Proceedings of the Royal Society of Edinburgh, **149**, 1175–1188, *2019*
DOI:10.1017/prm.2018.54

Finite time blow-up of complex solutions of the conserved Kuramoto–Sivashinsky equation in $\mathbb{R}^d$ and in the torus $\mathbb{T}^d$, $d \geqslant 1$

Léo Agélas
Department of Mathematics, IFP Energies Nouvelles,
1-4, avenue de Bois-Préau, F-92852 Rueil-Malmaison, France
(leo.agelas@ifpen.fr)

(MS received 30 January 2017; accepted 24 July 2017)

We consider complex-valued solutions of the conserved Kuramoto–Sivashinsky equation which describes the coarsening of an unstable solid surface that conserves mass and that is parity symmetric. This equation arises in different aspects of surface growth. Up to now, the problem of existence and smoothness of global solutions of such equations remained open in $\mathbb{R}^d$ and in the torus $\mathbb{T}^d$, $d \geqslant 1$. In this paper, we answer partially to this question. We prove the finite time blow-up of complex-valued solutions associated with a class of large initial data. More precisely, we show that there is complex-valued initial data that exists in every Besov space (and hence in every Lebesgue and Sobolev space), such that after a finite time, the complex-valued solution is in no Besov space (and hence in no Lebesgue or Sobolev space).

Keywords: Blow up; Surface growth; conserved Kuramoto–Sivashinsky

1. Introduction

In this paper, we consider the conserved Kuramoto–Sivashinsky (cKS) equation described by the following partial differential equation,

$$\partial_t v + \Delta^2 v + \Delta|\nabla v|^2 = 0 \tag{1.1}$$

with initial condition

$$v(0) = v_0 \tag{1.2}$$

on $\mathbb{R}^d$ with solutions vanishing at infinity as $|x| \to \infty$ or on the d-dimensional torus $\mathbb{T}^d \equiv \mathbb{R}^d/(2\pi\mathbb{Z})^d$, with periodic boundary conditions and in this case, we require, in addition, v_0 that is a periodic scalar function of period 2π with zero mean value, that is $\int_{\mathbb{T}^d} v_0(x)\,\mathrm{d}x = 0$.

The cKS equation models the step meandering instability on a surface characterized by the alternation of terraces with different properties [14]. It appeared as a model for the boundaries of terraces in the epitaxy of Silicon [14]. It also describes the growth of an amorphous thin film by physical vapour deposition [23, 24]-in this case, conserved dynamics are obtained by transforming to a frame that translates upward with constant velocity.

For simplicity of presentation, we consider the rescaled version (1.1) with a dimensional length-scales. Sometimes the equation is considered with a linear instability $+\Delta v$, which leads to the formation of hills, and the Kuramoto–Sivashinsky-type nonlinearity $-|\nabla v|^2$ leading to a saturation in the coarsening of hills (see [**21**, **24**]). Both terms are neglected here. They are lower order terms not important for questions regarding regularity and blow up. Furthermore, the equation is usually perturbed by space-time white noise referred as η (see for instance [**21**, **23**, **24**]), which we also neglect here, although many results do hold for the stochastic PDE also (see [**9**]).

Previous work shows that numerical simulations based on (1.1) can be well fitted to experimental data, and that (1.1) adequately describes the phenomena of coarsening and roughening that are characteristic for the growth of corresponding surfaces on intermediate timescales [**21**, **24**, **27**]. In particular, the characteristic statistical measures of the surface morphology such as the correlation length and the surface roughness calculated from the cKS model show very good agreement with available experimental data and, therefore, support the validity of this modelling approach (see [**21**] for more details).

Nevertheless, without the existence of a unique solution, there is no hope of guaranteeing that a numerical approximation is really an approximation in any meaningful sense, since it is not clear what is being approximated.

Thus, a crucial open problem for the cKS equation (1.1) is the fact that existence and uniqueness of global solutions is not known (see [**5**, **6**]) even in the one-dimensional case (see [**5**] and references therein).

For the one-dimensional case, the existence of global weak solutions on bounded domains has been established in [**8**, **27**]. The key point of the construction of global weak solutions lies on a L^2-energy estimate deriving from the fact that, in this case, the nonlinearity in (1.1) is orthogonal to the solution itself in the sense of L^2.

For the two-dimensional case, the situation seems even worse, as the existence of global weak solutions could only be established in H^{-1} using the non-standard Lyapunov function $\int_0^{2\pi} e^{v(x)}\,dx$ (see [**29**]).

However, up to now, the question of global regularity for the cKS equation (1.1) is still open (see [**5**, **6**] and references therein). Only existence, uniqueness and regularity of local solutions or global strong solutions with smallness condition on the initial data have been established in [**4**, **27**] with initial values in $W^{1,q}$ with $q \geqslant 2$ for $d = 1$ and $W^{1,4}$ for $d = 1, 2, 3$ and later improved in [**5**, **6**] for initial values in the critical Hilbert space $H^{d/2}$ or in a critical space of BMO-type.

The main difficulties for treating problem (1.1) are caused by the nonlinearity term $\Delta|\nabla v|^2$ and the lack of a maximum principle. Due to its nonlinear parts, there are more difficulties in establishing the existence of global strong solutions. Then in [**10**], numerical methods have been proposed for proving numerical existence, uniqueness and smoothness of global solutions of (1.1).

As in [**20**], in this paper, we omit the condition that $\widehat{v}$ is the Fourier transform of a real-valued solution v of (1.1) in the $d-$dimensional space and consider it in the space of all possible complex-valued functions.

In this situation, we answer to the existence and smoothness problem for the cKS equation (1.1) in the $d-$dimensional space, $d \geqslant 1$, by showing that for sufficiently large initial data, we get complex-valued solutions which blow-up in finite time. We

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thus extend to the whole domain $\mathbb{R}^d$ and the torus $\mathbb{T}^d$ for any $d \geqslant 1$ the finite time blow-up result obtained in [**7**] in the situation of complex-valued solutions, which was established only in the one-dimensional torus case.

More precisely, by borrowing the arguments used in [**22**], in our theorem 5.1 combined with corollary 5.2, we show that there is complex-valued initial data that exists in every Triebel-Lizorkin or Besov space (and hence in every Lebesgue and Sobolev space), such that after a finite time, the solution is in no Triebel-Lizorkin or Besov space (and hence in no Lebesgue or Sobolev space).

This finite time blow-up result may suggest as it was shown in [**1**], that a better taking into account of the main physical phenomena and a better approximation of terms related to them in the surface growth mathematical model can help to get existence and uniqueness of global strong solutions for such equations as the ones modelling epitaxy thin film growth.

