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Enumeration of parallel manipulators

Roberto Simoni*, Andrea Piga Carboni and Daniel Martins

Mechanical Engineering Department, Universidade Federal de Santa Catarina 88040-900, Florianópolis, SC, Brazil.

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SUMMARY
In this paper, we present a new method of enumeration of parallel manipulators with one end-effector. The method consists of enumerating all the manipulators possible with one end-effector that a single kinematic chain can originate. A very useful simplification for kinematic chain, mechanism and manipulator enumeration is their representation through graphs. The method is based on group theory where abstract structures are used to capture the internal symmetry of a structure in the form of automorphisms of a group. The concept used is orbits of the group of automorphisms of a colored vertex graph. The theory and some examples are presented to illustrate the method.

KEYWORDS: Kinematic chain; Mechanism; Manipulator; Group theory; Graph theory; Automorphism; Action; Orbit.

1. Introduction

The structural synthesis of kinematic chains consists of the generation of a complete list of kinematic chains that satisfy the general mobility criterion (1) without isomorphic and degenerate chains. This phase is also known as Grüebler synthesis, number synthesis, type synthesis or structural synthesis. In this phase of the project, a kinematic chain can be represented by the graph whose vertices correspond to the links of the chain and whose edges correspond to the joints of the chain. In graph theory terms, the structural synthesis of kinematic chains corresponds to the enumeration of graphs satisfying the general mobility criterion given by the equation

$$M = \lambda(v - e - 1) + e \tag{1}$$

where M is the graph mobility (i.e., kinematic chain), λ is the order of the screw system to which all the joint screws belong, v is the number of graph vertices (i.e., links), and e is the number of graph edges (i.e., joints).

Recently, Sunkari and Schmidt[1] have presented a synthesis method for planar kinematic chains based on group theory techniques. They examined the most efficient algorithms of isomorph-free exhaustive generation and used McKay's algorithm[2,3] for the generation of an isomorphism class representative in combination with degeneracy testing algorithms for the generation of a complete set of planar kinematic chains. Simoni *et al.*[4,5] have presented a variation of the Sunkari and Schmidt method.[1] We adapt the graph generator of McKay to use the degeneracy test of Martins and Carboni[6] that identifies the degeneracy of kinematic chains that operate in any screw system. Using this technique Simoni *et al.*[4] present new results for kinematic chain enumeration in other screw systems, i.e., for $\lambda \neq 3$.

Tuttle and coworkers[7–9] enumerated the kinematic chains and mechanisms systematically which reduced the need for isomorphism testing. The theory of symmetry groups is used successfully to eliminate isomorphic entities in the generation of bases and kinematic chains. Simoni *et al.*[4] used the concept of orbits of the group of automorphisms of noncolored vertex graphs and enumerated mechanisms for several screw systems.

Alizade and Bayram,[10] present a structural synthesis and classification of parallel manipulators with single and multiple platforms, where parallel manipulators are classifed according to their platform type(s) and the connections between them. This method determines simple structural groups for a given set of synthesis parameters and then a number of required actuators are added to the group to form the manipulator.

For certain synthesis parameters, the Alizade and Bayram method[10] finds one structure with the desired number and type of platforms (nonbinary links) and the number of binary links. After that, the number of binary links is distributed between the number of branches and legs originating only one manipulator for the specified parameters. Tsai[11] present a method of structural synthesis of parallel manipulators with a single platform, distributing the number of binary links between the number of legs of the manipulator.

The method of enumeration of manipulators that will be presented in this paper consists of enumerating all the manipulators which a single kinematic chain can originate. Some functional requirements such as mobility, number of links, number of joints, number of loops and redundancy are incorporated in the phase of structural synthesis of kinematic chains and the chains are enumerated using some methods of enumeration. All the manipulators are then enumerated using the method which we will present in Section 3.3. Finally, other functional requirements are incorporated, such as connectivity, degree-of-control, redundancy, etc., and the manipulators are classified.

In this paper, we present a method for the enumeration of all possible manipulators with one end-effector, not necessarily of the platform type, that a kinematic chain can originate. The next step is the systematization of the criteria of variety, connectivity, degree-of-control and redundancy,

* Corresponding author. E-mail: roberto.emc@gmail.com

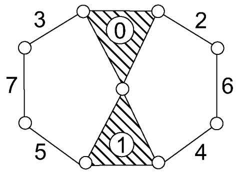

Fig. 1. Kinematic chain.

