London Mathematical Society ISSN 1461–1570

CROSS ORBITS

PETER ROWLEY

Abstract

This paper contains a variety of results about the action of Conway's largest simple group upon the crosses in the Leech lattice. These results are tailor-made for use in 'A Monster Graph, I' (*Proc. London Math. Soc.* (3) 90 (2005) 42–60), where a graph related to the Monster simple group is studied.

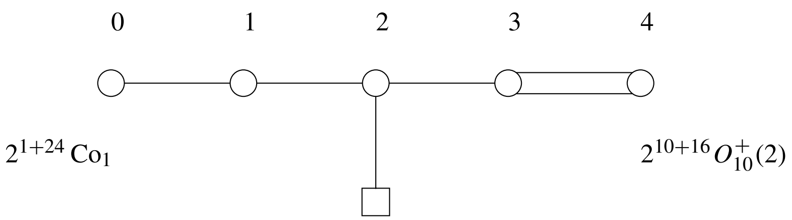
1. *Introduction*

In [8] a detailed investigation of a graph associated with the Monster simple group $\mathbb{M}$ is initiated. This graph has some

$$5,791,748,068,511,982,636,944,259,375$$

vertices and, indeed, the vertices may be identified with the class of 2B involutions in $\mathbb{M}$. For x in this conjugacy class, $C_{\mathbb{M}}(x) \cong 2_{+}^{1+24}\,\mathrm{Co}_1$. (Here and subsequently, with the exception of $\mathbb{M}$, we use ATLAS names and conventions for describing conjugacy classes and group structures.) From a different perspective, this graph is the point-line collinearity graph of Γ, the maximal 2-local geometry for $\mathbb{M}$ (see [6, 7]), and this is the view adopted in [8].

The geometry Γ has rank 5 and the following associated diagram.



Above the nodes, we have given the types of the objects, and below we give $\mathbb{M}_x$ for x of types 0 and 4 (for the remaining stabilizers, see [6]). The residue geometry for an object of type 0 (which we shall regard as the 'points' of Γ) is isomorphic to Σ, the maximal 2-local geometry for Conway's largest simple group, Co_1. This residue geometry is of paramount importance in [8], and it is the purpose of this paper to assemble a substantial arsenal of facts about Σ.

In Section 2 we set the stage for our later calculations. Section 2.1 introduces the Mathieu group M_{24} and the MOG, together with its various combinatorial accoutrements. These miraculous objects are ever-present in this paper. After a brief review of the Leech lattice in Section 2.2, we move on in the next subsection to discuss crosses and describe the maximal 2- local geometry for Co_1.

Received 21 February 2003, revised 18 June 2004; *published* 18 October 2004.
2000 Mathematics Subject Classification 20D08, 51E99.

Crosses occupy much of our attention as Σ_0, the points of Σ, may be identified with the set of crosses. Orbits of certain subgroups of M_{24} upon sextets, octads, 16-ads, triads and duums are described in Section 3 – we use this to bootstrap to the orbits of the subgroups $2^{11}M_{24}$, $2^{1+8}O_8^+(2)$, $2^{1+8}2^6A_8$, $2^{11}L_3(4)S_3$, and $2^{11}M_{12}2$ (of Co_1) on crosses. Such subgroups arise in our study of the Monster geometry in [8]. The analysis of the orbits of these subgroups on crosses occupies Section 4.

2. *Preliminaries*

2.1. *The MOG*

Throughout, Ω will denote a 24-element set that possesses a Steiner system $S(24, 8, 5)$ given by the MOG [4]. The blocks of this Steiner system will, as usual, be called *octads*. The MOG will be used frequently, to conjure up various octads and other related subsets of Ω. We name the elements of Ω as in [4]. So

$$\Omega = \begin{array}{|cc|cc|cc|} \hline \infty & 14 & 17 & 11 & 22 & 19 \\ 0 & 8 & 4 & 13 & 1 & 9 \\ 3 & 20 & 16 & 7 & 12 & 5 \\ 15 & 18 & 10 & 2 & 21 & 6 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \quad O_1 \quad & \quad O_2 \quad & \quad O_3 \quad \\ \hline \end{array},$$

with O_1, O_2 and O_3 the heavy bricks of the MOG (see [4]).

We may regard $P(\Omega)$, the power set of Ω, as a $GF(2)$-vector space, where vector addition is given by symmetric difference of sets. Then the subspace of $P(\Omega)$ spanned by all the octads is a twelve-dimensional subspace (see [1]); the subsets of Ω in this subspace are referred to as $\mathcal{C}$-*sets*. The $\mathcal{C}$-sets comprise $\emptyset$, Ω, the 759 octads, their complements (called 16-*ads*) and 2576 size-12 subsets (called *dodecads*).

A subset of Ω of size 2, 3 or 4 will be called, respectively, a *duad*, a *triad* or a *tetrad*. Two kinds of partitions of Ω – sextets and duums – will arise frequently later on. A *sextet* is a partition of Ω into six tetrads with the property that the union of any two of its tetrads yields an octad. It is a well-known fact that any tetrad of Ω is the tetrad of a unique sextet (see, for example, [4]). We shall specify a sextet by positioning the numbers $1, \ldots, 6$ (four of each) in the MOG. So the standard sextet, S_0, whose tetrads are the columns of the MOG, would be given by

$$\begin{array}{|cc|cc|cc|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}.$$

Of course,

$$\begin{array}{|cc|cc|cc|} \hline 1 & 3 & 2 & 5 & 4 & 6 \\ 1 & 3 & 2 & 5 & 4 & 6 \\ 1 & 3 & 2 & 5 & 4 & 6 \\ 1 & 3 & 2 & 5 & 4 & 6 \\ \hline \end{array}$$

also describes S_0. At certain points we find ourselves scrutinizing various properties of the 35 sextets in the MOG. So we need a systematic naming system for these sextets – S_{ij} will denote the sextet in the ith row and jth column of the MOG array. Thus S_{13} is just S_0.

A *duum* is a partition of Ω given by a dodecad and its complement (which is also a dodecad). Duums will be described by entering (twelve) 0's and (twelve) 1's in the MOG to indicate the two (complementary) dodecads. So, for example,

1	0	1	0	1	0
0	1	0	1	0	1
0	1	0	1	0	1
0	1	0	1	0	1

is a duum.

The sets of sextets, octads, 16-ads, triads and duums of Ω will be denoted, respectively, by $\mathcal{S}$, $\mathcal{O}$, $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{D}$. These sets will figure prominently in our investigations of Σ_0. Put $H = \mathrm{Stab}_{\mathrm{Sym}(\Omega)}(\mathcal{O})$. Then $H \cong M_{24}$, and we recall that $\mathrm{Stab}_H(X) \cong 2^6 : \hat{S}_6$, $2^4 : A_8$, $2^4 : A_8$, $L_3(4) : S_3$, $M_{12} : 2$ for X in, respectively, $\mathcal{S}$, $\mathcal{O}$, $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{D}$.

The combinatorial interplay between sextets, octads, 16-ads, triads and duums is very important in our understanding of particular orbits in Sections 3 and 4. Accordingly we set up notation to keep track of such matters. We shall view octads, 16-ads and triads as subsets of Ω (though they could equally be thought of as defining a partition of Ω).

Let Φ be a subset of Ω. Then

$$\mathcal{S}_{a_1^{i_1},\ldots,a_r^{i_r}}(\Phi) \qquad (\text{where } 0 < a_1 < a_2 \ldots)$$

consists of all sextets S such that (in some order): i_1 of the tetrads of S intersect Φ in a_1 elements, i_2 of the tetrads of S intersect Φ in a_2, and so on. If $i_j = 1$, then we write a_j instead of a_j^1. Clearly, $i_1 a_1 + \ldots + i_r a_r = |\Phi|$. For $k \in \mathbb{N} \cup \{0\}$, we put

$$\mathcal{O}_k(\Phi) = \{X \in \mathcal{O} : |X \cap \Phi| = k\};$$
$$\mathcal{A}_k(\Phi) = \{X \in \mathcal{A} : |X \cap \Phi| = k\};$$
$$\mathcal{T}_k(\Phi) = \{X \in \mathcal{T} : |X \cap \Phi| = k\}.$$

Also, $\mathcal{D}_{a_1,a_2}(\Phi)$ $(0 \leqslant a_1 \leqslant a_2)$ is the set of duums with the property that one of its dodecads intersects Φ in a_1 elements, and the complementary dodecad intersects Φ in a_2 elements (so $|\Phi| = a_1 + a_2$). In the case $a_1 = a_2$, we write $\mathcal{D}_{a_1^2}(\Phi)$ instead.

Now let $D \in \mathcal{D}$. So $D = \{\Phi_1, \Phi_2\}$, where Φ_1 and Φ_2 are dodecads such that $\Phi_1 \cup \Phi_2 = \Omega$. We define

$$\mathcal{O}_{a_1,a_2}(D) = \{X \in \mathcal{O} : |X \cap \Phi_i| = a_i,\ i = 1, 2\};$$
$$\mathcal{A}_{a_1,a_2}(D) = \{X \in \mathcal{A} : |X \cap \Phi_i| = a_i,\ i = 1, 2\};$$
$$\mathcal{T}_{a_1,a_2}(D) = \{X \in \mathcal{T} : |X \cap \Phi_i| = a_i,\ i = 1, 2\}.$$

In $\mathcal{O}_{a_1,a_2}(D)$, $\mathcal{A}_{a_1,a_2}(D)$ and $\mathcal{T}_{a_1,a_2}(D)$, we arrange the subscripts so that $a_1 \leqslant a_2$, and we will replace a_1, a_2 by a_1^2 if $a_1 = a_2$. A sextet is in one of the following sets: $\mathcal{S}_{(2.2)^4,(4.0)^2}(D)$, $\mathcal{S}_{(3.1)^6}(D)$ or $\mathcal{S}_{(2.2)^6}(D)$. The first set consists of all sextets S that have four tetrads intersecting both Φ_1 and Φ_2 in two elements, and each of the remaining tetrads is contained in either Φ_1 or Φ_2. The set $\mathcal{S}_{(3.1)^6}(D)$ contains all sextets S such that each tetrad of S intersects Φ_1 in one element and Φ_2 in three elements (or the other way round); $\mathcal{S}_{(2.2)^6}(D)$ is defined similarly. Excluding D, a duum is in one of the following sets: $\mathcal{D}_{(4.8)^2}(D)$ or $\mathcal{D}_{(6.6)^2}(D)$. The former set contains all duums $D' = \{\Phi'_1, \Phi'_2\}$ such that either $|\Phi_1 \cap \Phi'_1| = 4$ or $|\Phi_2 \cap \Phi'_1| = 4$ (so either $|\Phi_1 \cap \Phi'_2| = 8$ or $|\Phi_2 \cap \Phi'_2| = 8$); $\mathcal{D}_{(6.6)^2}(D)$ is defined similarly.

Finally, we deal with possible relationships between sextets – a sextet different from the sextet S is in one of $\mathcal{S}_{(3.1)^2,(1^4)^4}(S)$, $\mathcal{S}_{(2^2)^6}(S)$ and $\mathcal{S}_{(2.1^2)^4,(1^4)^2}(S)$. So, for example,

$\mathcal{S}_{(2^2)^6}(S)$ is the set of sextets S', all of whose tetrads meet the tetrads of S in either zero or two elements.

In some instances, we shall be interested in certain subsets of the above sets. For $\Phi \in \mathcal{T}$, we define

$$\mathcal{T}_0^+(\Phi) = \{X \in \mathcal{T}_0(\Phi) : X \cup \Phi \text{ is contained in an octad}\}$$

and

$$\mathcal{T}_0^-(\Phi) = \{X \in \mathcal{T}_0(\Phi) : X \cup \Phi \text{ is not contained in an octad}\}.$$

There will be times when we encounter permutations of M_{24} of cycle type $1^8 2^8$. Such involutions t fix (pointwise) an octad O, and define a 2^8 partition of $\Omega \setminus O$, say $\Phi_1, \dots, \Phi_8$. In this situation, we set

$$\mathcal{S}_{4^2;n}(O;t) = \{X \in \mathcal{S}_{4^2}(O) : \text{exactly } n \text{ of the } \Phi_i \text{ are contained in some tetrad of } X\}$$

and

$$\mathcal{S}_{(2^2)^4;n}(O;t) = \{X \in \mathcal{S}_{(2^2)^4}(O) : \text{exactly } n \text{ of the } \Phi_i \text{ are contained in some tetrad of } X\}.$$

Sometimes we employ ad-hoc notation for various subsets of the above sets (see particularly Section 3.2). For example, $\mathcal{O}_4^{(1)}(O_1)$ and $\mathcal{O}_4^{(2)}(O_1)$ indicate certain subsets of $\mathcal{O}_4(O_1)$.

2.2. *The Leech lattice*

Let $\mathbb{R}^{24}$ be spanned by the orthonormal basis $\{v_i : i \in \Omega\}$, where Ω is as given in Section 2.1. For $S \subseteq \Omega$, we define $v_S = \sum_{i \in S} v_i$ and for $x = (x_i)$, $y = (y_i) \in \mathbb{R}^{24}$, we use $x \cdot y$ to denote the inner product defined by $x \cdot y = \sum_{i=1}^{24} x_i y_i$. We shall use Λ to denote the Leech lattice. So, following [1], Λ is the lattice spanned by $v_\Omega - 4v_\infty$ and the vectors $2v_X$ $(X \in \mathcal{O})$. We use, frequently without mention, the following result to check whether vectors are in Λ.

THEOREM 2.1. *The vector* $x = (x_i) \in \mathbb{R}^{24}$ *is in* Λ *if and only if*

(i) *the co-ordinates* x_i *are all congruent modulo* 2*, to* d*, say;*

(ii) *the set of* i *for which* x_i *takes any given value modulo* 4 *is a* $\mathfrak{C}$*-set; and*

(iii) $\sum_{i=1}^{24} x_i \equiv 4d \pmod 8$.

Moreover, for $x, y \in \Lambda$*,* $x \cdot y$ *is a multiple of* 8 *and* $x \cdot x$ *is a multiple of* 16.

Proof. See [1, Theorem 2]. □

For $n \in \mathbb{N}$, Λ_n is the set $\{x \in \Lambda : x \cdot x = 16n\}$ and a vector in Λ_n is called a *type-n vector*. A vector in Λ will usually be described by displaying its ith co-ordinate $(i \in \Omega)$ in the ith position of the MOG – blank entries being read as 0. Sometimes, when v stands for a vector in Λ, for $i \in \Omega$ we use $\tilde{v}_i$ to denote the ith co-ordinate of v. This should not be confused with the basis vectors v_i. We also note that u_i^* $(i = 1, \dots, 24)$ refers to one of the vectors listed in Appendix A.

A *cross*, in the terminology of Curtis [3], is a set of forty-eight type-4 vectors in Λ with the property that for any two vectors x and y, either $x = -y$ or x and y are orthogonal (that is, $x \cdot y = 0$). Such sets of vectors are the principal object of study in this paper, and we shall think of them as consisting of twenty-four orthogonal type-4 vectors (so we do all our calculations up to sign). Let x be a type-4 vector of Λ. It is an important fact that x is contained in a unique cross, which henceforth we denote by $x^\times$. We shall call $\ell_0 = (8v_\infty)^\times$ the *standard cross*; note that $\ell_0 = (8v_i)^\times$ for all $i \in \Omega$. In some of our deliberations, we shall find it necessary to determine some (or all) of the other type-4 vectors in $x^\times$. These vectors may be calculated as follows. First write x as the sum of two type-2 vectors in Λ, say y and z (there are twenty-three different ways of doing this). Then the twenty-four vectors of $x^\times$ consist of x together with the twenty-three vectors $y - z$.

We shall be investigating the orbits of crosses under various subgroups of Co_1, Conway's largest simple group. This group is defined to be $\cdot 0/\pm 1$, where $\cdot 0$ is the group of all orthogonal transformations of $\mathbb{R}^{24}$ fixing the zero vector and leaving Λ invariant (note that Co_1 has an induced action on crosses, and indeed is transitive on the set of crosses). A permutation π of Ω may be extended to an orthogonal transformation of $\mathbb{R}^{24}$ by defining $v_i\pi = v_{i\pi}$ $(i \in \Omega)$. In addition, starting from a subset S of Ω, we also get an orthogonal transformation ε_S of $\mathbb{R}^{24}$ by defining

$$v_i\varepsilon_S = \begin{cases} v_i, & \text{if } i \notin S; \\ -v_i, & \text{if } i \in S. \end{cases}$$

By [1, Theorem 3], $\{\pi\varepsilon_C : \pi \in M_{24},\ C \in \mathcal{C}\}$ is a subgroup of $\cdot 0$ isomorphic to $2^{12} : M_{24}$ that projects to a subgroup of Co_1 isomorphic to $2^{11} : M_{24}$. Indeed, this group is the stabilizer in Co_1 of the standard cross ℓ_0.

Another source of elements in $\cdot 0$ is obtained by starting from a tetrad T of Ω. Letting S be the unique sextet of which T is a tetrad, we define

$$\zeta_T : \mathbb{R}^{24} \longrightarrow \mathbb{R}^{24}$$

as follows. Let $v \in \mathbb{R}^{24}$. For each tetrad T^* of S, we sum the coefficients of v in T^* and then subtract half this sum from each of the coefficients of v in T^*; finally, changing the sign of all the coefficients in T gives $v\zeta_T$. By [1], we see that $\zeta_T \in \cdot 0$ and, clearly, ζ_T is an involution. Elements of this kind will be employed in Section 4.

2.3. Σ_0 *and crosses*

As mentioned in Section 1, Σ_0 (the points of the Co_1 maximal 2-local geometry) may (and will) be identified with the set of crosses in the Leech lattice Λ. We shall describe the set of crosses from the point of view of ℓ_0. Before doing this, we need to introduce the idea of an *odd* or *even cross* defined on a sextet, as well as some further notation.

Let $S \in \mathcal{S}$. For each tetrad $T = \{i_1, i_2, i_3, i_4\}$ of S, we may define vectors of the form

$$\pm 4v_{i_1} \pm 4v_{i_2} \pm 4v_{i_3} \pm 4v_{i_4}.$$

Sometimes we shall describe a vector of this type by $(\pm 4)_T$. Now, if we require (for each tetrad of S) that the number of minuses be even, then we obtain (up to sign) twenty-four type-4 vectors, which is easily checked to be a cross. Such a cross is (usually) denoted by S_{even} and called the *even cross* (with underlying sextet S). If, on the other hand, for each tetrad of S we have an odd number of minuses, we denote the resulting cross by S_{odd} and refer to it as the *odd cross* (with underlying sextet S).

