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ON THE DECOMPOSITION OF CONTINUOUS MODULES

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ABSTRACT. We prove two theorems on continuous modules: *Decomposition Theorem*. A continuous module M has a decomposition, $M = M_1 \oplus M_2$, such that M_1 is essential over a direct sum $\sum \oplus_{i \in I} A_i$ of indecomposable summands A_i of M, and M_2 has no uniform submodules; and these data are uniquely determined by M up to isomorphism. *Direct Sum Theorem*. A finite direct sum $\sum \oplus_{i=1}^{n} A_i$ of indecomposable modules A_i is continuous if and only if each A_i is continuous and A_j-injective for all $j \neq i$.

1. **Introduction.** Utumi [11] introduced the concept of continuous rings, which was later generalized to quasicontinuous modules by Jeremy [3], and applied to continuous modules by Mohamed and Bouhy [5].

A module M is called *continuous* if (i) for every submodule A of M there exists a summand M_1 of M such that A is essential in M_1, and (ii) whenever a submodule A of M is isomorphic to a summand of M then A is itself a summand of M.

The concept of a continuous module is a generalization of that of a (quasi-)injective module. Mohamed and Singh [7] gave a decomposition theorem for dual continuous modules (the definition of which is dual to that of continuous modules), which was further improved by Mohamed and Müller [6]. So far, there have been no results on the decomposition of continuous modules. One purpose of this paper is to give such a theorem, namely:

THEOREM 1. *Let M be a continuous module. Then there exists a decomposition $M = M_1 \oplus M_2$, where M_1 is essential over an (automatically maximal) direct sum $\sum \oplus_{i \in I} A_i$ of indecomposable (hence uniform) summands of M, and M_2 has no non-zero uniform submodule. Moreover, these data are uniquely determined by M, up to isomorphism: if $M = M_1' \oplus M_2'$ is another decomposition such that M_1' is essential over a direct sum $\sum \oplus_{j \in J} A_j'$ of indecomposable summands of M, and M_2' has no non-zero uniform submodule, then $M_1 \cong M_1'$, $M_2 \cong M_2'$ and there is a bijection $\beta : I \to J$ such that $A_i \cong A'_{\beta(i)}$.*

REMARK. In the above situation, every direct sum of finitely many of the A_i is still a summand of M.

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While summands of continuous modules are continuous, the sum of continuous modules need not be continuous (cf. [5], 2.4 and 2.5). In fact, if $M \oplus M$ is continuous, then M becomes quasi-injective ([5], 3.6). Our second result contributes to this question:

THEOREM 2. *Let $M = \sum \oplus_{i=1}^{n} A_i$, where the A_i are indecomposable. Then M is continuous if and only if each A_i is continuous and A_j-injective for all $j \neq i$.*

REMARKS. This theorem dualizes Theorem 2 of Mohamed and Müller [6]. We note that a continuous module is a finite direct sum of indecomposable submodules if and only if it is of finite Goldie dimension.

2. **Preliminaries.** Our rings have identity elements but are not necessarily commutative. All modules are unitary right modules. A module A is said to be B-injective if every homomorphism from a submodule of B to A can be extended to B.

Goel and Jain [2] call a module M π-injective if for every pair of submodules M_1 and M_2 with $M_1 \cap M_2 = 0$, the projections $\pi_i : M_1 \oplus M_2 \to M_i$ can be lifted to endomorphisms of M. Every continuous module is π-injective. π-injective modules are the same as the quasi-continuous modules defined by Jeremy [3].

We say that a submodule N of a module M is essential if it intersects non-trivially with every non-zero submodule of M, and we write $N \subset' M$. We denote the injective hull, the unique maximal essential extension of M, by $E(M)$.

We list some of the properties of continuous modules which will be used:

(A) Every indecomposable continuous module is uniform ([5], Proposition 2.1).

(B) The endomorphism ring of an indecomposable continuous module is local ([5], Corollary 4.3).

(C) If M is π-injective and $E(M) = \sum \oplus_{i \in I} A_i$, then $M = \sum \oplus_{i \in I} (M \cap A_i)$ ([2], Theorem 1.1).

