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Convergence of equilibria for bending-torsion models of rods with inhomogeneities

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We prove that, in the limit of vanishing thickness, equilibrium configurations of inhomogeneous, three-dimensional non-linearly elastic rods converge to equilibrium configurations of the variational limit theory. More precisely, we show that, as $h \searrow 0$, stationary points of the energy $\mathcal{E}^h$, for a rod $\Omega_h \subset \mathbb{R}^3$ with cross-sectional diameter h, subconverge to stationary points of the Γ-limit of $\mathcal{E}^h$, provided that the bending energy of the sequence scales appropriately. This generalizes earlier results for homogeneous materials to the case of materials with (not necessarily periodic) inhomogeneities.

Keywords: elasticity; dimension reduction; homogenization; convergence of equilibria

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1. Introduction

The derivation of asymptotic models for two or three-dimensional elastic objects by lower-dimensional models has a long history, going back as far as to Bernoulli [**3**] and Euler [**7**]. Both considered thin rods but starting from a two-dimensional model instead of the three-dimensional one, as we study here. Since then a multitude of such models has been proposed, some incompatible with each other, as they usually depend on strong a priori assumptions. An in-depth study of the early history can be found in [**13**].

We start with the nonlinear three-dimensional model: Let $\Omega_h \subset \mathbb{R}^3$ be the reference configuration of a thin elastic body, with 'thickness' $h > 0$. The stored elastic energy of a deformation $y\colon \Omega_h \to \mathbb{R}^3$ is then given by

$$E^h(y) := \int_{\Omega_h} W(\nabla y(x))\,\mathrm{d}x,$$

where W is the *stored energy density*; typical assumptions on W are similar to (M1) –(M3), stated in § 2.2. We are interested in the limiting behaviour of E^h as $h \searrow 0$. One of the first results in terms of Γ-convergence were for sets of the form $\Omega_h := \omega \times (-h, h)$ with $\omega \subset \mathbb{R}^2$. Roughly speaking Γ-convergence is equivalent to the convergence of global minimizers y^h of E^h, possibly perturbed by a force term, to global minimizers of some limiting energy. For example in [12] *the theory for membranes*, that is, the limit for $h^{-1}E^h$ was obtained, in [8] the *bending theory for plates*, that is, for $h^{-3}E^h$, has been studied. The latter result contains, as a special case, the model proposed by Bernoulli and Euler. Further scalings $h^{-\alpha}E^h$ were later studied in [9]. In this present paper, we study rods with small cross-sectional diameter. So in our case, the reference configuration is $\Omega_h := (0, L) \times h\omega$ for some $L > 0$ and $\omega \subset \mathbb{R}^2$. The *bending-torsion theory for rods*, that is, the Γ-limit for $h^{-3}E^h$ was obtained by [15]. Under the additional assumption of a linear stress growth, the result was strengthened in [16] by proving that also stationary points y^h of E^h subconverge to stationary points of the Γ-limit.

All the previous mentioned results were obtained in the case of a single, homogeneous material. In [18] the first Γ-convergence result for a rod in this regime, that is, $h^{-3}E^h$, with inhomogeneities was proved. This was done under the assumption that the inhomogeneity was periodic, rapidly oscillating and only depending on the 'in-plane' variable $x_1 \in (0, L)$. All these additional assumptions can be dropped, as was shown in [14]. In the present paper, we extend the result of [14] by showing that also stationary points subconverge to stationary points of the Γ-limit.

In [5] the more linear case of $h^{-5}E^h$, called the *von Kármán model*, was studied, and Γ-convergence and convergence of stationary points was proved. This result, and the one presented here, heavily depend on methods developed in [14, 20].

