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Commutators Estimates on Triebel–Lizorkin Spaces

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Abstract. In this paper, we consider the behavior of the commutators of convolution operators on the Triebel–Lizorkin spaces $\dot{F}_p^{s,q}$.

1 Introduction

Let Ω be homogeneous of degree zero, integrable on the sphere S^{n-1} and satisfy the vanishing condition

$$\int_{S^{n-1}} \Omega(x')\,dx' = 0. \tag{1.1}$$

The following is the classical singular integral operator

$$Tf(x) = p.v. \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy. \tag{1.2}$$

Let $b \in \mathrm{BMO}(R^n)$, $\Omega \in L^\infty(R^n)$, define the commutators generalized by T and b as follows,

$$[b,T]f(x) = p.v. \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n}(b(x)-b(y))f(y)\,dy. \tag{1.3}$$

The result of $L^p(R^n)$ boundedness of $[b,T]$ can be found in [1, 4, 5, 10, 11]. We state the result of [1] which will be used in the following sections.

Theorem A [1] *Let T be a linear operator. If for all $w \in A_p(R^n)$, $\|Tf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$, then for $b \in \mathrm{BMO}(R^n)$ we have*

$$\|[b,T]f\|_{L^p(w)} \leq C\|b\|_{BMO}\|f\|_{L^p(w)}.$$

Youssfi [14] gives the boundedness on Besov space $\dot{B}_p^{s,q}(R^n)$ for the commutator $[b, R_j]$ using the paraproduct of Bony, where R_j is the Riesz transform.

The purpose of this paper is to establish the boundedness on Triebel–Lizorkin spaces $\dot{F}_p^{s,q}(R^n)$ for the commutator $[b,T]$.

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Theorem 1.1 *Let $b \in \mathrm{BMO}(R^n)$, $s > 0$ and $1 < p, q < \infty$. Let $[b, T]$ be defined as in* (1.3)*. Then the followings are equivalent.*

(a) *$[b, T]$ is bounded on $\dot{F}_p^{s,q}(R^n)$;*

(b) *$\pi_b \circ T^* - T^* \circ \pi_b$ is bounded on $\dot{F}_p^{s,q}(R^n)$;*

(c) $\left\| \left(\sum_{j\in Z} 2^{sjq} \left| [\triangle_j(b), T^*] S_{j-3}(f) \right|^q \right)^{1/q} \right\|_p \leq C \|f\|_{\dot{F}_p^{s,q}(R^n)}, s \geq 0.$

Here π_b is the paraproduct and T^* is the dual operator of T. $\triangle_j$ and S_{j-3} are convolution operators which will be defined in the next section.

Noting that T is bounded on $\dot{F}_p^{s,q}(R^n)$ for all $s \in R$ and $1 < p, q < \infty$ (see [2]), we have the following

Theorem 1.2 *Let $0 < s < \frac{n}{p}$ and $1 < p, q < \infty$. Let $[b, T]$ be defined as in* (1.3)*. If $b \in \dot{F}_{n/s}^{s,q}(R^n)$, then $[b, T]$ is bounded on $\dot{F}_p^{s,q}(R^n)$.*

Remark 1.3 If $b \in \mathrm{BMO}(R^n)$, we do not know whether π_b is bounded on $\dot{F}_p^{s,q}(R^n)$. But, when $b \in \dot{F}_{n/s}^{s,q}(R^n) \subset \mathrm{BMO}(R^n)$ with $0 < s < \frac{n}{p}$, we can show that π_b is bounded on $\dot{F}_p^{s,q}(R^n)$ (see Lemma 3.8).

In the one-dimensional case, as in [14], we replace T by the Hilbert transform. In this case we obtain

Theorem 1.4 *Let $b \in \mathrm{BMO}(R)$, $0 < s < 1$ and $1 < p, q < \infty$. Then the commutator $[b, H]$ is bounded on $\dot{F}_p^{s,q}(R)$ if and only if π_b is bounded on $\dot{F}_p^{s,q}(R)$.*

The paper is organized as follows. In Section 2 we give some preliminary definitions. The proofs of the theorems will be given in Section 3.

Throughout the paper, the letter C will denote (possibly different) constants that are independent of the essential variables.

2 Preliminaries

Let us begin with the definitions of Triebel–Lizorkin spaces $\dot{F}_p^{s,q}(R^n)$ (see [13]) and the paraproduct of Bony.

Let $\varphi \in C_0^\infty(R^n)$ be supported in the unit ball and satisfy $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. The function $\psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi)$ is in $C_0^\infty(R^n)$, supported by $\{\frac{1}{2} \leq |\xi| \leq 2\}$, and satisfies the identity

$$\sum_{j\in Z} \psi(2^{-j}\xi) = 1, \quad \text{for } \xi \neq 0.$$

We denote by $\triangle_j$ and S_j the convolution operators whose symbols are $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively. For $s \in R$ and $1 \leq p < \infty$, $1 \leq q \leq \infty$, the homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{s,q}(R^n)$ are defined by

$$\|f\|_{\dot{F}_p^{s,q}(R^n)} = \left\| \left(\sum_{j\in Z} 2^{jsq} |\triangle_j f|^q \right)^{\frac{1}{q}} \right\|_p, \tag{2.1}$$

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with the usual modification if $q = \infty$.