3. **Proof of Theorem 1.** Let M be continuous. Let $\{A_k : k \in K\}$ be the family of all indecomposable summands of M. We call a subset J of K *direct* if the sum $\sum_{j \in J} A_j$ is a direct sum. Consider the collection of all direct subsets of K, ordered by inclusion. An application of Zorn's Lemma yields a maximal direct subset I of K.

Since M is continuous, it can be decomposed as $M = M_1 \oplus M_2$ such that $\sum \oplus_{i \in I} A_i \subset' M_1$. Then M_2 does not contain any non-zero uniform submodule U, because otherwise there exists a summand V of M_2 such that $U \subset' V$. But then V is itself uniform (hence indecomposable), and the direct subset I can be enlarged, contradicting its maximality.

Now, let $M = M_1' \oplus M_2'$ be another decomposition, such that $\sum \oplus_{j \in J} A_j' \subset' M_1'$,

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the A'_j are indecomposable summands of M, and M'_2 has no non-zero uniform submodule. According to ([3], Theorem 7.1), the radical factor ring of endo(M) is von Neumann regular, and idempotents can be lifted. Therefore, Theorems 3 and 2 of [13] apply and show that M has the finite exchange property. Thus, $M=M_1\oplus M_2=M'_1\oplus M'_2$ yields decompositions $M_1=K_1\oplus L_1$ and $M_2=K_2\oplus L_2$ such that $M=M'_1\oplus K_1\oplus K_2$. It follows that $M'_2\cong K_1\oplus K_2$ and $M'_2\cong L_1\oplus L_2$ hold. As M_1 and M'_1 are essential over direct sums of uniform modules, each of their non-zero submodules contains a non-zero uniform submodule. This applies in particular to K_1 and L_2 if they are non-zero. But since these modules embed into M'_2 and M_2, respectively, which have no non-zero uniform submodules, we conclude $K_1=L_2=0$. We deduce $M_1=L_1\cong M'_1$ and $M_2=K_2\cong M'_2$.

Since M_1 is essential over $\sum\oplus_{i\in I}A_i$, and M'_1 is essential over $\sum\oplus_{j\in J}A'_j$, we obtain from $M_1\cong M'_1$ the isomorphism

$$E(\textstyle\sum\oplus_{i\in I}E(A_i))=E(\sum\oplus_{i\in I}A_i)=E(M_1)\cong E(M'_1)=E(\sum\oplus_{j\in J}E(A'_j)).$$

Corollary 4.1 of [12] states: "Any two representations of any injective module as the injective envelope of a direct sum of injective submodules have isomorphic refinements". As the $E(A_i)$ and $E(A'_j)$ are indecomposable, we obtain a bijection $\beta: I\to J$ such that $E(A_i)\cong E(A'_{\beta(i)})$.

We have $M=A_i\oplus P=A'_{\beta(i)}\oplus Q$, and the endomorphism ring of $A'_{\beta(i)}$ is local. ([10], Lemma 5.2) yields that $A'_{\beta(i)}$ is isomorphic to a summand of either A_i or P. In the first case, we obtain immediately $A'_{\beta(i)}\cong A_i$. In the second one, we have $P\cong A'_{\beta(i)}\oplus P'$ and therefore $M\cong A_i\oplus A'_{\beta(i)}\oplus P'$. Thus, $A_i\oplus A'_{\beta(i)}$ is continuous, and consequently π-injective. This fact, together with $E(A_i)\cong E(A'_{\beta(i)})$, implies $A_i\cong A'_{\beta(i)}$, by ([2], Proposition 1.11).

COROLLARY 3. *For a ring with right Krull dimension, every continuous module is essential over a (maximal) direct sum of indecomposable summands, which is unique up to isomorphism.*

Proof. Let $M=M_1\oplus M_2$, as in Theorem 1. Since every non-zero module contains a critical (hence uniform) submodule (cf. [9]), we conclude $M_2=0$.

The implications between (1) and (2) of the next corollary were proved by Matlis [4] and Papp [8], respectively.

