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# SOME REMARKS ON BLACKWELL–ROSS MARTINGALE INEQUALITIES

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Under a suitable condition on the conditional moment generating function of the martingale differences, an exponential supermartingale is used to generalize certain martingale inequalities due to Blackwell and Ross.

## 1. INTRODUCTION

Blackwell (cf. [2,3]) used game-theoretic methods to obtain the following inequalities. Let $\{S_n = \sum_1^n X_i,\ S_0 = 0,\ \mathcal{F}_n,\ n \geq 0\}$ be a supermartingale such that $|X_n| \leq 1$ and $E(X_n|\mathcal{F}_{n-1}) \leq -\gamma$ for all $n$ $(0 < \gamma < 1)$. Then, for any $a > 0$,

$$P(S_n \geq a \text{ for some } n \geq 1) \leq \left(\frac{1-\gamma}{1+\gamma}\right)^a. \tag{1}$$

In [3], he considered a martingale $\{S_n, \mathcal{F}_n\}$ where $|X_n| \leq 1$ and proved that for any positive constants $a$ and $b$,

$$P(S_n \geq a + bn \text{ for some } n \geq 1) \leq \exp(-2ab), \tag{2}$$

and for any positive $b$ $(0 < b < 1)$ and suitable $r$,

$$P(S_n \geq bn \text{ for some } n \geq m) \leq r^m \leq \exp(-mb^2/2). \tag{3}$$

Ross [5] used a supermartingale argument to extend (2) and (3) for martingales $S_n = \sum_1^n X_i$ when $-\alpha \leq X_n \leq \beta(\alpha, \beta > 0)$. These exponential inequalities result

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from the inherent subnormal character of the differences $X_n$. Therefore, it seems natural to widen the scope by assuming a suitable condition on the conditional moment generating function (mgf) $\phi_n(\theta) = E(\exp(\theta X_n)|\mathcal{F}_{n-1})$ for wider applications. Consequently, the boundedness condition of $X_n$'s is disposed and replaced by a subnormal condition on $\phi_n(\theta)$, leading to an exponential supermartingale, and the subsequent inequalities are easily derived. Thus, the purpose of this article is to generalize Blackwell–Ross inequalities for martingales (supermartingales) under a subnormal structure on $\phi_n(\theta)$ for wider scope and shorter and simpler proofs of earlier inequalities. Another variation of these inequalities is also given under a weaker condition leading to a sub-Poisson structure of $\phi_n(\theta)$. Thus, our exponential supermartingale approach unifies and generalizes the known inequalities for broader applications without the condition of bounded martingale (supermartingale) differences.

## 2. THE MAIN RESULTS

Let $\{S_n = \sum_1^n X_i, \mathcal{F}_n, n \geq 0\}$ be a martingale (supermartingale) with conditional mgf $\phi_n(\theta) = E(\exp(\theta X_n)|\mathcal{F}_{n-1})$ such that

$$\phi_n(\theta) \leq f(\theta) \leq \exp(-\gamma\theta + \lambda\theta^2) \quad (\lambda > 0, \gamma \geq 0),$$

where $f(\theta)$ is a continuous positive function with $f(0) = 1$.

Using a suitable exponential supermartingale, we derive the basic inequality

$$P(S_n \geq a + bn \text{ for some } n \geq m) \leq A^m \exp(-a(b+\gamma)/\lambda)$$

for a suitable $A \leq 1$. This inequality seems to be new under a subnormal structure that allows it to be true for martingales as well as supermartingales without the differences $X_n$'s being bounded. The generalized version of (3) given later also appears to be new. The given generalizations unify and simplify the earlier known inequalities. All of the inequalities are derived from the following well-known supermartingale inequality.

LEMMA: *Let $\{Z_n, \mathcal{F}_n, n \geq 0\}$ be a positive supermartingale. Then, for $m \geq 1$,*

$$P(\max_{n\geq m} Z_n \geq 1) \leq EZ_m \leq EZ_1 \leq EZ_0. \tag{4}$$

PROOF: Let $t = \inf\{n \geq m : Z_n \geq 1\}$ and $t(k) = \min(t,k)$, $k \geq m \geq 1$. Then $\{Z_{t(k)}, \mathcal{F}_k\}$ is a supermartingale by the optional stopping theorem and (4) follows from

$$P(Z_{t(k)} \geq 1) \leq EZ_{t(k)} \leq EZ_m \leq EZ_1 \leq EZ_0$$

by letting $k \to \infty$. Most of the subsequent inequalities are based on the following proposition, which is the basic generalized inequality described earlier. It generalizes known inequalities under less restrictive conditions by shorter and simpler proofs.

