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SHAPE THEOREMS FOR POISSON HAIL ON A BIVARIATE GROUND

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Abstract

We consider an extension of the Poisson hail model where the service speed is either 0 or ∞ at each point of the Euclidean space. We use and develop tools pertaining to sub-additive ergodic theory in order to establish shape theorems for the growth of the ice-heap under light tail assumptions on the hailstone characteristics. The asymptotic shape depends on the statistics of the hailstones, the intensity of the underlying Poisson point process, and on the geometrical properties of the zero speed set.

Keywords: Point process theory; Poisson rain; stochastic geometry; random closed set; time and space growth; shape; queueing theory; max-plus algebra; heaps; branching process; sub-additive ergodic theory

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1. Introduction

In this paper we revisit the Poisson hail growth model introduced in [1]. This model features independent and identically distributed (i.i.d.) pairs, each consisting of a compact random closed set (RACS) and a positive number, arriving on $\mathbb{R}^d$ according to a Poisson rain. Each pair is referred to as a *hailstone*; the RACS is referred to as the *footprint* of the hailstone and the positive number is its *height*. Each point of the Euclidean space is a server (in the queueing theory sense). The case studied in [1] is that with one type of servers.

The pure growth model is that where the service speed of each point of $\mathbb{R}^d$ is zero, and where the hailstones accumulate over time to form a random (ice) heap. This model can be seen as a simplified version of the so-called *diffusion limited aggregation* (DLA) model [8] with half-space initial condition. The main difference between this model and DLA lies in the fact that the hailstones fall in a privileged direction (e.g. according to gravity) in the former case rather than in a diffusive way in the latter.

The height of a tagged hailstone in this heap is the sum of its own height plus the maximum of the heights of all hailstones that arrived before and that have a footprint that intersects that of the tagged one. It was shown in [1] that when the dth power of the random diameters and the random heights have light-tailed distributions, i.e. have finite exponential moment, then the growth of the random heap of the pure growth model is asymptotically linear with time. This

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result was recently extended to certain heavy-tailed distributions [6]. In [1], the case where all servers have a constant positive service speed was also analyzed. The model with positive service speed is motivated by wireless communications: transmitters arrive according to a Poisson rain in the Euclidean plane. The footprint of an arrival is a spatial exclusion area which should be free of other transmitters during some random transmission time (the height of the arrival). The hard exclusion rule is simply obtained by a first-in–first-out (FIFO) serialization: an arriving transmitter should first wait for its exclusion area to be free of all those arrived before; it then transmits and finally leaves.

In this paper we consider a bivariate generalization of this model with two types of servers. All points in some subset of $\mathbb{R}^d$, called the *substrate*, have zero service speed, whereas service speed is infinite in the complement. For instance, when the substrate is limited to a single point (a special case that we refer to as the *stick model* below), hailstones get aggregated to the heap if their footprint intersects this point or the footprint of any earlier hailstone that is part of the heap, which is some analogue of DLA with an initial condition given by a point. As above, the main difference is that the diffusive and isotropic arrivals of DLA are replaced by pure gravitation. In the wireless setting alluded to above, this model allows one to evaluate the negative consequences of the FIFO rule. The substrate represents a customer with a very long transmission time (zero speed) and the complement normal operation (simplified to infinite speed). The bivariate model hence explains how congestion builds at the fluid scale in this FIFO model.

In this paper we study the asymptotic shape of this RACS when time tends to ∞ in this bivariate speed setting. Like the model of [1], this variant belongs to the class of infinite-dimensional max-plus linear systems [3]. Among the few instances of such systems studied in the past, the closest is the work on infinite tandem queueing networks [2]. The underlying structure of the max-plus recursion in [2] is a two-dimensional lattice. In contrast, here, the underlying structure of the recursion is random. Among common aspects, let us stress shape theorems. The lattice shape theorems in [2] are related to those in first-passage percolation [10], in the theory of lattice animals [5], [7]. Those of this paper pertain to first-passage percolation in random media. This topic was recently studied for certain random graphs such as the configuration model [4]. The shape theorems established in this paper are based on random structures of the Euclidean space, which stem from point process theory (Poisson rain) and stochastic geometry (RACS).

In Section 2 we provide the precise formulation of the model. In Section 3 we study the stick model alluded to above. In this case, Theorem 3.1 establishes a linear asymptotic growth for the maximum height of the heap in a convex set of directions. Also for the stick model, Theorem 3.2 establishes a linear asymptotic growth for the footprint of the heap. Both proofs rely on the version of the super additive ergodic theorem by Liggett [9]. Based on these results we are able to prove in Theorem 3.3 the existence of an asymptotic phase transition for the heap in the stick model.

The stick model is interesting not only because of its similarity with, e.g. the DLA, but also because it is instrumental in extending some of the previous results to more general substrates, as shown in the subsequent sections. The idea originates from [1] and, heuristically, it consists of reversing time and gravitation about a given point. Analogues of Theorem 3.1 are extended by this duality argument for compact substrates in Section 4, Theorem 4.1 and convex conical substrates in Section 5, Theorem 5.1. Let us emphasize that conic substrates are the basic cases we need to understand after performing the blow-up of a given profile which arises in the asymptotic analysis. The extension of these results to nonconvex conical substrates remains open.

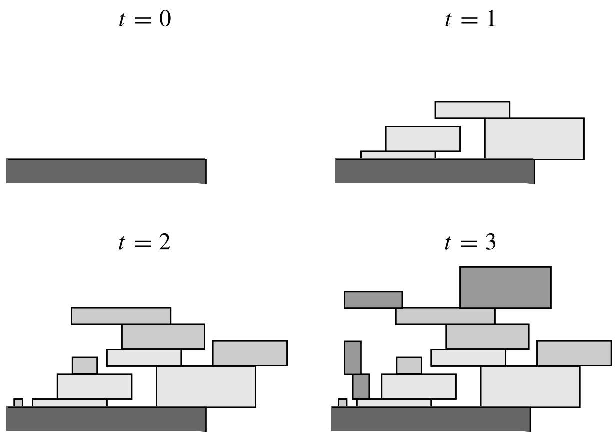

FIGURE 1: Evolution of a heap.

2. The model

We consider a queue where the servers are the points of $\mathbb{R}^d$. We distinguish two types of servers: $\mathcal{K}$ is the set of servers with a service speed equal to zero, and $\mathbb{R}^d \setminus \mathcal{K}$ is that of servers with a service speed equal to ∞. The customers are characterized by:

(i) a random closed set (RACS) of $\mathbb{R}^d$, such that the dth power of the diameter has a light-tailed distribution;

(ii) a random service time that is also light-tailed.

These customers arrive to the queue ($\mathbb{R}^d$) according to a Poisson rain with intensity λ.

Starting with an empty queue at time $t = 0$, a customer is queued if it hits $\mathcal{K}$ or if it hits an earlier customer which was already queued.

The ice-heap is a random set of $\mathbb{R}^d \times \mathbb{R}$, and the main questions of interest are about the growth of its height in various directions, and about the growth of its *spatial projection* (defined as its projection on $\mathbb{R}^d$), again in various directions. See Figure 1.

2.1. Precise formulation

Consider a homogeneous Poisson point process Φ in $\mathbb{R}^d \times \mathbb{R}$ with intensity $\lambda > 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then Φ can be seen as a simple counting measure, namely, as a sum of delta distributions at (different) points in $\mathbb{R}^d \times \mathbb{R}$. For every Borel set $A \subseteq \mathbb{R}^d \times \mathbb{R}$, $\Phi(A)$ counts the number of points that belong to the set A. By being Poisson homogeneous we mean the following:

(i) $\Phi(A)$ has a Poisson distribution with parameter $\lambda|A|$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d \times \mathbb{R}$;

(ii) given pairwise disjoint subsets $A_1, \ldots, A_n$ of $\mathbb{R}^d \times \mathbb{R}$, the random variables $\Phi(A_1), \ldots, \Phi(A_n)$ are independent.

This point process is independently marked. Each point comes with a pair of marks. These pairs are i.i.d. However, stochastic dependence within a pair is allowed. Let

$$\{(C_{(x,t)}, \sigma_{(x,t)})\}_{(x,t)\in\Phi}$$

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denote the marks. These are i.i.d. random pairs. The mark of point (x, t) consists of a compact RACS $C_{(x,t)}$ centered at the origin (e.g. the center of mass of the RACS is 0) and of a random variable $\sigma_{(x,t)}$ taking values in $\mathbb{R}^+$.