Define the Peetre maximal function (see [13])

$$(\triangle_j f)^{**}_\lambda(x) = \sup_{y \in R^n} \frac{|\triangle_j f(x-y)|}{(1+2^j|y|)^\lambda}, \quad \lambda > 0, j \in Z.$$

If $\lambda > \frac{n}{\min(p,q)}$, then

$$\|f\|_{\dot F^{s,q}_p(R^n)} \sim \Big\| \Big(\sum_{j\in Z} 2^{jsq} |(\triangle_j f)^{**}_\lambda|^q \Big)^{\frac{1}{q}} \Big\|_p. \tag{2.2}$$

When $p = \infty$, the homogeneous Triebel–Lizorkin spaces $\dot F^{s,q}_\infty(R^n)$ are defined by Carleson measures. We shall say that a sequence of positive Borel measures $(\nu_j)_{j\in Z}$ is a Carleson measure in $R^n \times Z$ if there exists a positive constant C such that $\sum_{j\ge k} \nu_j(B) \le C|B|$ for all $k \in Z$ and all Euclidean balls B with radius 2^{-k}, where $|B|$ is the Lebesgue measure of B. The norm of the Carleson measure $\nu = (\nu_j)_{j\in Z}$ is given by

$$\|\nu\| = \sup \Big\{ \frac{1}{|B|} \sum_{j\ge k} \nu_j(B) \Big\}, \tag{2.3}$$

where the supremum is taken over all $k \in Z$ and all balls B with radius 2^{-k}.

For $s \in R, 1 \le q \le \infty$, $\dot F^{s,q}_\infty(R^n)$ is the space of all distributions b for which the sequence $(2^{sjq}|\triangle_j(b)(x)|^q)_j$ is a Carleson measure (see [7]). The norm of b in $\dot F^{s,q}_\infty(R^n)$ is given by

$$\|f\|_{\dot F^{s,q}_\infty} = \sup \Big\{ \frac{1}{|B|} \sum_{j\ge k} \int_B 2^{sjq} |\triangle_j(b)(x)|^q \, dx \Big\}^{\frac{1}{q}}, \tag{2.4}$$

with the usual modification if $q = \infty$, where the supremum is taken over all $k \in Z$ and all balls B with radius 2^{-k}.

We can see $\dot F^{0,2}_p(R^n) = L^p(R^n)$, $\dot F^{0,2}_1(R^n) = H^1(R^n)$, and $\dot F^{0,2}_\infty(R^n) = \mathrm{BMO}(R^n) \subset \dot F^{0,\infty}_\infty(R^n)$.

The paraproduct of Bony between two functions f, g is defined by

$$\pi(g,f) = \pi_g(f) = \sum_{j\in Z} \triangle_j(g) S_{j-3}(f). \tag{2.5}$$

It is a well-known fact that for $b \in \dot F^{0,\infty}_\infty(R^n)$, π_b is bounded on $L^2(R^n)$ if and only if $b \in \dot F^{0,2}_\infty(R^n) = \mathrm{BMO}(R^n)$.

3 Proofs

The following lemma is well known, see [13].

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Lemma 3.1 *Let $\gamma > 1$ and $1 \le p < +\infty, 1 \le q \le \infty$. For any sequence $\{f_j\}_j$ of functions such that for each j, $\hat{f}_j$ is supported by $\{\gamma^{-1}2^j \le |\xi| \le \gamma 2^j\}$, we have*

$$\Big\| \sum_j f_j \Big\|_{\dot{F}^{s,q}_p(R^n)} \le C \Big\| \Big(\sum_j 2^{sjq} |f_j|^q \Big)^{\frac{1}{q}} \Big\|_p.$$

Lemma 3.2 *Let $g_j \in S'(R^n)(j \in Z)$, satisfying* $\operatorname{supp} \hat{g}_j \subset \{|\xi| \le 2^{j+1}\}$. *Denote $(g_j)^{**}_\lambda(x) = \sup_{z \in R^n} \frac{|g_j(x-z)|}{(1+2^j|z|)^\lambda}, \lambda > 0$. Then there holds*

$$(\partial^{(\alpha)} g_j)^{**}_\lambda(x) \le C 2^{j|\alpha|} (g_j)^{**}_\lambda(x).$$

Proof Choose a function $\theta \in S(R^n)$ such that $\hat{\theta}(\xi) = 1$ if $|\xi| \le 2$, then it is easy to see $g_j(x) = \theta_j * g_j(x)$, where $\theta_j(x) = 2^{jn}\theta(2^j x)$. Thus,

$$\begin{aligned}
\partial^{(\alpha)} g_j(x-y) &= \int_{R^n} \partial^{(\alpha)} \theta_j(x-y-z) g_j(z)\, dz \\
&= \int_{R^n} \partial^{(\alpha)} \theta_j(z-y) g_j(x-z)\, dz \\
&= \int_{R^n} \partial^{(\alpha)} \theta_j(z-y)(1+2^j|z|)^\lambda \frac{g_j(x-z)}{(1+2^j|z|)^\lambda}\, dz.
\end{aligned}$$

Note that $1 + 2^j|z| \le 1 + 2^j|y-z| + 2^j|y| \le (1+2^j|y-z|)(1+2^j|y|)$. Hence

$$\begin{aligned}
|\partial^{(\alpha)} g_j(x-y)| &\le \int_{R^n} |\partial^{(\alpha)}(\theta_j(z-y))| (1+2^j|y-z|)^\lambda (1+2^j|y|)^\lambda (g_j)^{**}_\lambda(x)\, dz \\
&= \int_{R^n} 2^{j|\alpha|} |(\partial^{(\alpha)}\theta)_j(z-y)| (1+2^j|y-z|)^\lambda (1+2^j|y|)^\lambda (g_j)^{**}_\lambda(x)\, dz.
\end{aligned}$$