Starting from an octad, a 16-ad, a triad or a duum, we may obtain (several) crosses. The general 'shape' of each of these is displayed below. In the following descriptions, the parities of the entries must be chosen so as to ensure that the vectors are in Λ.

$$\text{Octad:}\quad \left(\begin{array}{cccc|c} 6 & 2 & \ldots & 2 & \\ 2 & 6 & \ldots & 2 & 0 \\ 2 & 2 & \ldots & 6 & \\ \hline & & & & \\ & & 0 & & 2 \\ & & & & \end{array}\right);$$

$$\text{16-ad:}\quad \left(\begin{array}{c|cccccccc} & 4 & 4 & & & & & & \\ & -4 & -4 & & & & & & \\ & & & 4 & 4 & & & & \\ 2 & & & -4 & -4 & & & & \\ & & & & & \ddots & & & \\ & & & & & & & 4 & 4 \\ & & & & & & & -4 & -4 \\ \hline & & & & & & & & \\ 0 & & & & & 2 & & & \\ & & & & & & & & \end{array}\right);$$

$$\text{Triad:}\quad \left(\begin{array}{ccc|c} 5 & 3 & 3 & \\ 3 & 5 & 3 & 1 \\ 3 & 3 & 5 & \\ \hline & & & \text{five } 3\text{'s in each row} \\ & 1 & & \text{to make an octad with the trio;} \\ & & & \text{others } 1 \end{array}\right);$$

$$\text{Duum:}\quad \left(\begin{array}{cccc|cccc} 4 & & & & & & & \\ & 4 & & & & 2 & & \\ & & \ddots & & & & & \\ & & & 4 & & & & \\ \hline & & & & 4 & & & \\ & 2 & & & & 4 & & \\ & & & & & & \ddots & \\ & & & & & & & 4 \end{array}\right).$$

Each octad gives rise to 64 crosses, each 16-ad gives rise to $15.2^6.2$ crosses, and each triad and duum yields 2^{11} crosses. We shall speak of the *underlying* octad, 16-ad, triad or duum of a cross to mean, respectively, the octad, 16-ad, triad or duum used to define the cross.

For $\Xi \subseteq \mathcal{S}$,

$$\Xi^{\times} = \{S_{\mathrm{odd}}, S_{\mathrm{even}} : S \in \Xi\}$$

and we extend this notation to subsets of $\mathcal{O}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{D}$. So, for example, if $\Xi \subseteq \mathcal{O}$, then $\Xi^{\times}$ is the set of crosses whose underlying octad lies in Ξ.

For compatibility with [8], we use the $\alpha_i^j(\ )$ notation.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

DEFINITION 2.2. $\Sigma_0 = \{\ell_0\} \cup \alpha_1(\ell_0) \cup \alpha_2^1(\ell_0) \cup \alpha_2^2(\ell_0) \cup \alpha_3^1(\ell_0) \cup \alpha_3^2(\ell_0)$, where

$$\begin{aligned} \alpha_1(\ell_0) &= \mathcal{S}^{\times}; & \alpha_2^1(\ell_0) &= \mathcal{O}^{\times}; \\ \alpha_2^2(\ell_0) &= \mathcal{A}^{\times}; & \alpha_3^1(\ell_0) &= \mathcal{T}^{\times}; \\ \alpha_3^2(\ell_0) &= \mathcal{D}^{\times}. \end{aligned}$$

We remark that the lower subscript in $\alpha_i^j(\ell_0)$ gives the distance the points in $\alpha_i^j(\ell_0)$ are away from ℓ_0 in the point-line collinearity graph of Σ. We shall use $d(\ ,\ )$ to denote the distance function on this graph.

Set $G = \mathrm{Co}_1$. The next two results reveal certain aspects of the structure of Σ_0 (see [9]; an atlas of collinearity graphs of sporadic 2-local geometries is also currently in preparation).

THEOREM 2.3. (i) *The G_{ℓ_0}-orbits on Σ_0 are:*

$$\{\ell_0\}, \quad \alpha_1(\ell_0), \quad \alpha_2^1(\ell_0), \quad \alpha_2^2(\ell_0), \quad \alpha_3^1(\ell_0) \quad \textit{and} \quad \alpha_3^2(\ell_0).$$

(ii) *The following equalities hold.*

$$\begin{aligned} |\alpha_1(\ell_0)| &= 1771.2 = 2.7.11.23; \\ |\alpha_2^1(\ell_0)| &= 759.2^6 = 2^6.3.11.23; \\ |\alpha_2^2(\ell_0)| &= 759.15.2^6.2 = 2^7.3^2.5.11.23; \\ |\alpha_3^1(\ell_0)| &= \binom{24}{3}.2^{11} = 2^{14}.11.23; \\ |\alpha_3^2(\ell_0)| &= 1288.2^{11} = 2^{14}.7.23. \end{aligned}$$

THEOREM 2.4. *The stabilizers and point line distributions for the G_{ℓ_0}-orbits are given below (where, for $m \in \Sigma_0$, $G_{\ell_0 m}^{*m}$ is the induced action of $G_{\ell_0 m}$ upon the residue of m).*

(i) *For $m \in \alpha_1(\ell_0)$, $G_{\ell_0 m} \cong 2^{10}2^6\hat{S}_6$ and $G_{\ell_0 m}^{*m} \cong 2^6\hat{S}_6 = \mathrm{Stab}_{G_{\ell_0 m}^{*m}}(S_0)$, they are as follows.*

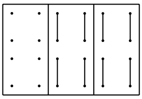
Orbit	Representative	Size	Point Distribution
$\{S_0\}$	S_0	1	$\{\ell_0\}\,2\alpha_1$
$\mathcal{S}_{(2^2)^6}(S_0)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}$	90	$3\alpha_1$
$\mathcal{S}_{(3.1)^2,(1^4)^4}(S_0)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 2 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 2 & 5 & 5 & 5 & 5 \\ 1 & 2 & 6 & 6 & 6 & 6 \\ \hline \end{array}$	240	$\alpha_1 2\alpha_2^1$
$\mathcal{S}_{(2.1^2)^4,(1^4)^2}(S_0)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 3 & 1 & 1 & 2 & 2 & 2 \\ 1 & 4 & 5 & 6 & 3 & 4 \\ 1 & 5 & 4 & 6 & 5 & 3 \\ 2 & 6 & 6 & 3 & 5 & 4 \\ \hline \end{array}$	1440	$\alpha_1 2\alpha_2^2$

(ii) *For $m \in \alpha_2^1(\ell_0)$, $G_{\ell_0 m} \cong 2^5 2^4 A_8$ and $G_{\ell_0 m}^{*m} \cong 2^4 A_8 = \mathrm{Stab}_{G_{\ell_0 m}^{*m}}(O_1)$, they are as follows.*

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Orbit	Representative	Size	Point Distribution
$\mathcal{S}_{4^2}(O_1)$	S_0	35	$\alpha_1 2\alpha_2^1$
$\mathcal{S}_{2^4}(O_1)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 1 & 1 & 1 & 1 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 5 \\ 3 & 3 & 3 & 3 & 6 & 5 \\ 4 & 4 & 4 & 4 & 6 & 5 \\ \hline \end{array}$	840	$\alpha_2^1 2\alpha_2^2$
$\mathcal{S}_{1^5,3}(O_1)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 3 & 1 & 1 & 6 & 5 & 3 \\ 4 & 1 & 2 & 5 & 4 & 5 \\ 2 & 6 & 3 & 2 & 4 & 3 \\ 1 & 5 & 4 & 2 & 6 & 6 \\ \hline \end{array}$	896	$\alpha_2^1 2\alpha_3^1$

(iii) *For* $x \in \alpha_2^2(\ell_0)$, $G_{\ell_0 m} \cong 2^4 2^4 2^3 L_3(2)$ *and* $G_{\ell_0 m}^{*m} \cong 2^4 2^3 L_3(2) = C_{G_{\ell_0 m}^{*m}}(t) \leqslant$ $\mathrm{Stab}_{G_{\ell_0 m}^{*m}}(O_1)$*, where* t *is the following involution, they are as follows.*



Orbit	Representative	Size	Point Distribution
$\mathcal{S}_{4^2;8}(O_1;t)$	S_0	7	$\alpha_1 2\alpha_2^2$
$\mathcal{S}_{4^2;0}(O_1;t)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 2 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 2 & 5 & 5 & 5 & 5 \\ 1 & 2 & 6 & 6 & 6 & 6 \\ \hline \end{array}$	28	$3\alpha_2^2$
$\mathcal{S}_{2^4;4}(O_1;t)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 2 & 1 & 1 & 5 & 2 & 4 \\ 6 & 3 & 1 & 4 & 2 & 5 \\ 3 & 2 & 5 & 3 & 5 & 6 \\ 1 & 6 & 4 & 3 & 4 & 6 \\ \hline \end{array}$	56	$\alpha_2^1 2\alpha_2^2$
$\mathcal{S}_{2^4;2}(O_1;t)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 5 & 2 & 4 & 1 & 3 & 3 \\ 5 & 2 & 1 & 4 & 6 & 6 \\ 6 & 3 & 5 & 5 & 1 & 4 \\ 3 & 6 & 2 & 2 & 1 & 4 \\ \hline \end{array}$	336	$3\alpha_2^2$
$\mathcal{S}_{2^4;0}(O_1;t)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 6 & 5 & 5 & 2 & 6 & 4 \\ 4 & 5 & 1 & 3 & 1 & 1 \\ 6 & 2 & 3 & 3 & 3 & 1 \\ 2 & 4 & 6 & 4 & 5 & 2 \\ \hline \end{array}$	448	$\alpha_2^2 2\alpha_3^2$
$\mathcal{S}_{1^5,3}(O_1;t)$	$\begin{array}{\|cc\|cc\|cc\|} \hline 3 & 1 & 1 & 6 & 5 & 3 \\ 4 & 1 & 2 & 5 & 4 & 5 \\ 2 & 6 & 3 & 2 & 4 & 3 \\ 1 & 5 & 4 & 2 & 6 & 6 \\ \hline \end{array}$	896	$\alpha_2^2 2\alpha_3^1$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

(iv) *For* $m \in \alpha_3^1(\ell_0)$, $G_{\ell_0 m} \cong L_3(4)S_3$ *and* $G^{*m}_{\ell_0 m} \cong L_3(4)S_3 = \mathrm{Stab}_{G^{*m}_{\ell_0 m}}(T)$ *where* T *is the triad*

$$\begin{array}{|c|c|c|}\hline \times & & \\ \times & & \\ \times & & \\ \hline \end{array},$$

they are as follows.

<table>
<tr><th>Orbit</th><th>Representative</th><th>Size</th><th>Point Distribution</th></tr>
<tr><td>$\mathcal{S}_3(T)$</td><td>S_0</td><td>21</td><td>$\alpha_2^1 2\alpha_3^1$</td></tr>
<tr><td>$\mathcal{S}_{1,2}(T)$</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}$</td><td>630</td><td>$\alpha_2^2 2\alpha_3^1$</td></tr>
<tr><td>$\mathcal{S}_{1^3}(T)$</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 1 & 1 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 & 3 & 2 \\ 5 & 5 & 5 & 5 & 3 & 2 \\ 6 & 6 & 6 & 6 & 3 & 2 \\ \hline \end{array}$</td><td>1120</td><td>$\alpha_3^2 2\alpha_3^1$</td></tr>
</table>

(v) *For* $m \in \alpha_3^2(\ell_0)$, $G_{\ell_0 m} \cong M_{12}2$ *and* $G^{*m}_{\ell_0 m} \cong M_{12}2 = \mathrm{Stab}_{G^{*m}_{\ell_0 m}}(D)$, *where* D *is the duum*

$$\begin{array}{|cc|cc|cc|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array},$$

they are as follows.

<table>
<tr><th>Orbit</th><th>Representative</th><th>Size</th><th>Point Distribution</th></tr>
<tr><td>$\mathcal{S}_{(2.2)^6}(D)$</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}$</td><td>396</td><td>$3\alpha_3^2$</td></tr>
<tr><td>$\mathcal{S}_{(2.2)^4,(4.0)^2}(D)$</td><td>$\begin{array}{|cc|cc|cc|}\hline 3 & 3 & 3 & 3 & 1 & 2 \\ 4 & 4 & 4 & 4 & 2 & 1 \\ 5 & 5 & 5 & 5 & 2 & 1 \\ 6 & 6 & 6 & 6 & 2 & 1 \\ \hline \end{array}$</td><td>495</td><td>$\alpha_2^2 2\alpha_3^2$</td></tr>
<tr><td>$\mathcal{S}_{(3.1)^6}(D)$</td><td>S_0</td><td>880</td><td>$\alpha_3^2 2\alpha_3^1$</td></tr>
</table>

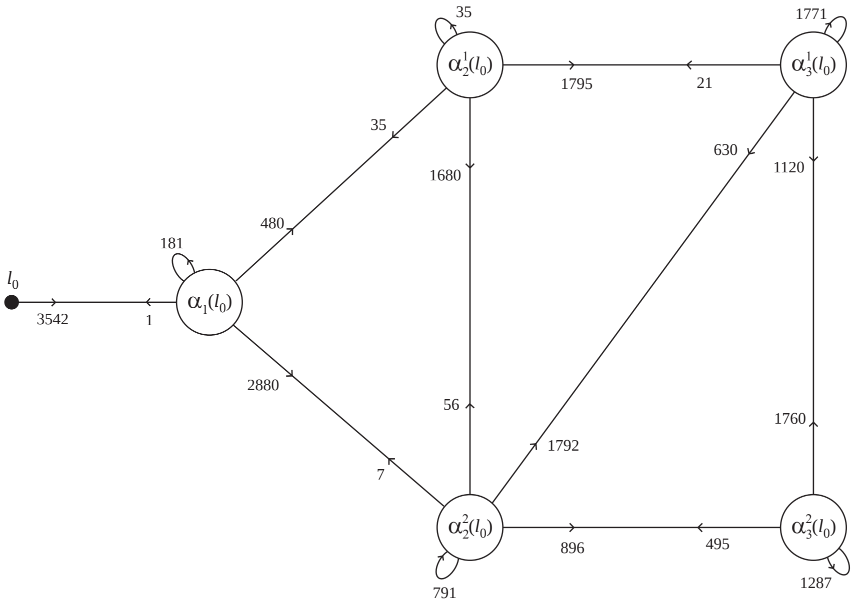


Figure 1: The collapsed adjacencies for the point-line collinearity graph of Σ

The collapsed adjacencies for the point-line collinearity graph of Σ are given in Figure 1. This graph also appears in [**5**, p. 164].

We pause to discuss the positioning of the 4, ± 4 entries of vectors that appear in crosses in $\alpha_2^2(\ell_0)$: that is, crosses which have an underlying 16-ad, say X. So $\Omega \setminus X \in \mathcal{O}$. There are sixteen of these vectors and they 'pair-up' in respect of the 4, ± 4 positions. Now choose any two positions in X, say i, j. Then, in M_{24}, there is a unique involution that fixes $\Omega \setminus X$ (pointwise) and interchanges i and j (see, for example, [**4**]), so giving rise to a 2^8 partition of X. This tells us where the 4, ± 4 entries of the eight 'pairs' of vectors in this cross must be located.

For $X \in \mathcal{A}$, we use $\mathcal{J}(X)$ to denote the fifteen involutions of M_{24} associated with X (that is, the involutions fixing $\Omega \setminus X$ point-wise). In Section 3.2 we shall examine subgroups L of M_{24} acting upon the set

$$\mathcal{A} \times \mathcal{J} = \{(X, t) : X \in \mathcal{A},\ t \in \mathcal{J}(X)\}.$$

Suppose that Ξ is an L-orbit of $\mathcal{A}$, and let $X \in \Xi$. We use $\Xi^{[n]}$ to denote an L-orbit on $\mathcal{A} \times \mathcal{J}$ for which L_X has an orbit of size n on $\mathcal{J}(X)$.

Suppose that Π is a subset of Σ_0. Then $[i_1, i_2, i_3, i_4, i_5]_\Pi$ denotes the set of all crosses ℓ in Σ_0 for which (exactly) i_1 (respectively, i_2, i_3, i_4, i_5) crosses in Π are in $\alpha_1(\ell)$ (respectively, $\alpha_2^1(\ell)$, $\alpha_2^2(\ell)$, $\alpha_3^1(\ell)$, $\alpha_3^2(\ell)$). Note that $i_1 + i_2 + i_3 + i_4 + i_5 = |\Pi|$. As we shall see, many cross orbits appear in this guise for various subsets of Σ_0.

The next result will be used repeatedly in Section 4.

LEMMA 2.5. *Let $\ell, m \in \Sigma_0$ and $w \in \Lambda_4$ be such that $m = w^{\times}$. The table below gives the inner products between v and w for each of the twenty-four vectors v such that $\ell = v^{\times}$. (So, for example, this applies when $\ell \in \alpha_2^1(m)$ $v \cdot w$ is equal to 0 for sixteen of the vectors v, ± 16 for seven of the vectors v and ± 48 for one vector,* **or** *$v \cdot w$ is equal to 0 for eight of the vectors v and ± 16 for sixteen vectors.)*

inner product	0	± 8	± 16	± 24	± 32	± 40	± 48
$\ell \in \alpha_1(m)$	20				4		
$\ell \in \alpha_2^1(m)$	16		7				1
	8		16				
$\ell \in \alpha_2^2(m)$	14		8		2		
	8		16				
$\ell \in \alpha_3^1(m)$		21		2		1	
		19		5			
$\ell \in \alpha_3^2(m)$	11		12		1		

Proof. Since G is transitive on Σ_0, we may take $m = 8v_{\infty}^{\times}$. Then Lemma 2.5 follows from the description of crosses given earlier in this subsection. □

REMARK. Using the data in Lemma 2.5, it can sometimes be troublesome to distinguish between the sets $\alpha_2^1(m)$ and $\alpha_2^2(m)$.