The paper is organized as follows: In § 2, we give some notations. In § 3, we introduce some Banach spaces. In § 4, we deal with local existence in the time of mild solutions in admissible spaces which contains the critical spaces. In § 5, we prove our theorem 5.1 with our corollary 5.2.

If we set $u = -v$ and $u_0 = -v_0$, we notice that u satisfies the following equivalent equation to (1.1):

$$\partial_t u + \Delta^2 u - \Delta|\nabla u|^2 = 0, \tag{1.3}$$

with initial condition

$$u(0) = u_0. \tag{1.4}$$

Then, without loss of generality, in what follows, we will consider equation (1.3) rather than (1.1).

2. Some notations

For any $x \in \mathbb{R}^d$, we denote by $\{x\}_+$ the vector having for components the values $\max\{x_m, 0\}$ for $1 \leqslant m \leqslant d$. We denote by $|\cdot|$ the modulus of a complex number. We denote by $\|\cdot\|$, the Euclidean norm on $\mathbb{C}^d$ defined for all $x \in \mathbb{C}^d$ by $\|x\| = \left(\sum_{1\leqslant m\leqslant d} |x_m|^2\right)^{1/2}$. We denote by $\|\cdot\|_\infty$, the infinity norm on $\mathbb{C}^d$ defined for all $x \in \mathbb{C}^d$ by $\|x\|_\infty = \max_{1\leqslant m\leqslant d} |x_m|$.

For $x \in \mathbb{C}^d$ and $r > 0$, let $B_r(x) = \{y \in \mathbb{C}^d : \|y - x\|_\infty \leqslant r\}$. Notice, here that the ball of $\mathbb{C}^d$ is defined with the norm $\|\cdot\|_\infty$ and not with the Euclidean norm of $\mathbb{C}^d$ as it is usually the case. This change is made in order to deal with the periodic case also.

For any $a \in \mathbb{R}$ and $r > 0$, we denote with the same notation $B_r(a)$ the ball $B_r(\mathcal{A})$ where $\mathcal{A} \in \mathbb{R}^d$ is such that for all $1 \leqslant m \leqslant d$, $\mathcal{A}_m = a$.

For any $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we say that $x \leqslant y$ (resp. $x \geqslant y$) if for all $1 \leqslant m \leqslant d$, $x_m \leqslant y_m$ (resp. $x_m \geqslant y_m$).

For any $x \in \mathbb{R}^d$ and $a \in \mathbb{R}$, we say that $x \leqslant a$ (resp. $x \geqslant a$) if for all $1 \leqslant m \leqslant d$, $x_m \leqslant a$ (resp. $x_m \geqslant a$).

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We use $X \lesssim Y$ to denote the estimate $X \leqslant CY$ with $C > 0$ a constant. For any function f defined on $\mathbb{R}^d \times \mathbb{R}^+$, for any $t \geqslant 0$, for a simplicity in the notation, we denote by $f(t)$ the function $x \longmapsto f(x,t)$ defined on $\mathbb{R}^d$.

For $\Omega_d = \mathbb{R}^d$ or $\mathbb{T}^d$, $d \in \mathbb{N}^*$, for any $f \in L^p(\Omega_d)$, with $1 \leqslant p \leqslant \infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p(\Omega_d)}$, the L^p-norm of f.

Given an absolutely integrable function $f \in L^1(\mathbb{R}^d)$, we define the Fourier transform $\widehat{f} : \mathbb{R}^d \longmapsto \mathbb{C}$ by the formula,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\xi} f(x)\,\mathrm{d}x,$$

and extend it to tempered distributions. For a function f which is periodic with period 1, and thus representable as a function on the torus $\mathbb{T}^d$, we define the discrete Fourier transform $\widehat{f} : \mathbb{Z}^d \longmapsto \mathbb{C}$ by the formula,

$$\widehat{f}(k) = \int_{\mathbb{T}^d} \mathrm{e}^{-ix\cdot k} f(x)\,\mathrm{d}x,$$

when f is absolutely integrable on $\mathbb{T}^d$, and extend this to more general distributions on $\mathbb{T}^d$.

3. Some Banach spaces

We denote by $\mathscr{S}(\mathbb{R}^d)$ the class of complex-valued tempered Schwartz functions on $\mathbb{R}^d$ and by $\mathscr{S}(\mathbb{T}^d)$ the set of all complex-valued, $2\pi-$periodic (in each component) and infinitely differentiable functions (on $\mathbb{T}^d$). Their dual space respectively $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{T}^d)$ are called the space of distributions.

In particular, any function $f \in \mathscr{S}(\mathbb{T}^d)$(resp. $\mathscr{S}'(\mathbb{T}^d)$) can be represented as $f(x) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{ik\cdot x}$ for any $x \in \mathbb{R}^d$ with $\sup\limits_{k\in\mathbb{Z}^d}(1+|k|)^m|\widehat{f}(k)| < \infty$ for any $m \in \mathbb{N}$ (resp. $m \in \mathbb{Z}^-$) (see [**25**] for more details).

For any $s \geqslant 0$, we denote by $\dot{H}^s(\mathbb{T}^d)$ the space of complex-valued, $2\pi-$periodic (in each component) functions with finite $\dot{H}^s(\mathbb{T}^d)-$norm given by $\|f\|_{\dot{H}^s(\mathbb{T}^d)} := \left(\sum_{k\in\mathbb{Z}^d} |k|^{2s}\,|\widehat{f}(k)|^2\right)^{1/2}$ (with the convention $0^0 = 1$). For any $p \geqslant 1$, we denote by $\mathscr{L}^p(\mathbb{R}^d)$ the space of complex-valued measurable functions of $L^p(\mathbb{R}^d)$ and by $\mathscr{L}^p(\mathbb{Z}^d)$ the space of $p-$summable complex-valued sequences of $\ell^p(\mathbb{Z}^d)$.

3.1. Besov spaces

In this subsection, we introduce the Besov spaces based on the dyadic unity partition of Littlewood-Paley decomposition (see [**3**, **11**, **12**, **28**] for more details). We detail the construction of these spaces due to the specific choice of our dyadic unity partition of Littlewood-Paley decomposition. To get the proof of Theorem 5.1, we take an arbitrary real-valued radial function φ in $\mathscr{S}(\mathbb{R}^d)$ but whose Fourier transform $\widehat{\varphi}$ is non-negative and is such that $\operatorname{supp}\widehat{\varphi} \subset B_1(3/2)$ and $\widehat{\varphi}(\xi) \geqslant 1/2$ for $\xi \in B_{1/2}(3/2)$, and define $\varphi_j(x) = 2^{jd}\varphi(2^jx)$ so that $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ for $j \in \mathbb{Z}$. We may assume, $\forall\xi \in \mathbb{R}^d\backslash\{0\}$, $\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j(\xi) = 1$.