THEOREM: *Let* $\{S_n = \sum_1^n X_i, \mathcal{F}_n, n \geq 0\}$ *be a martingale (supermartingale) such that the conditional moment generating function* $\phi_n(\theta)$ *satisfies*

$$\phi_n(\theta) \leq f(\theta) \leq \exp(-\gamma\theta + \lambda\theta^2), \qquad \gamma \geq 0, \lambda > 0, \theta > 0,$$

*where* $f(\theta)$ *is a continuous positive function such that* $f(0) = 1$. *Then for positive a and b,*

$$P(S_n \geq a + bn \text{ for some } n \geq m) \leq A^m \exp(-a(b+\gamma)/\lambda), \tag{5}$$

*where* $A = e^{-b\theta_0} f(\theta_0) \leq 1$ *and* $\theta_0 = (b+\gamma)/\lambda$. *Moreover,*

$$P(S_n \geq bn \text{ for some } n \geq m) \leq A_0^m \exp\left(-\frac{m(b+\gamma)^2}{4\lambda}\right), \tag{6}$$

*where* $A_0 = \exp(-(b-\gamma)\theta_0/2) f(\theta_0)$ *and* $\theta_0 = (b+\gamma)/2\lambda$.

PROOF: Let $Z_n(\theta) = \exp(\theta S_n - a\theta - bn\theta)$, $\theta > 0$. Clearly,

$$\begin{aligned} E(Z_n(\theta) | \mathcal{F}_{n-1}) &= Z_{n-1}(\theta)\exp(-b\theta)\phi_n(\theta) \leq Z_{n-1}(\theta)\exp(-b\theta) f(\theta) \\ &\leq Z_{n-1}(\theta)\exp(-(b+\gamma)\theta + \lambda\theta^2) \leq Z_{n-1}(\theta) \end{aligned}$$

if $\theta = \theta_0 = (b+\gamma)/\lambda$. Thus, $Z_n(\theta_0)$ is a positive supermartingale and, hence, (4) gives

$$P(S_n \geq a + bn \text{ for some } n \geq m) \leq EZ_m(\theta_0).$$

It is clear that $EZ_m(\theta_0) \leq f(\theta_0)e^{-b\theta_0} EZ_{m-1}(\theta_0) = AEZ_{m-1}(\theta_0)$ and, thus,

$$EZ_m(\theta_0) \leq AEZ_{m-1}(\theta_0) \leq A^2 EZ_{m-2}(\theta_0) \leq \cdots \leq A^m EZ_0(\theta_0) = A^m \exp(-a\theta_0),$$

and (5) follows. To see (6), let $g(s,n) = m(b-s) + sn$, $n \geq m$, $s \leq b$, and note that $bn \geq g(s,n)$ for every $n \geq m$, and a minimization consideration leads to the choice of $s = s_0 = (b-\gamma)/2$. Thus,

$$\begin{aligned} &P(S_n \geq bn \text{ for some } n \geq m) \\ &\quad \leq P\left(S_n \geq \frac{m(b+\gamma)}{2} + \frac{(b-\gamma)}{2} n \text{ for some } n \geq m\right) \\ &\quad \leq P\left(S_n \geq \frac{m(b+\gamma)}{2} + \frac{(b-\gamma)}{2} n \text{ for some } n \geq 1\right), \end{aligned}$$

and (5) gives

$$P(S_n \geq bn \text{ for some } n \geq m) \leq A_0^m \exp(-m(b+\gamma)^2/4\lambda),$$

where $A_0 = \exp(-\frac{1}{2}(b-\gamma)\theta_0) f(\theta_0)$ and $\theta_0 = (b+\gamma)/2\lambda$.

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*Remark 1:* If $b = 0$, it is possible to improve (5) by assuming that $f(\theta)$ is convex and $f'(0) < 0$. Using the convexity and negative slope at $\theta = 0$, it is easy to see that $f(\theta) = 1$ has a unique solution $\theta_1 \geq \theta_0 = \gamma/2\lambda$ $(\gamma > 0)$ defined earlier. Therefore, $Z_n(\theta_1)$ becomes a supermartingale and the bound in (5) becomes $\exp(-a\theta_1)$. This remark will be used later in connection with the special case leading to (1).