Thus,

$$\frac{|\partial^{(\alpha)} g_j(x-y)|}{(1+2^j|y|)^\lambda} \le C 2^{j|\alpha|} (g_j)^{**}_\lambda(x).$$

The lemma is completely proved. ∎

Lemma 3.3 *Let $h \in S(R^n), m \in N_0 = N \cup \{0\}, 0 \le s < m+1$. Set*

$$F_m(f)(x,y) = f(y) - \sum_{|\alpha| \le m} \frac{1}{\alpha!}(y-x)^\alpha (\partial^{(\alpha)} f)(x).$$

Then there exists a constant $C > 0$ such that

$$\Big\| \Big(\sum_{j \in Z} 2^{sjq} \Big[\int_{R^n} 2^{nj} |h\big(2^j(x-y)\big)|\, |F_m(S_j f)(x,y)|\, dy \Big]^q \Big)^{\frac{1}{q}} \Big\|_p \le C \|f\|_{\dot{F}^{s,q}_p(R^n)}.$$

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Proof If we set $g = S_j(f)$, then

$$|F_m(g)(x, x+z)| \le C|z|^{m+1}\Big[\sum_{|\alpha|=m+1}\int_0^1 |\partial^{(\alpha)} g(x+tz)|\,dt\Big].$$

Thus,

$$\begin{aligned}
&\int_{R^n} 2^{nj}|h(2^j z)|\,|F_m(S_j f)(x, x+z)|\,dz \\
&\quad\le C\int_{R^n} 2^{nj}|h(2^j z)|\,|z|^{m+1}\sum_{|\alpha|=m+1}\int_0^1 |\partial^{(\alpha)}(S_j f)(x+tz)|\,dtdz \\
&\quad= C\int_{R^n} 2^{nj}|h(2^j z)|\,|z|^{m+1}\sum_{|\alpha|=m+1}\int_0^1 \frac{|\partial^{(\alpha)}(S_j f)(x+tz)|}{(1+2^j|tz|)^\lambda}(1+2^j|tz|)^\lambda\,dtdz \\
&\quad\le C\int_{R^n} 2^{nj}|h(2^j z)|\,|z|^{m+1}\sum_{|\alpha|=m+1}\int_0^1 (\partial^{(\alpha)}(S_j f))^{**}_\lambda(x)(1+2^j|z|)^\lambda\,dtdz \\
&\quad\le C\sum_{|\alpha|=m+1}(\partial^{(\alpha)}(S_j f))^{**}_\lambda(x)2^{-j(m+1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in Z} 2^{sjq}\Big[\int_{R^n} 2^{nj}|h(2^j(x-y))||F_m(S_j f)(x,y)|\,dy\Big]^q\Big)^{\frac{1}{q}}\Big\|_p \\
&\qquad\le C\sum_{|\alpha|=m+1}\Big\|\Big(\sum_{j\in Z} 2^{sqj}|(\partial^{(\alpha)}(S_j f))^{**}_\lambda(x)|^q 2^{-jq(m+1)}\Big)^{\frac{1}{q}}\Big\|_p \\
&\qquad\le C\sum_{|\alpha|=m+1}\Big\|\Big(\sum_{j\in Z} 2^{sqj-jq(m+1)}|\sup_z\sum_{k\le j}\frac{|\partial^{(\alpha)}\triangle_k f|(x-z)}{(1+2^j|z|)^\lambda}|^q\Big)^{\frac{1}{q}}\Big\|_p \\
&\qquad\le C\sum_{|\alpha|=m+1}\Big\|\Big(\sum_{j\in Z} 2^{sqj-jq(m+1)}\Big[\sum_{k\le j}(\partial^{(\alpha)}\triangle_k f)^{**}_\lambda\Big]^q\Big)^{\frac{1}{q}}\Big\|_p.
\end{aligned}$$

Using the discrete Hardy inequality or discrete Hölder inequality, Lemma 3.2 and (2.2), Lemma 3.3 follows immediately. ■

Lemma 3.4 *Let $b \in \dot F^{0,\infty}_\infty(R^n)$ and $t > 0$. Then π_b is bounded on $\dot F^{-t,q}_p(R^n)$ for $1 \le p < \infty$, $1 \le q \le \infty$.*

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Proof We have

$$\Big\| \Big(\sum_{j\in Z} 2^{-tjq} |\triangle_j(b) S_{j-3}(f)|^q \Big)^{\frac{1}{q}} \Big\|_p \le \|b\|_{\dot{F}^{0,\infty}_{\infty}} \Big\| \Big(\sum_{j\in Z} 2^{-tjq} |S_{j-3}(f)|^q \Big)^{\frac{1}{q}} \Big\|_p$$
$$\le C \Big\| \Big(\sum_{j\in Z} 2^{-tjq} \Big(\sum_{k\le j} |\triangle_k(f)| \Big)^q \Big)^{\frac{1}{q}} \Big\|_p.$$