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For any $f \in \mathscr{S}'(\mathbb{R}^d)$ or $\mathscr{S}'(\mathbb{T}^d)$, we denote by $\Delta_j f$, $j \in \mathbb{Z}$, the function, $\Delta_j f := \varphi_j \star f$.

If $f \in \mathscr{S}'(\mathbb{R}^d)$, we notice that for all $x \in \mathbb{R}^d$ and for all $j \in \mathbb{Z}$,

$$\Delta_j f(x) = \mathcal{F}^{-1}(\widehat{\varphi}_j \widehat{f})(x) = \int_{\mathbb{R}^d} \widehat{\varphi}_j(\xi)\widehat{f}(\xi)\mathrm{e}^{i\xi\cdot x}\mathrm{d}\xi. \tag{3.1}$$

If $f \in \mathscr{S}'(\mathbb{T}^d)$, we notice that for all $x \in \mathbb{R}^d$ and for all $j \in \mathbb{Z}$,

$$\Delta_j f(x) = \sum_{k\in\mathbb{Z}^d} \widehat{\varphi}_j(k)\widehat{f}(k)\mathrm{e}^{ik\cdot x}. \tag{3.2}$$

Then a tempered distribution f belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ (resp. $\dot{B}^s_{p,q}(\mathbb{T}^d)$) modulo polynomials if and only if $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty$ (resp. $\|f\|_{\dot{B}^s_{p,q}(\mathbb{T}^d)} < \infty$) where for $\Omega_d = \mathbb{R}^d$ or $\mathbb{T}^d$,

$$\|f\|_{\dot{B}^s_{p,q}(\Omega_d)} := \begin{cases} \left(\displaystyle\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|^q_{L^p(\mathbb{R}^d)}\right)^{1/q} & \text{if } q < \infty \\ \displaystyle\sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^p(\mathbb{R}^d)} & \text{elsewhere,} \end{cases} \tag{3.3}$$

and $f = \sum_{j\in\mathbb{Z}} \Delta_j f \in \mathscr{S}'/\mathcal{P}_m$ where $\mathcal{P}_m$ is the space of polynomials of degree $\leqslant m$ and $m = [s - d/p]$, the integer part of $s - d/p$.

4. Local existence of mild solutions

In this section, we deal with local existence in the time of mild solutions in admissible spaces (see 4.0.2) which contains the critical spaces (see 4.0.3). It has been shown in [**6**] local existence in the time of mild solutions for the $d-$dimensional cKS equation (1.3) in the critical space $\mathcal{B} = \dot{B}^0_{\infty,\infty}(\mathbb{R}^d)$ and in [**5**] for the one-dimensional case in the critical space $\dot{H}^{1/2}(\mathbb{T})$. In this section, we extend their results to all critical spaces both on $\mathbb{R}^d$ and $\mathbb{T}^d$ for $d \in \mathbb{N}^*$. More precisely, we establish local existence in the time of mild solutions for the $d-$dimensional cKS equation (1.3) in admissible spaces.

We set $\Omega_d = \mathbb{R}^d$ or $\Omega_d = \mathbb{T}^d$ for the periodic case, $\mathfrak{F}_d = \mathbb{R}^d$ or $\mathfrak{F}_d = \mathbb{Z}^d$ for the periodic case. We set also ${\mathfrak{F}_d}^+ = (\mathbb{R}^+)^d$ or ${\mathfrak{F}_d}^+ = (\mathbb{Z}^+)^d$ for the periodic case. We set $A = \Delta^2$. In either $\mathbb{R}^d$ or $\mathbb{T}^d$, we let e^{-tA} for $t > 0$ be the usual biharmonic heat semigroup associated with the biharmonic heat equation $w_t + Aw = 0$ (see [**16**], for an explicit form of its solution on $\mathbb{R}^d$). From [**6**], the biharmonic heat kernel K on $\mathbb{R}^d$ is given for all $x \in \mathbb{R}^d$ and $t > 0$ by $K(x, t) := t^{-d/4}k(xt^{-1/4})$, where $k \in \mathscr{S}(\mathbb{R}^d)$ is the function such that $\widehat{k}(\xi) = e^{-|\xi|^4}$. After elementary computations, one has for any $1 \leqslant p \leqslant \infty$ and any multi-index $\alpha \in \mathbb{N}^d$

$$\|\partial^\alpha K(\cdot, t)\|_p \lesssim t^{-(((m_\alpha)/(4))+d/4(1-1/p))}, \tag{4.1}$$

where $m_\alpha = \sum_{i=1}^d \alpha_i$. Further, the biharmonic heat kernel $\mathcal{K}$ on $\mathbb{T}^d$ is obtained from K as follows: for all $x \in \mathbb{R}^d$, $t > 0$, $\mathcal{K}(x, t) = \sum_{m\in\mathbb{Z}^d} K(x + 2\pi m, t)$. Then,

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we start with the definition of mild solutions of cKS equation (1.3) obtained from Kato's semigroup approach [18].

Definition 4.0.1. We say that u is a mild solution of cKS equation (1.3) for the initial data u_0 if u is a solution to the equation $u = G_{u_0}(u)$ where $G_{u_0} : C([0,T],X) \to C([0,T],X)$, with X being a space of complex-valued tempered distributions on Ω_d: for a.e $t \in [0,T]$

$$G_{u_0}(u)(t) = e^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-s)A}\Delta|\nabla u(s)|^2\,\mathrm{d}s. \tag{4.2}$$

The Kato's semigroup approach used to find a fixed point of G_{u_0} is to show that G_{u_0} is a contraction mapping on $C([0,T],X)$ or on some subset of $C([0,T],X)$.

We introduce now the notion of admissible spaces.

Definition 4.0.2. We say that X is an admissible space if X is a complex-valued Banach space of tempered distributions on Ω_d satisfying:

(i) $\mathscr{S}(\Omega_d)$ is dense in X;
(ii) for all $f \in X$ and $\lambda > 0$, $\lambda^{1/4}\|\nabla e^{-\lambda A} f\|_\infty \lesssim \|f\|_X$.
(iii) only if $L^\infty(\Omega_d) \not\subset X$, for all $(f,g) \in X \times X$,

$$\|fg\|_X \lesssim \|f\|_\infty \|g\|_X + \|f\|_X \|g\|_\infty.$$

One can find in lemma X4 of [19] and corollary 2.54 of [2], some examples of Banach spaces satisfying property (iii). We continue with the notion of critical spaces.