*Special Cases:* Let $\{S_n = \sum_1^n X_i, \mathcal{F}_{n-1}, n \geq 0\}$ be a supermartingale where $-\alpha \leq X_n \leq \beta$ and $E(X_n|\mathcal{F}_{n-1}) \leq -\gamma$ $(\alpha > \gamma \geq 0, \beta > 0)$ for all $n$. For $\theta > 0$, let $m = (e^{\beta\theta} - e^{-\alpha\theta})/(\alpha + \beta)$ and

$$L(x) = \frac{\alpha}{\alpha+\beta}\, e^{\beta\theta} + \frac{\beta}{\alpha+\beta}\, e^{-\alpha\theta} + mx$$

be the line through $(-\alpha, e^{-\alpha\theta})$ and $(\beta, e^{\beta\theta})$. Due to convexity, $e^{\theta x} \leq L(x)$ and, hence,

$$\phi_n(\theta) = E(X_n|\mathcal{F}_{n-1}) \leq \frac{\alpha}{\alpha+\beta}\, e^{\beta\theta} + \frac{\beta}{\alpha+\beta}\, e^{-\alpha\theta} - m\gamma.$$

Letting $p = (\alpha - \gamma)/(\alpha + \beta)$ and $q = (\beta + \gamma)/(\alpha + \beta)$, we obtain

$$\phi_n(\theta) \leq e^{-\gamma\theta}(pe^{(\alpha+\beta)\theta q} + qe^{-(\alpha+\beta)p\theta}) = f(\theta). \tag{7}$$

By Lemma A.6 of Alon and Spencer [1, p. 235], we have

$$\phi_n(\theta) \leq \exp\left(-\gamma\theta + \frac{(\alpha+\beta)^2\theta^2}{8}\right).$$

Consequently, (5) gives

$$P(S_n \geq a + bn \text{ for some } n \geq m) \leq A^m \exp\left(-\frac{8a(b+\gamma)}{(\alpha+\beta)^2}\right), \tag{8}$$

where $A = f(\theta_0)e^{-b\theta_0}$, $\theta_0 = 8(b + \gamma)/(\alpha + \beta)^2$, and $f(\theta)$ is defined in (7). Similarly, (6) gives

$$P(S_n \geq bn \text{ for some } n \geq m) \leq A_0^m \exp\left(-\frac{2m(b+\gamma)^2}{(\alpha+\beta)^2}\right),$$

where $A_0 = \exp(-\frac{1}{2}(b - \gamma)\theta_0)f(\theta_0)$ and $\theta_0 = 4(b + \gamma)/(\alpha + \beta)^2$.

If $\gamma = 0$, then the preceding inequalities become the ones obtained by Ross [5]. Moreover, Blackwell's inequality (1) is also included in (8) with a different bound. Since $A \leq 1$ and setting $b = 0$ and $\alpha = \beta = 1$, (8) becomes

$$P(S_n \geq a \text{ for some } n \geq 1) \leq \exp(-2a\gamma),$$

as opposed to the bound $((1-\gamma)/(1+\gamma))^a$. However, as mentioned in Remark 1, we can improve the bound (5) as follows. Clearly, in this special case, $f(\theta)$ defined in (7) becomes