COROLLARY 2.6. *Suppose that $\ell, m \in \Sigma_0$, and let $w \in \Lambda_4$ be such that $m = w^{\times}$.*

(i) *If $v \cdot w = 0$ for at least seventeen vectors v with $v^{\times} = \ell$, then $\ell \in \alpha_1(m)$.*
(ii) *If $v \cdot w = \pm 32$ for at least three vectors v with $v^{\times} = \ell$, then $\ell \in \alpha_1(m)$.*
(iii) *If $v \cdot w \in \{\pm 8, \pm 24, \pm 40\}$ for at least one v with $v^{\times} = \ell$, then $\ell \in \alpha_3^1(m)$.*
(iv) *If $v \cdot w = \pm 48$ for a v such that $v^{\times} = \ell$, then $\ell \in \alpha_2^1(m)$.*

Proof. This is an easy consequence of Lemma 2.5. □

LEMMA 2.7. *Suppose that $\ell, m \in \alpha_1(\ell_0)$ and $\ell \neq m$, and let S_ℓ and S_m be, respectively, the underlying sextets for ℓ and m.*

(i) *If $S_m \in \{S_\ell\} \cup \mathcal{S}_{(2^2)^6}(S_\ell)$, then $m \in \alpha_1(\ell)$.*
(ii) *If $S_m \in \mathcal{S}_{(3.1)^2,(1^4)^4}(S_\ell)$, then $m \in \alpha_2^1(\ell)$.*
(iii) *If $S_m \in \mathcal{S}_{(2.1^2)^4,(1^4)^2}(S_\ell)$, then $m \in \alpha_2^2(\ell)$.*

Proof. Since G_{ℓ_0} is transitive on $\alpha_1(\ell_0)$, without loss of generality we may suppose that $S_\ell = S_0$, the standard sextet, and that ℓ is the even cross on S_0. Let

$$w_1 = 4v_{\infty} + 4v_0 + 4v_3 + 4v_{15} \qquad \text{and} \qquad w_2 = 4v_{14} + 4v_8 + 4v_{20} + 4v_{18}.$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Then $\ell = w_1^\times = w_2^\times$. Now $G_{\ell_0\ell} \cong 2^{11}2^6\hat{S}_6$ has four orbits on the sextets as follows:

$$\{S_\ell\}, \qquad \mathcal{S}_{(2^2)^6}(S_\ell), \qquad \mathcal{S}_{(3.1)^2,(1^4)^4}(S_\ell) \qquad \text{and} \qquad \mathcal{S}_{(2.1^2)^4,(1^4)^2}(S_\ell).$$

So we only need to check a representative from each of these orbits for S_m.

If $S_m = S_\ell$ (and so $m = (S_\ell)_{\mathrm{odd}}$), it is easy to see that $u \cdot w_1 = 0$ for twenty of the u, and $u \cdot w_1 = \pm 32$ for four of the u for which $u^\times = m$. When

$$S_m = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array},$$

we obtain the same conclusion (for an odd or even cross) and hence, by Lemma 2.5, part (i) holds.

For part (ii), we take

$$S_m = \begin{array}{|cc|cc|cc|} \hline 2 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 2 & 5 & 5 & 5 & 5 \\ 1 & 2 & 6 & 6 & 6 & 6 \\ \hline \end{array}.$$

Letting

$$w_3 = 4v_{14} + 4v_0 + 4v_3 + 4v_{15}$$

and

$$w_4 = -4v_{14} + 4v_0 + 4v_3 + 4v_{15},$$

we clearly have $u_1 \cdot w_3 = u_1 \cdot w_4 = 48$, whence, by Lemma 2.5, the odd and even crosses on S_m are both in $\alpha_2^1(\ell)$.

Finally, for part (iii) we choose

$$S_m = \begin{array}{|cc|cc|cc|} \hline 3 & 1 & 1 & 6 & 5 & 3 \\ 4 & 1 & 2 & 5 & 4 & 5 \\ 2 & 6 & 3 & 2 & 4 & 3 \\ 1 & 5 & 4 & 2 & 6 & 6 \\ \hline \end{array}.$$

Let $w_5 = 4v_{14} + 4v_8 + 4v_{15} + 4v_{17}$. Then $w_1 \cdot w_5 = 16$ and $w_2 \cdot w_5 = 32$. Since $d(\ell, m) \leqslant 2$, we see by consulting Lemma 2.5 that the even cross on S_m is in $\alpha_2^2(\ell)$. A similar argument shows that the odd cross on S_m is also in $\alpha_2^2(\ell)$. □

3. *Orbits of certain subgroups of* M_{24}

Because of the concrete description of Σ_0 given in Section 2.3, it is not surprising that we need information about orbits of certain subgroups of M_{24} on the sets $\mathcal{S}$, $\mathcal{O}$, $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{D}$. This information is gathered in this section – recall that $H = \mathrm{Stab}_{\mathrm{Sym}(\Omega)}(\mathcal{O})(\cong M_{24})$.

In Tables 1, 3, 4 and 6 of Assertions 3.1 to 3.4, the fourth column for each L-orbit Ξ of $\mathcal{A}$ indicates the sizes of the L_X-orbits on $\mathcal{J}(X)$ where $X \in \Xi$.

3.1. *Let* $L = \mathrm{Stab}_H(O_1)$*; so* $L \cong 2^4 : A_8$. *See Table* 1.

Table 1: $L \cong 2^4 : A_8$-orbits (see Assertion 3.1).

L-orbit	Size	Representative	
$\mathcal{S}_{4^2}(O_1)$	35	S_0	
$\mathcal{S}_{1^5,3}(O_1)$	896	$\begin{array}{\|cc\|cc\|cc\|} \hline 3 & 1 & 1 & 6 & 5 & 3 \\ 4 & 1 & 2 & 5 & 4 & 5 \\ 2 & 6 & 3 & 2 & 4 & 3 \\ 1 & 5 & 4 & 2 & 6 & 6 \\ \hline \end{array}$	
$\mathcal{S}_{2^4}(O_1)$	840	$\begin{array}{\|cc\|cc\|cc\|} \hline 1 & 1 & 1 & 1 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 5 \\ 3 & 3 & 3 & 3 & 6 & 5 \\ 4 & 4 & 4 & 4 & 6 & 5 \\ \hline \end{array}$	
$\mathcal{O}_8(O_1)$	1	O_1	
$\mathcal{O}_4(O_1)$	280	$\begin{array}{\|cc\|cc\|cc\|} \hline \times & & \times & & & \\ \times & & \times & & & \\ \times & & \times & & & \\ \times & & \times & & & \\ \hline \end{array}$	
$\mathcal{O}_2(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|} \hline \times & \times & \times & \times & \times & \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ \hline \end{array}$	
$\mathcal{O}_0(O_1)$	30	O_3	
$\mathcal{A}_0(O_1)$	1	$\Omega \setminus O_1$	15
$\mathcal{A}_4(O_1)$	280	$\begin{array}{\|cc\|cc\|cc\|} \hline & \times & & \times & \times & \times \\ & \times & & \times & \times & \times \\ & \times & & \times & \times & \times \\ & \times & & \times & \times & \times \\ \hline \end{array}$	15
$\mathcal{A}_6(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|} \hline & & & & & \times \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \hline \end{array}$	15
$\mathcal{A}_8(O_1)$	30	$\Omega \setminus O_3$	
$\mathcal{T}_3(O_1)$	56	$\{\infty, 14, 0\}$	
$\mathcal{T}_2(O_1)$	448	$\{\infty, 14, 17\}$	
$\mathcal{T}_1(O_1)$	960	$\{\infty, 17, 11\}$	
$\mathcal{T}_0(O_1)$	560	$\{4, 17, 11\}$	
$\mathcal{D}_{2,6}(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|} \hline 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \hline \end{array}$	
$\mathcal{D}_{4^2}(O_1)$	840	$\begin{array}{\|cc\|cc\|cc\|} \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$	

3.2. *Let* $K = \mathrm{Stab}_H(O_1)$ *and* $L = O_2(K)\,\mathrm{Stab}_K(\Phi)$ *where* $\Phi = \{17, 11\}$. *So* $L \cong 2^4 2^3 L_3(2)$. *See Tables* 2 *and* 3.

Table 2: $L \cong 2^4 2^3 L_3(2)$-orbits (see Assertion 3.2 and Table 3).

L-orbit	Size	Representative
$\mathcal{S}^{(1)}_{4^2}(O_1)$	7	$\begin{array}{\|cc\|cc\|cc\|} 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \end{array}$
$\mathcal{S}^{(2)}_{4^2}(O_1)$	28	S_0
$\mathcal{S}_{1^5,3}(O_1)$	896	$\begin{array}{\|cc\|cc\|cc\|} 3 & 1 & 1 & 6 & 5 & 3 \\ 4 & 1 & 2 & 5 & 4 & 5 \\ 2 & 6 & 3 & 2 & 4 & 3 \\ 1 & 5 & 4 & 2 & 6 & 6 \end{array}$
$\mathcal{S}^{(1)}_{2^4}(O_1)$	56	$\begin{array}{\|cc\|cc\|cc\|} 1 & 1 & 1 & 1 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 5 \\ 3 & 3 & 3 & 3 & 6 & 5 \\ 4 & 4 & 4 & 4 & 6 & 5 \end{array}$
$\mathcal{S}^{(2)}_{2^4}(O_1)$	336	$\begin{array}{\|cc\|cc\|cc\|} 2 & 5 & 3 & 4 & 1 & 6 \\ 3 & 4 & 2 & 5 & 1 & 1 \\ 2 & 5 & 3 & 4 & 6 & 1 \\ 3 & 4 & 2 & 5 & 6 & 6 \end{array}$
$\mathcal{S}^{(3)}_{2^4}(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|} 5 & 4 & 4 & 1 & 6 & 4 \\ 3 & 3 & 5 & 1 & 6 & 5 \\ 5 & 2 & 1 & 2 & 2 & 6 \\ 2 & 4 & 6 & 3 & 3 & 1 \end{array}$
$\mathcal{O}_8(O_1)$	1	O_1
$\mathcal{O}^{(1)}_4(O_1)$	56	$\begin{array}{\|cc\|cc\|cc\|} \times & \times & \times & \times & & \\ \times & \times & \times & \times & & \\ & & & & & \\ & & & & & \end{array}$
$\mathcal{O}^{(2)}_4(O_1)$	224	$\begin{array}{\|cc\|cc\|cc\|} \times & & \times & & & \\ \times & & \times & & & \\ \times & & \times & & & \\ \times & & \times & & & \end{array}$
$\mathcal{O}_2(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|} \times & \times & \times & \times & \times & \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \end{array}$
$\mathcal{O}^{(1)}_0(O_1)$	14	O_3
$\mathcal{O}^{(2)}_0(O_1)$	16	$\begin{array}{\|cc\|cc\|cc\|} & & \times & & \times & \\ & & \times & & \times & \\ & & \times & & \times & \\ & & \times & & \times & \end{array}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 3: $L \cong 2^4 2^3 L_3(2)$-orbits (*continued*; see Assertion 3.2 and Table 2).

L-orbit	Size	Representative	
$\mathcal{A}_0(O_1)$	1	$\Omega \setminus O_1$	$1+14$
$\mathcal{A}_4^{(1)}(O_1)$	56	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & \times & \times \\ & & & & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \hline\end{array}$	$3+12$
$\mathcal{A}_4^{(2)}(O_1)$	224	$\begin{array}{\|cc\|cc\|cc\|}\hline & \times & & \times & \times & \times \\ & \times & & \times & \times & \times \\ & \times & & \times & \times & \times \\ & \times & & \times & \times & \times \\ \hline\end{array}$	$3+4+8$
$\mathcal{A}_6(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & & \times \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \hline\end{array}$	$3+12$
$\mathcal{A}_8^{(1)}(O_1)$	14	$\Omega \setminus O_3$	$1+6+8$
$\mathcal{A}_8^{(2)}(O_1)$	16	$\begin{array}{\|cc\|cc\|cc\|}\hline \times & \times & & \times & & \times \\ \times & \times & & \times & & \times \\ \times & \times & & \times & & \times \\ \times & \times & & \times & & \times \\ \hline\end{array}$	$7+8$
$\mathcal{T}_3(O_1)$	56	$\{\infty, 0, 3\}$	
$\mathcal{T}_2(O_1)$	448	$\{\infty, 0, 17\}$	
$\mathcal{T}_1^{(1)}(O_1)$	64	$\{\infty, 17, 11\}$	
$\mathcal{T}_1^{(2)}(O_1)$	896	$\{\infty, 17, 4\}$	
$\mathcal{T}_0^{(1)}(O_1)$	112	$\{17, 11, 4\}$	
$\mathcal{T}_0^{(2)}(O_1)$	448	$\{17, 4, 16\}$	
$\mathcal{D}_{2,6}(O_1)$	448	$\begin{array}{\|cc\|cc\|cc\|}\hline 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \hline\end{array}$	
$\mathcal{D}_{4^2}^{(1)}(O_1)$	56	$\begin{array}{\|cc\|cc\|cc\|}\hline 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ \hline\end{array}$	
$\mathcal{D}_{4^2}^{(2)}(O_1)$	112	$\begin{array}{\|cc\|cc\|cc\|}\hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ \hline\end{array}$	
$\mathcal{D}_{4^2}^{(3)}(O_1)$	672	$\begin{array}{\|cc\|cc\|cc\|}\hline 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline\end{array}$	

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

3.3. *Let*

$$T = \begin{array}{|c|c|c|}\hline \times & & \\ \times & & \\ \times & & \\ & & \\ \hline\end{array}$$

and $L = \mathrm{Stab}_H(T)$. *So* $L \cong L_3(4) : S_3$*; see Tables 4 and 5.*

Table 4: $L \cong L_3(4)$-orbits (see Assertion 3.3 and Table 5).

L-orbit	Size	Representative	
$\mathcal{S}_3(T)$	21	S_0	
$\mathcal{S}_{1,2}(T)$	630	$\begin{array}{\|cc\|cc\|cc\|}\hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline\end{array}$	
$\mathcal{S}_{13}(T)$	1120	$\begin{array}{\|cc\|cc\|cc\|}\hline 1 & 1 & 1 & 1 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 5 \\ 3 & 3 & 3 & 3 & 6 & 5 \\ 4 & 4 & 4 & 4 & 6 & 5 \\ \hline\end{array}$	
$\mathcal{O}_3(T)$	21	O_1	
$\mathcal{O}_2(T)$	168	$\begin{array}{\|cc\|cc\|cc\|}\hline \times & \times & \times & \times & & \\ \times & \times & \times & \times & & \\ & & & & & \\ & & & & & \\ \hline\end{array}$	
$\mathcal{O}_1(T)$	360	$\begin{array}{\|cc\|cc\|cc\|}\hline \times & \times & \times & \times & \times & \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ \hline\end{array}$	
$\mathcal{O}_0(T)$	210	O_3	
$\mathcal{A}_0(T)$	21	$\Omega \setminus O_1$	15
$\mathcal{A}_1(T)$	168	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & \times & \times \\ & & & & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \hline\end{array}$	15
$\mathcal{A}_2(T)$	360	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & & \times \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \hline\end{array}$	1+14
$\mathcal{A}_3(T)$	210	$\Omega \setminus O_3$	$3 + 12$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

3.4. *Let $D = \{\Phi_1, \Phi_2\} \in \mathcal{D}$, where*

$$\Phi_1 = \begin{array}{|cc|cc|cc|}\hline \times & & \times & & \times & \\ & \times & & \times & & \times \\ & \times & & \times & & \times \\ & \times & & \times & & \times \\ \hline \end{array}$$

and $\Phi_2 = \Omega \setminus \Phi_1$.

Put $L = \mathrm{Stab}_H(D)$. So $L \cong M_{12}2$. See Table 6.

Table 5: $L \cong L_3(4)$-orbits (*continued*; see Assertion 3.3 and Table 4).

L-orbit	Size	Representative
$\mathcal{T}_3(T)$	1	T
$\mathcal{T}_2(T)$	63	$\begin{array}{\|cc\|cc\|cc\|}\hline \times & \times & & & & \\ \times & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$
$\mathcal{T}_1(T)$	630	$\begin{array}{\|cc\|cc\|cc\|}\hline \times & \times & & & & \\ & \times & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$
$\mathcal{T}_0^+(T)$	210	$\begin{array}{\|cc\|cc\|cc\|}\hline & \times & & & & \\ & \times & & & & \\ & \times & & & & \\ & & & & & \\ \hline \end{array}$
$\mathcal{T}_0^-(T)$	1120	$\begin{array}{\|cc\|cc\|cc\|}\hline & \times & \times & & & \\ & \times & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}$
$\mathcal{D}_{0,3}(T)$	280	$\begin{array}{\|cc\|cc\|cc\|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$
$\mathcal{D}_{1,2}(T)$	1008	$\begin{array}{\|cc\|cc\|cc\|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 6: $L \cong M_{12}2$-orbits (see Assertion 3.4).

<table>
<tr><th>L-orbit</th><th>Size</th><th>Representative</th><th></th></tr>
<tr><td>$\mathscr{S}_{(2.2)^4,(4.0)^2}(D)$</td><td>495</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 1 & 1 & 1 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 5 \\ 3 & 3 & 3 & 3 & 6 & 5 \\ 4 & 4 & 4 & 4 & 6 & 5 \\ \hline \end{array}$</td><td></td></tr>
<tr><td>$\mathscr{S}_{(3.1)^6}(D)$</td><td>880</td><td>S_0</td><td></td></tr>
<tr><td>$\mathscr{S}_{(2.2)^6}(D)$</td><td>396</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}$</td><td></td></tr>
<tr><td>$\mathscr{O}_{2,6}(D)$</td><td>264</td><td>$\begin{array}{|cc|cc|cc|}\hline \times & \times & \times & \times & \times & \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ \hline \end{array}$</td><td></td></tr>
<tr><td>$\mathscr{O}_{4^2}(D)$</td><td>495</td><td>O_1</td><td></td></tr>
<tr><td>$\mathscr{A}_{6,10}(D)$</td><td>264</td><td>$\begin{array}{|cc|cc|cc|}\hline & & & & & \times \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \\ \hline \end{array}$</td><td>15</td></tr>
<tr><td>$\mathscr{A}_{8^2}(D)$</td><td>495</td><td>$\Omega \setminus O_1$</td><td>1+2+12</td></tr>
<tr><td>$\mathscr{T}_{0,3}(D)$</td><td>440</td><td>$\begin{array}{|cc|cc|cc|}\hline \times & & & & & \\ & \times & & & & \\ & \times & & & & \\ & & & & & \\ \hline \end{array}$</td><td></td></tr>
<tr><td>$\mathscr{T}_{1,2}(D)$</td><td>1584</td><td>$\begin{array}{|cc|cc|cc|}\hline \times & & & & & \\ \times & & & & & \\ \times & & & & & \\ & & & & & \\ \hline \end{array}$</td><td></td></tr>
<tr><td>$\{D\}$</td><td>1</td><td>D</td><td></td></tr>
<tr><td>$\mathscr{D}_{(4.8)^2}(D)$</td><td>495</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$</td><td></td></tr>
<tr><td>$\mathscr{D}_{(6.6)^2}(D)$</td><td>792</td><td>$\begin{array}{|cc|cc|cc|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$</td><td></td></tr>
</table>

4. *Cross orbits*

Put $G = \mathrm{Co}_1$; we now investigate L-orbits on the crosses for certain subgroups L of G. Set $H = G_{\ell_0}/O_2(G_{\ell_0}) \cong M_{24}$.