Definition 4.0.3. We say that X is a critical space if X is an admissible space in the sense of definition 4.0.2 satisfying: for all $f \in X$, $\lambda > 0$, $\|f(\lambda\cdot)\|_X = \|f\|_X$ (called scale-invariance property).

Here, we give some examples of critical spaces in the sense of definition 4.0.3 such that the Lebesgue space $L^\infty(\Omega_d)$, the homogeneous Sobolev space $\dot{H}^{d/2}(\Omega_d)$ and the Besov spaces $\dot{B}^{d/p}_{p,\infty}(\Omega_d)$ with $0 < p \leqslant \infty$ (for property (iii) we refer to corollary 2.54 in [2] and lemma X4 in [19] and for property (ii) we refer to lemma 3.1 in [6] and lemma 2.4 in [2] to be adapted for our semigroup e^{-tA}). Notice that $\dot{B}^0_{\infty,\infty}(\Omega_d)$ is also a critical space in the sense of definition 4.0.3 since we do not have to ensure property (iii) due to the fact that $L^\infty(\Omega_d) \subset \dot{B}^0_{\infty,\infty}(\Omega_d)$.

Further, it can be shown (arguing similarly as in Frazier, Jawerth and Weiss in [13]) that all scale-invariant spaces of distributions (the distributions having the scale-invariance property), that also contain all Schwartz functions, are contained in the Besov space $\dot{B}^0_{\infty,\infty}$, namely any critical space $X \subset \dot{B}^0_{\infty,\infty}$.

Before to deal with the local existence in the time of mild solutions in admissible spaces, we need the following lemma,

Lemma 4.4. *Let X be an admissible space in the sense of the definition (4.0.2). Then for any $f \in X$,*

$$\lim_{\lambda\to 0^+} \lambda^{\frac{1}{4}}\|\nabla e^{-\lambda A} f\|_\infty = 0 \quad \textit{and} \quad \lim_{\lambda\to 0^+} \lambda^{1/4}\|\nabla e^{-\lambda A} f\|_X = 0.$$

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Proof. Let $Y = L^\infty(\Omega_d)$ or $Y = X$. Let $f \in X$ and $\epsilon > 0$. Since $\mathscr{S}(\Omega_d)$ is dense in X due to property (i), we get that there exists $g \in \mathscr{S}(\Omega_d)$ such that $\|g - f\|_X \leqslant \epsilon$. Further, we get for any $\lambda > 0$,

$$\lambda^{1/4}\|\nabla e^{-\lambda A} f\|_Y \leqslant \lambda^{1/4}\|\nabla e^{-\lambda A} g\|_Y + \lambda^{1/4}\|\nabla \mathrm{e}^{-\lambda A}(f-g)\|_Y. \tag{4.3}$$

Thanks to property (ii) if $Y = L^\infty(\Omega_d)$ and inequality (4.1) if $Y = X$ used with $p = 1$ and $\alpha \in \mathbb{N}^d$ s.t $|\alpha| = 1$, we deduce $\lambda^{1/4}\|\nabla e^{-\lambda A}(f-g)\|_Y \lesssim \|f-g\|_X$. Furthermore, for any $\lambda > 0$, we have also $\|\nabla e^{-\lambda A} g\|_Y \lesssim \|\nabla g\|_Y$. Hence, from (4.3), we get $\lambda^{\frac{1}{4}}\|\nabla e^{-\lambda A} f\|_Y \lesssim \lambda^{1/4}\|\nabla g\|_Y + \|f-g\|_X \lesssim \lambda^{1/4}\|\nabla g\|_Y + \epsilon$. We thus infer

$$\limsup_{\lambda\to 0^+} \lambda^{1/4}\|\nabla e^{-\lambda A} f\|_Y \lesssim \limsup_{\lambda\to 0^+} \lambda^{1/4}\|\nabla g\|_Y + \epsilon = \epsilon. \tag{4.4}$$

Since inequality 4.4 is valid for all $\epsilon > 0$, then we deduce $\limsup_{\lambda\to 0^+} \lambda^{1/4}\|\nabla e^{-\lambda A} f\|_Y = 0$ which implies $\lim_{\lambda\to 0^+} \lambda^{1/4}\|\nabla e^{-\lambda A} f\|_Y = 0$. Then, we conclude the proof. □

Let us turn now to the proof of local existence of mild solutions in admissible spaces.

To get these mild solutions, we proceed similarly as in [**5**–**7**, **15**, **17**]. As in [**6**], we introduce the map $\mathcal{V}$ defined by

$$\mathcal{V}(f,h)(t) := \int_0^t \mathrm{e}^{-(t-s)A}\Delta(\nabla f(s)\cdot\nabla h(s))\,\mathrm{d}s.$$

We notice that $G_{u_0}(u)(t) = e^{-tA}u_0 + \mathcal{V}(u,u)(t)$. We introduce also for $T > 0$ the space

$$\mathscr{M}_{T,X} := \{v \in C(]0,T] : \mathscr{S}'(\Omega_d)) \ : \|v\|_{\mathscr{M}_{T,X}} < \infty\},$$

equipped with the norm,

$$\|v\|_{\mathscr{M}_{T,X}} := \begin{cases} \sup\limits_{t\in]0,T]} t^{1/4}(\|\nabla v(t)\|_X + \|\nabla v(t)\|_\infty) & \text{if } L^\infty(\Omega_d) \not\subset X \\ \sup\limits_{t\in]0,T]} t^{1/4}\|\nabla v(t)\|_\infty & \text{otherwise.} \end{cases} \tag{4.5}$$

Then, we get the following proposition.