COROLLARY 4. *The following are equivalent for a ring R:*
(1) *R is right noetherian;*
(2) *every injective module is a direct sum of indecomposable modules;*
(3) *every continuous module is a direct sum of indecomposable modules.*

Proof. R right noetherian, implies that it has right Krull dimension. Thus, Corollary 3 yields $\sum\oplus_{i\in I}A_i\subset' M$, where the A_i are indecomposable continuous

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submodules. The $E(M)=E(\sum\oplus_{i\in I}A_i)=\sum\oplus_{i\in I}E(A_i)$ and the property (C) listed in the preliminaries gives $M=\sum\oplus_{i\in I}(E(A_i)\cap M)$. The converse follows trivially from the fact that every injective module is continuous.

4. **Proof of Theorem 2.** This proof involves the use of the following lemma, which is dual to Lemma 4 of [6].

LEMMA 5. *Let $N=\sum\oplus_{i\in I}A_i$, where each A_i is uniform and A_j-injective for all $j\neq i$. Then every non-essential submodule of N is contained in a proper summand of N.*

Proof. Let A be a non-essential submodule of N. Then $A\cap A_k=0$ for some k. Indeed, otherwise there are non-zero elements $a_i\in A\cap A_i$ for all i, and then the submodule $\sum\oplus_{i\in I}a_iR$ of A is essential in N, since the A_i are uniform. This contradicts the non-essentiality of A.

Consider the homomorphism $\sigma=1-\pi_k:A\to\sum\oplus_{i\neq k}A_i$, where π_k is the projection to A_k. This is a monomorphism, since $\sigma(a)=a-a_k=0$ implies $a=a_k\in A\cap A_k=0$. Now, as A_k is A_i-injective for all $i\neq k$, A_k is $\sum\oplus_{i\neq k}A_i$-injective, by ([1], Proposition 1.16). Hence there exists a homomorphism $\varphi:\sum\oplus_{i\neq k}A_i\to A_k$ such that $\varphi\sigma=\pi_k\mid A$. Thus, for any $a=\sum a_i\in A\subset\sum\oplus_{i\neq k}A_i$, we obtain $a_k=\pi_k(a)=\varphi\sigma(a)=\varphi(a-a_k)$. We define a submodule C of N by $C=\{b+\varphi(b):b\in\sum\oplus_{i\neq k}A_i\}$, and we claim $N=C\oplus A_k$. Since any $x\in N$ can be written as $x=b+x_k$, with $b\in\sum\oplus_{i\neq k}A_i$ and $x_k\in A_k$, we have

$$x=(b+\varphi(b))+(x_k-\varphi(b))\in C+A_k.$$

Hence, $N=C+A_k$. Furthermore, if $a_k\in C\cap A_k$, then $a_k=b+\varphi(b)$ for some $b\in\sum\oplus_{i\neq k}A_i$. Thus,

$$b=a_k-\varphi(b)\in\sum\oplus_{i\neq k}A_i\cap A_k=0,$$

and consequently $N=C\oplus A_k$.

The proof is finished once we show $A\subset C$. But $a=\sum a_i\in A\subset\sum\oplus_{i\in I}A_i$ implies $\sigma(a)=a-a_k\in\sum\oplus_{i\neq k}A_i$, and therefore

$$a=(a-a_k)+a_k=\sigma(a)+\varphi\sigma(a)\in C,$$

by definition of C.

Proof of Theorem 2. If $M=\sum\oplus_{i=1}^{n}A_i$ is continuous, then each A_i is continuous by ([5], Proposition 2.4) and A_j-injective for all $j\neq i$ by ([5], Corollary 3.5).

Conversely, let all A_i be continuous and A_j-injective for $j\neq i$. Then each A_i is uniform and has local endomorphism ring. Given a submodule A of M, then, among the decompositions of M into indecomposable summands (which are all isomorphic by the Krull–Schmidt–Azumaya Theorem), we choose one, $M=\sum\oplus_{i=1}^{n}A_i$, with a minimal t such that $A\subset\sum\oplus_{i=1}^{t}A_i$. We claim that this inclusion is essential. If not, by Lemma 5, A is contained in a proper summand S of $\sum\bigoplus_{i=1}^{t} A_i$. But then, the direct decomposition length of S is strictly smaller than t, in contradiction to the minimality of t. Thus, we have verified the first condition of continuity.