Since $t > 0$, let $0 < s < t$, then

$$\Big\| \Big(\sum_{j\in Z} 2^{-tjq} \Big(\sum_{k\le j} |\triangle_k(f)| \Big)^q \Big)^{\frac{1}{q}} \Big\|_p$$
$$\le C \Big\| \Big(\sum_{j\in Z} 2^{-tjq} \sum_{k\le j} 2^{-ksq} |\triangle_k(f)|^q \Big(\sum_{k\le j} 2^{ksq'} \Big)^{\frac{q}{q'}} \Big)^{\frac{1}{q}} \Big\|_p$$
$$\le C \Big\| \Big(\sum_{k\in Z} 2^{-tkq} |\triangle_k(f)|^q \Big)^{\frac{1}{q}} \Big\|_p$$
$$= C \|f\|_{\dot{F}^{-t,q}_p(R^n)}.$$

Thus the lemma follows. ■

Proposition 3.5 *Let $b \in \dot{F}^{0,\infty}_{\infty}(R^n)$, $s \ge 0$ and $1 < p, q < \infty$. Let T be a convolution operator and be bounded from $\dot{F}^{s,q}_p(R^n)$ to itself. Then $\pi_b \circ T$ is bounded from $\dot{F}^{s,q}_p(R^n)$ to itself if and only if there exists $C > 0$ such that*

$$\Big\| \Big(\sum_{j\in Z} 2^{sjq} |\triangle_j(b) S_{j-3}(Tf)|^q \Big)^{\frac{1}{q}} \Big\|_p \le C \|f\|_{\dot{F}^{s,q}_p(R^n)}.$$

Proof We follow the idea of [14]. First we prove "⇒". Note that if $F_j = \triangle_j(b) S_{j-3}(f)$, $\hat{F}_j$ is supported by $\{2^{j-2} \le |\xi| \le 2^{j+2}\}$. Then by Lemma 3.1,

$$\|\pi_b \circ T(f)\|_{\dot{F}^{s,q}_p(R^n)} = \Big\| \sum_{j\in Z} \triangle_j(b) S_{j-3}(Tf) \Big\|_{\dot{F}^{s,q}_p(R^n)}$$
$$\le C \Big\| \Big(\sum_{j\in Z} 2^{sjq} |\triangle_j(b) S_{j-3}(Tf)|^q \Big)^{\frac{1}{q}} \Big\|_p$$
$$\le C \|f\|_{\dot{F}^{s,q}_p(R^n)}.$$

To prove "⇐", let $X_j(f) = \triangle_j(b) S_{j-3}(Tf)$, and $Y_j(f) = \sum_{\nu=-2}^{2} \triangle_j(X_{j+\nu}(f))$, we only need to show

$$\Big\| \Big(\sum_{j\in Z} 2^{sjq} |X_j(f) - Y_j(f)|^q \Big)^{\frac{1}{q}} \Big\|_p \le C \|f\|_{\dot{F}^{s,q}_p(R^n)},$$

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since $\big\| \big(\sum_{j\in Z} 2^{sjq} |Y_j(f)|^q \big)^{\frac{1}{q}} \big\|_p = \|\pi_b \circ Tf\|_{\dot{F}_p^{s,q}(R^n)} \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}$. Since T is a bounded operator on $\dot{F}_p^{s,q}(R^n)$, without loss of the generality we set $T = \mathrm{Id}$.

Let $m = [s]$; first we consider the case $m = 0$, that is $0 \le s < 1$. Write

$$Y_j(f) - X_j(f) = A_j(f) + B_j(f),$$

where

$$A_j(f) = \sum_{\nu=-2}^{2} \big[\triangle_j(\triangle_{j+\nu}(b) S_{j+\nu-3}(f)) - \triangle_j(\triangle_{j+\nu}(b)) S_{j+\nu-3}(f) \big],$$

$$B_j(f) = \sum_{\nu=-2}^{2} \triangle_j(\triangle_{j+\nu}(b))[S_{j+\nu-3}(f) - S_{j-3}(f)].$$

Since $b \in \dot{F}_\infty^{0,\infty}(R^n)$ and $\sum_{\nu=-2}^{2} |S_{j+\nu-3}(f) - S_{j-3}(f)| = \sum_{\nu=-4}^{1} |\triangle_{j+\nu}(f)|$, it follows that

$$\begin{aligned} \Big\| \Big(\sum_{j\in Z} 2^{jsq} |B_j(f)|^q \Big)^{\frac{1}{q}} \Big\|_p &\le C \Big\| \Big(\sum_{j\in Z} 2^{jsq} \Big| \sum_{\nu=-2}^{2} \Big| S_{j+\nu-3}(f) - S_{j-3}(f) \Big| \Big|^q \Big)^{\frac{1}{q}} \Big\|_p \\ &\le C \Big\| \Big(\sum_{j\in Z} 2^{jsq} \Big(\sum_{\nu=-4}^{1} |\triangle_{j+\nu}(f)| \Big)^q \Big)^{\frac{1}{q}} \Big\|_p \\ &\le C\|f\|_{\dot{F}_p^{s,q}(R^n)}. \end{aligned}$$

We can see this holds for all $s \in R$. To estimate $A_j(f)$, we set $b_j = \triangle_{j+\nu}(b)$, $f_j = S_{j+\nu-3}(f)$ and forget about ν. Then

$$A_j(f)(x) = 2^{nj} \int_{R^n} \check{\psi}(2^j(x-y))(f_j(y) - f_j(x)) b_j(y)\, dy,$$

so that

$$|A_j(f)(x)| \le C\|b_j\|_\infty 2^{nj} \int_{R^n} \check{\psi}(2^j(x-y)) F_0(f_j)(x,y)\, dy.$$