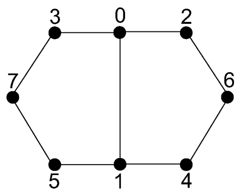

Fig. 2. Graph representation.

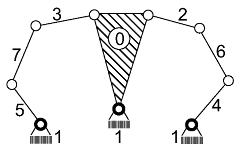

Fig. 3. Mechanism.

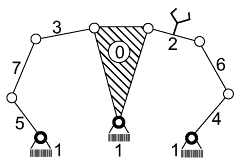

Fig. 4. Manipulator with one end-effector.

that are well established concepts,[6,12,13] for classification of the enumerated parallel manipulators.

2. Graph Representation of Mechanisms and Manipulators

In this paper we explore the number of parallel manipulators with one end-effector which a kinematic chain can originate. The exploration is carried out using graph and group theory. Therefore, we introduce the concepts of mechanism and manipulator and their representation in graphs through examples and figures.

Figure 1 shows a kinematic chain and Fig. 2 its graph representation. Figure 3 shows a mechanism (i.e., inversion of the kinematic chain) and Fig. 4 shows a manipulator with one end-effector originated from the kinematic chain in Fig. 1. In this paper, kinematic chains, mechanisms and manipulators are represented by graphs. This is a very useful simplification for analyzing the possible mechanisms or manipulators which the kinematic chain can originate.

A mechanism is a kinematic chain with one of its components (links) taken as a frame.[14] In terms of graph theory a mechanism corresponds to a graph with one of its vertices detached (colored) to represent the fixed link. Figure 5 shows the graph of the mechanism in Fig. 3 where the detached vertex represents the fixed link.

A generalized parallel manipulator is a closed-loop kinematic chain mechanism whose end-effector is linked to

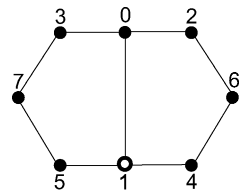

Fig. 5. Graph representation of mechanism.

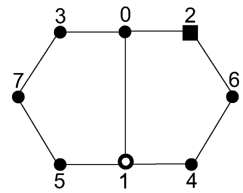

Fig. 6. Graph representation of manipulator.

the base by several independent kinematic chains.[15] In other words, a parallel manipulator is a kinematic chain with one of its components (links) taken as a frame and the other taken as an end-effector. In terms of graph theory a manipulator with one end-effector corresponds to a graph with two detached vertices (colored with distinct colors), one to represent the fixed link and the other to represent the end-effector. Figure 6 shows the graph of the manipulator in Fig. 4, where one of the detached links represents the base and the other represents the end-effector. If the manipulator possess more than one end-effector, more graph vertices must be detached to represent it.

Simoni *et al.*[4] used the concept of orbits of the group of automorphism of noncolored vertex graphs, of group theory, to enumerate all the possible inversions of a single kinematic chain. Using this technique, Simoni *et al.*[4] presented several new results in the enumeration of inversions of kinematic chains.

In this paper, we present an extension of the mechanism enumeration method for enumeration of parallel manipulators with one end-effector. For this we represent parallel manipulators by graphs with two of their vertices colored (detached), one to represent the base and the other to represent the end-effector, and use tools from group theory for enumeration of all the possible manipulators with one end-effector that a single kinematic chain can originate.

3. New Method for Enumeration of Parallel Manipulators

Our method for the enumeration of parallel manipulators consists of calculating orbits of the group of automorphism of colored vertex graphs and selecting all the possible distinct label listing of vertices (one to represent the base and the other to represent the end-effector) which can originate distinct manipulators.

Firstly, we present the fundamentals of group and graph theory and after that our method for the enumeration of parallel manipulators with one end-effector together with examples.

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3.1. Group and graph theory

Groups are abstract structures used in mathematics and science in general to capture the internal symmetry of a structure in the form of automorphisms of a group. Below we present the essential definitions of group theory found in the literature.[16–19]

Definition 1. *A group is a set G with a binary operation* $\bullet : G \times G \to G$ *that satisfies the following 3 axioms:*
(i). *Associativity: For all a, b, and c in G,*
$(a \bullet b) \bullet c = a \bullet (b \bullet c)$.
(ii). *Identity element: There is an element e in G such that for all a in G,* $e \bullet a = a \bullet e = a$.
(iii). *Inverse element: For each a in G, there is an element b in G such that* $a \bullet b = b \bullet a = e$*, where e is the identity element.*

Definition 2. *A set G' is a subgroup of a group G if it is a subset of G and is a group using the operation defined on G.*

Definition 3. *If X is a set and G is a group, then G acts on X if there is a function*

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

such that
(i). $(gh) \cdot x = g \cdot (h \cdot x)$ *for all g, h in the group G and x in the set X.*
(ii). $e \cdot x = x$ *for every element x in the set X (where e is the identity in G).*

We also call X a G-set if G acts on X.