PROPOSITION 4.5. *Let X be an admissible space in the sense of definition (4.0.2). Let $u_0 \in X$. Then there exists $T > 0$ such that there exists a unique mild solution $u \in \mathscr{M}_{T,X}$ to the cKS equation (1.3) for the initial data u_0. Moreover, $u \in C([0,T];X)$.*

Proof. Thanks to (4.1) used with $p = 1$ and $\alpha \in \mathbb{N}^d$ s.t $|\alpha| = 3$, we get for any $\lambda > 0$, $\|\nabla\Delta e^{-\lambda A} f\|_\infty \lesssim \lambda^{-3/4}\|f\|_\infty$ and $\|\nabla\Delta e^{-\lambda A} f\|_X \lesssim \lambda^{-3/4}\|f\|_X$, then by combining these inequalities with property (iii), we deduce that for all $(f,h) \in \mathscr{M}_{T,X} \times \mathscr{M}_{T,X}$

$$\|\mathcal{V}(f,h)\|_{\mathscr{M}_{T,X}} \lesssim \|f\|_{\mathscr{M}_{T,X}}\|h\|_{\mathscr{M}_{T,X}}. \tag{4.6}$$

Thanks to property (ii), we get for any $t > 0$, $t^{1/4}\|\nabla e^{-tA}u_0\|_\infty \lesssim \|u_0\|_X$ and thanks to (4.1) used with $p = 1$ and $\alpha \in \mathbb{N}^d$ s.t $|\alpha| = 1$ we get also $t^{1/4}\|\nabla e^{-tA}u_0\|_X \lesssim$

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$\|u_0\|_X$. We introduce v_0 the function defined on $\Omega_d \times [0, +\infty[$ by

$$v_0(t) := e^{-tA}u_0 \quad \text{for all } t \geqslant 0.$$

Therefore, for any $T > 0$, we have

$$\|v_0\|_{\mathscr{M}_{T,X}} \lesssim \|u_0\|_X.$$

Moreover, thanks to lemma 4.4, we get

$$\lim_{T \to 0} \|v_0\|_{\mathscr{M}_{T,X}} = 0. \tag{4.7}$$

We observe that for all $(f, g) \in \mathscr{M}_{T,X} \times \mathscr{M}_{T,X}$, $G_{u_0}(f) - G_{u_0}(g) = \mathcal{V}(f - h, f + h)$, then owing to (4.6), we deduce that there exists a constant $c_0 > 0$ such that

$$\|G_{u_0}(f) - G_{u_0}(g)\|_{\mathscr{M}_{T,X}} \leqslant c_0\|f - g\|_{\mathscr{M}_{T,X}}(\|f\|_{\mathscr{M}_{T,X}} + \|g\|_{\mathscr{M}_{T,X}}). \tag{4.8}$$

Thanks again to (4.6), we get also

$$\|G_{u_0}(f)\|_{\mathscr{M}_{T,X}} \leqslant \|v_0\|_{\mathscr{M}_{T,X}} + c_0\|f\|^2_{\mathscr{M}_{T,X}}. \tag{4.9}$$

For any $\rho > 0$, we denote by $\mathscr{M}_{T,X,\rho}$ the ball of centre 0 and radius ρ in $\mathscr{M}_{T,X}$, that is $\mathscr{M}_{T,X,\rho} := \{f \in \mathscr{M}_{T,X}; \|f\|_{\mathscr{M}_{T,X}} \leqslant \rho\}$. Thanks to (4.8) and (4.9), we obtain that G_{u_0} is a contraction mapping on $\mathscr{M}_{T,X,\rho}$ for some $T > 0$ if ρ satisfies

$$2c_0\rho < 1 \quad \text{and} \quad \|v_0\|_{\mathscr{M}_{T,X}} + c_0\rho^2 \leqslant \rho. \tag{4.10}$$

The inequalities in (4.10) are satisfied if and only if $1 - 4c_0\|v_0\|_{\mathscr{M}_{T,X}} > 0$ and $\rho < ((1 - \sqrt{1 - 4c_0\|v_0\|_{\mathscr{M}_{T,X}}})/(2c_0))$. However, thanks to (4.7), there exists $T_0 > 0$ such that $\|v_0\|_{\mathscr{M}_{T_0,X}} \leqslant ((1)/(8c_0))$ which implies $1 - 4c_0\|v_0\|_{\mathscr{M}_{T_0,X}} \geqslant 1/2$, therefore, with $\rho_0 = ((1 - \sqrt{1 - 4c_0\|v_0\|_{\mathscr{M}_{T_0,X}}})/(4c_0))$, we deduce that G_{u_0} is a contraction mapping on $\mathscr{M}_{T_0,X,\rho_0}$. Thanks to the contraction mapping theorem, we deduce that there exists an unique $u \in \mathscr{M}_{T_0,X,\rho_0}$ fixed point of G_{u_0}, i.e $u = G_{u_0}(u)$.

Since $u \in \mathscr{M}_{T_0,X}$ and $u = G_{u_0}(u)$, we deduce that $u \in C([0, T]; X)$ by using 4.1 and property (iii) (notice that if $L^\infty(\Omega_d) \subset X$, we do not need property (iii)). Then, we conclude the proof. □

5. Blow-up of complex-valued solutions of the cKS equation

The main ingredient of the proof of the existence of blowing-up solutions as in [**22**] consists in noticing that if the initial data has a positive Fourier transform, then that positivity is preserved for the solution at all further times. One can then use the Duhamel formulation of the solution and deduce a lower bound for the Fourier transform that blows up in finite time.

THEOREM 5.1. *Let $d \in \mathbb{N}^*$. Let $w \in \mathscr{S}(\Omega_d)$ such that $\widehat{w}$ is a real-valued function, $\widehat{w}$ is non-negative, has $\mathscr{L}^1-$norm equal to 1, and has support in $B_{1/2}(3/2) \cap \mathfrak{F}_d$ (so w is in every Triebel-Lizorkin or Besov space). Then if $A > 2^{16/15}$, and if u is a mild solution to cKS equation (1.3) whose Fourier transform is a non-negative real-valued*

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function supported in ${\mathfrak{F}_d}^+$*, with initial data* $u_0 = 2^7 Aw$*, then for any* $\sigma \in \mathbb{R}$*,* u *blows up in the Besov space* $\dot{B}^{\sigma}_{\infty,\infty}$*, at some time smaller than* $T_d := \log(2^{((1)/(15d^2))})$.

Proof. To get the proof, we proceed by using a contradiction argument. For this, we assume that there is $\sigma \in \mathbb{R}$ such that u do not blow-up in finite time in the Besov space $\dot{B}^{\sigma}_{\infty,\infty}$.

To get the contradiction, we adapt the construction of [**22**] to our case.

We begin by setting $w_n = w^{2^n}$ with $n \in \mathbb{N}$. We observe that $w_{n+1} = w_n^2$ and then $\widehat{w}_{n+1} = \widehat{w}_n \star \widehat{w}_n$. Since $\widehat{w}$ is non-negative, has $\mathscr{L}^1-$norm equal to 1 and is supported in $B_{1/2}(3/2) \cap \mathfrak{F}_d = \{\xi \in \mathfrak{F}_d : 1 \leqslant \xi \leqslant 2\}$, then by using an induction argument, we deduce that for all $n \in \mathbb{N}$, $\widehat{w}_n$ is also non-negative, has $\mathscr{L}^1-$norm equal to 1 and is supported in $\{\xi \in \mathfrak{F}_d : 2^n \leqslant \xi \leqslant 2^{n+1}\}$.

