \frac{1}{2} & \text{if } k(n) = \omega(\log n). \end{cases}$$

In particular, for $k = 1$, we obtain $\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}[\pi_{\tilde{\tau}_{n,1}} = \mathbf{1}] = \sqrt{\pi}/2$, which is a result of Kubicki and Morayne [17].

Proof. The corollary follows from Theorems 5.1, 5.2, and 5.3. $\square$

6. The probability of success of $\tau_{n,k}$ (unlabelled model)

In this section we analyse the asymptotic behaviour of the optimal algorithm for choosing the sink in the unlabelled model (when the selector does not know the length of the edges that appear in the induced graph). We do not know what the optimal algorithm is (denoted by $\tau_{n,k}$) in this case, however we are able to state quite accurate lower and upper bounds for the probability of its success.

Remark 6.1. (*The $k = n-1$ case.*) When $k = n-1$, we deal with the classical secretary problem, thus $\mathbb{P}[\pi_{\tau_{n,n-1}} = \mathbf{1}] \to 1/\mathrm{e}$ as $n \to \infty$ (see [19]) and also $\lim_{n\to\infty} n^{1/n}\mathbb{P}[\pi_{\tau_{n,n-1}} = \mathbf{1}] = 1/\mathrm{e}$. Throughout the rest of this section we always assume that $k < n-1$.

Theorem 6.1. *For π being a random permutation of vertices of P_n^k and $\tau_{n,k}$ being an optimal stopping time for P_n^k when looking for the sink in the unlabelled model, we have*

$$\limsup_{n\to\infty} n^{1/(k+1)}\, \mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}] \le \begin{cases} \Gamma\Bigl(1+\dfrac{1}{k+1}\Bigr) & \text{if } k \text{ is a constant,} \\ 1 & \text{if } k = o(\log n) \text{ and } k(n) \to \infty \text{ as } n \to \infty, \\ 1 - \dfrac{1}{2\mathrm{e}^{1/c}} & \text{if } k(n) = c\log n, \\ \frac{1}{2} & \text{if } k(n) = \omega(\log n). \end{cases}$$

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Proof. We have $\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}] \leq \mathbb{P}[\pi_{\tilde{\tau}_{n,k}} = \mathbf{1}]$, since when the lengths of the edges are known in the induced graph, we can take at least as efficient a decision as to when they are not known. Therefore, the result follows simply from Corollary 5.1. □

Since $\tau_{n,k}$ is optimal, it performs at least as well as any other stopping time. We will state the lower bound for the probability of success of $\tau_{n,k}$ by analysing the effectiveness of the following randomized algorithm τ_p^*.

- *Flip an asymmetric coin, having some probability p of coming down tails, n times.*
- *If it comes down tails M times, reject the first M elements.*
- *After this time pick the first element that is maximal in the induced graph. In other words, τ_p^* is equal to the first $j > M$ such that $\pi_j \in \max(P_{(j)})$.*
- *If no such j is found then define $\tau_p^* = n$.*

The randomization used in the above definition was introduced by Preater [21], who used the fact stated in Lemma 6.1 below. The algorithm itself was already presented in [12]. Here, we carry out a finer analysis of its probability of success.

Lemma 6.1. *Let $\pi \in S_n$ be a random permutation of vertices in V. Suppose that we have a coin that comes down tails with probability p. Let M denote the number of tails in n tosses. Then all vertices from V appear in $\{\pi_1, \pi_2, \dots, \pi_M\}$ with probability p independently.*

Throughout the rest of this section V_p^* will denote the set $\{\pi_1, \pi_2, \dots, \pi_M\}$ from Lemma 6.1. Let us also define the following sequence of the indicator random variables:

$$\tilde{X}_i^{(p)} = \begin{cases} 1 & \text{if } \{v_{i+1}, v_{i+2}, \dots, v_{i+k+1}\} \subseteq V_n \setminus V_p^*, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1 \leq i \leq n-k-1.$$

Let $\tilde{X}^{(p)} = \sum_{i=1}^{n-k-2} \tilde{X}_i^{(p)}$ and $\tilde{Y}^{(p)} = \sum_{i=2}^{n-k-1} \tilde{X}_i^{(p)}$. (Of course, $\tilde{X}^{(p)}$ and $\tilde{Y}^{(p)}$ both have the same distribution.)