Let

$$\xi_{(x,t)} := \operatorname{diam}(C_{(x,t)}) := \sup\{|y - z|\colon y, z \in C_{(x,t)}\}$$

be the diameter of set $C_{(x,t)}$. We assume that both random variables $\sigma_{(x,t)}$ and $\xi^d_{(x,t)}$ (the dth power of $\xi_{(x,t)}$) are *light-tailed*, in that

$$\mathbb{E}(\exp(c\xi^d_{(x,t)})) < \infty, \qquad \mathbb{E}(\exp(c\sigma_{(x,t)})) < \infty \tag{2.1}$$

for some constant $c > 0$ (note that the law of $\xi_{(x,t)}$ is the same for all (x, t) and that a similar observation holds for $\sigma_{(x,t)}$; so that there are only two conditions here).

The homogeneity assumption is reflected by the following *compatibility property*. Given the group of translations

$$T_{(x_0,t_0)}\colon (x, t) \mapsto (x, t) + (x_0, t_0)$$

of $\mathbb{R}^d \times \mathbb{R}$, there exists $S\colon \mathbb{R}^d \times \mathbb{R} \times \Omega \to \Omega$ measurable and satisfying the following properties.

(i) *Measure preserving.* For every $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$, $S_{(x_0,t_0)}\colon \Omega \to \Omega$ is measure preserving.

(ii) *Group property.* We have $S_{(x_0,t_0)} \circ S_{(x_1,t_1)} = S_{(x_0+x_1,t_0+t_1)}$ and $S_{(0,0)} = Id$.

(iii) *Compatibility.* We have $\Phi \circ S_{(x_0,t_0)}(A)(\omega) = \Phi(T_{(x_0,t_0)}A)(\omega),\ A \subset \mathbb{R}^d \times \mathbb{R}$.

We can then extend the sequence of marks to a random process $(C_{(x,t)}, \sigma_{(x,t)})$ defined on $\mathbb{R}^d \times \mathbb{R}$ and such that

$$(C_{(x,t)}, \sigma_{(x,t)}) = (C_{(0,0)}, \sigma_{(0,0)}) \circ S_{(x,t)} \quad \text{for all } (x, t).$$

Because of the Poisson and independence assumptions, there is no loss of generality in assuming that the flow S is ergodic. In particular, for every measurable $G \subseteq \Omega$ such that

$$\mathbb{P}(S^{-1}_{(0,t)} G \Delta G) = 0 \quad \text{for every } t \in \mathbb{R},$$

we have $\mathbb{P}(G) = 0$ or 1. Here $F \Delta G = (F \setminus G) \cup (G \setminus F)$ is the symmetric difference of F and G.

2.2. Height profile function

Let $H_{(x,t)}$ be the height of the heap at location $x \in \mathbb{R}^d$ at time $t \geq 0$. When the substrate $\mathcal{K}$ is the whole Euclidean space, the construction of this function and the identification of the conditions under which it is nondegenerate (e.g. not equal to $+\infty$ almost surely (a.s.) for all x and all $t > 0$) are one of the main achievements of [1]. This construction relies on a sequence of steps, all relying on the monotonicity properties of the dynamics. These steps, which include a discretization scheme, a percolation argument, and a branching upper bound, are combined to show that, under the foregoing tail and independence assumptions, $H_{(x,t)}$ is a.s. finite for all x and $t < \infty$.

The tail and independence assumptions are the same as in [1]. The finiteness of the height profile function for a substrate $\mathcal{K} \subset \mathbb{R}^d$ then follows from the monotonicity properties of this function with respect to (w.r.t.) the initial condition which is

$$H_{(x,0)} = \begin{cases} 0 & \text{if } x \in \mathcal{K}, \\ -\infty & \text{if } x \notin \mathcal{K} \end{cases}$$

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in place of $H_{(x,0)} \equiv 0$ in [1]. The construction of [1] also shows that for all x, the function $t \to H(x,t)$ is piecewise constant. Note that it here takes its values in $\mathbb{R}^+ \cup \{-\infty\}$. It will be assumed right-continuous. The left limit of $H_{(x,\cdot)}$ at t will be denoted by $H_{(x,t-)}$.

2.3. Stochastic differential equation

The dynamics can also be described by a stochastic differential equation which we briefly outline in this subsection (in spite of the fact that it will not be used below) as it is of independent interest.

Let N^x denote the Poisson point process of $\mathbb{R}^d \times \mathbb{R}$ of RACS intersecting location x, i.e.

$$N^x(B \times [a,b]) = \int_{B\times[a,b]} \mathbf{1}_{\{x \in C_{(y,s)}+y\}} \Phi(\mathrm{d}y\,\mathrm{d}s)$$

for all $a < b$ and B Borel sets of $\mathbb{R}^d$. For $t > u \geq 0$, if $H_{(x,u)} > 0$, then

$$H_{(x,t)} = H_{(x,u)} + \int_{\mathbb{R}^d\times[u,t]} \Big(\sigma_{(z,v)} + \sup_{y\in C_{(z,v)}+z} H_{(y,v)} - H_{(x,v-)}\Big) N^x(\mathrm{d}z\,\mathrm{d}v). \qquad (2.2)$$

The rationale is that at the first point of N^x, say (z,w) in $[u,t]$ if any, $H(x,u)$ is canceled by $H(x,w-)$ and the new value of $H(x,\cdot)$ is

$$H(x,w) = \sigma_{(z,w)} + \sup_{y\in C_{(z,w)}+z} H_{(y,w)}.$$

If $H_{(x,u)} = -\infty$, this equation still holds when interpreting $-\infty$ as a $-K$ with K large. For instance, in this case, at the first arrival of N^x,

$$\begin{aligned} H_{(x,t)} &= H_{(x,u)} + \sigma_{(z,w)} + \sup_{y\in C_{(z,w)}+z} (H_{(y,w)} - H_{(x,w-)}) \\ &= -K + \sigma_{(z,w)} + \sup_{y\in C_{(z,w)}+z} H_{(y,w)} + K \\ &= \sigma_{(z,w)} + \sup_{y\in C_{(z,w)}+z} H_{(y,w)}. \end{aligned}$$

If for all $y \in C_{(z,w)} + z$, $H_{(y,w)} = -\infty$, then $H_{(x,w)} = -\infty$ too. Otherwise, $H_{(x,w)} > 0$.

It follows from the construction summarized in the previous section that, under the foregoing tail and independence assumptions, (2.2) has a piecewise constant solution. All the results of this paper can hence be rephrased as properties of this stochastic differential equation.

2.4. Monotonicity

The proposed model is *monotone* in several arguments.

Monotonicity in $\mathcal{K}$. For two systems with the same data $(\Phi, \{C,\sigma\})$ but with initial substrates $\mathcal{K}^{(1)} \subseteq \mathcal{K}^{(2)}$, the associated heights $H^{(1)}$ and $H^{(2)}$ satisfy

$$H^{(1)}_{(x,t)} \leq H^{(2)}_{(x,t)} \quad \text{for every } (x,t) \in \mathbb{R}^d \times [0,\infty).$$

Similarly, there is monotonicity in t, in the σs and in the Cs.

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3. The stick model: $\mathcal{K} = \{0\}$

In this section we consider the $\mathcal{K} = \{0\}$ case and call it *the stick model*. Theorem 3.1 shows that there exists a finite asymptotic limit for the maximal height of the associated heap $H_{(x,t)}$ (referred to as the *stick heap* below) in any given convex set of directions. Theorem 3.2 shows that there exists a finite asymptotic limit for how far the spatial projection of the heap grows, measured with respect to a *set-gauge* to be defined.

3.1. Height growth

In this section we focus on the maximal height $\mathbb{H}_t^{(\Theta)}$ of the stick heap among all directions in a set of directions Θ, which is defined as follows.

Definition 3.1. For $\Theta \subseteq S_+^d := \{(x,h) \in \mathbb{R}^d \times (0,1] \colon |(x,h)| = 1\}$ nonempty,

$$\mathbb{H}_t^{(\Theta)} := \sup\{h \in [0,\infty) \colon \text{ there exists } x \in \mathbb{R}^d \text{ such that } (x,h) \in |(x,h)|\Theta,\ H_{(x,t)} \geq h\}.$$

In particular, if $\Theta = \{(0,1)\}$, the north pole of S_+^d, then $\mathbb{H}_t^{(\Theta)} = H_{(0,t)}$.

Since $H_{(0,t)} \geq 0$, the set where the supremum is evaluated in the last definition is nonempty as it always contains $h = 0$ (since $0\Theta = \{(0,0)\}$). This also implies that $\mathbb{H}_t^{(\Theta)} \geq 0$. See Figure 2.