Let $t \geqslant 0$. We will show now by induction that the proposition $\mathcal{P}(n)$ defined by (5.1) is true for all $n \in \mathbb{N}$.

We give just below the definition of the proposition $\mathcal{P}(n)$,

$$\mathcal{P}(n) := \{\widehat{u}(t) \geqslant A^{2^n} \alpha_n(t) \widehat{w}_n\}, \tag{5.1}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is the sequence of functions defined for all $s \geqslant 0$ by $\alpha_n(s) = 2^7 e^{-2^{n+4}d^2 s} \mathbf{1}_{\{s \geqslant t_n\}}$ with $\{t_n\}_{n\in\mathbb{N}}$ the sequence defined by $t_0 = 0$ and $t_n = ((\log(2))/(d^2)) \sum_{j=1}^{n+1} 2^{-4j}$ for all $n \geqslant 1$. Notice that the sequence $\{t_n\}_{n\in\mathbb{N}}$ is increasing and $\lim_{n\to\infty} t_n = \log(2^{((1)/(15d^2))}) = T_d$ which implies that for all $n \in \mathbb{N}$, $t_n < T_d$.

Let us show that the proposition $\mathcal{P}(n)$ ***is true for*** $n = 0$.

Since, we have

$$u = G_{u_0}(u), \tag{5.2}$$

where $G_{u_0}(u)$ is given by (4.2), then after taking the Fourier transform of equation (5.2), we deduce that for all $\xi \in \mathfrak{F}_d$,

$$\widehat{u}(\xi, t) = e^{-t|\xi|^4} \widehat{u}_0(\xi) - \int_0^t \mathrm{e}^{-(t-s)|\xi|^4} |\xi|^2 (\widehat{\nabla u} \star \widehat{\nabla u})(\xi, s) \,\mathrm{d}s. \tag{5.3}$$

We notice that $\widehat{\nabla u}(\xi, s) = i\xi \widehat{u}(\xi, s)$. Since $\widehat{u}(\cdot, s)$ is supported in ${\mathfrak{F}_d}^+$, then we get $\widehat{\nabla u}(\xi, s) = i\{\xi\}_+ \widehat{u}(\xi, s)$ and, therefore,

$$-i\widehat{\nabla u}(\xi, s) = \{\xi\}_+ \widehat{u}(\xi, s). \tag{5.4}$$

Since $\widehat{u}(\cdot, s)$ is non-negative, from (5.4) we get $-i\widehat{\nabla u}(\xi, s) \geqslant 0$ and, therefore, we deduce,

$$\begin{aligned} (\widehat{\nabla u} \star \widehat{\nabla u})(\xi, s) &= -((-i\widehat{\nabla u}) \star (-i\widehat{\nabla u}))(\xi, s) \\ &\leqslant 0. \end{aligned} \tag{5.5}$$

Therefore, from (5.3), we get $\widehat{u}(\xi, t) \geqslant e^{-t|\xi|^4} \widehat{u}_0(\xi)$ which gives us $\widehat{u}(\xi, t) \geqslant 2^7 A e^{-t|\xi|^4} \widehat{w}(\xi)$ and then $\widehat{u}(\xi, t) \geqslant 2^7 A e^{-2^4 d^2 t} \widehat{w}(\xi)$ since $|\xi| \leqslant \sqrt{d} \|\xi\|_\infty$ and $\widehat{w}$ is supported in $\{\xi \in \mathfrak{F}_d : 1 \leqslant \xi \leqslant 2\}$. Hence, we get that the proposition $\mathcal{P}(0)$ is true.

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Let us assume that the proposition $\mathcal{P}(n)$ ***is true for a given*** $n \in \mathbb{N}$***. Then, let us show that*** $\mathcal{P}(n+1)$ ***will also be true***.

From (5.3), since $\widehat{u}_0 = 2^7 A\widehat{w} \geqslant 0$, we have,

$$\widehat{u}(\xi,t) \geqslant -\int_0^t \mathrm{e}^{-(t-s)|\xi|^4}|\xi|^2(\widehat{\nabla u}\star\widehat{\nabla u})(\xi,s)\,\mathrm{d}s. \tag{5.6}$$

Since $\mathcal{P}(n)$ is true, then from (5.4), we get $-i\widehat{\nabla u}(\xi,s) \geqslant \{\xi\}_+ A^{2^n}\alpha_n(s)\widehat{w}_n(\xi)$. Since $\widehat{w}_n$ is supported in $\{\xi \in \mathfrak{F}_d : 2^n \leqslant \xi \leqslant 2^{n+1}\}$, then we get $-i\widehat{\nabla u}(\xi,s) \geqslant 2^n A^{2^n}\alpha_n(s)\widehat{w}_n(\xi) \geqslant 0$ (which means that each component of the vector is $>2^n A^{2^n}\alpha_n(s)\widehat{w}_n(\xi) \geqslant 0$) and, therefore, we deduce,

$$\begin{aligned}(\widehat{\nabla u}\star\widehat{\nabla u})(\xi,s) &= -((-i\widehat{\nabla u})\star(-i\widehat{\nabla u}))(\xi,s)\\ &\leqslant -d(2^n A^{2^n}\alpha_n(s))^2(\widehat{w}_n\star\widehat{w}_n)(\xi)\\ &= -d(2^n A^{2^n}\alpha_n(s))^2\widehat{w}_{n+1}(\xi).\end{aligned} \tag{5.7}$$

Using (5.7), from (5.6), we deduce

$$\widehat{u}(\xi,t) \geqslant \int_0^t e^{-(t-s)|\xi|^4}|\xi|^2 d(2^n A^{2^n}\alpha_n(s))^2\widehat{w}_{n+1}(\xi)\,ds. \tag{5.8}$$

Since $\widehat{w}_{n+1}$ is supported in $\{\xi \in \mathfrak{F}_d : 2^{n+1} \leqslant \xi \leqslant 2^{n+2}\}$, then we get $2^{2(n+1)}d \leqslant |\xi|^2 \leqslant 2^{2(n+2)}d$, hence from (5.8) we get,

$$\begin{aligned}\widehat{u}(\xi,t) &\geqslant d^2\,2^{4n+2}A^{2^{n+1}}\widehat{w}_{n+1}(\xi)\int_0^t \mathrm{e}^{-(t-s)2^{4(n+2)}d^2}\,\alpha_n(s)^2\mathrm{d}s\\ &= 2^{14}d^2\,2^{4n+2}A^{2^{n+1}}\widehat{w}_{n+1}(\xi)\int_0^t \mathrm{e}^{-(t-s)2^{4(n+2)}d^2}e^{-2^{n+5}d^2 s}\mathbf{1}_{\{t_n\leqslant s\leqslant t\}}\mathrm{d}s\\ &\geqslant 2^{14}d^2\,2^{4n+2}A^{2^{n+1}}\widehat{w}_{n+1}(\xi)\mathrm{e}^{-2^{n+5}d^2 t}\mathbf{1}_{\{t\geqslant t_n\}}\int_{t_n}^t e^{-(t-s)2^{4(n+2)}d^2}\,\mathrm{d}s\\ &= 2^8 A^{2^{n+1}}\widehat{w}_{n+1}(\xi)e^{-2^{n+5}d^2 t}\mathbf{1}_{\{t\geqslant t_n\}}(1-e^{-2^{4(n+2)}d^2(t-t_n)}).\end{aligned}$$