4.1. We repeat, to some extent, Theorem 2.3.

THEOREM 4.1. *Let $L = G_{\ell_0} \cong 2^{11} : M_{24}$. Then the L-orbits on crosses are as shown in Table 7.*

Table 7: $L \cong 2^{11} : M_{24}$-orbits (see Theorem 4.1).

Orbit	Size	Representative
$\{\ell_0\}$	1	$\ell_0 = 8v_\infty^\times$
$\alpha_1(\ell_0)$	$1771.2 = 2.7.11.23$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ \hline\end{array}^{\times}$
$\alpha_2^1(\ell_0)$	$759.2^6 = 2^6.3.11.23$	$\begin{array}{\|cc\|cc\|cc\|}\hline 6 & -2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ \hline\end{array}^{\times}$
$\alpha_2^2(\ell_0)$	$759.15.2^6.2 = 2^7.3^2.5.11.23$	$\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 2 & 4 & 4 & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ \hline\end{array}^{\times}$
$\alpha_3^1(\ell_0)$	$\binom{24}{3}.2^{11} = 2^{14}.11.23$	$\begin{array}{\|cc\|cc\|cc\|}\hline 5 & -1 & -1 & -1 & -1 & -1 \\ 3 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}^{\times}$
$\alpha_3^2(\ell_0)$	$1288.2^{11} = 2^{14}.7.23$	$\begin{array}{\|cc\|cc\|cc\|}\hline -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ \hline\end{array}^{\times}$

4.2. Put $g = \varepsilon_{O_1}$. Observe that $g \in O_2(G_{\ell_0})$ – g is sometimes referred to as an *octad involution of* G. Set $L = C_G(g)$. Then $L \cong 2^{1+8}O_8^+(2)$ (see [2]). We define $\mathcal{JP}$ to be the following set of crosses: ℓ_0, all odd and even crosses based on the (35) MOG sextets together with the crosses (in $\alpha_2^1(\ell_0)$) based on the octad O_1. We note that the latter set is given as follows.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

$$\left\{ \left.\begin{array}{|cc|cc|cc|}\hline 6 & & & & & \\ \pm 2^7 & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}\right.^{\times} \; : \; \text{odd number of instances of '}-2\text{' in } O_1 \setminus \{\infty\} \right\}$$

Thus $|\mathcal{JP}| = 1 + 35.2 + 64 = 135$. In fact, $\mathcal{JP}$ may be identified with the set of isotropic points of the natural module for $O_8^+(2)$; hence the choice of notation.

THEOREM 4.2. *The L-orbits on crosses are as shown in Table 8.*

Moreover,

$$[7, 8, 120, 0, 0]_{\mathcal{JP}} = \mathcal{S}_{2^4}(O_1)^{\times} \cup \mathcal{O}_0(O_1)^{\times} \cup (\Omega \setminus O_1)^{\times} \cup \mathcal{A}_8(O_1)^{[7]\times};$$

$$[1, 10, 60, 64, 0]_{\mathcal{JP}} = \mathcal{S}_{1^5,3}(O_1)^{\times} \cup \mathcal{O}_4(O_1)^{\times} \cup \mathcal{A}_4(O_1)^{[3]\times} \cup \mathcal{T}_3(O_1)^{\times};$$

$$[0, 1, 30, 72, 32]_{\mathcal{JP}} = \mathcal{O}_2(O_1)^{\times} \cup \mathcal{T}_0(O_1)^{\times} \cup \mathcal{T}_2(O_1)^{\times} \cup \mathcal{A}_6(O_1)^{\times} \cup \mathcal{D}_{2,6}(O_1)^{\times};$$

$$[0, 0, 15, 64, 56]_{\mathcal{JP}} = \mathcal{D}_{4^2}(O_1)^{\times} \cup \mathcal{T}_1(O_1)^{\times} \cup \mathcal{A}_4(O_1)^{[12]\times} \cup \mathcal{A}_8(O_1)^{[8]\times}.$$

Table 8: $L \cong 2^{1+8}O_8^+(2)$-orbits (see Theorem 4.2).

Orbit	Size	Representative
$\mathcal{JP}$	135	ℓ_0
$[7, 8, 120, 0, 0]_{\mathcal{JP}}$	$32,400 = 2^4.3^4.5^2$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline 4 & 4 & 4 & 4 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}\right.^{\times}$
$[1, 10, 60, 64, 0]_{\mathcal{JP}}$	$241,920 = 2^8.3^3.5.7$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline & 4 & 4 & & & \\ & 4 & & & & \\ & & & & & \\ 4 & & & & & \\ \hline \end{array}\right.^{\times}$
$[0, 1, 30, 72, 32]_{\mathcal{JP}}$	$3,870,720 = 2^{12}.3^3.5.7$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline \end{array}\right.^{\times}$
$[0, 0, 15, 64, 56]_{\mathcal{JP}}$	$4,147,200 = 2^{11}.3^4.5^2$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ \hline \end{array}\right.^{\times}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Proof. Put $K = L_{\ell_0}$. First we observe that $K \cong 2^{11}.2^4 A_8$. Since K contains all the sign changes on Golay code sets, for Ξ a $\mathrm{Stab}_H(O_1)$-orbit of sextets, octads, triads or duums, $\Xi^\times$ will be a K-orbit. The case of Ξ being a $\mathrm{Stab}_H(O_1)$-orbit of 16-ads must be approached with care on account of the possible positions of the 4, ± 4 entries. However, note that by using sign changes we can interchange the 4, 4 and 4, -4 entries of a 16-ad type cross.

It is straightforward to check that for a tetrad T of Ω, we have $\zeta_T \in L$, provided that the unique sextet containing T is one of the 35 MOG sextets. By choosing such appropriate examples of T, we now analyse how the K-orbits of crosses fuse. Let

$$T_0 = \begin{array}{|cc|cc|cc|} \hline & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \\ \hline \end{array}\ ; \quad T_1 = \begin{array}{|cc|cc|cc|} \hline & & \times & \times & \times & \times \\ & & & & & \\ & & & & & \\ & & & & & \\ \hline \end{array}\ ;$$

$$T_2 = \begin{array}{|cc|cc|cc|} \hline & & & \times & & \\ & & & \times & & \\ & & & & & \times \\ & & & & & \times \\ \hline \end{array}\ ; \quad T_3 = \begin{array}{|cc|cc|cc|} \hline & & \times & \times & & \\ & & & & & \\ & & & & & \\ & & & & \times & \times \\ \hline \end{array}\ ;$$

$$T_4 = \begin{array}{|cc|cc|cc|} \hline & & \times & & \times & \\ & & & \times & & \times \\ & & & & & \\ & & & & & \\ \hline \end{array}\ ; \quad T_5 = \begin{array}{|cc|cc|cc|} \hline & & & & \times & \times \\ & & & & \times & \times \\ & & & & & \\ & & & & & \\ \hline \end{array}\ .$$

4.2.1. *$\mathcal{JP}$ is an L-orbit.*

Since $(8v_\infty)\zeta_{T_0} = 4v_\infty - 4v_0 - 4v_3 - 4v_{15}$, $(8v_\infty^\times)\zeta_{T_0}$ is an odd cross based on the standard sextet. Let $u_1 = 4v_{14} + 4v_0 + 4v_3 + 4v_{15}$. Then $u_1^\times \in \mathcal{JP}$ and

$$(u_1^\times)\zeta_{T_0} = \begin{array}{|cc|c|c|} \hline -6 & 2 & & \\ -2 & -2 & & \\ -2 & -2 & & \\ -2 & -2 & & \\ \hline \end{array}^{\,\times} \in \mathcal{JP}.$$

In view of Assertion 3.1, we see that $\mathcal{JP}$ is contained in an L-orbit, whence, as $[L : L_{\ell_0}] = 135$, Assertion 4.2.1 follows.

4.2.2. $\mathcal{S}_{2^4}(O_1)^\times \cup \mathcal{O}_0(O_1)^\times \cup (\Omega \setminus O_1)^\times \cup \mathcal{A}_8(O_1)^{[7]\times}$ *is contained in an L-orbit.*

Let $u_2 = 4v_\infty + 4v_{14} + 4v_{17} + 4v_{11}$ and $w_2 = 4v_{22} + 4v_9 + 4v_5 + 4v_6$. Then $u_2^\times = w_2^\times$ and we have the following crosses:

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

$$(w_2^{\times})\zeta_{T_0} = \left|\begin{array}{cc|cc|cc} \hline &&&& 2 & 6 \\ &&&& -2 & 2 \\ &&&& -2 & 2 \\ &&&& -2 & 2 \\ \hline \end{array}\right|^{\times} \in \mathcal{O}_0(O_1)^{\times};$$

$$(u_2^{\times})\zeta_{T_1} = \left|\begin{array}{cc|cc|cc} \hline 2 & 2 &&& 4 & 4 \\ -2 & -2 &&&& \\ -2 & -2 &&&& \\ -2 & -2 &&&& \\ \hline \end{array}\right|^{\times} \in (\Omega \setminus O_1)^{\times};$$

$$(w_2^{\times})\zeta_{T_2} = \left|\begin{array}{cc|cc|cc} \hline &&& 4 & 2 & 2 \\ &&& 4 & -2 & 2 \\ && -2 & -2 && \\ && -2 & -2 && \\ \hline \end{array}\right|^{\times} \in \mathcal{A}_8(O_1)^{[7]\times}.$$

Using Assertion 3.1, we deduce that 4.2.2 holds.

4.2.3. *$\mathcal{S}_{1^5,3}(O_1)^{\times} \cup \mathcal{O}_4(O_1)^{\times} \cup \mathcal{A}_4(O_1)^{[3]\times} \cup \mathcal{T}_3(O_1)^{\times}$ is contained in an L-orbit.*

Let

$$u_3 = \left|\begin{array}{cc|cc|cc} \hline & 4 & 4 &&& \\ & 4 &&&& \\ &&&&& \\ 4 &&&&& \\ \hline \end{array}\right|.$$

Then $u_3^{\times}$ is the even cross based on

$$\left|\begin{array}{cc|cc|cc} \hline 3 & 1 & 1 & 6 & 5 & 3 \\ 4 & 1 & 2 & 5 & 4 & 5 \\ 2 & 6 & 3 & 2 & 4 & 3 \\ 1 & 5 & 4 & 2 & 6 & 6 \\ \hline \end{array}\right|.$$

So $u_3^{\times} \in \mathcal{S}_{1^5,3}(O_1)^{\times}$. Now

$$(u_3)\zeta_{T_0} = \left|\begin{array}{cc|cc|cc} \hline -2 && 2 &&& \\ -2 && -2 &&& \\ -2 & -4 & -2 &&& \\ 2 & -4 & -2 &&& \\ \hline \end{array}\right|;$$

$$(u_3)\zeta_{T_3} = \left|\begin{array}{cc|cc|cc} \hline & -2 & -2 & 2 && \\ & -2 &&&& \\ & -6 &&&& \\ -2 &&&& 2 & 2 \\ \hline \end{array}\right| = w_3;$$

$$(w_3)\zeta_{T_4} = \left|\begin{array}{cc|cc|cc} \hline 3 & 1 & 1 & 1 & -1 & -1 \\ 3 & 1 & -1 & -1 & -1 & -1 \\ 3 & -3 & -1 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 & 1 & 1 \\ \hline \end{array}\right|.$$

Hence $(w_3^{\times})\zeta_{T_4} \in \alpha_3^1(\ell_0)$ and the unique octad containing the ± 3 positions is O_1, whence the triad underlying $\zeta_{T_4}(w_3^{\times})$ is contained in O_1. Together with $(u_3^{\times})\zeta_{T_0} \in \mathcal{A}_4(O_1)^{[3]\times}$, $(u_3^{\times})\zeta_{T_3} \in \mathcal{O}_4(O_1)^{\times}$ and Assertion 3.1, this gives Assertion 4.2.3.

4.2.4. *$\mathcal{O}_2(O_1)^\times \cup \mathcal{T}_0(O_1)^\times \cup \mathcal{T}_2(O_1)^\times \cup \mathcal{A}_6(O_1)^\times \cup \mathcal{D}_{2,6}(O_1)^\times$ is contained in an L-orbit.*

Set

$$u_4 = \begin{array}{|cc|cc|cc|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline\end{array}\,.$$

Then $u_4^\times \in \mathcal{O}_2(O_1)^\times$ and, from

$$(u_4)\zeta_{T_0} = \begin{array}{|cc|cc|cc|}\hline 3 & -1 & 1 & 1 & 1 & 3 \\ -3 & 1 & -1 & -1 & -1 & 1 \\ -3 & 1 & -1 & -1 & -1 & 1 \\ -3 & 1 & -1 & -1 & -1 & 1 \\ \hline\end{array}\,,$$

we infer that $(u_4^\times)\zeta_{T_0} \in \mathcal{T}_0(O_1)^\times$. Thus $\mathcal{O}_2(O_1)^\times \cup \mathcal{T}_0(O_1)^\times$ is contained in an L-orbit.

Let

$$w_4 = \begin{array}{|cc|cc|cc|}\hline 3 & 1 & -1 & 1 & 1 & -1 \\ -3 & -1 & 1 & -1 & 3 & 1 \\ -1 & 1 & 3 & 1 & -1 & 1 \\ -1 & 1 & 3 & 1 & -1 & 1 \\ \hline\end{array}\,;$$

so $w_4^\times \in \mathcal{T}_0(O_1)^\times$. Since

$$(w_4)\zeta_{T_0} = \begin{array}{|cc|cc|cc|}\hline 4 & & -4 & & & 2 \\ -2 & -2 & -2 & -2 & 2 & \\ & & & & -2 & \\ & & & & -2 & \\ \hline\end{array}\,,$$

and

$$(u_4)\zeta_{T_5} = \begin{array}{|cc|cc|cc|}\hline 4 & -4 & & & & 2 \\ -2 & -2 & -2 & -2 & 2 & \\ & & & & -2 & \\ & & & & -2 & \\ \hline\end{array}\,,$$

we deduce that $\mathcal{O}_2(O_1)^\times \cup \mathcal{T}_0(O_1)^\times \cup \mathcal{A}_6(O_1)^\times$ is contained in an L-orbit.

Let

$$w_5 = \begin{array}{|cc|cc|cc|}\hline 1 & -3 & -3 & -1 & -1 & 1 \\ -3 & 1 & 1 & -1 & -1 & 1 \\ -3 & 1 & 1 & -1 & -1 & 1 \\ -3 & 1 & 1 & -1 & -1 & 1 \\ \hline\end{array}$$

and

$$w_6 = \begin{array}{|cc|cc|cc|}\hline -1 & -3 & -1 & 1 & 1 & 3 \\ -3 & -1 & 1 & -1 & -1 & 1 \\ -3 & -1 & 1 & -1 & -1 & 1 \\ -3 & -1 & 1 & -1 & -1 & 1 \\ \hline\end{array}\,.$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Noting that $w_5^\times, w_6^\times \in \mathcal{T}_0(O_1)^\times$, and that

$$(w_5)\zeta_{T_0} = \begin{array}{|cc|cc|cc|}\hline 5 & -3 & -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

and

$$(w_6)\zeta_{T_0} = \begin{array}{|cc|cc|cc|}\hline 4 & & -2 & 2 & 2 & \\ 2 & 2 & & & & 2 \\ 2 & 2 & & & & 2 \\ 2 & 2 & & & & 2 \\ \hline \end{array},$$

we conclude that 4.2.4 holds.

4.2.5. $\mathcal{D}_{4^2}(O_1)^\times \cup \mathcal{T}_1(O_1)^\times \cup \mathcal{A}_4(O_1)^{[12]\times} \cup \mathcal{A}_8(O_1)^{[8]\times}$ *is contained in an L-orbit.*

Let

$$u_5 = \begin{array}{|cc|cc|cc|}\hline -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ \hline \end{array}.$$

Then $u_5^\times \in \mathcal{D}_{4^2}(O_1)^\times$ and we check that

$$(u_5)\zeta_{T_0} = \begin{array}{|cc|cc|cc|}\hline -1 & -1 & 1 & -3 & 1 & 3 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ \hline \end{array}.$$

Hence $(u_5^\times)\zeta_{T_0} \in \alpha_3^1(\ell_0)$ with $\{\infty, 17, 22\}$ as its underlying triad. So $(u_5^\times)\zeta_{T_0} \in \mathcal{T}_1(O_1)^\times$. For

$$w_7 = \begin{array}{|cc|cc|cc|}\hline -1 & 1 & 1 & -1 & 3 & 3 \\ -1 & -3 & 1 & -1 & -1 & -1 \\ -1 & -3 & 1 & -1 & -1 & -1 \\ -1 & -3 & 1 & -1 & -1 & -1 \\ \hline \end{array},$$

we have $w_7^\times \in \mathcal{T}_1(O_1)^\times$ and

$$(w_7)\zeta_{T_1} = \begin{array}{|cc|cc|cc|}\hline 4 & 2 & 2 & 4 & & \\ & 2 & 2 & & & \\ & 2 & 2 & & & \\ & 2 & 2 & & & \\ \hline \end{array}.$$

Therefore $(w_7^\times)\zeta_{T_1} \in \mathcal{A}_4(O_1)^{[12]\times}$.

Taking

$$w_8 = \begin{array}{|cc|cc|cc|}\hline 2 & & 2 & -2 & & 4 \\ & -2 & & & -2 & 2 \\ & -2 & & & 2 & 2 \\ & -2 & & & 2 & 2 \\ \hline \end{array},$$

we note that $w_8^\times \in \mathcal{D}_{4^2}(O_1)^\times$.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Since

$$(w_8^{\times})\zeta_{T_1} = \left[\begin{array}{cc|cc|cc} 4 & & & 4 & 2 & -2 \\ & & & & -2 & 2 \\ & & -2 & -2 & & \\ & & -2 & -2 & & \end{array}\right]^{\times} \in \mathcal{A}(O_1)^{[8]\times},$$

using Assertion 3.1 again, we obtain Assertion 4.2.5.