Theorem 6.2. *For k being a constant, π being a random permutation of vertices of P_n^k, and $p = 1 - \delta_k n^{-1/(k+1)}$, where $\delta_k = (k+2)^{-1/(k+1)}$, we have*

$$\liminf_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq \left(\frac{1}{k+2}\right)^{1/(k+1)} \frac{k+1}{k+2}.$$

Proof. Note that if $\mathbf{1} \notin V_p^*$, $v_2 \in V_p^*$, and $\tilde{Y}^{(p)} = 0$, then $\mathbf{1}$ is the only element that comes as a maximal one in an induced graph after time M. Those three events are independent. Since $\mathbb{E}[\tilde{Y}^{(p)}] \leq n(1-p)^{k+1}$ and, by Markov's inequality, $\mathbb{P}[\tilde{Y}^{(p)} = 0] \geq 1 - n(1-p)^{k+1}$, by Lemma 6.1 we obtain

$$\begin{aligned} \mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] &\geq \mathbb{P}[\mathbf{1} \notin V_p^*, v_2 \in V_p^*, \tilde{Y}^{(p)} = 0] \\ &= \mathbb{P}[\mathbf{1} \notin V_p^*]\mathbb{P}[v_2 \in V_p^*]\mathbb{P}[\tilde{Y}^{(p)} = 0] \\ &\geq (1-p)p(1 - n(1-p)^{k+1}). \end{aligned} \tag{6.1}$$

Since $p = 1 - \delta_k n^{-1/(k+1)}$, where $\delta_k = (k+2)^{-1/(k+1)}$, we have

$$n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq \delta_k(1 - \delta_k n^{-1/(k+1)})(1 - \delta_k^{k+1})$$
$$\to \left(\frac{1}{k+2}\right)^{1/(k+1)} \frac{k+1}{k+2} \quad \text{as } n \to \infty. \qquad \square$$

Theorem 6.3. *For $k = k(n) \to \infty$ as $n \to \infty$ such that $k(n) = o(\log n)$, π being a random permutation of vertices of P_n^k, and $p = 1 - \delta_n n^{-1/(k+1)}$, where δ_n is a function such that $\delta_n = 1 - 1/o(k(n))$ and $\delta_n \to 1$ as $n \to \infty$, we have*

$$\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq 1.$$

Proof. Recall that $\psi_{n,k+1}(p)$ is the probability that P_n^{k+1} percolates if each vertex is open with probability p, independently from all other vertices. We can associate the event of the vertex being open in the percolation model with the event of the vertex being in V_p^* in optimal stopping model. Then $\psi_{n,k+1}(p) = p^2\mathbb{P}[\tilde{X}^{(p)} = 0]$, where p^2 corresponds to the probability that $\mathbf{1}$ and v_n both belong to V_p^* (or, equivalently, are both open). Since $\tilde{X}^{(p)}$ and $\tilde{Y}^{(p)}$ have the same distribution, by (6.1) and Lemma 4.3, we obtain

$$\begin{aligned}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] &\geq \mathbb{P}[\mathbf{1} \notin V_p^*]\mathbb{P}[v_2 \in V_p^*]\mathbb{P}[\tilde{X}^{(p)} = 0] \\ &= (1-p)p\frac{\psi_{n,k+1}(p)}{p^2} \\ &\geq (1-p)p(1-(1-p)^{k+1})^{n-k-1}.\end{aligned}$$

Since $p = 1 - \delta_n n^{-1/(k+1)}$, $n^{-1/(k+1)} \to 0$ as $n \to \infty$ and $n/\delta_n^{k+1} \to \infty$ as $n \to \infty$, we have

$$n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq \delta_n(1 - \delta_n n^{-1/(k+1)})\left(1 - \frac{\delta_n^{k+1}}{n}\right)^{n-k-1} \to 1 \quad \text{as } n \to \infty. \qquad \square$$

Theorem 6.4. *For $k = k(n) = \omega(\log n)$, π being a random permutation of vertices of P_n^k, and $p = 1/\mathrm{e}$, we have*

$$\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq \frac{1}{\mathrm{e}}.$$

Proof. Since $\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1} \mid \mathbf{1} \in V_p^*] = 0$, by Lemma 6.1 we have

$$\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] = \mathbb{P}[\mathbf{1} \notin V_p^*]\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1} \mid \mathbf{1} \notin V_p^*] = (1-p)\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1} \mid \mathbf{1} \notin V_p^*]. \tag{6.2}$$