Since $\|b_j\|_\infty \le C\|b\|_{\dot{F}_\infty^{0,\infty}(R^n)}$, by virtue of Lemma 3.3, we obtain

$$\Big\| \sum_{j\in Z} 2^{jsq} |A_j(f)|^q \Big\|_p \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

Now we have proved that for $0 \le s < 1$,

$$\Big\| \Big(\sum_{j\in Z} 2^{sjq} |\triangle_j(b) S_{j-3}(Tf)|^q \Big)^{\frac{1}{q}} \Big\|_p \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

Let $\eta(x) \in S(R^n)$ be a function supported by

$$\{\gamma^{-1} \leq |\xi| \leq \gamma\}, \quad (\gamma > 1),$$

and let L_j be the convolution operator whose symbol is $\eta(2^{-j}\xi)$. Then after replacing ψ by η in the above proof, it is not hard to show that for $0 \leq s < 1$,

$$\Big\| \Big(\sum_{j \in Z} 2^{sjq} |L_j(b) S_{j-3}(Tf)|^q \Big)^{\frac{1}{q}} \Big\|_p \leq C \|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

Now suppose that the proposition holds for all $m' < m$ and the above inequality holds for any arbitrary function η. We have $Y_j(f) - X_j(f) = A_j(f) + B_j(f)$ and

$$\Big\| \Big(\sum_{j \in Z} 2^{jsq} |B_j(f)|^q \Big)^{\frac{1}{q}} \Big\|_p \leq C \|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

We only need to estimate $A_j(f)$. Write $A_j(f) = C_j(f) + D_j(f)$, where

$$C_j(f) = \sum_{\nu=-2}^{2} 2^{nj} \int_{R^n} \check{\phi}(2^j(x-y)) \triangle_{j+\nu}(b)(y) F_m(f_j)(x,y)\, dy,$$

$$D_j(f) = \sum_{\nu=-2}^{2} \sum_{0<|\alpha|\leq m} \frac{2^{nj}}{\alpha!} (\partial^{(\alpha)} f_j)(x) \int_{R^n} \check{\phi}(2^j(x-y)) \triangle_{j+\nu}(b)(y)(y-x)^{\alpha}.$$

Since $\|b_j\|_\infty \leq C\|b\|_{\dot{F}_\infty^{0,\infty}(R^n)}$, by virtue of Lemma 3.3 we have

$$\Big\| \Big(\sum_{j \in Z} 2^{jsq} |C_j(f)|^q \Big)^{\frac{1}{q}} \Big\|_p \leq C \|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

On the other hand, if we forget again the index ν, $D_j(f)$ takes the form

$$D_j(f)(x) = \sum_{0<|\alpha|\leq m} D_j^{\alpha}(f)(x),$$

where

$$D_j^{\alpha}(f)(x) = \frac{2^{nj}}{\alpha!} S_{j-3}(\partial^{(\alpha)} f)(x) \int_{R^n} \check{\psi}(2^j(x-y))(x-y)^{\alpha} \triangle_j(b)(y)\, dy.$$

Following from Lemma 3.4 and the hypothesis of this proposition, using the interpolation theorem we can get that π_b is bounded form $\dot{F}_p^{s-|\alpha|,q}(R^n)$ to $\dot{F}_p^{s-|\alpha|,q}(R^n)$ for $|\alpha| \leq m$. Using the induction hypothesis we obtain, for $|\alpha| \neq 0$, that

$$\Big\| \Big(\sum_{j \in R} 2^{(s-|\alpha|)jq} |R_{j,\alpha}(b) S_{j-3}(g)|^q \Big)^{\frac{1}{q}} \Big\|_p \leq C \|g\|_{\dot{F}_p^{s-|\alpha|,q}},$$

where

$$R_{j,\alpha}(b)(x) = \frac{2^{nj}}{\alpha!}\int_{R^n} \check{\psi}(2^j(x-y))2^{j|\alpha|}(x-y)^\alpha \triangle_j(b)(y)\,dy.$$

But

$$D_j^\alpha(f)(x) = 2^{-j|\alpha|}R_{j,\alpha}(b)(x)S_{j-3}(\partial^{(\alpha)}f)(x).$$

Thus

$$\Big\|\Big(\sum_{j\in Z} 2^{jsq}|D_j(f)|^q\Big)^{\frac{1}{q}}\Big\|_p \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

So the proposition is now completely proved. ■

Now let us prove Theorem 1.1. First we state the following important result (see [9]).

Theorem 3.6 *Let $1 < q < \infty$. Let T be a linear operator. If for all $w \in A_1$, $\|Tf\|_{L^q(w)} \le C\|f\|_{L^q(w)}$ and $\|Tf\|_{L^{q'}(w)} \le C\|f\|_{L^{q'}(w)}$, then for all $1 < p < \infty$,*

$$\Big\|\Big(\sum_{j\in Z}|Tf_j|^q\Big)^{\frac{1}{q}}\Big\|_p \le C\Big\|\Big(\sum_{j\in Z}|f_j|^q\Big)^{\frac{1}{q}}\Big\|_p.$$

Proof If $q \le p$, choose $u(x) \in L^{(\frac{p}{q})'}(R^n)$ satisfying $\|u\|_{L^{(\frac{p}{q})'}} = 1$, such that

$$\Big\|\Big(\sum_{j\in Z}|Tf_j|^q\Big)^{\frac{1}{q}}\Big\|_p^q = \sum_{j\in Z}\int_{R^n}|Tf_j(x)|^q u(x)\,dx.$$