Definition 3.2. A set $\Theta \subseteq S_+^d$ is convex if for all $\theta_1, \theta_2 \in \Theta$, and $s \in [0,1]$,

$$s\theta_1 + (1-s)\theta_2 \in |s\theta_1 + (1-s)\theta_2|\Theta.$$

Note that if Θ is convex then for all $a, b \geq 0$ and $\theta_1, \theta_2 \in \Theta$, we have

$$a\theta_1 + b\theta_2 \in |a\theta_1 + b\theta_2|\Theta.$$

Theorem 3.1. *For all $\Theta \subseteq S_+^d$ convex and closed, there exists a nonnegative constant $\gamma^{(\Theta)}$ such that*

$$\lim_{t\to\infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \lim_{t\to\infty} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \gamma^{(\Theta)} < \infty,$$

where the first limit holds both in the a.s. and the L_1 sense.

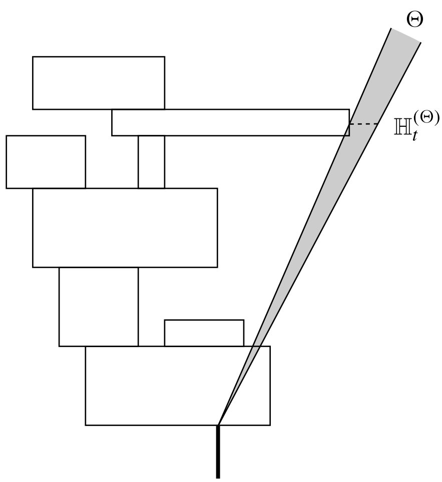

FIGURE 2: Definition of $\mathbb{H}_t^{(\Theta)}$ with $\mathcal{K} = \{0\}$.

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Before proving this theorem, we give a few preliminary lemmas.

The following lemma is a direct consequence of the independence of the Poisson rain in disjoint sets and of homogeneity. In this lemma, $\Phi \cap B$ denotes the set of points of Φ that belong to B.

Lemma 3.1. *Let $X \colon \Omega \to \mathbb{R}^d$ be a random variable which is independent of*

$$\{\Phi \cap B, (C_{(y,s)}, \sigma_{(y,s)}) \colon (y, s) \in B \cap \Phi, B \in \mathcal{B}(\mathbb{R}^d \times (0, \infty))\}.$$

Then, for every $\Theta \subseteq S_+^d$, the stochastic process $\{\mathbb{H}_t^{(\Theta)} \circ S_{(X,0)}, t > 0\}$ has the same law as $\{\mathbb{H}_t^{(\Theta)}, t > 0\}$ and it is independent of the σ-algebra generated by X and

$$\{\Phi \cap B, (C_{(y,s)}, \sigma_{(y,s)}) \colon (y, s) \in B \cap \Phi, B \in \mathcal{B}(\mathbb{R}^d \times (-\infty, 0])\}.$$

Lemma 3.2. *We have*

$$\sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(S_+^d)}}{t} < \infty. \tag{3.1}$$

The proof of Lemma 3.2 is quite close to the proof of [1, Theorem 2]. In order to make this paper self-contained, we provide a proof in the appendix.

Lemma 3.3. *For all $0 \le t_1 < t_2$ and $x, y \in \mathbb{R}^d$, the stick heap satisfies the following inequality:*

$$H_{(x+y,t_2)} \ge H_{(x,t_1)} + H_{(y,t_2-t_1)} \circ S_{(x,t_1)}.$$

Proof. For $t \ge t_1$, let $\tilde{H}_{(z,t)}$ be constructed as explained above with the initial condition

$$\tilde{H}_{(z,t_1)} := \begin{cases} H_{x,t_1} & \text{if } z = x, \\ -\infty & \text{if } z \ne x. \end{cases}$$

Then, by monotonicity $H_{(x+y,t)} \ge \tilde{H}_{(x+y,t)}$ and it suffices to show that $\tilde{H}_{(x+y,t)} = H_{(x,t_1)} + H_{(y,t-t_1)} \circ S_{(x,t_1)}$.

If $H(x, t_1) = -\infty$ then $\tilde{H}(z, t_1) = -\infty$ for all z, and it follows that $\tilde{H}(z, t) = -\infty$ for all z and all $t \ge t_1$. If $H(x, t_1)$ is nonnegative then the process $\tilde{H}(z, t_1)$ is nothing more than the process $H(z, t)$ shifted by $H(x, t_1)$ in space and by t_1 in time. So, in both cases, it satisfies the relation $\tilde{H}(x + y, t) = H(x, t_1) + H(y, t - t_1) \circ S(x, t_1)$ indeed. □

Proof of Theorem 3.1. Let $X_t \in \mathbb{R}^d$ be such that

$$(X_t, \mathbb{H}_t^{(\Theta)}) \in |(X_t, \mathbb{H}_t^{(\Theta)})|\Theta, \qquad H_{(X_t,t)} \ge \mathbb{H}_t^{(\Theta)}.$$

The existence of such an X_t is obtained from the proof of [1, Corollary 1].

This proof shows that at time t, not only the height, but also the diameter of the heap is a.s. finite. (Later on we will also prove an upper bound for this diameter in Lemma 3.4.) Therefore, with probability 1, we can find at least one X_t that satisfies the above properties. There could be more than one and, in order for X_t to be a random variable (i.e. a measurable function), we may, for instance, take the smallest X_t in the lexicographical order.

For $0 \le t_1 \le t_2$, let

$$\mathbb{H}_{t_1,t_2}^{(\Theta)} := \mathbb{H}_{t_1-t_2}^{(\Theta)} \circ S_{(X_{t_1},t_1)}.$$

In order to prove that the limit in the theorem exists and is a.s. constant, we use the super-additive ergodic theorem of Liggett; see [9]. We have to verify that the following properties hold.

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(i) *Super-additivity.* For $t_2 > t_1 \geq 0$,

$$\mathbb{H}_{t_2}^{(\Theta)} \geq \mathbb{H}_{t_1}^{(\Theta)} + \mathbb{H}_{t_1,t_2}^{(\Theta)}.$$

(ii) For $t_2 > t_1 \geq 0$, the joint distribution of $\{\mathbb{H}_{t_2,t_2+k}^{(\Theta)}, k > 0\}$ is the same as that of $\{\mathbb{H}_{t_1,t_1+k}^{(\Theta)}, k > 0\}$.

(iii) For $k > 0$, $\{\mathbb{H}_{nk,(n+1)k}^{(\Theta)}, n > 0\}$ is a stationary process.

(iv) The bound for the expectation is

$$\sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} < \infty.$$

To prove (i), let $t_2 > t_1 \geq 0$ be fixed and let

$$V = \{(x, h) \in \mathbb{R}^d \times (0, \infty) : (x, h) \in |(x, h)|\Theta, h \leq H_{(x,t_2-t_1)} \circ S_{(X_{t_1},t_1)}\}.$$

For $(x, h) \in V$, by the convexity of Θ, we have

$$(X_{t_1} + x, \mathbb{H}_{t_1}^{(\Theta)} + h) \in |(X_{t_1} + x, \mathbb{H}_{t_1}^{(\Theta)} + h)|\Theta. \tag{3.2}$$

Moreover,

$$\mathbb{H}_{t_1}^{(\Theta)} + h \leq H_{(X_{t_1},t_1)} + H_{(x,t_2-t_1)} \circ S_{(X_{t_1},t_1)} \leq H_{(X_{t_1}+x,t_2)}, \tag{3.3}$$

where we used Lemma 3.3 in the last inequality. By combining (3.2) and (3.3), it follows that $\mathbb{H}_{t_2}^{(\Theta)} \geq \mathbb{H}_{t_1}^{(\Theta)} + h$, which implies the super-additive inequality after taking the supremum of h over $(x, h) \in V$.

To prove (ii) we go back to the definition of $\mathbb{H}_{t_i,t_i+k}^{(\Theta)}$ for $i = 1, 2$,

$$\{\mathbb{H}_{t_i,t_i+k}^{(\Theta)}, k > 0\} = \{\mathbb{H}_k^{(\Theta)} \circ S_{(X_{t_i},t_i)}, k > 0\}.$$

By Lemma 3.1, it follows that both families of random variables have the same joint distribution as $\{\mathbb{H}_k^{(\Theta)}, k > 0\}$.

To prove (iii) it is enough to check that, for $k > 0$ fixed, the random variables $\{\mathbb{H}_{nk,(n+1)k}^{(\Theta)}, n > 0\}$ are i.i.d. By definition,

$$\mathbb{H}_{nk,(n+1)k}^{(\Theta)} = \mathbb{H}_k^{(\Theta)} \circ S_{(X_{nk},nk)}.$$

Using Lemma 3.1 once again, it follows that $\mathbb{H}_{nk,(n+1)k}^{(\Theta)}$ is distributed as $\mathbb{H}_k^{(\Theta)}$. Then the independence property follows again from Lemma 3.1.