Now, for any events B and C we will write, for short, $\mathbb{P}_{\mathbf{1}}[B]$ instead of $\mathbb{P}[B \mid \mathbf{1} \notin V_p^*]$ and $\mathbb{P}_{\mathbf{1}}[B \mid C]$ instead of $\mathbb{P}[B \mid \mathbf{1} \notin V_p^* \cap C]$. For $s = 2, \ldots, k+2$, let B_s denote the event that v_s is the topmost (apart from $\mathbf{1}$) vertex that belongs to V_p^*, i.e. for all t, $2 \leq t < s$, $v_t \notin V_p^*$ and $v_s \in V_p^*$. Let $\tilde{Z}_s = \sum_{i=s}^{n-k-1} \tilde{X}_i^{(p)}$ (in particular, $\tilde{Z}_2 = \tilde{Y}^{(p)}$). Note that the events $\mathbf{1} \notin V_p^*$, B_s, and $\tilde{Z}_s = 0$ are independent. Moreover, $\mathbb{P}_{\mathbf{1}}[\pi_{\tau_p^*} = \mathbf{1} \mid B_s \cap \tilde{Z}_s = 0] = 1/(s-1)$ and $\mathbb{P}[\tilde{Z}_s = 0] \geq \mathbb{P}[\tilde{Z}_2 = 0]$ for any $s = 3, \ldots, k+2$. Since $\tilde{X}^{(p)}$ and $\tilde{Y}^{(p)}$ have the same distribution and $\mathbb{P}[\tilde{X}^{(p)} = 0] = \psi_{n,k+1}(p)/p^2$ (compare proof of Theorem 6.3), by Lemma 4.3 we obtain, for $k = k(n) = \omega(\log n)$,

$$\mathbb{P}[\tilde{Z}_2 = 0] = \mathbb{P}[\tilde{Y}^{(p)} = 0] \geq (1-(1-p)^{k+1})^{n-k-1} \to 1 \quad \text{as } n \to \infty. \tag{6.3}$$

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Therefore,

$$\begin{aligned}
\mathbb{P}_{\mathbf{1}}[\pi_{\tau_p^*} = \mathbf{1}] &\geq \sum_{s=2}^{k+2} \mathbb{P}_{\mathbf{1}}[\pi_{\tau_p^*} = \mathbf{1} \mid B_s \cap \tilde{Z}_s = 0]\mathbb{P}_{\mathbf{1}}[B_s \cap \tilde{Z}_s = 0] \\
&= \sum_{s=2}^{k+2} \frac{1}{s-1} \mathbb{P}[B_s]\mathbb{P}[\tilde{Z}_s = 0] \\
&\geq \mathbb{P}[\tilde{Z}_2 = 0] \sum_{s=2}^{k+2} \frac{1}{s-1}(1-p)^{s-2} p \\
&= \mathbb{P}[\tilde{Y}^{(p)} = 0] \frac{p}{1-p} \sum_{s=1}^{k+1} \frac{(1-p)^s}{s} \\
&\to -\frac{p}{1-p} \log p \quad \text{as } n \to \infty. \qquad (6.4)
\end{aligned}$$

Finally, for $p = 1/\mathrm{e}$, by (6.2) and (6.4),

$$\liminf_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq \liminf_{n\to\infty} n^{1/(k+1)} \bigg(-(1-p) \frac{p}{1-p} \log p \bigg) = -p \log p = \frac{1}{\mathrm{e}},$$

completing the proof. □

Theorem 6.5. *For $k = k(n) = c \log n$, π being a random permutation of vertices of P_n^k, and $p = 1 - \delta/n^{1/(k+1)}$, where $\delta = \mathrm{e}^{1/c}(1 - 1/\mathrm{e})$ for $c > (\log(\mathrm{e}/(\mathrm{e}-1)))^{-1}$, and δ is a constant arbitrarily close to 1 for $c \leq (\log(\mathrm{e}/(\mathrm{e}-1)))^{-1}$, we have*

$$\liminf_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq \begin{cases} (1 - \mathrm{e}^{1/c}) \log(1 - \mathrm{e}^{-1/c}) & \text{if } c \leq (\log(\mathrm{e}/(\mathrm{e}-1)))^{-1}, \\ \mathrm{e}^{1/c - 1} & \text{if } c > (\log(\mathrm{e}/(\mathrm{e}-1)))^{-1}. \end{cases}$$

Proof. We follow the idea of the proof of Theorem 6.4 and, by (6.2) and (6.4) and the fact that p tends to $1 - \delta/\mathrm{e}^{1/c}$ from below, we obtain

$$\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \geq (1-p)\mathbb{P}[\tilde{Y}^{(p)} = 0] \frac{p}{1-p} \sum_{s=1}^{k+1} \frac{(1-p)^s}{s} \geq p\mathbb{P}[\tilde{Y}^{(p)} = 0] \sum_{s=1}^{k+1} \frac{(\delta/\mathrm{e}^{1/c})^s}{s}.$$