Pick $1 < r < (\frac{p}{q})'$, then for a.e. $x \in R^n$, we have $M(u^r)(x) < \infty$, thus $(M(u^r))^{\frac{1}{r}} \in A_1$. Using the weighted L^q estimate for operator T, there follows

$$\int_{R^n}|Tf_j(x)|^q u(x)\,dx \le \int_{R^n}|Tf_j(x)|^q(M(u^r))^{\frac{1}{r}}\,dx \le C\int_{R^n}|f_j(x)|^q(M(u^r))^{\frac{1}{r}}\,dx.$$

Thus,

$$\begin{aligned}\sum_{j\in Z}\int_{R^n}|Tf_j(x)|^q u(x)\,dx &\le C\sum_{j\in Z}\int_{R^n}|f_j(x)|^q(M(u^r))^{\frac{1}{r}}\,dx\\ &\le C\Big\|\sum_{j\in Z}|f_j|^q\Big\|_{L^{\frac{p}{q}}}\|(M(u^r))^{\frac{1}{r}}\|_{L^{(\frac{p}{q})'}}\\ &\le C\Big\|\Big(\sum_{j\in Z}|f_j|^q\Big)^{\frac{1}{q}}\Big\|_p^q.\end{aligned}$$

Therefore if $q \le p$, $\|(\sum_{j\in Z}|Tf_j|^q)^{\frac{1}{q}}\|_p \le C\|(\sum_{j\in Z}|f_j|^q)^{\frac{1}{q}}|_p$ holds, then by its duality, if $q \ge p$, the result is also true. So the proof is complete. ■

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Remark 3.7 If we replace the operator T in Theorem 3.6 with a sequence of linear operators $\{T_j\}_{j\in Z}$, we can see that the same result holds for T_j provided that $\|T_j f\|_{L^q(w)} \le C\|f\|_{L^q(w)}$ and $\|T_j f\|_{L^{q'}(w)} \le C\|f\|_{L^{q'}(w)}$ for all $j \in Z$ and $w \in A_1$.

Proof of Theorem 1.1 First we prove $(c) \Leftrightarrow (b)$. The part (c) $\Rightarrow$ (b) is similar to Proposition 3.5.

(b) $\Rightarrow$ (c): As in the proof of Proposition 3.5, let

$$X_j(f) = [\triangle_j(b), T]S_{j-3}(f),$$

and

$$Y_j(f) = \sum_{\nu=-2}^{2} \triangle_j(X_{j+\nu}(f)).$$

Since $\|(\sum_{j\in Z} 2^{jsq}|Y_j(f)|^q)^{\frac{1}{q}}\|_p = \|(\pi \circ T - T \circ \pi)f\|_{\dot{F}_p^{s,q}(R^n)} \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}$, we only need to prove

$$\Big\| \Big(\sum_{j\in Z} 2^{jsq}|Y_j(f) - X_j(f)|^q \Big)^{\frac{1}{q}} \Big\|_p \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

We proceed by induction on $m = [s]$. For $m = 0$, we have $0 \le s < 1$. Write $Y_j(f) - X_j(x) = A_j(f) + B_j(f)$, where

$$A_j(f) = \sum_{\nu=-2}^{2} \big\{ \triangle_j([\triangle_{j+\nu}(b), T]S_{j+\nu-3}(f)]) - [\triangle_j(\triangle_{j+\nu}(b)), T]S_{j+\nu-3}(f) \big\},$$

$$B_j(f) = \sum_{\nu=-2}^{2} \big\{ [\triangle_j(\triangle_{j+\nu}(b)), T](S_{j+\nu-3}(f) - S_{j-3}(f)) \big\}.$$

Since T is bounded on $L^p(w)$ for all $w \in A_p$, $1 < p < \infty$ (see [6]), by Theorem A and Remark 3.7, it follows that

$$\Big\| \Big(\sum_{j\in Z} 2^{sjq}|B_j|^q \Big)^{\frac{1}{q}} \Big\|_p \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

Also, using Remark 3.7 and the similar proof in Proposition 3.5, we can get

$$\Big\| \Big(\sum_{j\in Z} 2^{jsq}|A_j|^q \Big)^{\frac{1}{q}} \Big\|_p \le C\|f\|_{\dot{F}_p^{s,q}(R^n)}.$$

Thus we complete the proof of (c)$\Leftrightarrow$ (b).

In fact, from the above proof we can see that if $b \in \dot{F}_\infty^{0,\infty}(R^n)$ and $s \ge 0, 1 < p, q < \infty$, we also have (b)$\Leftrightarrow$ (c).

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Next let us prove (a)$\Leftrightarrow$ (b): For $f, g \in S(R^n)$, we have

$$\langle [b, T]f, g\rangle = \langle b, T(f)g - fT^*(g)\rangle.$$

Since $\|[b, T]f\|_p \le C\|f\|_p$ for $b \in BMO$, by the proof of [3, Theorem III.1] we can see that $F = T(f)g - fT^*(g) \in H^1$. Thus

$$\langle b, F\rangle = \sum_{j\in Z} \langle \triangle_j(b), F\rangle.$$

As in the proof of [14, Theorem 7], we can rewrite $\langle b, F\rangle$ in the form

$$\langle b, F\rangle = A_1 + A_2 + A_3 + A_4$$

where

$$\begin{aligned}
A_1 &= -\langle [\pi_b, T^*]f, g\rangle,\\
A_2 &= \langle [\pi_b, T]g, f\rangle,\\
A_3 &= -\sum_{\nu=-2}^{2}\sum_{j\in Z}\Big\{ \langle [\triangle_{j+\nu}(b), T^*](S_{j-3}(f) - S_{j+\nu-3}(f)), \triangle_j g\rangle\\
&\qquad - \langle [\triangle_{j+\nu}(b), T](S_{j-3}(g) - S_{j+\nu-3}(g)), \triangle_j f\rangle\Big\},\\
A_4 &= -\sum_{\nu=-2}^{2}\sum_{\mu=-2}^{2}\sum_{l\in Z}\langle [S_{l-\nu+2}b, T^*](\triangle_l f), \triangle_{l+\mu} g\rangle.
\end{aligned}$$