Definition 4. *The symmetric group on a set X, denoted by* S_X *or* $Sym(X)$*, is the group whose underlying set is the set of all bijective functions from X to X, in which the group operation is that of composition of functions.*

The symmetric group on the finite set $X = \{1, 2, \ldots, n\}$ is denoted as S_n and all $\sigma \in S_n$ will be denoted by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Permutations can also be represented by a binary matrix operation. For instance,

$$\sigma = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

can be represented as:

$$\begin{bmatrix} b \\ a \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Subgroups of S_n are called permutation groups.

The set of graph vertices $V_n = \{1, 2, 3, \ldots, n\}$ form a permutation group and the definitions above can be applied.

Example 1. *Figure 7 shows the graph of a Stephenson kinematic chain and Figs. 8 and 9 show the action of* σ_1 *and* σ_2 *in G, respectively, on the labels of the Stephenson graph (see Fig. 7), where*

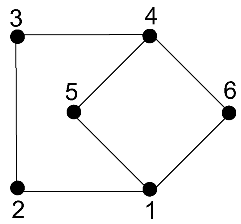

Fig. 7. Graph representation.

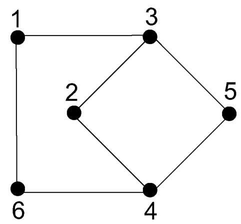

Fig. 8. σ_1 action.

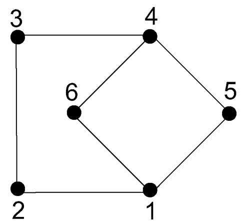

Fig. 9. σ_2 action.

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 6 & 2 \end{pmatrix} = (134)(256) \quad \textit{and}$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix} = (14)(23)(56).$$

Definition 5. *Let* G_1 *and* G_2 *be two groups. A homomorphism of* G_1 *in* G_2 *is an application* $\phi : G_1 \to G_2$ *such that, for all x and y in* G_1

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y).$$

If ϕ *is bijective, the application is an isomorphism. An isomorphism* ϕ *is an automorphism if* $G_1 = G_2$.

In terms of graph theory, two graphs H and H', with graph vertices $V_n = \{1, 2, \ldots, n\}$, are said to be isomorphic if there is a permutation σ of V_n such that $\{x, y\}$ is in the set of graph edges $E(H)$ if and only if $\{\sigma(x), \sigma(y)\}$ is in the set of graph edges $E(H')$.

An automorphism of a graph is a graph isomorphism with itself, i.e., a mapping of the vertices of a given graph H from the vertices of H such that the resulting graph is isomorphic with H. The sets of these permutations which map the graph into itself form a group called the group of automorphisms of the graph. This group of automorphisms is said to be a vertex-induced group. The group of automorphisms of the graph is a subgroup of the symmetric group and contains all the possible

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permutations of the vertices that preserve the adjacency. The group of automorphisms of a graph characterizes its symmetries and are therefore very useful for determining some of its properties.

The McKay algorithm[2,20] is, to the authors' knowledge, the best algorithm for computing graph automorphisms and isomorphisms.

Definition 6. *Consider a group G acting on a set X. The orbit of the point $x \in X$ is denoted by*

$$\mathcal{O}_x = \{g \cdot x \mid g \in G\}.$$

The orbit of a point x in the set X is the set of elements of the set X to which the point x can be moved by the elements of the group G. The set of orbits of the set X under the action of the group G form a partition of the set X. The associated equivalence relation is defined by $x \sim y$ if and only if there exists an element g in the group G such that $g \cdot x = y$. The orbits are equivalence classes under this relation; two elements x and y are equivalent if and only if their orbits are the same, i.e., $\mathcal{O}_x = \mathcal{O}_y$.

The action of the group of automorphisms of the graph permutes the graph vertices. If a graph vertex of the label x is moved by the action of an element of the group of automorphisms to a vertex of the label y, then x and y are in the same orbit, i.e., $\mathcal{O}_x = \mathcal{O}_y$. For graphs, the equivalence relation is associated with the symmetry of their vertices. If the vertices of labels x and y are in the same orbit they possess the same properties of symmetry in the graph. The orbit of a graph vertex corresponds to the set of vertices for which the vertex is moved by the action of the group of automorphisms of the graph.