Finally, (iv) results from the upper bound given by Lemma 3.2. □

3.2. Spatial projection

Definition 3.3. For $t \geq 0$, let F_t be the spatial projection of the heap; namely, the RACS of $\mathbb{R}^d$ which is the union of all the RACS added to the heap up to time t:

$$F_t := \{x \in \mathbb{R}^d : H_{(x,t)} \geq 0\}.$$

If the sets $C(x, t)$ are a.s. connected then so is F_t. However, if the sets $C(x, t)$ are a.s. convex then F_t has no reason to be convex.

In general, F_t is not necessarily a RACS. However, under the light-tailedness assumptions (2.1) we have the following lemma.

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Lemma 3.4. *For all finite t, F_t is a RACS and*

$$\sup_{t>0} \frac{\mathbb{E}(\operatorname{diam}(F_t))}{t} < \infty. \tag{3.4}$$

Proof. The proof is an application of Lemma 3.2, which follows the ideas in the proof of [1, Corollary 1].

It holds that F_t is a RACS as a consequence of the upper bound branching process constructed for F_t in the proof of Lemma 3.2. This branching process has a.s. finitely many offspring in each generation. This implies that for all finite $t > 0$, only a finite number of RACS $C_{(x,s)}$ may contribute to F_t.

We now prove (3.4). First note that the set F_t does not depend on the heights. However, we will make use of them in the following way. Assume that $\sigma_{(x,t)} = \xi_{(x,t)} = \operatorname{diam}(C_{(x,t)})$. We now show that under this assumption,

$$4 \sup_{x\in\mathbb{R}^d} H_{(x,t)} \geq \operatorname{diam}(F_t).$$

For every $x \in \mathbb{R}^d$ such that $H_{(x,t)} \geq 0$, there exists an integer n and some set of points $(x_1, t_1), \dots, (x_n, t_n) \in \mathbb{R}^d \times [0, t)$ such that the following hold:

(i) $(x_i, t_i) \in \operatorname{supp} \Phi$ for $i = 1, \dots, n$;

(ii) $0 \leq t_i < t_{i+1} < t$ for $i = 1, \dots, (n-1)$;

(iii) $x \in x_n + C_{(x_n,t_n)}$ and $H_{(x,s)} = H_{(x,t_n)}$ for $s \in [t_n, t)$;

(iv) for $i = 1, \dots, (n-1)$, there exists $y_i \in x_{i+1} + C_{(x_{i+1},t_{i+1})} \cap x_i + C_{(x_i,t_i)}$ such that $H_{(y_i,s)} = H_{(y_i,t_i)}$ for $s \in [t_i, t_{t+1})$;

(v) $0 \in x_1 + C_{(x_1,t_1)}$ and $H_{(0,s)} = 0$ for $s \in [0, t_1)$.

Therefore,

$$|x| \leq |x - x_n| + \sum_{i=1}^{n-1} |x_{i+1} - x_i| + |x_1| \leq 2 \sum_{i=1}^{n} \operatorname{diam}(C_{(x_i,t_i)}) = 2H_{(x,t)}.$$

Maximizing over $\{x \in \mathbb{R}^d : H_{(x,t)} \geq 0\}$ and applying Lemma 3.2 concludes the proof. □

Definition 3.4. Given a direction $v \in S^{d-1}$ and a closed set $A \subseteq \mathbb{R}^d$ containing the origin, let

$$D_t^{(A,v)} := \inf\{r \in [0, \infty) : (A + rv) \cap F_t = \varnothing\},$$

where the infimum of an empty set is ∞.

Here are a few examples. If $A = \{0\}$ then $D_t^{(A,v)}$ can be interpreted as the *internal growth of F_t in the v direction at time t*. It is also the contact distance with free space in the v-direction. Other interesting cases arise when $A = \{x \in \mathbb{R}^d : x \cdot v \geq 0\}$ or $A = \{x \in \mathbb{R}^d : x = \alpha v, \alpha \geq 0\}$; then $D_t^{(A,v)}$ can be interpreted as the *external growth of F_t in the v direction*. These cases are covered in Theorem 3.2 and illustrated in Figure 3.

Definition 3.5. The pair (v, A), where $v \in S^{d-1}$ is a direction and $A \subseteq \mathbb{R}^d$ a closed set, forms a *set-gauge* if

(i) A contains the origin and for every $a \in A$, $A + a \subseteq A$,

(ii) $-v$ does not belong to the closed convex hull of A.

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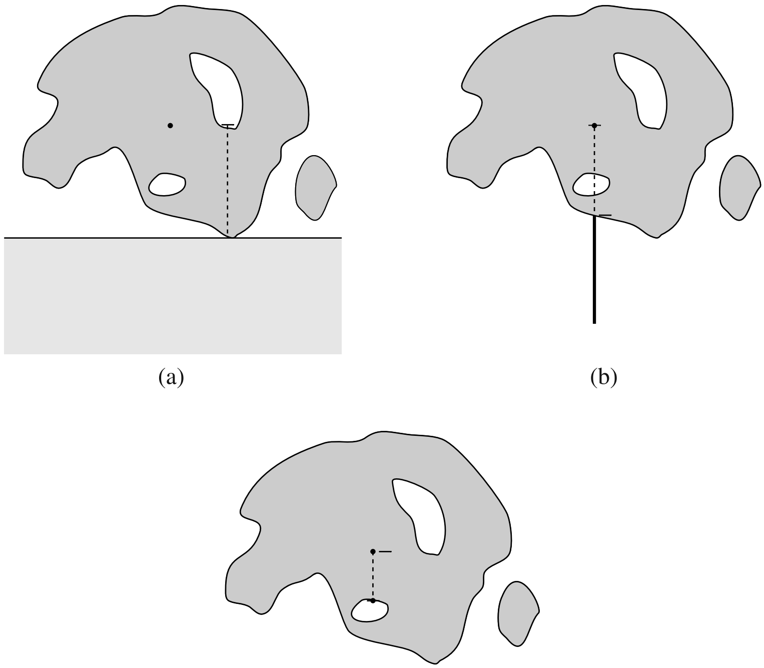

(a) (b) (c)

FIGURE 3: Different set-gauges measuring the spatial growth of F_t. The direction of v is south. (a) is the $A = \{x \in \mathbb{R}^d : x \cdot v \geq 0\}$ case, (b) is the $A = \{x \in \mathbb{R}^d : x = \alpha v, \alpha \geq 0\}$ case, (c) is the $A = \{0\}$ case.

The three above examples are set-gauges. Here are other examples.

If A is a closed convex cone of $\mathbb{R}^d$, different from $\mathbb{R}^d$, and $-v \notin A$, then (v, A) forms a set-gauge.

If (v, A) forms a set-gauge then (v, B), where $B := \bigcup_{r>0}(A + rv)$ also forms a set-gauge. In this case

$$D_t^{(B,v)} = \sup\{r \in [0, \infty) \colon (A + rv) \cap F_t \neq \varnothing\}.$$

Note that for all set-gauges (v, A), $D_t^{(A,v)}$ is a.s. finite. This follows from the property that F_t is a.s. compact and the assumption that $-v$ does not belong to the convex hull of A.

Our main result is the following theorem.

Theorem 3.2. *Given a direction $v \in S^{d-1}$ and a closed set $A \subseteq \mathbb{R}^d$, such that (v, A) forms a set-gauge, there exists a nonnegative constant $\phi = \phi^{A,v}$ such that*

$$\lim_{t\to\infty} \frac{D_t^{(A,v)}}{t} = \lim_{t\to\infty} \frac{\mathbb{E}D_t^{(A,v)}}{t} = \sup_{t>0} \frac{\mathbb{E}D_t^{(A,v)}}{t} =: \phi < \infty,$$

where the first limit is both a.s. and in L_1.

Proof. Once again the proof relies on the distributional super-additive ergodic theorem. Let $X_t \in \mathbb{R}^d$ be a random variable such that

$$X_t \in (A + D_t^{(A,v)} v) \cap F_t.$$

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The existence of a finite X_t satisfying this relation follows from the fact that F_t is compact and A is closed. It also uses the fact that $-v$ does not belong to the convex hull of A. There is no reason to have uniqueness. However, we can use the same construction as in the proof of Theorem 3.1 to cope with multiple solutions.