To get estimates for $|A_2|, |A_3|$ and $|A_4|$ we use the fact that $b \in \mathrm{BMO}(R^n), s > 0$ and Remark 3.7. Indeed, for $l \in Z$ we have $S_l b \in \mathrm{BMO}(R^n)$ and $\|S_j b\|_{\mathrm{BMO}(R^n)} \le C\|b\|_{\mathrm{BMO}(R^n)}$. Note that T^* is bounded on $L^p(w)$ for all $w \in A_p$. So by Hölder's inequality, Theorem A and Remark 3.7, we get

$$\begin{aligned}
&\Big\|\sum_{l\in Z}\langle [S_{l-\nu+2}b, T^*](\triangle_l f), \triangle_{l+\mu} g\rangle\Big\|\\
&\le \Big\|\Big(\sum_{l\in Z} 2^{slq}|[S_{l-\nu+2}b, T^*](\triangle_l f)|^q\Big)^{\frac{1}{q}}\Big\|_p \Big\|\Big(\sum_{l\in Z} 2^{-slq'}|\triangle_{l+\mu} g|^{q'}\Big)^{\frac{1}{q'}}\Big\|_{p'}\\
&\le C\|b\|_{\mathrm{BMO}(R^n)}\Big\|\Big(\sum_{l\in Z} 2^{slq}|\triangle_l f|^q\Big)^{\frac{1}{q}}\Big\|_p \|g\|_{\dot F^{-s,q'}_{p'}(R^n)}\\
&= C\|f\|_{\dot F^{s,q}_p(R^n)}\|g\|_{\dot F^{-s,q'}_{p'}(R^n)}
\end{aligned}$$

This shows that

$$|A_4| \le C\|f\|_{\dot F^{s,q}_p(R^n)}\|g\|_{\dot F^{-s,q'}_{p'}(R^n)}.$$

To estimate $|A_2|$, by Lemma 3.4 and the duality, we see that $\pi_b \circ T$ and $T \circ \pi_b$ are bounded on $\dot{F}_{p'}^{-s,q'}(R^n)$ for $s > 0$ and $b \in \dot{F}_{\infty}^{0,\infty}(R^n)$. The duality between $\dot{F}_p^{s,q}(R^n)$ and $\dot{F}_{p'}^{-s,q'}(R^n)$ now guarantees that

$$|A_2| \leq C\|f\|_{\dot{F}_p^{s,q}(R^n)}\|g\|_{\dot{F}_{p'}^{-s,q'}(R^n)}.$$

To estimates $|A_3|$ we only consider the former term since the later is similar, observing that by Hölder's inequality and Remark 3.7

$$\begin{aligned}
&\Big\|\sum_{j\in Z}\langle[\triangle_{j+\nu}(b),T^*](S_{j-3}(f)-S_{j+\nu-3}(f)),\triangle_j g\rangle\Big\| \\
&\qquad\leq C\|b\|_{\dot{F}_{\infty}^{0,\infty}(R^n)}\Big\|\Big(\sum_{j\in Z}2^{sjq}|S_{j-3}(f)-S_{j+\nu-3}(f)|^q\Big)^{\frac{1}{q}}\Big\|_p\|g\|_{\dot{F}_{p'}^{-s,q'}(R^n)} \\
&\qquad\leq C\Big\|\Big(\sum_{j\in Z}2^{sjq}|\sum_{\nu=-4}^{1}\triangle_{j+\nu}f|^q\Big)^{\frac{1}{q}}\Big\|_p\|g\|_{\dot{F}_{p'}^{-s,q'}(R^n)} \\
&\qquad\leq C\|f\|_{\dot{F}_p^{s,q}(R^n)}\|g\|_{\dot{F}_{p'}^{-s,q'}(R^n)}.
\end{aligned}$$

Thus under the assumptions of Theorem 1.1, we obtain the equivalence between

$$|\langle b,F\rangle| \leq C\|f\|_{\dot{F}_p^{s,q}(R^n)}\|g\|_{\dot{F}_{p'}^{-s,q'}(R^n)}$$

and

$$|\langle[\pi_b,T^*]f,g\rangle| \leq C\|f\|_{\dot{F}_p^{s,q}(R^n)}\|g\|_{\dot{F}_{p'}^{-s,q'}(R^n)}.$$

Thus the proof of Theorem 1.1 is finished. ∎

To prove Theorem 1.2, we need the following lemma.