3.2. Orbits of noncolored vertex graphs and corresponding mechanisms

Using the tools of group theory presented above we can obtain the inversions (i.e., mechanisms) of a kinematic chain choosing a representative of each orbit of the group of automorphism of a noncolored vertex graph. The number of orbits is equal to the number of mechanisms that the graph (i.e., kinematic chain) can originate. To ascertain which are the possible choices for the fixed link there only needs to be chosen a representative of each orbit.[4]

Example 2. *Figure 10 shows a planar kinematic chain with mobility three ($M = 3$) and two loops, the chain is represented by a labeled, noncolored graph (without vertices detached) as shown in Fig. 11 which will be called H. The group of automorphisms of graph H possesses four elements:* $\sigma_1 = (0)(1)(2)(3)(4)(5)(6)(7)$, $\sigma_2 = (23)(45)(67)$,

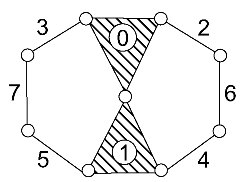

Fig. 10. Kinematic chain.

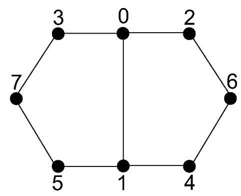

Fig. 11. Graph representation.

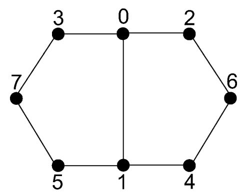

Fig. 12. σ_1 action.

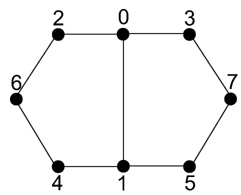

Fig. 13. σ_2 action.

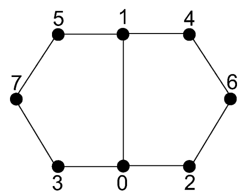

Fig. 14. σ_3 action.

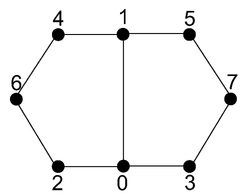

Fig. 15. σ_4 action.

$\sigma_3 = (01)(24)(35)$, *and* $\sigma_4 = (01)(25)(34)(67)$. *The action of the group of automorphisms on graph H is shown in Figs. 12, 13, 14, and 15 respectively.*

The orbit of vertex 0 is equal to the orbit of vertex 1, i.e., $\mathcal{O}_0 = \mathcal{O}_1 = \{0, 1\}$, *the orbit of vertex 2 is equal to the orbit of vertices 3, 4 and 5, i.e.,* $\mathcal{O}_2 = \mathcal{O}_3 = \mathcal{O}_4 = \mathcal{O}_5 = \{2, 3, 4, 5\}$, *and the orbit of vertex 6 is equal to the orbit of vertex 7, i.e.,* $\mathcal{O}_6 = \mathcal{O}_7 = \{6, 7\}$*; therefore, there are three orbits of the group of automorphisms, i.e.,* $\{0, 1\}$, $\{2, 3, 4, 5\}$, *and* $\{6, 7\}$.

The possible mechanisms for the kinematic chain shown in Fig. 10 are obtained by choosing a representative of each orbit of the group of automorphism induced by associated noncolored graph vertices, for example 0, 2, and 6.

The number of orbits of the group of automorphisms (i.e., 3) is equal to the number of mechanisms that the kinematic chain can originate. The links that are in the same orbit originate identical mechanisms, i.e., the changing of

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a fixed link does not cause different characteristic in the movement of the mechanism in relation to the fixed link. The changing of a fixed link, for links that are in different orbits, leads to different characteristic in the movement of the mechanism originating distinct mechanisms for the kinematic chain.

3.3. Orbits of colored vertex graphs and corresponding parallel manipulators with one end-effector

For enumeration of the possible parallel manipulators with one end-effector for a kinematic chain we use colored vertex graphs. The method of enumeration of all the possible manipulators with one end-effector for a kinematic chain consists of calculating orbits of the group of automorphisms of colored vertex graphs which represent the inversions. With the proposed technique for the enumeration of inversions in Section 3.2, we enumerate all the possible choices of the fixed link. To enumerate all the possible parallel manipulators with one end-effector which can be originated by a single kinematic chain we only need to enumerate all the possible choices of the end-effector for each inversion. The simplest way to do this is to color one vertex (which originates the inversion) of each time and to calculate the orbits of the group of automorphisms of a colored vertex graph (with the vertex that originates the inversion colored). The vertex (link) that represents the inversion will be considered as a base for the manipulator and a representative of each orbit of the colored graph will represent the end-effector.