However, for all $t \geqslant t_{n+1}$, we have $1-e^{-2^{4(n+2)}d^2(t-t_n)} \geqslant 1/2$, since $t_{n+1}-t_n \geqslant ((\log(2))/(d^2))\,2^{-4(n+2)}$. Then, we deduce,

$$\begin{aligned}\widehat{u}(\xi,t) &\geqslant 2^7 A^{2^{n+1}}\widehat{w}_{n+1}(\xi)e^{-2^{n+5}d^2 t}\,\mathbf{1}_{\{t\geqslant t_{n+1}\}}\\ &= A^{2^{n+1}}\widehat{w}_{n+1}(\xi)\alpha_{n+1}(t).\end{aligned}$$

Then, we deduce that the proposition $\mathcal{P}(n+1)$ is true.

By induction, we thus deduce that for all $n \in \mathbb{N}$ ***and for all*** $\xi \in \mathfrak{F}_d$,

$$\widehat{u}(\xi,t) \geqslant A^{2^n}\widehat{w}_n(\xi)\alpha_n(t). \tag{5.9}$$

Thanks to (5.9), we have for all $j \in \mathbb{N}$, for all $\xi \in \mathfrak{F}_d$,

$$\widehat{\varphi}_j(\xi)\widehat{u}(\xi,T_d) \geqslant A^{2^j}\widehat{\varphi}_j(\xi)\widehat{w}_j(\xi)\alpha_j(T_d).$$

From § 3.1, we notice that $\widehat{\varphi}_j(\xi) \geqslant 1/2$ for all $\xi \in \{\zeta \in \mathfrak{F}_d : 2^j \leqslant \zeta \leqslant 2^{j+1}\}$ which is the support of $\widehat{w}_j$ and moreover, $\widehat{w}_j$ is non-negative, then we deduce that for all

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$\xi \in \mathfrak{F}_d$, $\widehat{\varphi}_j(\xi)\widehat{w}_j(\xi) \geqslant 1/2\widehat{w}_j(\xi)$. Therefore, we infer that for all $j \in \mathbb{N}$, for all $\xi \in \mathfrak{F}_d$,

$$\widehat{\varphi}_j(\xi)\widehat{u}(\xi, T_d) \geqslant \frac{1}{2} A^{2^j} \widehat{w}_j(\xi)\alpha_j(T_d) \geqslant 0. \tag{5.10}$$

Since for any $j \in \mathbb{Z}$, $\widehat{\varphi}_j\widehat{u} \geqslant 0$ (thanks to $\widehat{\varphi} \geqslant 0$ and $\widehat{u} \geqslant 0$), then thanks to (3.1) and (3.2), we deduce that for all $j \in \mathbb{Z}$, $\Delta_j u(0, T_d) = \|\Delta_j u(T_d)\|_{L^\infty(\mathbb{R}^d)}$. Moreover, we observe also that $\Delta_j u(0, T_d) = \|\widehat{\varphi}_j\widehat{u}(T_d)\|_{\mathscr{L}^1(\mathfrak{F}_d)}$. Then we infer,

$$\begin{aligned} \|u(T_d)\|_{\dot{B}^\sigma_{\infty,\infty}(\Omega_d)} &:= \sup_{j\in\mathbb{Z}} 2^{j\sigma}\|\Delta_j u(T_d)\|_{L^\infty(\mathbb{R}^d)} \\ &= \sup_{j\in\mathbb{Z}} 2^{j\sigma}\|\widehat{\varphi}_j\widehat{u}(T_d)\|_{\mathscr{L}^1(\mathfrak{F}_d)}. \end{aligned}$$

Thanks to (5.10), we infer,

$$\|u(T_d)\|_{\dot{B}^\sigma_{\infty,\infty}(\Omega_d)} \geqslant \frac{1}{2} \sup_{j\in\mathbb{N}} 2^{j\sigma} A^{2^j} \alpha_j(T_d)\|\widehat{w}_j\|_{\mathscr{L}^1(\mathfrak{F}_d)}.$$

However, for any $j \in \mathbb{N}$, $\|\widehat{w}_j\|_{L^1(\mathfrak{F}_d)} = 1$ and since for all $j \in \mathbb{N}$, $T_d > t_j$, we get $\alpha_j(T_d) = 2^7 e^{-((2^{j+4}\log(2))/(15))} = 2^7(e^{-((16\log(2))/(15))})^{2^j}$. Then, we deduce that,

$$\|u(T_d)\|_{\dot{B}^\sigma_{\infty,\infty}(\Omega_d)} \geqslant 2^6 \sup_{j\in\mathbb{N}} 2^{j\sigma}(Ae^{-((16\log(2))/(15))})^{2^j}.$$

Therefore, we deduce that if $A > e^{((16\log(2))/(15))} = 2^{16/15}$ then $\|u(T_d)\|_{\dot{B}^\sigma_{\infty,\infty}(\Omega_d)} = \infty$, which yields to a contradiction and thus we conclude the proof. □

Corollary 5.2. *Let $d \in \mathbb{N}^*$. Let $\Omega_d = \mathbb{R}^d$ or $\mathbb{T}^d$ for the periodic case. Let w be as in theorem 5.1 and let $A > 2^{16/15}$. Let X be a critical space and $T_d := \log(2^{((1)/(15d^2))})$. Then there is no mild solution $u \in C([0, T_d]; X)$ to the cKS equation (1.3) for the initial data $u_0 = 2^7 Aw$.*

Proof. For the proof, we borrow some arguments used in [**22**]. Suppose for a contradiction that there is a mild solution $u \in C([0, T_d]; X)$. We point out that since $u_0 \in \mathscr{S}(\Omega_d) \subset X$ thanks to property (i), thanks to proposition 4.5 we get the local existence in the time of a unique mild solution of (1.3) for the critical space X. To get a contradiction, we need to use theorem 5.1.