Let

$$S_1 = \left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 5 \\ 3 & 3 & 3 & 3 & 6 & 5 \\ 4 & 4 & 4 & 4 & 6 & 5 \end{array}\right]$$

and let ℓ be the even cross with S_1 as its underlying sextet. So $\ell \in \mathcal{S}_{2^4}(O_1)^{\times}$ and $\ell \in \alpha_1(\ell_0)$. Set $I_1 = \{(4, 4), (5, 4), (6, 4)\}$ and $I_2 = \{(1, 3), (1, 4), (2, 4), (3, 4)\}$. By inspection,

$$S_1 \in \begin{cases} \mathcal{S}_{(2^2)^6}(S_{ij}), & \text{for } (i, j) \in I_1; \\ \mathcal{S}_{(3.1)^2,(1^4)^4}(S_{ij}), & \text{for } (i, j) \in I_2; \\ \mathcal{S}_{(2.1^2)^4,(1^4)^2}(S_{ij}), & \text{for all other } (i, j). \end{cases}$$

Hence, by Lemma 2.7, we have the next assertion.

4.2.6. *Let m be a cross in $\mathcal{JP}$, where m is either an odd or an even cross on the sextet MOG S_{ij}. Then the following statements hold.*

(i) *If $(i, j) \in I_1$, then $m \in \alpha_1(\ell)$.*

(ii) *If $(i, j) \in I_2$, then $m \in \alpha_2^1(\ell)$.*

(iii) *If $(i, j) \notin I_1 \cup I_2$, then $m \in \alpha_2^2(\ell)$.*

Let u be one of the vectors in the cross ℓ, and let

$$w = \left[\begin{array}{cc|cc|cc} 6 & & & & & \\ (\pm 2)^7 & & & & & \\ & & & & & \\ & & & & & \end{array}\right].$$

So $w^{\times} \in \mathcal{JP}$. Let $T = \{\infty, 14, 17, 11\}$. If $u \neq (\pm 4)_T$, then $w \cdot u \in \{0, \pm 16\}$. For $u = (\pm 4)_T$ (recall that ℓ is assumed to be even) we check that $w \cdot u$ equals ± 32 (twice) or ± 16 (twice). Consulting Lemma 2.5 we deduce that $w^{\times} \in \alpha_2^2(\ell)$. Taking this with 4.2.6, we infer that $\ell \in [7, 8, 120, 0, 0]_{\mathcal{JP}}$. Together, 4.2.1 and 4.2.2 imply that 4.2.7 holds.

4.2.7. $\mathcal{S}_{2^4}(O_1)^{\times} \cup \mathcal{O}_0(O_1)^{\times} \cup (\Omega \setminus O_1)^{\times} \cup \mathcal{A}_8(O_1)^{[7]\times} \subseteq [7, 8, 120, 0, 0]_{\mathcal{JP}}$.

Now let

$$S_2 = \left[\begin{array}{cc|cc|cc} 6 & 1 & 1 & 2 & 3 & 6 \\ 5 & 1 & 4 & 3 & 5 & 3 \\ 4 & 2 & 6 & 4 & 5 & 6 \\ 1 & 3 & 5 & 4 & 2 & 2 \end{array}\right]$$

and let ℓ stand for the even cross with underlying sextet S_2. Note that $\ell \in \mathcal{S}_{1^5,3}(O_1)^{\times}$.

Set $I_3 = \{(1,4), (3,5), (4,2), (5,6), (6,1)\}$. Again scrutinizing the MOG, we see that

$$S \in \begin{cases} \mathcal{S}_{(3.1)^2,(1^4)^4}(S_{ij}), & \text{when } (i,j) \in I_3; \\ \mathcal{S}_{(2.1^2)^4,(1^4)^2}(S_{ij}), & \text{otherwise.} \end{cases}$$

So Lemma 2.7 gives us Assertion 4.2.8.

4.2.8. *Let m be an odd or even cross on the sextet MOG S_{ij}. Then $m \in \alpha_2^1(\ell)$ if $(i,j) \in I_3$ and $m \in \alpha_2^2(\ell)$ if $(i,j) \notin I_3$.*

Let m be a cross on O_1, and let w be a vector in m. Let $T_1 = \{20, 11, 21, 6\}$. Since $O_1 \cap T_1 = \{20\}$, for $u = (\pm 4)_{T_1}$ we have $w \cdot u = \pm 8$ whence, by Lemma 2.5, $m \in \alpha_3^1(l)$. Therefore, by 4.2.8, $\ell \in [1, 10, 60, 64, 0]_{\mathcal{JP}}$. Hence, using 4.2.1 and 4.2.3, we see that Assertion 4.2.9 holds.

4.2.9. $\mathcal{S}_{1^5,3}(O_1)^\times \cup \mathcal{O}_4(O_1)^\times \cup \mathcal{A}_4(O_1)^{[3]\times} \cup \mathcal{T}_3(O_1)^\times \subseteq [1, 10, 60, 64, 0]_{\mathcal{JP}}$.

Let

$$u = \left|\begin{array}{cc|cc|cc} 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \end{array}\right|$$

and put $\ell = u^\times$.

Let $m \in \mathcal{JP}$. Suppose that m is an odd or even cross on the MOG sextet S. If S has a tetrad T such that $|T \cap O| = 1$ where

$$O = \left|\begin{array}{cc|cc|cc} \times & \times & \times & \times & \times & \\ & & & & & \times \\ & & & & & \times \\ & & & & & \times \end{array}\right|,$$

then for $w \in (\pm 4)_T$, we have $w \cdot u = \pm 24$ or ± 8. In view of Lemma 2.5 we must have $m \in \alpha_3^1(\ell)$. Surveying the MOG reveals that twenty of the sextets there have this property, and the remaining fifteen each have tetrads that cut O in 2^4. Suppose now that S is a sextet of the latter kind and let $T_1, \ldots, T_6$ be the tetrads of S. Without loss we may suppose that $\infty \in T_1$. Then we check that, for $i \neq 1$, $u \cdot (\pm 4)_{T_i}$ is equal to 0 or ± 16. A further calculation shows that for the four vectors w' with support T_1, we see $u \cdot w'$ is equal to ± 32 (twice) or ± 16 (twice). Consequently, by Lemma 2.5, $m \in \alpha_2^2(\ell)$. So we have shown that 4.2.10 holds.

4.2.10. *Forty of the crosses on MOG sextets are in $\alpha_3^1(\ell)$ and thirty of the crosses on MOG sextets are in $\alpha_2^2(\ell)$.*

Now we consider the case when $m = w^\times$, where

$$w = \left|\begin{array}{c|c|c} 6 & & \\ (\pm 2)^7 & & \\ & & \\ & & \end{array}\right|^\times.$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

If the v_{14} co-ordinate of w is -2, then $u \cdot w = 6.6+2.2 = 40$, which, by Lemma 2.5, implies that $m \in \alpha_3^1(\ell)$. So we now assume that the v_{14} co-ordinate of w is 2. Then $u \cdot w = 32$. It may be checked for any vector w' of m with $w' \neq w$, that $|u \cdot w'| \leqslant 24$ and hence $m \in \alpha_3^2(\ell)$ by Lemma 2.5. Thus $\ell \in [0, 1, 30, 72, 32]_{\mathcal{J}\mathcal{P}}$. So, by 4.2.4, Assertion 4.2.11 holds.

4.2.11. $\mathcal{O}_2(O_1)^{\times} \cup \mathcal{T}_0(O_1)^{\times} \cup \mathcal{T}_2(O_1)^{\times} \cup \mathcal{A}_6(O_1)^{\times} \cup \mathcal{D}_{2,6}(O_1)^{\times} \subseteq [0, 1, 30, 72, 32]_{\mathcal{J}\mathcal{P}}$.

Next we examine $\ell = u^{\times} \in \mathcal{D}_{4^2}(O_1)^{\times}$ where

$$u = \left|\begin{array}{cc|cc|cc} -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \end{array}\right|,$$

and we let m be a cross based on the sextet MOG S_{ij}. Set

$$\Phi = \left|\begin{array}{cc|cc|cc} \times & & \times & & \times & \\ & \times & & \times & & \times \\ & \times & & \times & & \times \\ & \times & & \times & & \times \end{array}\right|.$$

Suppose that S_{ij} possesses a tetrad T such that $|T \cap \Phi| = 1$. Set $w = (\pm 4)_T$. If $14 \notin T$, then $u \cdot w = \pm 8$ while if $14 \in T$, then $u \cdot w \in \{\pm 8, \pm 24\}$. So, using Lemma 2.5, we see that $m \in \alpha_3^1(\ell)$. Thus, by inspection, Assertion 4.2.12 holds.

4.2.12. *$m \in \alpha_3^1(\ell)$ if either $(i, j) = (1, 1)$ or $1 \leqslant i \leqslant 3$, $2 \leqslant j$.*

Put $I_4 = \{(4, 2), (5, 2), (6, 2), (4, 5), (5, 5), (6, 5)\}$ and $I_5 = \{(i, j) | i > 3\} \setminus I_4$. Again perusing the MOG, we learn that Assertion 4.2.13 holds.

4.2.13. (i) *The tetrads of S_{ij} contained in $\Omega \setminus O_1$ will intersect Φ in two elements when $(i, j) \in I_5$.*

(ii) *Suppose that $(i, j) \in I_4$. Then the tetrads of S_{ij} in $\Omega \setminus O_1$ intersect Φ in $2^2 4^1$.*

We also note that Assertion 4.2.14 holds.

4.2.14. *If T is a tetrad of S_{ij} with $T \subseteq \Omega \setminus O_1$ and $|T \cap \Phi| = 2$, then $(\pm 4)_T \cdot u$ is equal to 0 (twice) and ± 16 (twice), where $(\pm 4)_T$ are vectors in an even or odd cross on S_{ij}.*

We now prove the following assertion.

4.2.15. *If $(i, j) \in I_4$, then $S_{ij_{\text{even}}} \in \alpha_2^2(\ell)$ and $S_{ij_{\text{odd}}} \in \alpha_3^2(\ell)$.*

Let $(i, j) \in I_4$. By 4.2.13(ii), there is a tetrad T of S_{ij} with $T \subseteq (\Omega \setminus O_1) \cap \Phi$. Then

$$(\pm 4)_{T_{\text{even}}} \cdot \ell = \begin{cases} 0 & (3 \text{ times}), \\ 32 & (\text{once}), \end{cases} \quad \text{and}$$
$$(\pm 4)_{T_{\text{odd}}} \cdot \ell = \pm 16 \ \ (4 \text{ times}).$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Now we observe that there is a tetrad T_1 of S_{ij} with $T_1 \subseteq O_1$, $|T_1 \cap \Phi| = 2$ and $14 \notin T_1$. So $(\pm 4)_{T_1} \cdot \ell$ is equal to 0 (twice) and ± 16 (twice). In the case of $S_{ij_{\text{even}}}$ we see, using 4.2.14, that $w \cdot \ell = 0$ for $4 + 2 + 2 + 3 + 2 = 13$ vectors w in the cross m. So Lemma 2.5 implies that $m = S_{ij_{\text{even}}} \in \alpha_2^2(\ell)$. For $T_2 = O_1 \setminus T_1$, we check that $(\pm 4)_{T_{2odd}} \cdot \ell$ is equal to 0 (once), ± 16 (twice) and ± 32 (once). Hence, by Lemma 2.5, $m = S_{ij_{\text{odd}}} \in \alpha_3^2(\ell)$, so proving 4.2.15.

4.2.16. *If $(i, j) \in I_5$, then $m \in \alpha_3^2(\ell)$.*

By 4.2.14, for each of the four tetrads T in $\Omega \setminus O_1$ we have $(\pm 4)_T \cdot u = 0$ (twice) and $(\pm 4)_T \cdot u = \pm 16$ (twice). For T a tetrad contained in O_1, we check that

$$(\pm 4)_T \cdot u = \begin{cases} \left.\begin{array}{l} 0 \text{ (twice)}, \\ \pm 16 \text{ (twice)}, \end{array}\right\} & \text{if } 14 \notin T, \text{ and} \\ \left.\begin{array}{l} 0 \text{ (once)}, \\ \pm 16 \text{ (twice)}, \\ \pm 32 \text{ (once)}, \end{array}\right\} & \text{if } 14 \in T. \end{cases}$$

Therefore, using Lemma 2.5, Assertion 4.2.16 holds.

There is one further sextet to examine.

4.2.17. *$S_{31_{\text{even}}} \in \alpha_2^2(\ell)$ and $S_{31_{\text{odd}}} \in \alpha_3^2(\ell)$.*

Let T be a tetrad of S_{31} contained in O_1. If $14 \in T$, then we always get $(\pm 4)_T \cdot u = \pm 16$. Meanwhile, if $14 \notin T$, then

$$(\pm 4)_{T_{\text{odd}}} \cdot u = \begin{cases} 0 \text{ (three times)}, \\ \pm 32 \text{ (once)}, \end{cases} \qquad \text{and}$$
$$(\pm 4)_{T_{\text{even}}} \cdot u = \pm 16 \text{ (four times)}.$$

Hence, by 4.2.14 and Lemma 2.5, $S_{31_{\text{odd}}} \in \alpha_3^2(\ell)$. For w any vector in $S_{31_{\text{even}}}$, $w \cdot u = 0, \pm 16$, whence $S_{31_{\text{even}}} \in \alpha_2^1(\ell) \cup \alpha_2^2(\ell)$. From Appendix A,

$$u^{\times} = u_{13}^{*\times} \qquad \text{and} \qquad u_{13}^{*} \cdot (-4v_{17} - 4v_{11} - 4v_{22} - 4v_{19}) = 32$$

and so we infer, using Lemma 2.5, that $S_{31_{\text{even}}} \in \alpha_2^2(\ell)$.

We now let m be a cross based on O_1. So we have $m = \tilde{v}^{\times}$ where $\tilde{v}_\infty = 6$ and $\tilde{v}_i = 0$ for all $i \notin O_1$.

4.2.18. *Suppose that either $\tilde{v}_8 = \tilde{v}_{20} = \tilde{v}_{18} = 2$, or that exactly two of $\tilde{v}_8$, $\tilde{v}_{20}$ and $\tilde{v}_{18}$ are equal to -2. Then $m \in \alpha_3^1(\ell)$.*

If $\tilde{v}_8 = \tilde{v}_{20} = \tilde{v}_{18} = 2$, then

$$u \cdot v = -8 \begin{cases} \text{if } v_{14} = 2; \\ \text{if } v_{18} = -2. \end{cases}$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Also, if exactly two of $\tilde{v}_8$, $\tilde{v}_{20}$ and $\tilde{v}_{18}$ are equal to -2, then

$$u \cdot v = \begin{cases} -8 & \text{if } v_{14} = 2, \\ -24 & \text{if } v_{14} = -2. \end{cases}$$

Hence, using Lemma 2.5, we see that $m \in \alpha_3^1(\ell)$.

4.2.19. *Let $w \in u^\times (= \ell)$ and assume that $w_i = 0, \pm 2$ for $i \in O_1$. Then $|w \cdot v| \leqslant 24$. Further, if there exists a $w' \in u^\times$ such that $w' \cdot v \in \{0, \pm 16\}$, then $|w \cdot v| \leqslant 16$.*

Since $\tilde{v}_\infty = 6$, $\tilde{v}_i = \pm 2$ for $i \in O_1 \setminus \{\alpha\}$ and $\tilde{v}_i = 0$ for all $i \notin O_1$,

$$|w \cdot v| \leqslant 6.2 + 2.2 + 2.2 + 2.2 = 24.$$

If we have a w' in $u^\times$ with the given properties then, by Lemma 2.5, $m \notin \alpha_3^1(\ell)$ and hence $|w \cdot v| \neq 24$. Thus $|w \cdot v| \leqslant 16$, so giving us 4.2.19.

4.2.20. *Let $w \in u^\times$ and suppose that $|w \cdot v| \geqslant 32$. Then $w \in \{u = u_1^*, u_2^*, u_3^*, u_4^*, u_{17}^*\}$.*

By 4.2.19. we must have $w_j = \pm 4$ for some $j \in O_1$. If $j \in \{\infty, 8, 20, 18\}$, then the only way to obtain $|w \cdot v| \geqslant 32$ is to have $j = \infty$. So $w = u_{17}$. Thus 4.2.20 holds.

We shall describe vectors in $v^\times$ by

$$\boxed{\begin{array}{cc} 6 & 2 \\ & \pm \\ & \pm \\ & \pm \end{array}} \quad \text{or} \quad \boxed{\begin{array}{cc} 6 & -2 \\ & \pm \\ & \pm \\ & \pm \end{array}},$$

the sign in the indicated position meaning that the entry is ± 2. For co-ordinates 0, 3 and 15, all possibilities for ± 2 are allowed (subject to the resulting vector being in the Leech lattice). So, for example,

$$\boxed{\begin{array}{cc} 6 & 2 \\ & - \\ & - \\ & - \end{array}}$$

stands for any of the vectors:

$$\boxed{\begin{array}{cc} 6 & 2 \\ 2 & -2 \\ 2 & -2 \\ 2 & -2 \end{array}}, \quad \boxed{\begin{array}{cc} 6 & 2 \\ 2 & -2 \\ -2 & -2 \\ -2 & -2 \end{array}}, \quad \boxed{\begin{array}{cc} 6 & 2 \\ -2 & -2 \\ 2 & -2 \\ -2 & -2 \end{array}}, \quad \text{or} \quad \boxed{\begin{array}{cc} 6 & 2 \\ -2 & -2 \\ -2 & -2 \\ 2 & -2 \end{array}}.$$

4.2.21. *Suppose that v is of type*

$$\boxed{\begin{array}{cc} 6 & 2 \\ & - \\ & - \\ & - \end{array}}.$$

Then $m \in \alpha_2^1(\ell) \cup \alpha_2^2(\ell)$ when $\tilde{v}_0 = \tilde{v}_3 = \tilde{v}_{15} = 2$ and $m \in \alpha_3^2(\ell)$ otherwise.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

For

$$v = \boxed{\begin{array}{rr} 6 & 2 \\ 2 & -2 \\ 2 & -2 \\ 2 & -2 \end{array}},$$

we calculate that $v \cdot u_{17}^* = -16$, and that $|v \cdot u_i^*| \leqslant 16$ for $i = 1, 2, 3, 4$. Note that $m \notin \alpha_1(\ell)$ because of Theorem 2.4. Hence, by 4.2.20 and Lemma 2.5, $m = v^\times \in \alpha_2^1(\ell) \cup \alpha_2^2(\ell)$. For v with $\{\tilde{v}_0, \tilde{v}_3, \tilde{v}_{15}\} \neq \{2\}$ we have $v \cdot u_{17}^* = -32$ and, for $i = 1, 2, 3, 4$, we have $v \cdot u_i^* \in \{0, \pm 16\}$. Therefore, using 4.2.20 and Lemma 2.5 again, we deduce that $m = v^\times \in \alpha_3^2(\ell)$.