For $0 \le t_1 \le t_2$, let

$$D_{t_1,t_2}^{(A,v)} := D_{t_2-t_1}^{(A,v)} \circ S_{X_{t_1},t_1}.$$

By Lemma 3.1, properties analogous to properties (ii) and (iii) in the proof of Theorem 3.1 do hold. We now prove the super-additivity and the boundedness of the expectations.

In order to prove the super-additive inequality, it is enough to show that, for every $r < D_{t_1}^{(A,v)} + D_{t_1,t_2}^{(A,v)}$,

$$(A + rv) \cap F_{t_2} \neq \varnothing. \tag{3.5}$$

If $r < D_{t_1}^{(A,v)}$, this follows from the monotonicity of F_t w.r.t. time and from the definition of $D_{t_1}^{(A,v)}$. Now let $r = D_{t_1}^{(A,v)} + r'$ with $r' \in [0, D_{t_1,t_2}^{(A,v)})$. From the definition of F_t,

$$F_{t_2-t_1} \circ S_{X_{t_1},t_1} + X_{t_1} \subseteq F_{t_2}.$$

From the definition of a set-gauge and the property $X_{t_1} \in A + D_{t_1}^{(A,v)} v$,

$$A + X_{t_1} + r'v \subseteq A + rv.$$

From the definition of $D_{t_1,t_2}^{(A,v)}$, for $r' < D_{t_1,t_2}^{(A,v)}$,

$$(A + r'v) \cap (F_{t_2-t_1} \circ S_{X_{t_1},t_1}) \neq \varnothing,$$

which implies that $(A + X_{t_1} + r'v) \cap (F_{t_2-t_1} \circ S_{X_{t_1},t_1} + X_{t_1}) \neq \varnothing$ and (3.5) follows from the last two inclusions.

Now we prove the boundedness of expectations. Given that A and v form a gauge there exists a hyperplane given by $P = \{x \in \mathbb{R}^d : x \cdot w = 0\}$, with $w \in S^{d-1}$, that separates $-v$ and A, i.e.

(i) $v \cdot w > 0$,

(ii) $a \cdot w \ge 0$ for every $a \in A$.

Then, letting $A' = \{x \in \mathbb{R}^d : x \cdot w \ge 0\}$, and using the monotonicity inherited from the fact that $A \subseteq A'$, we have

$$D_t^{(A,v)} \le D_t^{(A',w)} \le \frac{\operatorname{diam}(F_t)}{|v \cdot w|}.$$

Finally, applying Lemma 3.4, we obtain

$$\limsup_{t\to\infty} \frac{\mathbb{E} D_t^{(A,v)}}{t} \le \frac{\limsup_{t\to\infty} \mathbb{E} \operatorname{diam}(F_t)/t}{|v \cdot w|} < \infty.$$

So the proof is complete. □

We now focus on the gauges with $A = \{0\}$. Our aim is to prove that, under an extra assumption on the RACS,

$$\lim_{t\to\infty} \frac{D_t^{(A,v)}}{t} > 0.$$

This will in turn imply that for every $x \in \mathbb{R}^d$, the time that it takes for F_t to hit x is a.s. finite.

Lemma 3.5. *Assume that the intensity of* Φ *is positive and that, with a positive probability, the footprint has a nonempty interior that contains the origin. Then, for* $A = \{0\}$ *and* $v \in S^{d-1}$*, we have*

$$\lim_{t\to\infty} \frac{D_t^{(A,v)}}{t} > 0.$$

Proof. By Theorem 3.2,

$$\lim_{t\to\infty} \frac{D_t^{(A,v)}}{t} = \sup_{t>0} \frac{\mathbb{E}D_t^{(A,v)}}{t} \geq \mathbb{E}D_1^{(A,v)}.$$

From the lemma conditions, there is a positive r such that, with positive probability, C contains the ball B_r with radius r centered at the origin. By the thinning property of the Poisson point process, we may consider only the Poisson rain (with a smaller, but positive intensity) with RACS that include B_r. Then, using the monotonicity mentioned earlier, we may take $C = B_r$. In the latter case, it is not difficult to see that

$$\mathbb{P}\bigg(D_1^{(A,v)} > \frac{r}{2}\bigg) \geq \mathbb{P}(\Phi(B_{r/2} \times (0,1]) > 0) > 0.$$

Then $\mathbb{E}D_1^{(A,v)} > 0$ and the result follows. □

Definition 3.6. Given $K \subseteq \mathbb{R}^d$ let $\tau(K)$ denote the time it takes for F_t to cover K, i.e.

$$\tau(K) := \inf\{t \in [0,\infty] \colon K \setminus F_t = \varnothing\}.$$

Remark 3.1. It holds that $\tau(K)$ is a stopping time in the sense that $\{\tau(K) \leq t\}$ belongs to the σ-algebra generated by

$$\{\Phi \cap B, (C_{(y,s)}, \sigma_{(y,s)}) \colon (y,s) \in \Phi \cap B, B \in \mathcal{B}(\mathbb{R}^d \times [0,t])\}.$$

Moreover, $\{\tau(K) \leq t\}$ is independent of the σ-algebra of subsequent events generated by

$$\{\Phi \cap B, (C_{(y,s)}, \sigma_{(y,s)}) \colon (y,s) \in \Phi \cap B, B \in \mathcal{B}(\mathbb{R}^d \times [t,\infty))\}.$$

Corollary 3.1. *Assume that the intensity of* Φ *is positive and that, with a positive probability, the footprint has a nonempty interior that contains the origin. Then, for all bounded sets* $K \subseteq \mathbb{R}^d$*,* $\tau(K)$ *is a.s. finite.*

Proof. It suffices to assume, by the same reasoning as in the previous proof, that the footprint C is a ball of fixed radius $r > 0$, sufficiently small. Let $\{x_1, \dots, x_n\} \subseteq K \setminus \{0\}$ such that

$$K \subseteq \bigcup_{i=1}^{n} B_{r/2}(x_i).$$

Denote also $v_i = x_i/|x_i|$ for $i = 1, \dots, n$.

Consider now $C' := B_{r/2}$ and F'_t, $D_t'^{(A,v)}$ constructed from C' and $A = \{0\}$. By Lemma 3.5, it follows that, for $i = 1, \dots, n$,

$$\tau'_i := \inf\{t \in [0,\infty] \colon D_t'^{(A,v_i)} \geq |x_i|\} < \infty \quad \text{a.s.}$$

By construction of C', if $x_i \in F'_t$ then also $B_{r/2}(x_i) \subseteq F_t$, which concludes the proof. □

3.3. Phase transition

From Theorem 3.1 there exists a growth rate $\gamma_{\theta_w(\phi)}$ in the direction

$$\theta_w(\phi) := \sin(\phi)e_{d+1} + \cos(\phi)w \in S^d_+, \qquad w \in S^{d-1},\ \phi \in \left[0, \frac{\pi}{2}\right].$$

For fixed $w \in S^{d-1}$, $\gamma_{\theta_w(\cdot)}$ admits the following phase transition.

Theorem 3.3. *Assume that with a positive probability, the footprint has a nonempty interior that contains the origin. Then, for $w \in S^{d-1}$ there exists an angle, $\phi_*(w) \in (0, \pi/2)$ such that $\gamma_{\theta_w(\phi)}$ is positive for any $\phi \in (\phi_*(w), \pi/2]$ and $\gamma_{\theta_w(\phi)} = 0$ for any $\phi \in [0, \phi_*(w))$.*

Proof. Let us first show that if $\gamma_{\theta_w(\phi)} > 0$ then for all $\hat{\phi} \in (\phi, \pi/2)$, $\gamma_{\theta_w(\hat{\phi})} > 0$. For any $t > 0$, let x_t be defined by $t = x_t \tan\phi$ and let $\hat{t} = x_t \tan\hat{\phi}$. Then

$$H_{(x_t w, \hat{t})} \geq H_{(x_t w, t)} + H_{(0, \hat{t}-t)} \circ S_{(x_t w, t)},$$

so that

$$\gamma_{\theta_w(\hat{\phi})} = \lim \frac{H_{(x_t w, \hat{t})}}{\hat{t}} \geq \gamma_{\theta_w(\phi)} \frac{\tan\phi}{\tan\hat{\phi}} > 0.$$

Now let

$$\phi_*(w) = \inf\left\{\phi \in \left[0, \frac{\pi}{2}\right] : \gamma_{\theta_w(\phi)} > 0\right\}.$$

If $\phi = \pi/2$ then $\gamma_{\theta_w(\phi)} > 0$. Hence, $\phi_*(w)$ is well defined. It follows from the last monotonicity property that it is the threshold above which $\gamma_{\theta_w(\phi)} > 0$.