Lemma 3.8 *Let $0 < s < \frac{n}{p}$ and $1 < p,q < \infty$. If $b \in \dot{F}_{\frac{n}{s}}^{s,q}(R^n)$, then π_b is bounded on $\dot{F}_p^{s,q}(R^n)$.*

Proof Note that $\|\pi_b(f)\|_{\dot{F}_p^{s,q}(R^n)} = \big\|\sum_j \triangle_j(b)S_{j-3}(f)\big\|_{\dot{F}_p^{s,q}(R^n)}$. Since the Fourier transform of $\triangle_j(b)S_{j-3}(f)$ is supported by $\{2^{j-2} \leq |\xi| \leq 2^{j+2}\}$. Therefore by Lemma 3.1 and Hölder's inequality,

$$\begin{aligned}
\|\pi_b(f)\|_{\dot{F}_p^{s,q}(R^n)} &\leq C\Big\|\Big(\sum_{j\in Z}2^{sjq}|\triangle_j(b)S_{j-3}(f)|^q\Big)^{\frac{1}{q}}\Big\|_p \\
&\leq C\Big\|\Big(\sum_{j\in Z}2^{sjq}|\triangle_j(b)|^q\Big)^{\frac{1}{q}}\Big\|_{\frac{n}{s}}\big\|\sup_j|S_{j-3}(f)|\big\|_{\frac{pn}{n-sp}} \\
&\leq C\|b\|_{\dot{F}_{\frac{n}{s}}^{s,q}(R^n)}\|f\|_{\frac{pn}{n-sp}}.
\end{aligned}$$

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Since $\dot{F}_p^{s,q}(R^n) \hookrightarrow \dot{F}^{0,r}_{\frac{pn}{n-sp}}(R^n)$ for any $1 < r < \infty$. Thus we get

$$\|\pi_b(f)\|_{\dot{F}_p^{s,q}(R^n)} \leq C\|f\|_{\dot{F}_p^{s,q}(R^n)}. \qquad \blacksquare$$

Proof of Theorem 1.2 From Lemma 3.8, we know that π_b is bounded on $\dot{F}_p^{s,q}(R^n)$ under the assumptions of Theorem 1.2. By Lemma 3.4, we see that π_b is bounded on $\dot{F}_p^{-t,q}(R^n)$ for all $t > 0$, $1 < p, q < \infty$. Using the interpolation theorem we obtain that π_b is bounded on $L^2(R^n)$, thus $b \in \mathrm{BMO}(R^n)$. Hence Theorem 1.2 follows immediately from Theorem 1.1 and the $\dot{F}_p^{s,q}(R^n)$ boundedness of T^*. $\blacksquare$

Finally, let us prove Theorem 1.4. The "only if part" is a corollary of Theorem 1.1. To prove the converse we need the following lemma.

Lemma 3.9 *Let $b \in \dot{F}^{0,\infty}_{\infty}$, $0 < s < 1$ and $1 < p, q < \infty$. Then $H \circ \pi_b - \pi_{H(b)}$ is bounded on $\dot{F}_p^{s,q}(R)$.*

Proof Set $S = H \circ \pi_b - \pi_{H(b)}$. It has been shown in [14] that the kernel $K(x, y)$ of S restricted to the set $\Omega = \{(x, y) \in R \times R, x \neq y\}$ is C^∞ and satisfies

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\alpha,\beta}|x - y|^{1-|\alpha|-|\beta|} \quad \text{for } \alpha, \beta \in N \text{ and } x \neq y.$$

On the other hand S satisfies the weak boundedness property [8] and $S(1) = 0$. Thus the lemma follows from the boundedness of singular integral operators on Triebel–Lizorkin spaces (see [8, 12]). $\blacksquare$

Proof of Theorem 1.4 In fact, the proof is a minor modification of [14, Theorem 3]. Choose the function ψ in Section 2 to be radial and real-valued such that the functions $\triangle_j(f)$ and $S_j(f)$ are real-valued. Thus without loss of generality, we may assume that b is real-valued by virtue of Proposition 3.5.

By Lemma 3.9, the operator $\pi_{H(b)} - H \circ \pi_b$ is bounded on $\dot{F}_p^{s,q}(R)$. Moreover, Theorem 1.1 guarantees that $[\pi_b, H] = -[\pi_b, H^*]$ is bounded on $\dot{F}_p^{s,q}(R)$. It follows that $\pi_{H(b)} - \pi_b \circ H$ is bounded on $\dot{F}_p^{s,q}(R)$. Noticing that $H^2 = -\mathrm{Id}$, we obtain that $\pi_b + \pi_{H(b)} \circ H$ is bounded on $\dot{F}_p^{s,q}(R)$.

Write

$$\begin{aligned}\triangle_j(b + iH(b))S_{j-3}\big(f - iH(H(f))\big) &= \triangle_j(b)S_{j-3}(f) + \triangle_j(H(b))S_{j-3}(H(f)) \\ &\quad + i\big[\triangle_j(H(b))S_{j-3}(f) - \triangle_j(b)S_{j-3}(H(f))\big].\end{aligned}$$

Then the operator $\pi_{(b+iH(b))} \circ (\mathrm{Id} - iH)$ is bounded on $\dot{F}_p^{s,q}(R)$. By Proposition 3.5, we obtain

$$\Big\| \Big(\sum_{j\in Z} 2^{jsq}|\triangle_j(b + iH(b))S_{j-3}(f - iH(f))|^q\Big)^{\frac{1}{q}} \Big\|_p \leq C\|f\|_{\dot{F}_p^{s,q}(R)}$$

for all $f \in S(R)$.

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Note that

$$|\triangle_j(b)S_{j-3}(f)| \le |\triangle_j(b+iH(b))S_{j-3}(f-iH(f))|.$$

Thus the proof is finished. ■

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