4.2.22. *Suppose that v is of type*

$$\boxed{\begin{array}{rr} 6 & -2 \\ & - \\ & - \\ & - \end{array}}.$$

Then $m \in \alpha_2^2(\ell)$ when $\tilde{v}_0 = \tilde{v}_3 = \tilde{v}_{15} = -2$, and $m \in \alpha_3^2(\ell)$ otherwise.

If

$$v = \boxed{\begin{array}{rr} 6 & -2 \\ -2 & -2 \\ -2 & -2 \\ -2 & -2 \end{array}},$$

then we see that $u_1^* \cdot v = u_{17}^* \cdot v = -32$, whence $m = v^\times \in \alpha_2^2(\ell)$ by Lemma 2.5.

Now suppose that $\{\tilde{v}_0, \tilde{v}_3, \tilde{v}_{15}\} \neq \{-2\}$. We calculate that

$$u_1^* \cdot v = -32, \qquad u_{17}^* \cdot v = -16 \qquad \text{and} \qquad u_i^* \cdot v \neq \pm 32 \quad \text{for } i = 2, 3, 4.$$

So 4.2.20 and Lemma 2.5 imply that $m = v^\times \in \alpha_3^2(\ell)$.

4.2.23. *Suppose that v is of type*

$$\boxed{\begin{array}{rr} 6 & 2 \\ & + \\ & + \\ & - \end{array}}$$

(by which we mean that the '$-$' can be in any of the three positions).

Let

$$v' = \boxed{\begin{array}{rr} 6 & 2 \\ -2 & -2 \\ 2 & 2 \\ -2 & 2 \end{array}}, \qquad v'' = \boxed{\begin{array}{rr} 6 & -2 \\ -2 & 2 \\ -2 & -2 \\ 2 & 2 \end{array}} \qquad \text{and} \qquad v''' = \boxed{\begin{array}{rr} 6 & 2 \\ 2 & 2 \\ -2 & 2 \\ -2 & -2 \end{array}}.$$

Then $v'^\times, v''^\times, v'''^\times \in \alpha_2^2(\ell)$, and the remainder are in $\alpha_3^2(\ell)$.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

First we observe that for v with $\{\tilde{v}_0, \tilde{v}_3, \tilde{v}_{15}\} \neq \{2\}$, we have $v \cdot u_{17}^* = -32$. Further, we have

$$v' \cdot u_3^* = -32, \quad v'' \cdot u_4^* = -32 \quad \text{and} \quad v''' \cdot u_2^* = 32,$$

which implies that $v'^{\times}, v''^{\times}, v'''^{\times} \in \alpha_2^2(\ell)$. By calculation we see for $v \neq v', v'', v'''$ that $v \cdot u_i^* \neq \pm 32$ ($i = 1, 2, 3, 4$) and so for such v, we see that $v^{\times} \in \alpha_3^2(\ell)$, by 4.2.20 and Lemma 2.5.

Now we consider v when $\{\tilde{v}_0, \tilde{v}_3, \tilde{v}_{15}\} = \{2\}$. Then

$$\boxed{\begin{array}{cr} 6 & 2 \\ 2 & -2 \\ 2 & 2 \\ 2 & 2 \end{array}} \cdot u_i^* = \begin{cases} -32, & i = 3, \\ 0, \pm 16, & i = 1, 2, 4, 17; \end{cases}$$

$$\boxed{\begin{array}{cr} 6 & 2 \\ 2 & 2 \\ 2 & -2 \\ 2 & 2 \end{array}} \cdot u_i^* = \begin{cases} -32, & i = 4, \\ 0, \pm 16, & i = 1, 2, 3, 17; \end{cases}$$

$$\boxed{\begin{array}{cr} 6 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & -2 \end{array}} \cdot u_i^* = \begin{cases} 32, & i = 2, \\ 0, \pm 16, & i = 1, 3, 4, 17. \end{cases}$$

Therefore by 4.2.20 and Lemma 2.5, when $\{\tilde{v}_0, \tilde{v}_3, \tilde{v}_{15}\} = \{2\}$, $v^{\times} \in \alpha_3^2(\ell)$. This completes the proof of 4.2.23.

4.2.24. *Suppose that v is of type*

$$\boxed{\begin{array}{cr} 6 & -2 \\ & + \\ & + \\ & - \end{array}},$$

and let

$$v'' = \boxed{\begin{array}{cr} 6 & -2 \\ 2 & -2 \\ 2 & 2 \\ 2 & -2 \end{array}} \qquad \text{and} \qquad v''' = \boxed{\begin{array}{rr} 6 & -2 \\ 2 & 2 \\ 2 & -2 \\ -2 & 2 \end{array}}.$$

Then $v'^{\times}, v''^{\times}, v'''^{\times} \in \alpha_2^1(\ell) \cup \alpha_2^2(\ell)$ and the remainder are in $\alpha_3^2(\ell)$.

Calculating that, when $\tilde{v}_0 = \tilde{v}_3 = \tilde{v}_{15} = -2$, $v \cdot u_{17}^* = -32$ and $v \cdot u_i^* = 0, \pm 16$ for $i = 1, 2, 3, 4$, Assertion 4.2.20 and Lemma 2.5 yield that $m = v^{\times} \in \alpha_3^2(\ell)$.

Now assume that $\{\tilde{v}_0, \tilde{v}_3, \tilde{v}_{15}\} = \{2, -2\}$. Then $v \cdot u_{17}^* = v \cdot u_1^* = -16$. We check, for $v \notin \{v', v'', v'''\}$, that $v \cdot u_j^* = \pm 32$ for each exactly one $j \in \{2, 3, 4\}$, and $v \cdot u_i^* = 0, \pm 16$ for $i \in \{2, 3, 4\} \setminus \{j\}$, while for $v \in \{v', v'', v'''\}$, we see that $v \cdot u_i^* = -16$ for each $i \in \{2, 3, 4\}$. Consequently, again using 4.2.20 and Lemma 2.5, we have verified 4.2.24.

Our next step is to remove the ambiguity in Assertions 4.2.21 and 4.2.24.

4.2.25. (i) *For*

$$v = \begin{bmatrix} 6 & 2 \\ 2 & -2 \\ 2 & -2 \\ 2 & -2 \end{bmatrix},$$

we have $m = v^\times \in \alpha_2^2(\ell)$.

(ii) *Let*

$$v' = \begin{bmatrix} 6 & -2 \\ -2 & 2 \\ 2 & 2 \\ 2 & -2 \end{bmatrix}, \qquad v'' = \begin{bmatrix} 6 & -2 \\ 2 & -2 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \textit{and} \qquad v''' = \begin{bmatrix} 6 & -2 \\ 2 & 2 \\ 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

Then $v'^\times, v''^\times, v'''^\times \in \alpha_2^2(\ell)$.

Let

$$w = \begin{bmatrix} 2 & 6 \\ -2 & 2 \\ -2 & 2 \\ -2 & 2 \end{bmatrix}.$$

Observing that $w^\times = v^\times$ and that $w \cdot u_1^* = 32$, we see that Assertion 4.2.21 and Lemma 2.5 show that $m = v^\times \in \alpha_2^2(\ell)$. So part (i) holds.

Now, $v'^\times = w'^\times$, $v''^\times = w''^\times$, $v'''^\times = w'''^\times$, where

$$w' = \begin{bmatrix} 2 & -6 \\ 2 & -2 \\ -2 & -2 \\ -2 & 2 \end{bmatrix}, \qquad w'' = \begin{bmatrix} 2 & -6 \\ -2 & 2 \\ 2 & -2 \\ -2 & -2 \end{bmatrix}, \qquad \text{and} \qquad w''' = \begin{bmatrix} 2 & -6 \\ -2 & -2 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

We readily see that $w' \cdot u_1^* = w'' \cdot u_1^* = w''' \cdot u_1^* = -32$ whence, by 4.2.24 and Lemma 2.5, $v'^\times, v''^\times, v'''^\times \in \alpha_2^2(\ell)$, which proves Assertion 4.2.25.

Counting up, we have:

$$\begin{aligned} \text{in } \alpha_2^2(\ell) :\ & 6 \text{ (by 4.2.15)} + 1 \text{ (by 4.2.17)} + 1 \text{ (by 4.2.22)} + 3 \text{ (by 4.2.23)} + 4 \text{ (by 4.2.25)} \\ & = 15; \\ \text{in } \alpha_3^1(\ell) :\ & 16 \times 2 \text{ (by 4.2.12)} + 32 \text{ (by 4.2.18)} \\ & = 64; \\ \text{in } \alpha_3^2(\ell) :\ & 1(\ell_0) + 6 \text{ (by 4.2.15)} + 12 \times 2 \text{(by 4.2.16)} + 1 \text{ (by 4.2.17)} + 3 \text{ (by 4.2.21)} \\ & +3 \text{ (by 4.2.22)} + 9 \text{ (by 4.2.23)} + 9 \text{ (by 4.2.24)} \\ & = 56. \end{aligned}$$

So at long last we have shown that $m \in [0, 0, 15, 64, 56]_{\mathcal{JP}}$. Hence the following assertion holds.

4.2.26. $\mathcal{D}_{4^2}(O_1)^\times \cup \mathcal{T}_1(O_1)^\times \cup \mathcal{A}_4(O_1)^{[12]\times} \cup \mathcal{A}_8(O_1)^{[8]\times} \subseteq [0, 0, 15, 64, 56]_{\mathcal{JP}}$.

Thus we see that the containments in Assertions 4.2.7, 4.2.9, 4.2.11 and 4.2.26 are equalities, from which Theorem 4.2 follows.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

4.3. Just as in Section 4.2, we put $g = \varepsilon_{O_1}$. Let $\mathcal{S}_2^2$ denote the following set of seven sextets:

$$\left\{ S_{44} = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}, \quad S_{54} = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ \hline \end{array}, \right.$$

$$S_{56} = \begin{array}{|cc|cc|cc|} \hline 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ \hline \end{array}, \quad S_{31} = \begin{array}{|cc|cc|cc|} \hline 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 1 & 4 & 4 & 4 & 4 \\ 2 & 1 & 5 & 5 & 5 & 5 \\ 2 & 1 & 6 & 6 & 6 & 6 \\ \hline \end{array},$$

$$S_{41} = \begin{array}{|cc|cc|cc|} \hline 1 & 2 & 3 & 3 & 4 & 4 \\ 2 & 1 & 4 & 4 & 3 & 3 \\ 1 & 2 & 5 & 5 & 6 & 6 \\ 1 & 2 & 6 & 6 & 5 & 5 \\ \hline \end{array}, \quad S_{51} = \begin{array}{|cc|cc|cc|} \hline 1 & 2 & 3 & 3 & 4 & 4 \\ 1 & 2 & 5 & 5 & 6 & 6 \\ 2 & 1 & 4 & 4 & 3 & 3 \\ 1 & 2 & 6 & 6 & 5 & 5 \\ \hline \end{array},$$

$$\left. S_{61} = \begin{array}{|cc|cc|cc|} \hline 1 & 2 & 3 & 3 & 4 & 4 \\ 1 & 2 & 5 & 5 & 6 & 6 \\ 1 & 2 & 6 & 6 & 5 & 5 \\ 2 & 1 & 4 & 4 & 3 & 3 \\ \hline \end{array} \right\}.$$

Now we define

$$\mathcal{P} = \left\{ \ell_0, S_{\text{even}}, S_{\text{odd}} : S \in \mathcal{S}_2^2 \right\}.$$

Clearly, $|\mathcal{P}| = 15$ and we shall see that $\mathcal{P}$ may be identified with the 1-spaces of a four-dimensional vector space over GF(2).

Set $L = \operatorname{Stab}_{C_G(g)}(\mathcal{P})$. Observe that $L_{\ell_0}/O_2(G_{\ell_0})$, in addition, to stabilizing O_1, may be written as $O_2(\operatorname{Stab}_H(O_1))\operatorname{Stab}_H(\{11, 17\})$. Consequently, $L_{\ell_0} \cong 2^{11}2^4 2^3 L_3(2)$ $(L_{\ell_0}/O_2(G_{\ell_0}) = \operatorname{Stab}_H O_1 \cap \operatorname{Stab}_H \{17, 11\})$. Put $\bar{K} = L_{\ell_0}/O_2(G_{\ell_0}) \cong 2^4 2^3 L_3(2)$.

THEOREM 4.3. *We have $L \cong 2^{1+8}2^6 A_8$ and the L-orbits on crosses are as shown in Tables 9 and 10.*

Proof. Clearly, L-orbits must be the union of certain L_{ℓ_0}-orbits, which are given in 3.2. Also, as $L \leqslant C_G(g)$ by definition of L, unions of appropriate L-orbits must yield the $C_G(g)$-orbits already calculated in Section 4.2. We also observe that $\zeta_T \in L$ for any tetrad which is a tetrad of a sextet belonging to $\mathcal{S}_2^2$. Set $T_1 = \{17, 11, 22, 19\}$, $T_2 = \{3, 20, 15, 18\}$ and $T_3 = \{17, 11, 21, 6\}$. From 3.2, L_{ℓ_0} acts transitively on $\mathcal{P} \setminus \{\ell_0\}$. Now $(8v_\infty)\zeta_{T_2} = 4v_\infty - 4v_{14} - 4v_0 - 4v_8$ and therefore $\mathcal{P}$ is an L-orbit. Because $[L : L_{\ell_0}] = 15$, the index of $L/O_2(C_G(g))$ in $C_G(g)/O_2(C_G(g)) \cong O_8^+(2)$ is 135 and so $L \cong 2^{1+8}2^6 A_8$ by [2].

By 3.2, $\mathcal{S}_{4^2}^{(2)}(O_1)$ is a $\bar{K}$-orbit of size 28 with representative S_0. Now

$$(v_\infty + v_0 + v_3 + v_{15})\zeta_{T_1} = \begin{array}{|cc|cc|cc|} \hline 2 & -6 & & & & \\ 2 & -2 & & & & \\ 2 & -2 & & & & \\ 2 & -2 & & & & \\ \hline \end{array}$$

and since L contains all the sign changes, we find that $\mathcal{S}_{4^2}^{(2)}(O_1)^\times \cup \{O_1\}^\times$ is contained in an L-orbit. Noting that this set together with $\mathcal{P}$ gives $\mathcal{J}\mathcal{P}$, we see that $\mathcal{S}_{4^2}^{(2)}(O_1)^\times \cup \{O_1\}^\times$ is an L-orbit and it is easy to check that it is contained in $[7, 8, 0, 0, 0]_{\mathcal{P}}$.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 9: $L \cong 2^{1+8}2^6 A_8$-orbits (see Theorem 4.3 and Table 10).

Orbit	Size	Representative
$\mathcal{JP}$		
$\mathcal{P}$	15	ℓ_0
$[7, 8, 0, 0, 0]_{\mathcal{P}}$	120	$S_{0_{\text{even}}}$
$[7, 8, 120, 0, 0]_{\mathcal{JP}}$		
$[7, 0, 8, 0, 0]_{\mathcal{P}}$	$2^4.3.5$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & 4 & 4 & 4 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \hline\end{array}^{\times}$
$[1, 0, 14, 0, 0]_{\mathcal{P}}$	$2^7.3.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & & & & \\ & & 4 & & & 4 \\ 4 & & & & & \\ & & & & & \\ \hline\end{array}^{\times}$
$[3, 4, 8, 0, 0]_{\mathcal{P}}$	$2^5.3.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & & & & \\ & & 4 & & & \\ 4 & & & & & \\ & & 4 & & & \\ \hline\end{array}^{\times}$
$[0, 1, 14, 0, 0]_{\mathcal{P}}$	$2^{10}.3.5$	$\begin{array}{\|cc\|cc\|cc\|}\hline & & 6 & & -2 & \\ & & 2 & & 2 & \\ & & 2 & & 2 & \\ & & 2 & & 2 & \\ \hline\end{array}^{\times}$
$\Psi_1 = [1, 10, 60, 64, 0]_{\mathcal{JP}}$		
$[1, 2, 12, 0, 0]_{\mathcal{P}}$	$2^8.3.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline & 4 & 4 & & & \\ & 4 & & & & \\ & & & & & \\ 4 & & & & & \\ \hline\end{array}^{\times}$
$[0, 1, 6, 8, 0]_{\mathcal{P}} \cap \Psi_1$	$2^{11}.3.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 6 & & -2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & \\ \hline\end{array}^{\times}$
$\Psi_2 = [0, 1, 30, 72, 32]_{\mathcal{JP}}$		
$[0, 1, 6, 8, 0]_{\mathcal{P}} \cap \Psi_2$	$2^{12}.3.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline\end{array}^{\times}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

We next look within $[7, 8, 120, 0, 0]_{\mathscr{J}\mathscr{P}}$ for L-orbits. Set $w_1 = 4v_\infty + 4v_{14} + 4v_{17} + 4v_{11}$. Then $w_1^\times \in \mathscr{S}_{2^4}^{(1)}(O_1)^\times$ and

$$(w_1^\times)\zeta_{T_1} = \left.\begin{array}{|cc|cc|cc|}\hline 2 & 2 & & & 4 & 4 \\ -2 & -2 & & & & \\ -2 & -2 & & & & \\ -2 & -2 & & & & \\ \hline \end{array}\right.^{\times} \in \mathscr{A}_0(O_1)^{[1]\times}$$

Therefore, using Lemma 2.5, we deduce that the next assertion holds.