$$\frac{1-\gamma}{2}e^{\theta}+\frac{1+\gamma}{2}e^{-\theta},$$

and the unique nonzero solution of $f(\theta)=1$ is

$$\theta_1=\ln\left(\frac{1-\gamma}{1+\gamma}\right)$$

and the bound $\exp(-a\theta_1)$ in Remark 1 produces (1). Also, it should be noted that (1) gives a sharper bound than $\exp(-2a\gamma)$. In fact, $((1-\gamma)/(1+\gamma))^a<\exp(-2a\gamma)$, and the bound in (1) is attained. For example, let $X_1,X_2,\ldots$ be iid random variables such that $P(X_1=-1)=q=(1+\gamma)/2$ and $P(X_1=1)=p=[(1-\gamma)/2]\ (0<\gamma<1)$ so that $EX_1=-\gamma$. Set $S_n=\sum_1^n X_i$ and let $f(a)=P(S_n\geq a$ for some $n\geq 1)=P(S_n=a$ for some $n\geq 1)$, where $a$ is a positive integer. A simple conditioning argument shows that $f(a)=pf(a-1)+qf(a+1)$. Since $f(a)<1$, the only nontrivial solution is $f(a)=(p/q)^a=((1-\gamma)/(1+\gamma))^a$. It is also possible to find an analogue of the first part of (3) as follows. Consider $Z_n(\theta)=\exp(\theta S_n-bn\theta)$, $\theta>0$, and note that $E(Z_n(\theta)|\mathcal{F}_{n-1})=Z_{n-1}(\theta)e^{-b\theta}\phi_n(\theta)$. From (7), we have

$$E(Z_n(\theta)|\mathcal{F}_{n-1})\leq Z_{n-1}(\theta)\exp(-b\theta)f(\theta)=Z_{n-1}(\theta)\psi(\theta),$$

where $\psi(\theta)=p\exp((\beta-b)\theta)+q\exp(-(\alpha+b)\theta)$, $\alpha>\gamma\geq 0$, and $b\leq\beta$. It is easy to see that $\psi(\theta)$ is convex and $\psi(0)=1$ and $\psi'(0)=-\gamma-b<0$. Hence, there exists $\theta^*>0$ such that $\psi(\theta^*)=1$, and there is $\theta_1$ $(0<\theta_1<\theta^*)$ that minimizes $\psi(\theta)$; it occurs at

$$\theta_1=\frac{1}{(\alpha+\beta)}\ln\frac{(\beta+\gamma)(\alpha+b)}{(\alpha-\gamma)(\beta-b)}.$$

Also, note that $\rho=\psi(\theta_1)<\psi(\theta_0)\leq 1$, where $\theta_0=8(b+\gamma)/(\alpha+\beta)^2$. Thus, $EZ_n(\theta_1)\leq\rho EZ_{n-1}(\theta_1)$, and it follows that $EZ_m(\theta_1)\leq\rho^m$. Hence, we obtain

$$P(S_n\geq bn \text{ for some } n\geq m)\leq\rho^m\leq A_0^m\exp\left(-\frac{2m(b+\gamma)^2}{(\alpha+\beta)^2}\right),$$

where the second bound is the same as obtained earlier by the use of Alon and Spencer's inequality for $\psi(\theta)$. This is the generalized version of (3) for nonsymmetrically bounded martingales (supermartingales), and the first bound $\rho^m$ is certainly new. Also, it should be remarked that the left-hand probability is zero if $b>\beta$. In the special case $\alpha=\beta=1$ and $\gamma=0$, our minimizing $\theta_1$ becomes

$$\theta_1 = \frac{1}{2}\ln\frac{(1+b)}{(1-b)}, \quad 0<b<1$$

and

$$\rho = \psi(\theta_1) = 1/[(1+b)^{1+b}(1-b)^{1-b}]^{1/2},$$

which produces Blackwell's inequality (3).

## 3. ANOTHER EXTENSION

We now consider a supermartingale $\{S_n = \sum_1^n X_i, \mathcal{F}_n, n \geq 0\}$ such that $X_n \leq \beta$ ($\beta > 0$) for all $n$ (i.e., the differences $X_n$ are bounded above only). With no loss of generality, we assume that $\beta = 1$. For $\theta > 0$ and $x \leq 1$, we will use the inequality

$$e^{\theta x} \leq 1 + \theta x + x^2(e^{\theta} - \theta - 1). \tag{9}$$

Inequality (9) is obvious from $e^{\theta x} = \sum_{k=0}^{\infty}(\theta^k x^k/k!)$ when $0 \leq x \leq 1$. That (9) is true for $x \leq 0$ follows from the elementary inequality $e^u \leq 1 + u + \frac{1}{2}u^2$, which follows from $e^t \geq 1 + t$ when integrated from $u$ to zero. Replacing $u$ by $\theta x$ ($x \leq 0, \theta > 0$) and using the fact that $e^{\theta} - \theta - 1 \geq \frac{1}{2}\theta^2$, (9) follows. Now let $v_n = E(X_n^2 | \mathcal{F}_{n-1})$ and note from (9) that

$$\phi_n(\theta) = E(e^{\theta X_n} | \mathcal{F}_{n-1}) \leq 1 + v_n(e^{\theta} - \theta - 1) \leq \exp(v_n(e^{\theta} - \theta - 1)). \tag{10}$$