For $p = 1 - \delta/n^{1/(k+1)}$, $k = k(n) = c \log n$, and $\delta \in (0, 1)$, we have (compare (6.3))

$$\mathbb{P}[\tilde{Y}^{(p)} = 0] \geq (1 - (1-p)^{k+1})^{n-k-1} = \bigg(1 - \frac{\delta^{k+1}}{n}\bigg)^{n-k-1} \to 1 \quad \text{as } n \to \infty.$$

Therefore, for $\delta \in (0, 1)$, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} n^{1/(k+1)} \mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \\
&\quad \geq \liminf_{n\to\infty} n^{1/(k+1)} \bigg(1 - \frac{\delta}{n^{1/(k+1)}}\bigg)\bigg(1 - \frac{\delta^{k+1}}{n}\bigg)^{n-k-1} \sum_{s=1}^{k+1} \frac{(\delta/\mathrm{e}^{1/c})^s}{s} \\
&\quad = \mathrm{e}^{1/c}\bigg(1 - \frac{\delta}{\mathrm{e}^{1/c}}\bigg)\bigg(-\log\bigg(1 - \frac{\delta}{\mathrm{e}^{1/c}}\bigg)\bigg).
\end{aligned}$$

TABLE 1: Values of the lower bounds for $\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}]$ and the upper bounds for $\limsup_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}]$.

	Lower bound	Upper bound
k is constant	$(k+2)^{-1/(k+1)}\dfrac{k+1}{k+2}$	$\Gamma\left(1+\dfrac{1}{k+1}\right)$
$k(n)\to\infty$ and $k(n)=o(\log n)$	1	1
$k(n)=c\log n$	$c \le \left(\log\dfrac{\mathrm{e}}{\mathrm{e}-1}\right)^{-1}$, $(1-\mathrm{e}^{1/c})\log(1-\mathrm{e}^{-1/c})$ $c > \left(\log\dfrac{\mathrm{e}}{\mathrm{e}-1}\right)^{-1}$, $\mathrm{e}^{1/c-1}$	$1-\dfrac{1}{2\mathrm{e}^{1/c}}$
$k(n)=\omega(\log n)$	$\dfrac{1}{\mathrm{e}}$	$\frac{1}{2}$

Setting $\delta = \mathrm{e}^{1/c}(1-1/\mathrm{e})$ for $c > (\log(\mathrm{e}/(\mathrm{e}-1)))^{-1}$, we have

$$\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] \ge \mathrm{e}^{1/c}\frac{1}{\mathrm{e}}\left(-\log\left(\frac{1}{\mathrm{e}}\right)\right) = \mathrm{e}^{1/c-1}.$$

Now, let $c \le (\log(\mathrm{e}(\mathrm{e}-1)))^{-1}$. Since we can choose δ to be a constant arbitrarily close to 1,

$$\begin{aligned}\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}] &\ge \mathrm{e}^{1/c}\left(1-\frac{1}{\mathrm{e}^{1/c}}\right)(-\log(1-\mathrm{e}^{-1/c}))\\ &= (1-\mathrm{e}^{1/c})\log(1-\mathrm{e}^{-1/c}).\end{aligned}$$ □

Remark 6.2. Since $\tau_{n,k}$ is optimal, we have $\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}] \ge \mathbb{P}[\pi_{\tau_p^*} = \mathbf{1}]$ and the lower bounds from Theorems 6.2–6.5 apply also to $\tau_{n,k}$.

See Table 1 for values of the lower bounds for $\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}]$ and the upper bounds for $\liminf_{n\to\infty} n^{1/(k+1)}\mathbb{P}[\pi_{\tau_{n,k}} = \mathbf{1}]$ for the whole range of k.

7. Remarks and open questions

The optimal strategy $\tilde{\tau}_{n,k}$ has a very simple description. This is an open question if one can characterize a natural, larger class of directed graphs for which the stopping time defined by $\tilde{\tau}_{n,k}$ is optimal.

The possibility of translating an optimal stopping problem into the percolation theory language was very convenient here. Crucial was the observation that $\tilde{\tau}_{n,k}$ stops the search of P_n^k if and only if P_n^{k+1} percolates and an analysis of site percolation process on P_n^{k+1} leads to the proof of the asymptotic behaviour of $\tilde{\tau}_{n,k}$. It is interesting whether there are other natural families of graphs for which a similar scheme works.

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