4.3.1. $\Pi_1 = \mathscr{S}_{2^4}^{(1)}(O_1)^\times \cup \mathscr{A}_0(O_1)^{[1]\times} \subseteq [7, 0, 8, 0, 0]_{\mathscr{P}}$, *and* Π_1 *is contained in an* L*-orbit.*

Table 10: $L \cong 2^{1+8}2^6 A_8$-orbits (*continued*; see Theorem 4.3 and Table 9).

Orbit	Size	Representative
$[0, 0, 3, 8, 4]_{\mathscr{P}}$	$2^{15}.3.5.7$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & 2 & 2 & 2 & \\ -2 & 2 & & & & 2 \\ 2 & 2 & & & & 2 \\ 2 & 2 & & & & 2 \\ \hline \end{array}\right.^{\times}$
$[0, 0, 15, 64, 56]_{\mathscr{J}\mathscr{P}}$		
$\mathscr{A}_4^{(2)}(O_1)^{[4]\times} \cup \mathscr{T}_1^{(1)}(O_1)^\times$ $\subseteq [0, 0, 7, 8, 0]_{\mathscr{P}}$	$2^{14}.3.5$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline 2 & & 2 & & 4 & 4 \\ 2 & & 2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & \\ \hline \end{array}\right.^{\times}$
$\mathscr{A}_4^{(2)}(O_1)^{[8]\times} \cup \mathscr{T}_1^{(2)}(O_1)^\times$ $\cup \mathscr{D}_{4^2}^{(3)}(O_1)^\times$ $\subseteq [0, 0, 1, 8, 6]_{\mathscr{P}}$	$2^{15}.3.5.7$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 4 & 2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & 4 \\ 2 & & 2 & & & \\ \hline \end{array}\right.^{\times}$
$\mathscr{A}_4^{(1)}(O_1)^{[12]\times} \cup \mathscr{A}_8^{(1)}(O_1)^{[8]\times}$ $\cup \mathscr{D}_{4^2}^{(1)}(O_1)^\times$ $\subseteq [0, 0, 7, 0, 8]_{\mathscr{P}}$	$2^{11}.3.5.7$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline & 4 & 4 & & 2 & 2 \\ & & & & 2 & 2 \\ & & & & 2 & 2 \\ & & & & 2 & 2 \\ \hline \end{array}\right.^{\times}$
$\mathscr{A}_8^{(2)}(O_1)^{[8]\times} \cup \mathscr{D}_{4^2}^{(2)}(O_1)^\times$ $\subseteq [0, 0, 1, 0, 14]_{\mathscr{P}}$	$2^{14}.3.5$	$\left.\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & 2 & 4 & 2 & \\ & & 2 & & 2 & \\ & & 2 & & 2 & \\ & & 2 & & 2 & \\ \hline \end{array}\right.^{\times}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

For $w_2 = 4v_{19} + 4v_{12} + 4v_{21} + 4v_6$ we have $w_2^\times \in \mathcal{S}_{2^4}^{(2)}(O_1)^\times$, and

$$(w_2^\times)\zeta_{T_2} = \left[\begin{array}{|cc|cc|cc|} & & & & -2 & 2 \\ & & & & -2 & -2 \\ & & & & -2 & -6 \\ & & & & -2 & -2 \end{array}\right]^\times \in \mathcal{O}_0^{(1)}(O_1)^\times,$$

and

$$(w_2^\times)\zeta_{T_1} = \left[\begin{array}{|cc|cc|cc|} & & 2 & 2 & 2 & -2 \\ & & & & & \\ & & -2 & -2 & 2 & -2 \\ & & -4 & -4 & & \end{array}\right] \in \mathcal{A}_8^{(1)}(O_1)^{[1]\times}.$$

With the aid of Lemma 2.5, this gives the following assertion.

4.3.2. $\Pi_2 = \mathcal{S}_{2^4}^{(2)}(O_1)^\times \cup \mathcal{O}_0^{(1)}(O_1)^\times \cup \mathcal{A}_8^{(1)}(O_1)^{[1]\times} \subseteq [3, 4, 8, 0, 0]_{\mathcal{P}}$, *and* Π_2 *is contained in an* L*-orbit.*

Now let $w_3 = 4v_\infty + 4v_3 + 4v_4 + 4v_9$ and $w_4 = 4v_{11} + 4v_{13} + 4v_{22} + 4v_{12}$, and observe that $w_3^\times, w_4^\times \in \mathcal{S}_{2^4}^{(3)}(O_1)^\times$. Then

$$(w_3^\times)\zeta_{T_1} = \left[\begin{array}{|cc|cc|cc|} 2 & -2 & & & & \\ -2 & -2 & & -4 & -4 & \\ 2 & -2 & & & & \\ -2 & -2 & & & & \end{array}\right]^\times \in \mathcal{A}_0(O_1)^{[14]\times},$$

and

$$(w_4^\times)\zeta_{T_2} = \left[\begin{array}{|cc|cc|cc|} & & -4 & & 2 & -2 \\ & & -4 & & -2 & -2 \\ & & & & 2 & -2 \\ & & & & -2 & -2 \end{array}\right]^\times \in \mathcal{A}_8^{(1)}(O_1)^{[6]\times},$$

whence, using Lemma 2.5, we see that 4.3.3 holds.

4.3.3. $\Pi_3 = \mathcal{S}_{2^4}^{(3)}(O_1)^\times \cup \mathcal{A}_0(O_1)^{[14]\times} \cup \mathcal{A}_8^{(1)}(O_1)^{[6]\times} \subseteq [1, 0, 14, 0, 0]_{\mathcal{P}}$ *and* Π_3 *is contained in an* L*-orbit.*

Set

$$w_5 = \left[\begin{array}{|cc|cc|cc|} & & 6 & & 2 & \\ & & -2 & & 2 & \\ & & 2 & & 2 & \\ & & 2 & & 2 & \end{array}\right]; \; w_5^\times \in \mathcal{O}_0^{(2)}(O_1)^\times.$$

Since

$$(w_5^\times)\zeta_{T_1} = \left[\begin{array}{|cc|cc|cc|} & & -2 & 4 & 2 & 4 \\ & & -2 & & 2 & \\ & & & -2 & & -2 \\ & & & -2 & & -2 \end{array}\right]^\times \in \mathcal{A}_8^{(2)}(O_1)^{[7]\times},$$

we have Assertion 4.3.4.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

4.3.4. *$\Pi_4 = \mathcal{O}_0^{(2)}(O_1)^\times \cup \mathcal{A}_8^{(2)}(O_1)^{[7]\times} \subseteq [0, 1, 14, 0, 0]_{\mathcal{P}}$ and Π_3 is contained in an L-orbit.*

We know that $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 = [7, 8, 120, 0, 0]_P$, and from Assertions 4.3.1–4.3.4 we see that Π_1, Π_2, Π_3 and Π_4 are clearly all L-orbits with all containments being equalities.

Next we examine $[1, 10, 60, 64, 0]_{\mathcal{JP}}$. Let $w_6 = 4v_{14} + 4v_8 + 4v_{15} + 4v_{17}$. So $w_6^\times \in \mathcal{S}_{15,3}(O_1)^\times$,

$$(w_6^\times)\zeta_{T_2} = \left[\begin{array}{cc|cc|cc} -4 & & 2 & -2 & & \\ -4 & & -2 & -2 & & \\ 2 & 2 & & & & \\ -2 & & & & & \end{array}\right]^\times \in \mathcal{A}_4^{(1)}(O_1)^{[3]\times},$$

and

$$(w_6^\times)\zeta_{T_3} = \left[\begin{array}{cc|cc|cc} & -2 & -2 & 2 & & \\ & -2 & & & & \\ & -6 & & & & \\ -2 & & & & 2 & 2 \end{array}\right]^\times \in \mathcal{O}_4^{(1)}(O_1)^\times.$$

So we get Assertion 4.3.5.

4.3.5. *$\Pi_5 = \mathcal{S}_{15,3}(O_1)^\times \cup \mathcal{A}_4^{(1)}(O_1)^{[3]\times} \cup \mathcal{O}_4^{(1)}(O_1)^\times \subseteq [1, 2, 12, 0, 0]_{\mathcal{P}}$ and Π_5 is contained in an L-orbit.*

Put

$$w_7 = \left[\begin{array}{cc|cc|cc} 6 & & -2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & \end{array}\right].$$

So $w_7^\times \in \mathcal{O}_4^{(2)}(O_1)^\times$.

From

$$(w_7^\times)\zeta_{T_3} = \left[\begin{array}{cc|cc|cc} 1 & -1 & 1 & -1 & -1 & -1 \\ -3 & -1 & 1 & -1 & -1 & -1 \\ -3 & -1 & 1 & -1 & -1 & -1 \\ 1 & -5 & 1 & -1 & -1 & -1 \end{array}\right]^\times \in \mathcal{T}_3(O_1)^\times$$

and

$$(w_7^\times)\zeta_{T_2} = \left[\begin{array}{cc|cc|cc} 2 & -4 & -2 & & & \\ -2 & -4 & 2 & & & \\ & 2 & & -2 & & \\ & 2 & & -2 & & \end{array}\right]^\times \in \mathcal{A}_4^{(2)}(O_1)^{[3]\times},$$

we obtain Assertion 4.3.6.

4.3.6. *$\Pi_6 = \mathcal{O}_4^{(2)}(O_1)^\times \cup \mathcal{T}_3(O_1)^\times \cup \mathcal{A}_4^{(2)}(O_1)^{[3]\times} \subseteq [0, 1, 6, 8, 0]_{\mathcal{P}}$ and Π_6 is contained in an L-orbit.*

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Since $\Pi_5 \cup \Pi_6 = [1, 10, 60, 64, 0]_{\mathcal{J}\mathcal{P}}$, Assertions 4.3.5 and 4.3.6 imply that Π_5 and Π_6 are L-orbits.

We now aim to break the $C_G(g)$-orbit $[0, 1, 30, 72, 32]_{\mathcal{J}\mathcal{P}}$ into L-orbits; as we shall see, there are two L-orbits here. From 3.2 , we know that $\mathcal{O}_2(O_1)$ is a $\bar{K}$-orbit, and

$$\begin{array}{|cc|cc|cc|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline \end{array}^{\times} \in \mathcal{O}_2(O_1)^{\times}.$$

Now

$$\zeta_{T_1} : \begin{array}{|cc|cc|cc|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline \end{array}^{\times} \longmapsto \begin{array}{|cc|cc|cc|}\hline 3 & -1 & 1 & 1 & 1 & 3 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ \hline \end{array}^{\times}$$

$$= w^{\times} \in \mathcal{T}_0(O_1)^{\times},$$

with the corresponding triad being $\{17, 11, 22\}$. There are two $\bar{K}$-orbits on $\mathcal{T}(O_1)$ which may be distinguished as follows. For $\mathcal{T}_0^{(1)}(O_1)$ the triad intersects the tetrads of the sextets in $\mathcal{S}_2^2$ in 3 (once), and in 2.1 (six times), while for $\mathcal{T}_0^{(2)}(O_1)$ the tetrad intersects the tetrads of the sextets in $\mathcal{S}_2^2$ in 2.1 (three times), and in 1^3 (four times). From this we deduce that $w^{\times} \in \mathcal{T}_0^{(1)}(O_1)^{\times}$. Next we note that

$$\zeta_{T_2} : \begin{array}{|cc|cc|cc|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline \end{array}^{\times} \longmapsto \begin{array}{|cc|cc|cc|}\hline 4 & -4 & & & & -2 \\ -2 & -2 & -2 & -2 & -2 & \\ & & & & -2 & \\ & & & & -2 & \\ \hline \end{array}^{\times}$$

$$= u^{\times} \in \mathcal{A}_6(O_1)^{\times}.$$

From 3.2, $\mathcal{A} \times \mathcal{J}$ for $X \in \mathcal{A}_6(O_1)$ breaks as $3 + 12$. Noting that here the corresponding involution interchanges $\{\infty, 14\}$ and $\{17, 11\}$ and interchanges no other pair in O_1, we see that $u^{\times} \in \mathcal{A}_6(O_1)^{[3]\times}$. Thus Assertion 4.3.7 holds.

4.3.7. $\quad \Pi_7 = \mathcal{O}_2(O_1)^{\times} \cup \mathcal{T}_0^{(1)}(O_1)^{\times} \cup \mathcal{A}_6(O_1)^{[3]\times}$ *is contained in an L-orbit.*

Now, starting with

$$w = \begin{array}{|cc|cc|cc|}\hline 4 & & 2 & 2 & 2 & \\ -2 & 2 & & & & 2 \\ 2 & 2 & & & & 2 \\ 2 & 2 & & & & 2 \\ \hline \end{array}$$

(so $w^{\times} \in \mathcal{D}_{2,6}(O_1)^{\times}$), we see that

$$(w^{\times})\zeta_{T_1} = \begin{array}{|cc|cc|cc|}\hline -1 & -1 & 1 & 1 & 1 & 3 \\ -3 & -3 & -1 & -1 & -1 & 1 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ 1 & -3 & -1 & -1 & -1 & 1 \\ \hline \end{array}^{\times} \in \mathcal{T}_0(O_1)^{\times}$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

and

$$(w^\times)\zeta_{T_3} = \left|\begin{array}{rr|rr|rr} 1 & -3 & 1 & 1 & 1 & -1 \\ -5 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 3 & 1 \end{array}\right|^\times \in \mathcal{T}_2(O_1)^\times.$$

The underlying triad of $(w^\times)\zeta_{T_1}$ is $\{1, 7, 10\}$, whence we conclude that $(w^\times)\zeta_{T_1} \in \mathcal{T}_0^{(2)}(O_1)^\times$. Let

$$u = \left|\begin{array}{rr|rr|rr} 1 & -1 & 1 & 1 & 1 & 3 \\ -3 & -1 & -1 & -1 & -1 & 1 \\ -3 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -3 & -3 & 1 & -1 \end{array}\right|.$$

Then $u^\times \in \mathcal{T}_2(O_1)^\times$ and

$$(u^\times)\zeta_{T_1} = \left|\begin{array}{cc|cc|cc} 2 & 2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ 4 & & & & 4 & 2 \end{array}\right|^\times \in \mathcal{A}_6(O_1)^{[12]\times}.$$

Hence, the next assertion holds.

4.3.8. $\Pi_8 = \mathcal{T}_0^{(2)}(O_1)^\times \cup \mathcal{T}_2(O_1)^\times \cup \mathcal{D}_{2,6}(O_1)^\times \cup \mathcal{A}_6(O_1)^{[12]\times}$ *is contained in an* L*-orbit.*

Using Lemma 2.5, it may be shown that $\Pi_7 \subseteq [0, 1, 6, 8, 0]_{\mathcal{P}}$ and $\Pi_8 \subseteq [0, 0, 3, 8, 4]_{\mathcal{P}}$. Since $\Pi_7 \cup \Pi_8 = [0, 1, 30, 72, 32]_{\mathcal{J}\mathcal{P}}$, we deduce that $\Pi_7 = [0, 1, 6, 8, 0]_{\mathcal{P}}$ and $\Pi_8 = [0, 0, 3, 8, 4]_{\mathcal{P}}$ are L-orbits.

Finally, we split $[0, 0, 15, 64, 56]_{\mathcal{J}\mathcal{P}}$ into L-orbits.

Employing Lemma 2.5 yet again, we may verify the containments in the first column. Put

$$\Pi = \mathcal{A}_4^{(2)}(O_1)^{[4]} \cup \mathcal{T}_1^{(1)}(O_1)^\times,$$

and note that $\Pi = [0, 0, 7, 8, 0]_{\mathcal{P}} \cap [0, 0, 15, 64, 56]_{\mathcal{J}\mathcal{P}}$.

So, since

$$\zeta_{T_3} : \left|\begin{array}{cc|cc|cc} 2 & & 2 & & 4 & -4 \\ 2 & & 2 & & & \\ 2 & & 2 & & & \\ 2 & & 2 & & & \end{array}\right|^\times \longmapsto \left|\begin{array}{rr|rr|rr} -1 & -1 & -1 & 1 & 3 & -5 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -3 & 1 & -1 & 1 & 1 \end{array}\right|^\times,$$

we deduce that Π is an L-orbit.

For $\Pi = \mathcal{A}_4^{(1)}(O_1)^{[12]\times} \cup \mathcal{A}_8^{(1)}(O_1)^{[8]\times} \cup \mathcal{D}_{4^2}^{(1)}(O_1)^\times$, we observe that

$$\left|\begin{array}{cc|cc|cc} 2 & 2 & 2 & 2 & & \\ 2 & 2 & 2 & 2 & 4 & \\ & & & & & \\ & & & 4 & & \end{array}\right|^\times \in \mathcal{A}_4^{(1)}(O_1)^{[12]\times}$$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

and

$$\left.\begin{array}{|cc|cc|cc|}\hline & 4 & 4 & & 2 & 2 \\ & & & & 2 & 2 \\ & & & & 2 & 2 \\ & & & & 2 & 2 \\ \hline\end{array}\right.^{\times} \in \cup \mathcal{A}_8^{(1)}(O_1)^{[8]\times}.$$

Now

$$\zeta_{T_1} : \left.\begin{array}{|cc|cc|cc|}\hline 2 & 2 & 2 & 2 & & \\ 2 & 2 & 2 & 2 & 4 & \\ & & & & & \\ & & & 4 & & \\ \hline\end{array}\right.^{\times} \longmapsto \left.\begin{array}{|cc|cc|cc|}\hline & & & & 2 & 2 \\ & & & & & -4 \\ -2 & -2 & & & & \\ -2 & -2 & -2 & 2 & -2 & -2 \\ \hline\end{array}\right.^{\times}$$

and

$$\zeta_{T_1} : \left.\begin{array}{|cc|cc|cc|}\hline & 4 & 4 & & 2 & 2 \\ & & & & 2 & 2 \\ & & & & 2 & 2 \\ & & & & 2 & 2 \\ \hline\end{array}\right.^{\times} \longmapsto \left.\begin{array}{|cc|cc|cc|}\hline & 2 & & 4 & 2 & 2 \\ -2 & & -2 & -2 & & \\ -2 & & -2 & -2 & & \\ -2 & & -2 & -2 & & \\ \hline\end{array}\right.^{\times}$$

yield that Π is an L-orbit.