With this technique all the manipulators with one end-effector that the kinematic chain can originate are enumerated. Having established the possible choices of a base, for each colored base (colored graph vertex) the group of automorphisms of colored vertex graph captures the internal symmetries of graph and supplies the information through the orbits of the group. In the case of colored graphs, the group of automorphisms captures equivalence between the graph vertices in relation to the colored vertices. The vertices that are in the same orbit originate identical manipulators with one end-effector. Now we present some examples of the method.

Example 3. Enumeration of planar manipulators with one end-effector: *In example 2, we enumerated the inversions of the kinematic chain in Fig. 10, i.e.,* 0, 2, *and* 6*. Now, we enumerate all the possible manipulators with one end-effector for the kinematic chain in Fig. 10.*

We begin with inversion 0*. The graph vertex of label* 0 *in Fig. 11 is colored as shown in Fig. 16 and the orbits of the group of automorphisms of the colored graph are calculated. The group of automorphisms of the graph with vertex* 0 *colored possesses two elements;* $\sigma_1 = (0)(1)(2)(3)(4)(5)(6)(7)$ *and* $\sigma_2 = (23)(45)(67)$*. Therefore, the orbits of the group of automorphisms are;* $\mathcal{O}_0 = \{0\}$*,* $\mathcal{O}_1 = \{1\}$*,* $\mathcal{O}_2 = \mathcal{O}_3 = \{2, 3\}$*,* $\mathcal{O}_4 = \mathcal{O}_5 = \{4, 5\}$*, and* $\mathcal{O}_6 = \mathcal{O}_7 = \{6, 7\}$*.*

The vertex of label 2 *in Fig. 11 is then colored as shown in Fig. 17. In this case the group of automorphisms of the graph with vertex* 2 *colored possesses only one element, i.e., the identity* $\sigma_1 = (0)(1)(2)(3)(4)(5)(6)(7)$*. Thus, the number of orbits is equal to the number of vertices.*

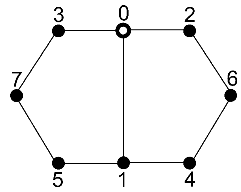

Fig. 16. Vertex 0 colored.

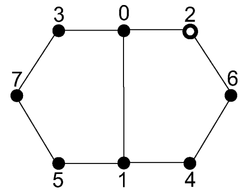

Fig. 17. Vertex 2 colored.

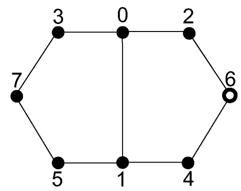

Fig. 18. Vertex 6 colored.

Finally, the vertex of label 6 *in Fig. 11 is colored as shown in Fig. 18. In this case the group of automorphisms of the graph with vertex* 6 *colored possesses two elements;* $\sigma_1 = (0)(1)(2)(3)(4)(5)(6)(7)$ *and* $\sigma_2 = (01)(24)(35)$*. Orbits are;* $\mathcal{O}_0 = \mathcal{O}_1 = \{0, 1\}$*,* $\mathcal{O}_2 = \mathcal{O}_4 = \{2, 4\}$*,* $\mathcal{O}_3 = \mathcal{O}_5 = \{3, 5\}$*,* $\mathcal{O}_6 = \{6\}$*, and* $\mathcal{O}_7 = \{7\}$*.*

With this technique, we enumerate all the possible string listings of vertices that can originate distinct manipulators selecting the colored vertex (inversion) and a vertex of each orbit of the group of automorphisms of the graph with colored vertices, where the string listings $x|y$ *represent the two colored vertices of the graph, i.e., one manipulator where* x *is the fixed link and* y *is the end-effector.*

Table I shows the list of parallel manipulators with one end-effector that the kinematic chain in Fig. 10 can originate. Column 1 shows the orbits of the noncolored graph, column 2 shows the possible inversions (i.e., one representative of each orbit of the noncolored graph), column 3 shows the orbits of the colored graph where the colored vertex is the vertex that originates the inversion shown in column 2, and column 4 shows the possible manipulators with one end-effector for the kinematic chain in Fig. 10. In column 4, the manipulator with one end-effector is originated from one representative of each orbit of the noncolored graph (i.e., inversion) to be the base and one representative of each orbit of the colored graph to be the end-effector. Using this technique, we enumerate 15 distinct parallel manipulators with one end-effector that the kinematic chain in Fig. 10 can originate.