It remains to prove that this threshold is nondegenerate.

Let us first prove that it is positive. Let s_w denote the spatial growth rate in direction w and h the vertical growth rate. Both s_w and h are positive and finite. So if the angle ϕ is smaller than $\arctan(h/s_w) > 0$, then $\gamma_{\theta_w(\phi)} = 0$.

Let us now show that $\phi_*(w) < \pi/2$.

For all $n \in \mathbb{N}$, let $x_n = (nr/2)w$ (here $r > 0$ such that with positive probability, $B_r \subseteq C$). Let Π_n be the Poisson rain of RACSs that contain a ball of radius r centered at x_n. Let $t_0 \equiv T_0 > 0$ be the first time or arrival of a RACS of Π_0 and, for each $n = 0, 1, \dots$, let $T_{n+1} = T_n + t_{n+1}$ be the first arrival time after T_n of a RACS of Π_{n+1}. The random variables t_n are i.i.d. exponential with mean, say, $b > 0$. Also, $H_{(x_n, T_n)}$ is not smaller than the sum of $(n+1)$ i.i.d. random variables with distribution $H_{(0,T_0)}$. Since $\mathbb{E}H_{(0,T_0)} > 0$, it follows that $\liminf H_{(x_i,T_i)}/T_i > 0$ a.s. Further, $T_i/x_i \to 2b/r < \infty$, so $\phi_*(w) \leq \arctan(2b/r) < \pi/2$. □

4. The model with $\mathcal{K}$ compact

In this section we study the growth of the heap starting with a compact substrate $\mathcal{K} \subseteq \mathbb{R}^d$, in some convex set of directions Θ.

4.1. Asymptote at $0 \in \mathcal{K}$

In this section we fix $0 \in \mathcal{K}$. Let $\mathcal{K}^{(0)} = \{0\}$. Whenever $\mathbb{H}_t^{(\Theta)}$ is computed with respect to $\mathcal{K}^{(0)}$ (respectively $\mathcal{K}$) we denote it by $\mathbb{H}_t^{(\Theta,0)}$ (respectively $\mathbb{H}_t^{(\Theta)}$). An analogous notation is used for all the other possible constructions. Given a constant $M \geq 0$, the measure preserving transformation $S_{(0,M)}$ of Ω to itself is denoted by S_M.

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In the next lemma $\tau := \tau(\mathcal{K})$ denotes the time it takes for $F_t^{(0)}$ to cover the whole set $\mathcal{K}$.

Lemma 4.1. *For all* $\Theta \subseteq S_+^d$ *closed and for all* $M, t \geq 0$*, the following inequalities hold on* $\{\tau \leq M\}$*:*

$$\mathbb{H}_{M+t}^{(\Theta,0)} \geq \mathbb{H}_t^{(\Theta)} \circ S_M \geq \mathbb{H}_t^{(\Theta,0)} \circ S_M.$$

Proof. On $\tau \leq M$, $\mathcal{K} \subseteq F_M^{(0)}$, so that for every $x \in \mathbb{R}^d$, $H_{(x,M)}^{(0)} \geq H_{(x,0)} \circ S_M$. This implies that $H_{(x,M+t)}^{(0)} \geq H_{(x,t)} \circ S_M$ for all $t > 0$ by the monotonicity in the construction of H. The leftmost inequality then follows.

The rightmost inequality is just a consequence of the monotonicity w.r.t. the initial substrates $\mathcal{K}^{(0)} \subseteq \mathcal{K}$. □

Lemma 4.2. *Under the assumptions of Corollary 3.1 and Theorem 3.1, if* $0 \in \mathcal{K}$*, we have*

$$\lim_{t\to\infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \lim_{t\to\infty} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} = \gamma^{(\Theta)} < \infty,$$

where the first limit is both in the a.s. and the L_1 *sense.*

Note that the rightmost term is the one corresponding to $\mathcal{K}^{(0)}$; this tells us that asymptotically, the heaps starting at $\mathcal{K}$ or $\mathcal{K}^{(0)}$ behave similarly in terms of directional shape.

Proof. By Lemma 4.1, Theorem 3.1, and the fact that S_M is measure preserving, we have

$$\lim_{t\to\infty} \frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} \mathbf{1}_{\{\tau \leq M\}} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \mathbf{1}_{\{\tau \leq M\}} \quad \text{a.s.}$$

Hence,

$$\mathbb{P}\bigg(\lim_{t\to\infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg) = \mathbb{P}\bigg(\lim_{t\to\infty} \frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg) \geq \mathbb{P}(\tau \leq M).$$

Since $M > 0$ is arbitrary and τ is finite a.s., we obtain the a.s. convergence of $\mathbb{H}_t^{(\Theta)}/t$ to the announced limit.

Now we proceed to show the convergence in L_1. By Lemma 4.1,

$$\bigg(\frac{\mathbb{H}_{M+t}^{(\Theta,0)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg)\mathbf{1}_{\{\tau \leq M\}} \geq \bigg(\frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg)\mathbf{1}_{\{\tau \leq M\}}.$$

Then

$$\mathbb{E}\bigg|\frac{\mathbb{H}_{M+t}^{(\Theta,0)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg| \geq \mathbb{E}\bigg|\bigg(\frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg)\mathbf{1}_{\{\tau \leq M\}}\bigg|.$$

By the independence property given in Remark 3.1, and using again that S_M is measure preserving,

$$\mathbb{E}\bigg|\frac{\mathbb{H}_{M+t}^{(\Theta,0)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg| \geq \mathbb{E}\bigg|\frac{\mathbb{H}_t^{(\Theta)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t}\bigg|\mathbb{P}(\tau \leq M).$$

Choose M sufficiently large so that $\mathbb{P}(\tau \leq M) > 0$. Then letting $t \to \infty$ concludes the proof thanks to the L^1 convergence of Theorem 3.1. □

4.2. Asymptote at $0 \notin \mathcal{K}$

In this section we assume that $0 \notin \mathcal{K} \neq \varnothing$. We use the following notation: $\mathcal{K}^{(0)} = \{0\}$ and $\mathcal{K}^{(1)} = \mathcal{K} \cup \{0\}$. Whenever $\mathbb{H}_t^{(\Theta)}$ is computed with respect to $\mathcal{K}^{(0)}$ (respectively $\mathcal{K}^{(1)}$ or $\mathcal{K}$), we denote it by $\mathbb{H}_t^{(\Theta,0)}$ (respectively $\mathbb{H}_t^{(\Theta,1)}$ or $\mathbb{H}_t^{(\Theta)}$). We use analogous notation for all other possible constructions.

In the next lemma, $\tau := \tau(\mathcal{K}^{(0)})$ is the time it takes for F_t to hit the origin.

Lemma 4.3. *For all closed $\Theta \subseteq S_+^d$ and all $M, t \geq 0$, the following inequalities hold on $\{\tau \leq M\}$:*

$$\mathbb{H}_{t+M}^{(\Theta,1)} \geq \mathbb{H}_{t+M}^{(\Theta)} \geq \mathbb{H}_t^{(\Theta,0)} \circ S_M.$$

Proof. The proof is very similar to the proof of Lemma 4.1. On $\tau \leq M$, $\mathcal{K}^{(0)} \subseteq F_M$; therefore, for every $x \in \mathbb{R}^d$, $H_{(M,x)} \geq H_{(0,x)}^{(0)} \circ S_M$, which implies the second inequality. The first inequality is a consequence of the monotonicity. □

Using this lemma (instead of Lemma 4.1) and the same ideas as in the proof of Lemma 4.2, leads to the following theorem.

Theorem 4.1. *Under the assumptions of Corollary 3.1 and Theorem 3.1, in all cases ($0 \in \mathcal{K}$ or $0 \notin \mathcal{K}$),*

$$\lim_{t\to\infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \lim_{t\to\infty} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} = \gamma^{(\Theta)} < \infty,$$

with the first limit holding both a.s. and in the L_1 sense.

5. The model with $\mathcal{K}$ a convex cone and its generalizations

In this section the substrate is first a convex cone of $\mathbb{R}^d$ with its vertex at the origin, and then an object similar to such a cone but more general.

Definition 5.1. Given $\mathcal{C} \subseteq \mathbb{R}^d$ a closed convex cone with vertex at the origin, we define $\Theta(\mathcal{C}) \subseteq S_+^d$ to be the following subset of S_+^d (see Definition 3.1): $\Theta(\mathcal{C}) := S_+^d \cap (\mathcal{C} \times \mathbb{R})$.