Now define $Z_n(\theta) = \exp(\theta S_n - a\theta - b\theta \sum_1^n v_i)$, $\theta > 0$. Then $E(Z_n(\theta)|\mathcal{F}_{n-1}) = Z_{n-1}(\theta)\exp(-bv_n\theta)\phi_n(\theta)$, and it follows from (10) that

$$E(Z_n(\theta)|\mathcal{F}_{n-1}) \leq Z_{n-1}(\theta)\exp(v_n\psi(\theta)),$$

where $\psi(\theta) = e^{\theta} - (1+b)\,\theta - 1$. We set $\psi(\theta) = 0$, which has a unique solution $\theta_0 = \delta(b) > 0$, so that $Z_n(\theta_0)$ is a supermartingale. Hence, we obtain from (4) that

$$P\left(S_n \geq a + b\sum_1^n v_i \text{ for some } n \geq 1\right) \leq \exp(-a\delta(b)). \tag{11}$$

If we suppose that the $v_i$'s are constants, then

$$\begin{aligned} P&\left(S_n \geq b\sum_1^n v_i \text{ for some } n \geq m\right) \\ &\leq P\left(S_n \geq \frac{1}{2}b\sum_1^m v_i + \frac{1}{2}b\sum_1^n v_i \text{ for some } n \geq 1\right) \\ &\leq \exp\left(-\frac{b}{2}\,\delta(b/2)\textstyle\sum_1^m v_i\right). \end{aligned} \tag{12}$$

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Properties of $\delta(b)$ have been noted in Lemma 1 of [4] (e.g., $\ln(1+b) < \delta(b) < 2b$). These inequalities become useful for certain convergence problems. Clearly, if $S_n$ is a martingale, then (12) continues to hold for $|S_n|$ when the bound is multiplied by a factor of 2, and obviously $S_n/\sum_{i=1}^{n} v_i$ converges to zero *a.s.* provided $\sum_{i=1}^{n} v_i \to \infty$ as $n \to \infty$. As an example, let $P(X_n = -1 | \mathcal{F}_{n-1}) = 1/(n+1)$, $P(X_n = 0 | \mathcal{F}_{n-1}) = 1 - [2/(n+1)]$, and $P(X_n = 1 | \mathcal{F}_{n-1}) = 1/(n+1)$, $n \geq 1$. Then $v_n = 2/(n+1)$, and $S_n/\sum_{i=1}^{n} v_i$ converges to zero with probability 1, and the denominator is $\sim \ln(n)$ (as opposed to $n$ in the usual strong law of large numbers).

*Remark 2:* If $X_n \leq \beta$ ($\beta > 0$), then replacing $a$ by $a\beta$ and $b$ by $b\beta$ in (11) and (12), the inequalities continue to hold. In addition, if $E(X_n | \mathcal{F}_{n-1}) \leq -\gamma$ ($\gamma > 0$), then (11) continues to hold when $a$ is replaced by $(a + \gamma)$.

*Remark 3:* A final remark is that when $0 < \theta < 1$, then $\psi(\theta) = e^{\theta} - \theta - 1 = \sum_2^{\infty} (\theta^k/k!) \leq \frac{1}{2}\theta^2(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots) \leq \frac{3}{4}\theta^2$. Now set $Z_n(\theta) = \exp(\theta S_n - a\theta - b\theta \sum_1^n v_i)$, where $v_i$ has been defined earlier. Clearly, $E(Z_n(\theta) | \mathcal{F}_{n-1}) \leq Z_{n-1}(\theta) \exp(-b\theta + \frac{3}{4}\theta^2)$. Hence, letting $\theta_0 = 4b/3$, it follows that if $b \leq \frac{3}{4}$, then (4) implies

$$P\left(S_n \geq a + b\sum_1^n v_i \text{ for some } n \geq 1\right) \leq \exp(-a\theta_0) \leq \exp(-4ab/3) < \exp(-ab).$$

Moreover, if $\{S_n, \mathcal{F}_n\}$ is a martingale, then this inequality still hods with $v_i$ as the conditional variance of the martingale difference $X_i$.

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