Figure 19 shows some results of Table I, the kinematic chain in Fig. 10 on the first level, the mechanisms derived from this chain (i.e., 0, 2, *and* 6*) on the second level, and the manipulators with one end-effector for the first mechanism (inversion* 0*), i.e.,* 0|1, 0|2, 0|4, *and* 0|6 *on the third level.*

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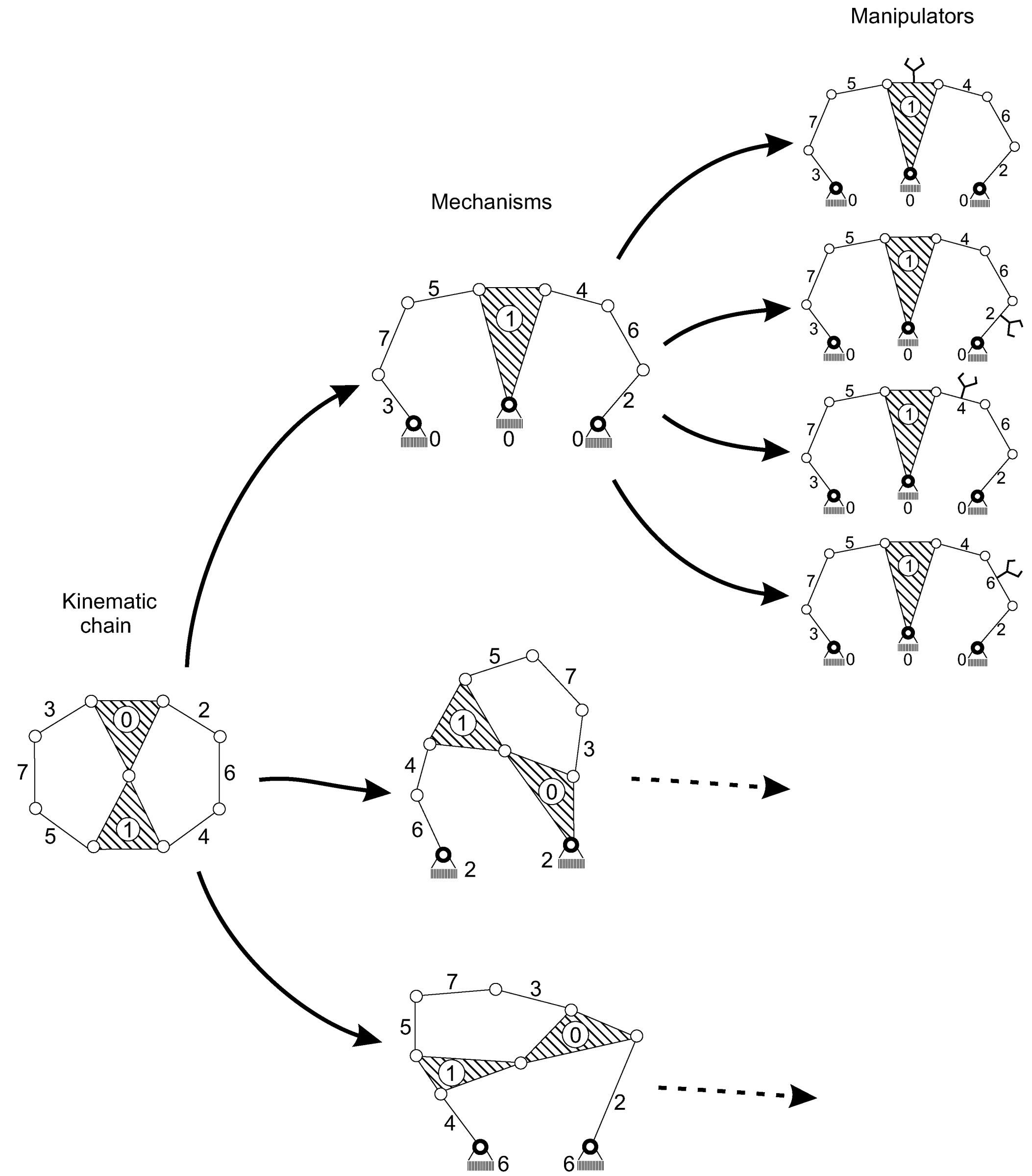

Fig. 19. Representation of the method for the results of Table I.