For the proofs of this section, we use yet another property of the model, which is some form of invariance by time reversal. Consider the reflection

$$R\colon (x, t) \mapsto (x, -t).$$

Because the Poisson rain is invariant in law by R, and because the marks are i.i.d., there exists a measure preserving $V : \Omega \to \Omega$ which is compatible with R, i.e.

$$(\Phi \circ V)(A) = \Phi(RA), \qquad C_{(x,t)} \circ V = C_{R(x,t)}, \qquad \sigma_{(x,t)} \circ V = \sigma_{R(x,t)}.$$

In the following theorem, $\mathbb{H}_t^{(\Theta,0)}$ and $\mathbb{H}_t^{(\Theta,1)}$ are the heights computed when starting with the substrate $\mathcal{K}^{(0)} := \{0\}$ or $\mathcal{K}^{(x)} := \{0, x\}$, respectively, whereas $H_{(x,t)}$ is the height at x when starting with the substrate $\mathcal{K} = \mathcal{C}$.

Theorem 5.1. *Under the assumptions of Corollary 3.1, for all closed convex cones $\mathcal{K} \subseteq \mathbb{R}^d$ and all $x \in \mathbb{R}^d$,*

$$\lim_{t\to\infty} \frac{\max(0, H_{(x,t)})}{t} = \lim_{t\to\infty} \frac{\mathbb{E}\max(0, H_{(x,t)})}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta(\mathcal{K}),0)}}{t} = Z < \infty,$$

where the first limit is in L_1.

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Proof. We split the proof into three cases.

Case 1: x is the vertex of the cone. Without loss of generality, the vertex is assumed to be at the origin. The key observation is the following duality between the dynamics starting with $\mathcal{K}$ and $\mathcal{K}^{(0)}$:

$$H_{(0,t)} = \mathbb{H}_{(0,t)}^{(\Theta(\mathcal{K}),0)} \circ V \circ S_{(0,t)},$$

see the supporting Figure 4. Once this is established, the L_1 limit results from the fact that $S_{(0,t)} \circ V$ is measure preserving and therefore both sides are equivalent in distribution.

We first prove that $H_{(0,t)} \le \mathbb{H}_{(0,t)}^{(\Theta(\mathcal{K}),0)} \circ V \circ S_{(0,t)}$. Consider the set of points

$$(x_0, t_0), \dots, (x_n, t_n) \in \mathbb{R}^d \times [0, t)$$

'connecting' 0 with its height at t. Specifically, these satisfy

(i) $(x_i, t_i) \in \operatorname{supp} \Phi$ for $i = 0, \dots, n$,

(ii) $0 \le t_i < t_{i+1} < t$ for $i = 0, \dots, (n-1)$,

(iii) $0 \in C_{(x_n,t_n)}$ and $H_{(0,s)} = H_{(0,t_n)}$ for $s \in [t_n, t]$,

(iv) there exists $y_i \in C_{(x_{i+1},t_{i+1})} \cap C_{(x_i,t_i)}$ such that $H_{(y_i,s)} = H_{y_i,t_i}$ for $s \in [t_i, t_{t+1})$ and $i = 0, \dots, (n-1)$,

(v) there exists $z \in C_{(x_0,t_0)} \cap \mathcal{K}$ and $H_{(z,s)} = 0$ for $s \in [0, t_0)$.

Now let $(\tilde{x}_i, \tilde{t}_i) = T_{(0,t)} R(x_{n-i}, t_{n-i}) = (x_{n-i}, t - t_{n-i})$ and $\tilde{y}_i = y_{n-i}$. Then, by the compatibility properties, these quantities satisfy

(i) $(\tilde{x}_i, \tilde{t}_i) \in \operatorname{supp} \Phi \circ V \circ S_{(0,t)}$ for $i = 0, \dots, n$,

(ii) $0 \le t_i < t_{i+1} < t$ for $i = 0, \dots, (n-1)$,

(iii) $z \in C_{(\tilde{x}_n,\tilde{t}_n)}$ and $H_{(z,s)}^{(0)} \circ V \circ S_{(0,t)} = H_{(z,t_n)}^{(0)} \circ V \circ S_{(0,t)}$ for $s \in [\tilde{t}_n, t]$,

(iv) $\tilde{y}_{i+1} \in C_{(\tilde{x}_{i+1},\tilde{t}_{i+1})} \circ V \circ S_{(0,t)} \cap C_{(\tilde{x}_i,\tilde{t}_i)} \circ V \circ S_{(0,t)}$ such that $H_{(\tilde{y}_i,s)}^{(0)} \circ V \circ S_{(0,t)} = H_{\tilde{x}_i,\tilde{t}_i}^{(0)} \circ V \circ S_{(0,t)}$ for $s \in [\tilde{t}_i, \tilde{t}_{t+1})$ and $i = 0, \dots, (n-1)$,

(v) $0 \in C_{(\tilde{x}_1,\tilde{t}_1)}$ and $H_{(0,s)}^{(0)} \circ V \circ S_{(0,t)} = 0$ for $s \in [0, \tilde{t}_0)$.

Given that $z \in \mathcal{K}$ and $\mathcal{K}$ is the convex cone $\mathcal{C}$, then, for any $h > 0$,

$$\frac{(z, h)}{|(z, h)|} \in \Theta(\mathcal{K}).$$

Then

$$\mathbb{H}_{(0,t)}^{(\Theta(\mathcal{K}),0)} \circ V \circ S_{(0,t)} \ge H_{(z,t)}^{(0)} \circ V \circ S_{(0,t)} = \sum_{i=0}^{n} \sigma_{(\tilde{x}_i,\tilde{t}_i)} \circ V \circ S_{(0,t)} = \sum_{i=0}^{n} \sigma_{(x_i,t_i)} = H_{(0,t)}.$$

The proof of the inequality in the other direction is similar to the previous one but starting with the dynamics of $\mathbb{H}_{(0,t)}^{(\Theta(\mathcal{K}),0)} \circ V \circ S_{(0,t)}$.

Case 2: $x \in \mathcal{K}$. The result in this case is obtained by comparison with the growth of the vertex studied in case 1. From the monotonicity w.r.t. the initial substrates,

$$H_{(0,t)}^{(0)} \circ S_{(x,0)} \le H_{(x,t)}. \tag{5.1}$$

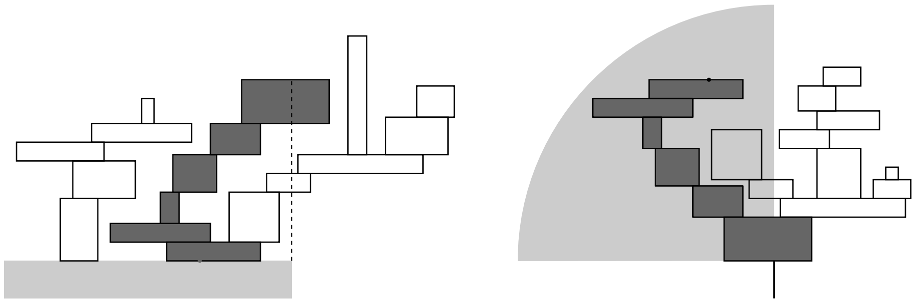

FIGURE 4: Visualization of the duality argument in the proof of Theorem 5.1 in the $x = 0$ case.

On the other hand, using $\mathcal{K}^{(x)} = \{0, x\}$, we have the following identity:

$$H_{(x,t)} \leq \mathbb{H}_t^{(\Theta(\mathcal{K}),1)} \circ V \circ S_{(0,t)}. \tag{5.2}$$

To prove the last relation, we again use a set of points $(x_0, t_0), \ldots, (x_n, t_n) \in \mathbb{R}^d \times [0, t)$ connecting x with its height. As before

(i) $(x_i, t_i) \in \operatorname{supp} \Phi$ for $i = 0, \ldots, n$,

(ii) $0 \leq t_i < t_{i+1} < t$ for $i = 0, \ldots, (n-1)$,

(iii) $0 \in C_{(x_n,t_n)}$ and $H_{(0,s)} = H_{(0,t_n)}$ for $s \in [t_n, t]$,

(iv) there exists $y_i \in C_{(x_{i+1},t_{i+1})} \cap C_{(x_i,t_i)}$ such that $H_{(y_i,s)} = H_{y_i,t_i}$ for $s \in [t_i, t_{t+1})$ and $i = 0, \ldots, (n-1)$,

(v) there exists $z \in C_{(x_0,t_0)} \cap \mathcal{K}$ and $H_{(z,s)} = 0$ for $s \in [0, t_0)$.