In order to use theorem 5.1, **we have to show that for all $s \in [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.**

The main ingredient to prove this property of the solution u is to propagate this property from the initial data u_0 to the local solution built from a fixed point argument on some interval $[0, t_0]$, $t_0 > 0$, identify this local solution to u thanks to the uniqueness and re-iterate this procedure with the new initial data $u(t_0)$ and continue until reaching the time T_d.

Therefore, we begin by showing that: for any $t \in [0, T_d[$, if $\widehat{u}(t)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$, then there exists some $\epsilon(t) > 0$ such that for all $s \in [t, t + \epsilon(t)] \cap [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.

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Thanks to proposition 4.5, we get that for every $t \in [0, T_d[$ there is a number $\epsilon(t) > 0$ depending on $u(t)$ such that there is an unique mild solution $v_t \in C([t, t+\epsilon(t)], X)$ to equation (1.3) with $v_t(t) = u(t)$, obtained as the fixed point of the iterated sequence $\{v^{(n)}(\cdot - t)\}_{n\in\mathbb{N}}$ defined by $v^{(0)} = 0$ and for all $n \in \mathbb{N}$ by

$$v^{(n+1)}(\sigma) = G_{u(t)}(v^n)(\sigma) \quad \text{for all } \sigma \in [0, \epsilon(t)], \tag{5.11}$$

where $G_{u(t)}(v^n)$ is defined from (4.2). Then, after taking the Fourier transform of (5.11), we obtain for all $\sigma \in [0, \epsilon(t)]$ and $\xi \in \mathbb{R}^d$,

$$\widehat{v^{(n+1)}}(\xi, \sigma) = e^{-\sigma|\xi|^4}\widehat{u}(\xi, t) - \int_0^\sigma \mathrm{e}^{-(\sigma-\tau)|\xi|^4}|\xi|^2(\widehat{\nabla v^{(n)}} \star \widehat{\nabla v^{(n)}})(\xi, \tau)\,\mathrm{d}\tau. \tag{5.12}$$

Thus, if $\widehat{u}(t)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$ then from equation (5.12) just above, by using an induction argument, we infer that for all $n \in \mathbb{N}$, for all $\sigma \in [0, \epsilon(t)]$, $\widehat{v^{(n)}}(\sigma)$ is also a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$ which implies that for all $s \in [t, t+\epsilon(t)]$, $\widehat{v}_t(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.

As a consequence of the uniqueness of the mild solutions, we get that for all $s \in [t, t+\epsilon(t)[\cap[0, T_d]$, $u(s) = v_t(s)$.

Therefore, we deduce that for every $t \in [0, T_d[$, if $\widehat{u}(t)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$ then for all $s \in [t, t+\epsilon(t)] \cap [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.

Let us show now that for all $s \in [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.

Since $u(0) = u_0 = 2^7 Aw$ and $\widehat{w}$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$, then we infer that for all $s \in [0, \epsilon(0)] \cap [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$, where we recall that $\epsilon(0) > 0$.

If $\epsilon(0) \geqslant T_d$, then we get that for all $s \in [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.

Otherwise $\epsilon(0) < T_d$. Then, we consider the set $\mathscr{P}$ defined by

$$\mathscr{P} := \{t_0 \in]0, T_d]; \quad \text{for all } s \in [0, t_0], \widehat{u}(s) \geqslant 0 \quad \text{and} \quad \widehat{u}(s) \quad \text{is supported in } {\mathfrak{F}_d}^+\}.$$

We observe that $\epsilon(0) \in \mathscr{P}$ and, therefore, $\mathscr{P}$ is non empty. We thus consider $t_\infty := \sup \mathscr{P}$ then we get $\epsilon(0) \leqslant t_\infty \leqslant T_d$ and we have also that *for all $s \in [0, t_\infty[$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.*

Let us show that $t_\infty = T_d$. For this, let us assume that $t_\infty < T_d$.

We show first that $\widehat{u}(t_\infty) \geqslant 0$ and $\widehat{u}(t_\infty)$ is supported in ${\mathfrak{F}_d}^+$.

Indeed, due to the definition of t_∞, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ of elements of $\mathscr{P}$ such that $t_\infty = \lim_{n\to\infty} t_n$. Since $u \in C([0, T_d]; X)$ and $\mathscr{S}(\Omega_d) \subset X$ thanks to property (i), we deduce that for all $\varphi \in \mathscr{S}(\Omega_d)$, $\langle u(t_\infty), \widehat{\varphi}\rangle = \lim_{n\to\infty}\langle u(t_n), \widehat{\varphi}\rangle$. However, for any $s \in [0, T_d]$, we have $\langle \widehat{u}(s), \varphi\rangle = \langle u(s), \widehat{\varphi}\rangle$ for all $\varphi \in \mathscr{S}(\Omega_d)$. Then, we infer that for all $\varphi \in \mathscr{S}(\Omega_d)$, $\langle \widehat{u}(t_\infty), \varphi\rangle = \lim_{n\to\infty}\langle \widehat{u}(t_n), \varphi\rangle$ which implies that for all $\varphi \in \mathscr{S}(\Omega_d)$, $\varphi \geqslant 0$, $\langle \widehat{u}(t_\infty), \varphi\rangle \geqslant 0$ due to the fact that for all $n \in \mathbb{N}$, $t_n \in \mathscr{P}$. *This means that $\widehat{u}(t_\infty) \geqslant 0$ and similarly we show that $\widehat{u}(t_\infty)$ is supported in ${\mathfrak{F}_d}^+$.*

As a consequence, we get *for all $s \in [t_\infty, t_\infty + \epsilon(t_\infty)] \cap [0, T_d]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$*, with $\epsilon(t_\infty) > 0$. Then by gathering the results in italics, with $t_\star := \min(T_d, t_\infty + \epsilon(t_\infty)) > t_\infty$, we thus obtain that for all $s \in [0, t_\star]$, $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$ and this contradicts the definition of t_∞. Then we deduce that $t_\infty = T_d$ which means that for all $s \in [0, T_d]$ $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$.

As a result, we finally get that for all $s \in [0, T_d]$ $\widehat{u}(s)$ is a non-negative real-valued function supported in ${\mathfrak{F}_d}^+$. Then by theorem 5.1, we get that u blows up in any Triebel-Lizorkin space or Besov space, at some time smaller than T_d and, in particular, u blows up in X at some time smaller than T_d, which leads to a contradiction and then we conclude the proof. □

Acknowledgements

The author would like to thank the anonymous referee for his helpful comments and Professor Dirk Blömker for his helpful discussions and suggestions.

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