We choose always the vertex of the lowest label in each orbit to represent the mechanism and/or the manipulator, but the choice could be another. Therefore, if the vertices are in the same orbit they have exactly the same kinematic and structural characteristics as the mechanism or manipulator. For example, in line 1 of Table I, we choose the vertex of label 0 (see column 2) to represent the inversion but we could choose the vertex of label 1. The orbits of the colored graph with one of the two vertices colored (i.e., 0 or 1) will be the same as that shown in column 3 and, consequently, the manipulators indicated in column 4 will have the same kinematic characteristics. The manipulator 0|6 shown in Fig. 20 is the same as 1|7.

Note that the vertices that are in the same orbit as the group of automorphisms of the noncolored vertex graph only originate one manipulator with one end-effector because the base-end-effector change does not cause alterations in the kinematic and structural characteristics of the manipulator and therefore the manipulator only appears once in Table I,

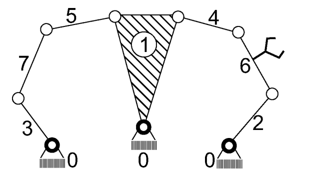

Fig. 20. Base 0, end-effector 6.

for example 0|1. The vertices that are in the same orbits as the group of automorphisms of different noncolored vertex graphs appear twice on the list of manipulators, for example 0|6 and 6|0 (see Figs. 20 and 21). They possess totally different kinematic and structural characteristics. Often the manipulators originated by the same two vertices appear camouflaged, as is the case of 2|7 and 6|3.

If the vertices are in different orbits to the group of automorphisms of a noncolored vertex graph then the base-end-effector change does not originate manipulators with

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Table I. Results of the enumeration of manipulators for the kinematic chain in Fig. 10.

1	2	3	4
Orbit of non-colored graph	Inversions	Orbit of colored graph	Manipulators
		0	–
		1	0|1
0, 1	0	2, 3	0|2
		4, 5	0|4
		6, 7	0|6
		0	2|0
		1	2|1
		2	–
		3	2|3
2, 3, 4, 5	2	4	2|4
		5	2|5
		6	2|6
		7	2|7
		0, 1	6|0
		2, 4	6|2
6, 7	6	3, 5	6|3
		6	–
		7	6|7
Total number of manipulators			$\Sigma = 15$

different structural characteristics and therefore they appear twice on the list of manipulators.

The results presented in Table I are new and therefore we do not have references for comparison.

Example 4. Enumeration of planar manipulators with one end-effector: *Figure 22 shows a planar kinematic chain with a mobility of three (i.e., $M = 3$), ten links (i.e., $n = 10$) and variety zero (i.e., $V = 0$). The graph of the chain is shown in Fig. 23.*

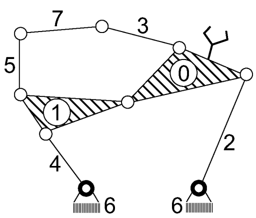

Fig. 21. Base 6, end-effector 0.

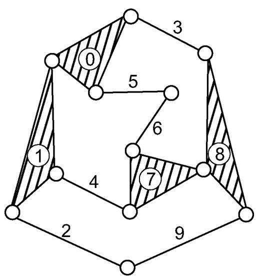

Fig. 22. Kinematic chain.

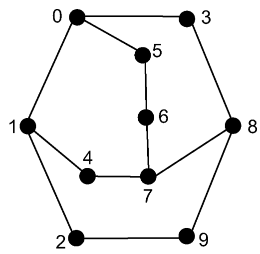

Fig. 23. Graph representation.

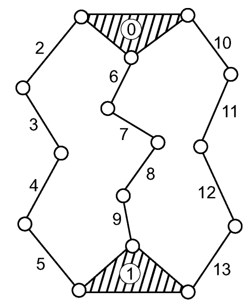

Fig. 24. Kinematic chain.

The orbits of the group of automorphisms of the noncolored vertex graph (see Fig. 23) are:

- *{0, 1, 7, 8},*
- *{2, 5, 6, 9} and*
- *{3, 4}*

which originates three inversions; 0, 2, and 3. Applying our method, coloring the vertex that originates the inversion and calculating the orbits of the group of automorphisms we have that:

- *for vertex 0 colored, the orbits are {0}, {1}, {2}, {3}, {4}, {5},{6}, {7}, {8}, {9},*
- *for vertex 2 colored, the orbits are {0}, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9} and*
- *for vertex 3 colored, the orbits are {0, 8}, {1, 7}, {2, 6}, {3}, {4}, {5, 9}.*

Table II shows the possible manipulators with one end-effector for the kinematic kinematic chain in Fig. 22.