To establish the second condition, namely that every monomorphism $f: B \to M$ from a summand B of M splits, we consider first the case of an indecomposable summand B. Here, by the Krull–Schmidt–Azumaya Theorem, there is an isomorphism $\varphi: A_k \cong B$, for some k. We consider the maps

$$A_k \overset{\varphi}{\cong} B \xrightarrow{f} M = \sum \bigoplus_{i=1}^{n} A_i \xrightarrow{\pi_i} A_i,$$

where the π_i are the projections. As f is a monomorphism, we have $\bigcap_{i=1}^{n} \ker(\pi_i f) = 0$. Since B is uniform, we conclude $\ker(\pi_i f) = 0$ for some i. For this i, $\pi_i f\varphi: A_k \to A_i$ is a monomorphism. Now, in case $i = k$, $\pi_i f\varphi$ is an isomorphism by the indecomposability and continuity of A_i. On the other hand if $i \neq k$, then A_k is A_i-injective and therefore $\pi_i f\varphi$ splits. By indecomposability, again, $\pi_i f\varphi$ is an isomorphism. Consequently in both cases, $\pi_i f$ and $(\pi_i f)^{-1}$ are isomorphisms.

We define $g: M \to B$ by $g \mid A_i = (\pi_i f)^{-1}$ and $g \mid \sum\bigoplus_{j\neq i} A_j = 0$. We claim that $gf = 1_B$, the identity map on B. For any $b \in B$ we may write $f(b) = x + y$ with $x \in A_i$ and $y \in \sum\bigoplus_{j\neq i} A_j$. Then,

$$\begin{aligned} gf(b) &= g(x) + g(y) = (\pi_i f)^{-1}(x) = (\pi_i f)^{-1}(\pi_i(x+y)) \\ &= (\pi_i f)^{-1}(\pi_i f(b)) = b. \end{aligned}$$

Therefore, $f: B \to M$ splits, as claimed.

Now, we consider a monomorphism $f: B \to M$ from an arbitrary summand B of M, and we proceed by induction over the direct decomposition length l of B. $l = 1$ is the special case dealt with above. If $l \geq 2$, then we write $B = B_1 \oplus B_2$ where B_1 has length $l-1$ and B_2 is indecomposable. Let e_i denote the injections $B_i \to B$. The monomorphism $fe_1: B_1 \to M$ splits by assumption of induction; hence $M = \mathrm{im}(fe_1) \oplus M_2$. Let ρ_i $(i = 1, 2)$ denote the two corresponding projections. Clearly, $\rho_2 fe_2: B_2 \to M_2 \subset M$ is a monomorphism. As B_2 is indecomposable, it splits and we obtain $M_2 = \mathrm{im}(\rho_2 fe_2) \oplus M_3$.

Using $\mathrm{im}(\rho_1) = \mathrm{im}(fe_1) \subset \mathrm{im}(f)$, we calculate for every $b \in B$ that

$$\rho_2 fe_2(b) = fe_2(b) - \rho_1 fe_2(b) \in \mathrm{im}(f)$$

and

$$f(b) = fe_1(b) + fe_2(b) = fe_1(b) + \rho_1 fe_2(b) + \rho_2 fe_2(b) \in \mathrm{im}(fe_1) + \mathrm{im}(\rho_2 fe_2).$$

We conclude that $\mathrm{im}(f) = \mathrm{im}(fe_1) \oplus \mathrm{im}(\rho_2 fe_2)$ holds, and therefore

$$M = \mathrm{im}(fe_1) \oplus M_2 = \mathrm{im}(fe_1) \oplus \mathrm{im}(\rho_2 fe_2) \oplus M_3 = \mathrm{im}(f) \oplus M_3,$$

which proves the second condition.

REMARK. It is still an open question whether an infinite direct sum of indecomposable continuous modules, each injective with respect to all the others, is continuous, even over a noetherian ring. We have been able to prove that over a right noetherian ring, a module $M=\sum\bigoplus_{i=1}^{\infty} A_i$ is continuous provided the A_i are indecomposable, continuous and A_j-injective for $j\neq i$, and M is π-injective.

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