When we now consider $(\tilde{x}_i, \tilde{t}_i) = T_{(0,t)} \circ R(x_{n-i}, t_{n-i}) = (x_{n-i}, t - t_{n-i})$ and $\tilde{y}_i = y_{n-i}$, it follows that there exists a path of RACS starting at x and finishing at $(z, H_{(x,t)}) \in \Theta(\mathcal{K})$. By the very definition of $\mathbb{H}_t^{(\Theta(\mathcal{K}),1)} \circ V \circ S_{(0,t)}$ this then implies (5.2).

The desired limit then follows from (5.1) and (5.2), the result of case 1 and Lemma 4.2. Note that the L_1 limit is hence the same for all points $x \in \mathcal{K}$.

Case 3: $x \notin \mathcal{K}$. Let $x \notin \mathcal{K}$ and let u be the Euclidean distance from x to $\mathcal{K}$, so that $|x - y| = u$ for some $y \in \mathcal{K}$. On the segment $[x, y]$, choose points $x_0 = y, x_1, \ldots, x_m = x$ equidistantly, where m is the smallest integer that exceeds $2u/r$. Consider shifted versions of $\mathcal{K}$, say $\mathcal{K}_0 = \mathcal{K}, \mathcal{K}_1, \ldots, \mathcal{K}_m$ such that for any $i \geq 1$, $\mathcal{K}_i \supset \mathcal{K}_{i-1}$, and $\mathcal{K}_i$ includes the points $x_0, \ldots, x_i$ and does not include the points $x_{i+1}, \ldots, x_m$. Then we show the convergence of $\max(0, H_{(x_i,t)})$ to Z in L_1 using an induction argument: if the convergence holds for x_{i-1}, then it also holds for x_i. Because of that, we may assume without loss of generality that $m = 1$, so $y = x_0$, $x = x_1$, and $|x - y| \leq r/2$.

Let $\widetilde{\mathcal{K}} = \mathcal{K}_1$ and let $\tilde{H}_{(y,t)}$ be the height associated with the cone $\widetilde{\mathcal{K}}$. Let ε be a positive number.

First, we show that

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{H_{(y,t)}}{t} > Z + \varepsilon\right) = 0 \tag{5.3}$$

and that the random variables $\max(0, H(y, t))/t$ are uniformly integrable. By monotonicity (see Section 2.4), we have $H_{(y,t)} \leq \tilde{H}_{(y,t)}$ and, in view of the previous cases, $\max(0, \tilde{H}_{(y,t)})/t \to Z$ in L_1 and, therefore, in probability. Therefore, both (5.3) and uniform integrability of $H_{(y,t)}/t$ follow.

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Secondly, we show that

$$\lim_{t\to\infty} \mathbb{P}\biggl(\frac{H_{(y,t)}}{t} < Z - \varepsilon\biggr) = 0. \tag{5.4}$$

Indeed, let $\Pi_{x,y}$ be a stream of RACSs that contain a ball of radius r that covers both x and y. By our assumptions, this is a homogeneous Poisson process of positive intensity, say ν. For each t, let $t - \eta_t$ be the last arrival of such a RACS before t. Clearly, the random variable η_t has an exponential distribution with parameter ν. Further, $H_{(y,t)} \geq H_{(x,t-\eta_t)}$ a.s., so for any $T > 0$,

$$\mathbb{P}\biggl(\frac{H_{(y,t)}}{t} < Z - \varepsilon\biggr) \leq \mathbb{P}(\eta_t > T) + \mathbb{P}\biggl(\frac{H_{(x,t-T)}}{t} < Z - \varepsilon\biggr) \to \mathrm{e}^{-\nu T} \quad \text{as } t \to \infty.$$

Letting $T \to \infty$ leads to (5.4).

Finally (5.3) and (5.4) imply the convergence in probability $H_{(x,t)}/t \to Z$ and, further, uniform integrability implies the L_1-convergence of $\max(0, H_{(x,t)})$ to Z. □

Definition 5.2. We say that $\mathcal{K} \subseteq \mathbb{R}^d$ is similar to the closed convex cone $\mathcal{K}^{(c)} \subseteq \mathbb{R}^d$ if there exist two vectors $v_\pm \in \mathbb{R}^d$ such that $\mathcal{K}^{(c)} + v_- \subseteq \mathcal{K} \subseteq \mathcal{K}^{(c)} + v_+$.

Remark 5.1. Note that if $\mathcal{K}$ is a convex cone then it is trivially similar to itself. Also, if $\mathcal{K}$ is similar to the convex cones $\mathcal{K}_1^{(c)}$ and $\mathcal{K}_2^{(c)}$ then $\mathcal{K}_1^{(c)} = \mathcal{K}_2^{(c)}$ by the geometry of the convex cones.

By monotonicity we obtain the following corollary from Theorem 5.1.

Corollary 5.1. *Assume that the hypothesis of Corollary 3.1 holds. Given that $\mathcal{K} \subseteq \mathbb{R}^d$ is similar to a closed convex cone $\mathcal{K}^{(c)}$ with vertex at the origin, for all $x \in \mathcal{K}$,*

$$\lim_{t\to\infty} \frac{\max(H_{(x,t)}, 0)}{t} = \lim_{t\to\infty} \frac{\mathbb{E}\max(H_{(x,t)}, 0)}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta(\mathcal{K}^{(c)}),0)}}{t} < \infty,$$

where the first limit is in L_1.

Appendix

Proof of Lemma 3.2. The proof leverages the ideas developed in the proof of [1, Theorem 2]. We use the same discretization of time and space as in the proof of this theorem to show the following.

(i) There exists a branching process constructed from an i.i.d. family of random variables $\{(v_i, s_i)\}_i$ with light-tails. For a given i, v_i denotes the number of offsprings of i and s_i denotes the (common) *height* of its offspring.

(ii) For $n \in \mathbb{N}$, let $h(n)$ denote the maximum height of this branching process at generation n, namely the maximum, over all lineages, of the sum of the heights of all generations in the lineage. Then, in order to prove (3.1), it suffices to prove that $\mathbb{E}h(n) \leq Cn$ for every $n > 0$ for some finite C.

For $n \in \mathbb{N}$, let d_n denote the number of individuals of generation n in this branching process. For $a > 0$, let

$$D(a) := \bigcup_{n\geq 1} \{d_n > a^n\}, \qquad \bar{D}(a) := \Omega \setminus D(a).$$

Let $a_m = Cm$, $m \in \mathbb{N}$. From Chernoff's inequality, for some sufficiently large constant $C > 0$,

$$\mathbb{P}(D(a_m)) \leq 2^{-m}.$$

From Chernoff's inequality again, it follows that for all $i \in \mathbb{N}$, $\delta > 0$, and $c_m > 0$ to be fixed, we obtain,

$$\mathbb{P}\biggl(\biggl\{\frac{h(n)}{n} > (c_m + i)\biggr\} \cap \bar{D}(a_m)\biggr) \leq (a_m \mathbb{E}(\mathrm{e}^{\delta s})\mathrm{e}^{-\delta c_m})^n \mathrm{e}^{-\delta n i},$$

where s is a typical height. Therefore,

$$\begin{aligned}\mathbb{E}\biggl(\frac{h(n)}{n}\biggr) &= \sum_{m\geq 1} \mathbb{E}\biggl(\frac{h(n)}{n}\mathbf{1}_{\{\bar{D}(a_m)\setminus\bar{D}(a_{m-1})\}}\biggr), \\ &\leq \sum_{m\geq 1}\biggl(\sum_{i\geq 0}\mathbb{P}\biggl(\biggl\{\frac{h(n)}{n} > (c_m+i)\biggr\} \cap \bar{D}(a_m)\biggr)\biggr) + \mathbb{P}(\bar{D}(a_{m-1}))c_m, \\ &\leq \sum_{m\geq 1}\biggl(\frac{a_m \mathbb{E}(\mathrm{e}^{\delta s})\mathrm{e}^{-\delta c_m}}{1-\mathrm{e}^{-\delta}}\biggr)^n + 2(2^{-m}c_m).\end{aligned}$$

Now we fix δ sufficiently small such that $\mathbb{E}(\mathrm{e}^{\delta s}) < \infty$. Recalling that $a_m = Cm$, in order to conclude the proof, it suffices to construct c_m independent of n, such that

$$(Cm\mathrm{e}^{-\delta c_m})^n \leq 2^{-m}, \qquad \sum_{m\geq 1} 2^{-m} c_m < \infty,$$

where C is a constant independent of n. The last bound is satisfied for $c_m = Bm$ for any $B > 0$. However, for B sufficiently large $C\mathrm{e}^{-\delta B m} \leq 4^{-m}$, which concludes the proof. □

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