Example 5. Enumeration of spatial manipulators with one end-effector: *Figure 24 shows a spatial kinematic chain with $M = 6$ and $n = 14$ enumerated by Tischler* et al.[21] *and Simoni* et al.[22] *as one of the most promising candidates for the design of robotic fingers. The graph of the chain is shown in Fig. 25.*

Table II. Results of the enumeration of planar manipulators with one end-effector.

Inversion	Manipulators	Total number
0	0|1; 0|2; 0|3; 0|4; 0|5; 0|6; 0|7; 0|8; 0|9	9
2	2|0; 2|1; 2|3; 2|4; 2|5; 2|6; 2|7; 2|8; 2|9	9
3	3|0; 3|1; 3|2; 3|4; 3|5	5
Total number of manipulators		$\Sigma = 23$

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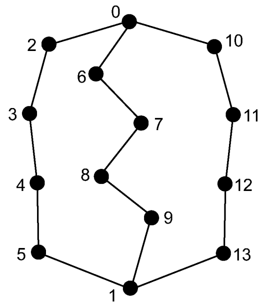

Fig. 25. Graph representation.

The orbits of the group of automorphisms of a noncolored vertex graph (see Fig. 23) are:

- *{0, 1},*
- *{2, 5, 6, 9, 10, 13} and*
- *{3, 4, 7, 8, 11, 12}.*

which originates three inversions; 0, 2, and 3. Applying our method, coloring the vertex that originates the inversion and calculating the orbits of the group of automorphisms we have that:

- *for vertex 0 colored, the orbits are {0}, {1}, {2, 6, 10}, {3, 7, 11}, {4, 8, 12}, {5, 9, 13},*
- *for vertex 2 colored, the orbits are {0}, {1}, {2}, {3}, {4}, {5}, {6, 10}, {7, 11}, {8, 12}, {9, 13} and*
- *for vertex 3 colored, the orbits are {0}, {1}, {2}, {3}, {4}, {5}, {6, 10}, {7, 11}, {8, 12}, {9, 13}.*

Table III shows the possible manipulators with one end-effector for the kinematic chain in Fig. 24.

3.4. Implementation and discussion

McKay[2,3] implemented the Nauty (No AUTomorphisms, Yes?) which is a set of very efficient C language procedures for determining the group of automorphism of a colored vertex graph. It provides this information in the form of a set of generators, the size of the group, and the orbits of the group.

Our implementation consists of adapting the Nauty program of McKay[2,3] to calculate the orbits of noncolored and colored vertex graphs, which represent kinematic chains (or mechanisms), in algorithmic form, as we present in the examples. In this way we can enumerate all the possible parallel manipulators with one end-effector that a set of kinematic chain can originate.

This method of enumeration of parallel manipulators with one end-effector is presented for the first time and can be applied to a high number of graphs (i.e., kinematic chains).

Table III. Results of the enumeration of spatial manipulators with one end-effector.

Inversion	Manipulators	Total number
0	0|1; 0|2; 0|3; 0|4; 0|5	5
2	2|0; 2|1; 2|3; 2|4; 2|5; 2|6; 2|7; 2|8; 2|9	9
3	3|0; 3|1; 3|2; 3|4; 3|5; 3|6; 3|7; 3|8; 3|9	9
Total number of manipulators		$\Sigma = 23$

We are working on a more general method to enumerate manipulators with more than one end-effector. The next step is the elaboration of classification criteria for the enumerated manipulators, because generally the number is very great and it is difficult to analyze the individual merits of each manipulator. These criteria depend on the requirements of each task.

4. Conclusions

A new method for the enumeration of all the possible parallel manipulators with one end-effector that one kinematic chain can originate was presented. This method uses the concept of orbits of the group of automorphisms of colored vertex graphs. To the best of the authors' knowledge, this is the first method for enumeration of all the possible manipulators which a kinematic chain can originate. In the implementation of the method we use the Nauty program of McKay which determines the group of automorphisms of colored vertex graphs very quickly.

Future work will be carried out to extend the method to enumerate parallel manipulators with more than one end-effector. Another stage is the elaboration of criteria for the classification of the manipulators because the number of parallel manipulators which each chain can originate is generally very great and it is difficult to analyze the individual merits of each manipulator.

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