We now show that

$$\Pi = \mathcal{A}_4^{(2)}(O_1)^{[8]\times} \cup \mathcal{T}_1^{(2)}(O_1)^{\times} \cup \mathcal{D}_{4^2}^{(3)}(O_1)^{\times}$$

is an L-orbit. Let $\ell \in \Pi$. Since $\ell \in [0, 0, 1, 8, 6]_{\mathcal{P}}$, there is a unique cross in $\mathcal{P}$, say m, such that $m \in \alpha_2^2(\ell)$. Hence $L_\ell \leqslant L_m$; consequently, as $[L : L_m] = |\mathcal{P}| = 15$, we know that 15 divides $[L : L_\ell]$. Since

$$\begin{aligned} |\mathcal{A}_4^{(2)}(O_1)^{[8]\times}| &= 2^{11}.7, \\ |\mathcal{T}_1^{(2)}(O_1)^{\times}| &= 2^{18}.7, \quad \text{and} \\ |\mathcal{D}_{4^2}^{(3)}(O_1)^{\times}| &= 2^{16}.3.7, \end{aligned}$$

this forces the L-orbit of ℓ to be equal to Π, thereby completing the proof of Theorem 4.3.

THEOREM 4.4. *Let $L \leqslant G_{\ell_0}$ be such that $O_2(G_{\ell_0}) \leqslant L$ and $L/O_2(G_{\ell_0}) = \mathrm{Stab}_H\, \Phi$ where $\Phi = \{\infty, 0, 3\}$. Thus $L \cong 2^{11}L_3(4)S_3$, and the L-orbits on crosses are as shown in Tables 11, 12 and 13.*

THEOREM 4.5. *Let $L \leqslant G_{\ell_0}$ be such that $O_2(G_{\ell_0}) \leqslant L$ and $L/O_2(G_{\ell_0}) = \mathrm{Stab}_H(D)$, where D is the duum*

$$\begin{array}{|cc|cc|cc|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline\end{array}\,.$$

So $L \cong 2^{11}M_{12}2$, and the L-orbits on crosses are shown in Tables 14 and 15.

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 11: $L \cong 2^{11}L_3(4)S_3$-orbits (see Theorem 4.4 and Tables 12 and 13).

Orbit	Size	Representative
$\{\ell_0\}$	1	ℓ_0
$\alpha_1(\ell_0)$:		
$\mathcal{S}_3(\Phi)^\times$	$21.2 = 2.3.7$	$\left[\begin{array}{cc\|cc\|cc} 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \end{array}\right]^\times$
$\mathcal{S}_{1,2}(\Phi)^\times$	$630.2 = 2^4.3^2.5.7$	$\left[\begin{array}{cc\|cc\|cc} 4 & 4 & & & & \\ 4 & 4 & & & & \\ & & & & & \\ & & & & & \end{array}\right]^\times$
$\mathcal{S}_{1^3}(\Phi)^\times$	$1120.2 = 2^6.5.7$	$\left[\begin{array}{cc\|cc\|cc} 4 & 4 & 4 & 4 & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right]^\times$
$\alpha_2^1(\ell_0)$:		
$\mathcal{O}_3(\Phi)^\times$	$21.2^6 = 2^6.3.7$	$\left[\begin{array}{cc\|cc\|cc} 6 & -2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \end{array}\right]^\times$
$\mathcal{O}_2(\Phi)^\times$	$168.2^6 = 2^9.3.7$	$\left[\begin{array}{cc\|cc\|cc} 6 & -2 & 2 & 2 & & \\ 2 & 2 & 2 & 2 & & \\ & & & & & \\ & & & & & \end{array}\right]^\times$
$\mathcal{O}_1(\Phi)^\times$	$360.2^6 = 2^9.3^2.5$	$\left[\begin{array}{cc\|cc\|cc} 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \end{array}\right]^\times$
$\mathcal{O}_0(\Phi)^\times$	$210.2^6 = 2^7.3.5.7$	$\left[\begin{array}{cc\|cc\|cc} & & 6 & -2 & & \\ & & 2 & 2 & & \\ & & 2 & 2 & & \\ & & 2 & 2 & & \end{array}\right]^\times$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 12: $L \cong 2^{11}L_3(4)S_3$-orbits (*continued*; see Theorem 4.4 and Tables 11 and 13).

Orbit	Size	Representative
$\alpha_2^2(\ell_0)$:		
$\mathcal{A}_0(\Phi)^{\times}$	$21.15.2^6.2 = 2^7.3^2.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline & & 2 & 2 & 2 & 2 \\ & & 2 & 2 & 2 & 2 \\ & & 2 & 2 & 2 & 2 \\ & & 2 & 2 & 2 & 2 \\ \hline \end{array}^{\times}$
$\mathcal{A}_1(\Phi)^{\times}$	$168.15.2^6.2 = 2^{10}.3^2.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & 2 & 2 \\ & & & & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}^{\times}$
$\mathcal{A}_2(\Phi)^{[1]\times}$	$360.2^6.2 = 2^{10}.3^2.5$	$\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 2 & 2 & 2 & 2 & \\ 4 & & & & & 2 \\ 4 & & & & & 2 \\ & & & & & 2 \\ \hline \end{array}^{\times}$
$\mathcal{A}_2(\Phi)^{[14]\times}$	$360.14.2^6.2 = 2^{11}.3^2.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 2 & 2 & 2 & 2 & \\ 4 & & & & & 2 \\ & & & & & 2 \\ 4 & & & & & 2 \\ \hline \end{array}^{\times}$
$\mathcal{A}_3(\Phi)^{[3]\times}$	$210.3.2^6.2 = 2^8.3^2.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & 2 & 2 & & \\ 4 & & 2 & 2 & & \\ & & 2 & 2 & & \\ & & 2 & 2 & & \\ \hline \end{array}^{\times}$
$\mathcal{A}_3(\Phi)^{[12]\times}$	$210.12.2^6.2 = 2^{10}.3^2.5.7$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & 4 & 2 & 2 & & \\ & & 2 & 2 & & \\ & & 2 & 2 & & \\ & & 2 & 2 & & \\ \hline \end{array}^{\times}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 13: $L \cong 2^{11}L_3(4)S_3$-orbits (*continued*; see Theorem 4.4 and Tables 11 and 12).

Orbit	Size	Representative
$\alpha_3^1(\ell_0)$:		
$\mathcal{T}_3(\Phi)^\times$	$1.2^{11} = 2^{11}$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline 5 & 1 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}\right.^{\times}$
$\mathcal{T}_2(\Phi)^\times$	$63.2^{11} = 2^{11}.3^2.7$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline 5 & -3 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}\right.^{\times}$
$\mathcal{T}_1(\Phi)^\times$	$630.2^{11} = 2^{12}.3^2.5.7$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline 5 & -3 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}\right.^{\times}$
$\mathcal{T}_0^+(\Phi)^\times$	$210.2^{11} = 2^{12}.3.5.7$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}\right.^{\times}$
$\mathcal{T}_0^-(\Phi)^\times$	$1120.2^{11} = 2^{16}.5.7$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline 1 & 5 & -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}\right.^{\times}$
$\alpha_3^2(\ell_0)$:		
$\mathcal{D}_{0,3}(\Phi)^\times$	$280.2^{11} = 2^{14}.5.7$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline -2 & 4 & 2 & & 2 & \\ 2 & & & 2 & 2 & \\ 2 & & & 2 & 2 & \\ & 2 & & 2 & & 2 \\ \hline \end{array}\right.^{\times}$
$\mathcal{D}_{1,2}(\Phi)^\times$	$1008.2^{11} = 2^{15}.3^2.7$	$\left.\begin{array}{\|rr\|rr\|rr\|} \hline -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & 2 & \\ & 2 & & 2 & 2 & \\ & 2 & & 2 & & 2 \\ \hline \end{array}\right.^{\times}$

Table 14: $L \cong 2^{11}M_{12}2$ (see Theorem 4.5 and Table 15).

Orbit	Size	Representative
$\{\ell_0\}$	1	ℓ_0
$\alpha_1(\ell_0)$:		
$\mathcal{S}_{(2.2)^4,(4.0)^2}(D)^{\times}$	$495.2 = 2.3^2.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & 4 & \\ & & & & & 4 \\ & & & & & 4 \\ & & & & & 4 \\ \hline\end{array}^{\times}$
$\mathcal{S}_{(3.1)^6}(D)^{\times}$	$880.2 = 2^5.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ 4 & & & & & \\ \hline\end{array}^{\times}$
$\mathcal{S}_{(2.2)^6}(D)^{\times}$	$396.2 = 2^3.3^2.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 4 & 4 & & & & \\ 4 & 4 & & & & \\ & & & & & \\ & & & & & \\ \hline\end{array}^{\times}$
$\alpha_2^1(\ell_0)$:		
$\mathcal{O}_{2,6}(D)^{\times}$	$264.2^6 = 2^9.3.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 6 & -2 & 2 & 2 & 2 & \\ & & & & & 2 \\ & & & & & 2 \\ & & & & & 2 \\ \hline\end{array}^{\times}$
$\mathcal{O}_{4^2}(D)^{\times}$	$495.2^6 = 2^6.3^2.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 6 & -2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ \hline\end{array}^{\times}$
$\alpha_2^2(\ell_0)$:		
$\mathcal{A}_{6,10}(D)^{\times}$	$264.15.2^6.2 = 2^{10}.3^2.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline & & & & & 2 \\ 2 & 2 & 2 & 2 & 2 & \\ 2 & 2 & 2 & 2 & 2 & \\ 2 & 2 & 2 & 2 & 2 & \\ \hline\end{array}^{\times}$
$\mathcal{A}_{8^2}(D)^{[1]\times}$	$495.2^6.2 = 2^7.3^2.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 2 & 4 & & 4 & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ \hline\end{array}^{\times}$
$\mathcal{A}_{8^2}(D)^{[2]\times}$	$495.2.2^6.2 = 2^8.3^2.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 2 & 4 & 4 & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ \hline\end{array}^{\times}$
$\mathcal{A}_{8^2}(D)^{[12]\times}$	$495.12.2^6.2 = 2^9.3^3.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 2 & 2 & 4 & & & \\ 2 & 2 & & 4 & & \\ 2 & 2 & & & & \\ 2 & 2 & & & & \\ \hline\end{array}^{\times}$

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press

Table 15: $L \cong 2^{11}M_{12}2$ (*continued*; see Theorem 4.5 and Table 14).

Orbit	Size	Representative
$\alpha_3^1(\ell_0)$:		
$\mathcal{T}_{0,3}(D)$	$440.2^{11} = 2^{14}.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}^{\times}$
$\mathcal{T}_{1,2}(D)$	$1584.2^{11} = 2^{15}.3^2.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline 5 & 1 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}^{\times}$
$\alpha_3^2(\ell_0)$:		
$D^{\times}$	$1.2^{11} = 2^{11}$	$\begin{array}{\|cc\|cc\|cc\|}\hline -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ \hline\end{array}^{\times}$
$\mathcal{D}_{4,8}(D)^{\times}$	$495.2^{11} = 2^{11}.3^2.5.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline -2 & 4 & 2 & & 2 & \\ 2 & & & 2 & 2 & \\ 2 & & & 2 & 2 & \\ & 2 & & 2 & & 2 \\ \hline\end{array}^{\times}$
$\mathcal{D}_{6^2}(D)^{\times}$	$792.2^{11} = 2^{14}.3^2.11$	$\begin{array}{\|cc\|cc\|cc\|}\hline -2 & 4 & 2 & & 2 & \\ 2 & & 2 & & & 2 \\ 2 & & 2 & & & 2 \\ & 2 & & 2 & & 2 \\ \hline\end{array}^{\times}$

Appendix A.

The twenty-four vectors given below are those in the cross $u_1^{*\times}$.

$$u_1^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & 4 & 2 & & 2 & \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ & 2 & & 2 & & 2 \\ \hline\end{array} \qquad u_{13}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & -4 & -2 & & -2 \\ 2 & & 2 & & -2 & \\ 2 & & 2 & & -2 & \\ 2 & & 2 & & -2 & \\ \hline\end{array}$$

$$u_2^* \quad \begin{array}{|cc|cc|cc|}\hline 2 & & 2 & & 2 & \\ 4 & 2 & & -2 & & -2 \\ & 2 & & 2 & & 2 \\ & -2 & & -2 & & -2 \\ \hline\end{array} \qquad u_{14}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & 2 & & -2 \\ -2 & & 2 & -4 & 2 & \\ -2 & & -2 & & 2 & \\ 2 & & 2 & & -2 & \\ \hline\end{array}$$

$$u_3^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & -2 & & -2 & \\ & 2 & & 2 & & 2 \\ -4 & -2 & & 2 & & 2 \\ & -2 & & -2 & & -2 \\ \hline\end{array} \qquad u_{15}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & 2 & & -2 \\ 2 & & 2 & & -2 & \\ -2 & & 2 & -4 & 2 & \\ -2 & & -2 & & 2 & \\ \hline\end{array}$$

$$u_4^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & -2 & & -2 & \\ & -2 & & -2 & & -2 \\ & 2 & & 2 & & 2 \\ -4 & -2 & & 2 & & 2 \\ \hline\end{array}$$

$$u_5^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & 2 & -4 & 2 & \\ & -2 & & -2 & & 2 \\ & -2 & & -2 & & 2 \\ & -2 & & -2 & & 2 \\ \hline\end{array}$$

$$u_6^* \quad \begin{array}{|cc|cc|cc|}\hline 2 & & 2 & & -2 & \\ & -2 & 4 & 2 & & 2 \\ & 2 & & 2 & & -2 \\ & -2 & & -2 & & 2 \\ \hline\end{array}$$

$$u_7^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & -2 & & 2 & \\ & 2 & & 2 & & -2 \\ & 2 & -4 & -2 & & -2 \\ & -2 & & -2 & & 2 \\ \hline\end{array}$$

$$u_8^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & -2 & & 2 & \\ & -2 & & -2 & & 2 \\ & 2 & & 2 & & -2 \\ & 2 & -4 & -2 & & -2 \\ \hline\end{array}$$

$$u_9^* \quad \begin{array}{|cc|cc|cc|}\hline 2 & & -2 & & -2 & 4 \\ & 2 & & -2 & & 2 \\ & 2 & & -2 & & 2 \\ & 2 & & -2 & & 2 \\ \hline\end{array}$$

$$u_{10}^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & 2 & & -2 & \\ & 2 & & -2 & -4 & -2 \\ & -2 & & 2 & & -2 \\ & 2 & & -2 & & 2 \\ \hline\end{array}$$

$$u_{11}^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & 2 & & -2 & \\ & 2 & & -2 & & 2 \\ & 2 & & -2 & -4 & -2 \\ & -2 & & 2 & & -2 \\ \hline\end{array}$$

$$u_{12}^* \quad \begin{array}{|cc|cc|cc|}\hline -2 & & 2 & & -2 & \\ & -2 & & 2 & & -2 \\ & 2 & & -2 & & 2 \\ & 2 & & -2 & -4 & -2 \\ \hline\end{array}$$

$$u_{16}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & 2 & & -2 \\ -2 & & -2 & & 2 & \\ 2 & & 2 & & -2 & \\ -2 & & 2 & -4 & 2 & \\ \hline\end{array}$$

$$u_{17}^* \quad \begin{array}{|cc|cc|cc|}\hline -4 & -2 & & 2 & & 2 \\ 2 & & 2 & & 2 & \\ 2 & & 2 & & 2 & \\ 2 & & 2 & & 2 & \\ \hline\end{array}$$

$$u_{18}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & 2 & & 2 \\ 2 & -4 & -2 & & -2 & \\ -2 & & -2 & & -2 & \\ 2 & & 2 & & 2 & \\ \hline\end{array}$$

$$u_{19}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & 2 & & 2 \\ 2 & & 2 & & 2 & \\ 2 & -4 & -2 & & -2 & \\ -2 & & -2 & & -2 & \\ \hline\end{array}$$

$$u_{20}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & 2 & & 2 \\ -2 & & -2 & & -2 & \\ 2 & & 2 & & 2 & \\ 2 & -4 & -2 & & -2 & \\ \hline\end{array}$$

$$u_{21}^* \quad \begin{array}{|cc|cc|cc|}\hline & -2 & & 2 & 4 & 2 \\ -2 & & 2 & & -2 & \\ -2 & & 2 & & -2 & \\ -2 & & 2 & & -2 & \\ \hline\end{array}$$

$$u_{22}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & -2 & & 2 \\ -2 & & 2 & & 2 & -4 \\ -2 & & 2 & & -2 & \\ 2 & & -2 & & 2 & \\ \hline\end{array}$$

$$u_{23}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & -2 & & 2 \\ 2 & & -2 & & 2 & \\ -2 & & 2 & & 2 & -4 \\ -2 & & 2 & & -2 & \\ \hline\end{array}$$

$$u_{24}^* \quad \begin{array}{|cc|cc|cc|}\hline & 2 & & -2 & & 2 \\ -2 & & 2 & & -2 & \\ 2 & & -2 & & 2 & \\ -2 & & 2 & & 2 & -4 \\ \hline\end{array}$$

References

1. J. H. Conway, 'Three lectures on exceptional groups', *Finite simple groups* (ed. M. B. Powell and G. Higman, Academic Press, New York, 1971). 220, 222, 223
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups* (Oxford Univ. Press, London, 1985). 237, 251
3. R. T. Curtis, 'On the Matthieu group M_{24} and related topics', Ph.D. Thesis, University of Cambridge, 1972. 222
4. R. T. Curtis, 'A new combinatorial approach to M_{24}', *Math. Proc. Cambridge Philos. Soc.* 79 (1976) 25–42. 220, 228
5. A. A. Ivanov, *Geometry of sporadic groups I, Petersen and tilde geometries*, Encyclopedia Math. Appl. 76 (Cambridge Univ. Press, Cambridge, 1999). 228
6. M. A. Ronan and S. D. Smith, '2-Local geometries for some sporadic groups', *Finite groups* (Santa Cruz Conf. 1979), Proc. Sympos. Pure Math. 37 (Amer. Math. Soc., Providence, RI, 1980) 283–289. 219
7. M. A. Ronan and G. Stroth, 'Minimal parabolic geometries for the sporadic groups', *European J. Comb.* 5 (1984) 59–91. 219
8. P. Rowley, 'A Monster Graph I', *Proc. London Math. Soc.*, to appear. 219, 220, 224
9. Y. Segev, 'On the uniqueness of the Co_1 2-local geometry', *Geom. Dedicata* 25 (1988) 159–219. 225

Peter Rowley peter.j.rowley@manchester.ac.uk

Department of Mathematics
UMIST
PO Box 88
Manchester M60 1QD
United Kingdom

https://doi.org/10.1112/S146115700000111X Published online by Cambridge University Press