Now we turn to the precise mathematical description. Let $L > 0$ and let $\omega \subset \mathbb{R}^2$ be open, bounded, connected. The (scaled) energy of a non-homogeneous rod with length L and cross-section $h\omega$ and external forces $g \in L^2((0, L), \mathbb{R}^3)$, deformed by $\widetilde{y}\colon [0, L] \times h\omega \to \mathbb{R}^3$, is given by

$$\widetilde{\mathcal{E}}^h(y) := \frac{1}{h^4}\int_{(0,L)\times h\omega} W^h\left(x_1, \frac{x'}{h}, \nabla y(x)\right)\mathrm{d}x - \frac{1}{h^2}\int_{(0,L)\times h\omega} g(x_1)\cdot y(x)\,\mathrm{d}x.$$

The hypotheses on the elastic energy density $W^h\colon (0, L) \times \omega \times \mathbb{R}^{3\times 3} \to [0, \infty)$ are listed in § 2.2. We perform the usual change of variables $(x_1, x_2, x_3) \mapsto (x_1, hx_2, hx_3)$. Then the rod Ω_h is transformed to $\Omega := \Omega_1$, the deformation $\widetilde{y}$ becomes $y\colon [0, L] \times \omega \to \mathbb{R}^3$ and the energy transforms to

$$\mathcal{E}^h(y) := \frac{1}{h^2}\int_{(0,L)\times\omega} W^h(x, \nabla_h y(x))\,\mathrm{d}x - \int_{(0,L)\times\omega} g(x_1)\cdot y(x)\,\mathrm{d}x, \tag{1.1}$$

where $\nabla_h = (\partial_1, (1/h)\partial_2, (1/h)\partial_3)$. As already mentioned, in [14] the Γ-convergence of $\mathcal{E}^h$ along a subsequence to a limiting functional $\mathcal{E}^0$ was proved. This limit is given by

$$\mathcal{E}^0(y, d_2, d_3) := \begin{cases} \int_0^L Q_1^0(x_1, R^{\mathrm{T}}(x_1)R'(x_1)) - g(x_1)\cdot y(x_1)\,\mathrm{d}x_1 & \text{if } (y, d_2, d_3) \in \mathcal{A}, \\ \infty & \text{else,} \end{cases} \tag{1.2}$$

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where Q_1^0 is a quadratic form in the second argument, which will be introduced in proposition 2.4; the class of limiting deformations $\mathcal{A}$ is given by

$$\mathcal{A} := \{(y, d_2, d_3) \in W^{2,2}((0,L),\mathbb{R}^3) \times W^{1,2}((0,L),\mathbb{R}^3) \times W^{1,2}((0,L),\mathbb{R}^3) : (y' \mid d_2 \mid d_3) \in W^{1,2}((0,L),\mathrm{SO}(3))\}, \tag{1.3}$$

and $R = (y' \mid d_2 \mid d_3)$ is the rotation associated with $(y,\, d_2,\, d_3)$.

Formally, the first variation of the energy functional $\mathcal{E}^h$ in direction $\psi\colon [0,\, L] \times \omega \to \mathbb{R}^3$ is given by

$$\mathrm{D}\mathcal{E}^h(y)[\psi] := \frac{1}{h^2}\int_{(0,L)\times\omega} \mathrm{D}W^h(x, \nabla_h y(x)) : \nabla_h \psi \,\mathrm{d}x - \int_{(0,L)\times\omega} g(x_1)\cdot\psi(x)\,\mathrm{d}x. \tag{1.4}$$

For the first integral to be well-defined, however, we need to impose a strong growth condition on $\mathrm{D}W^h$ (see (M3) in § 2.2). Deformations y satisfying $\mathrm{D}\mathcal{E}^h(y)[\psi] = 0$ for all test functions ψ are said to be stationary points. If we impose the boundary condition $y(0,\, x') = (0,\, hx')$ for $x' \in \omega$, then the natural class of test functions are smooth maps, which vanish on $\{0\} \times \omega$; we denote this class by $C^\infty_{\mathrm{bdy}}(\overline{\Omega},\, \mathbb{R}^3)$. Another notion of stationary points exists, introduced by J. Ball in [**1**], which requires a significant weaker condition on $\mathrm{D}W$, allowing for physical growth, that is, $W(F) \to \infty$ if $\det F \searrow 0$ and $W(F) = \infty$ if $\det F \leqslant 0$. In [**6**] the convergence of such stationary points for the von Kármán rod (for homogeneous materials) was shown. Due to the highly inhomogeneous material, we will need the stronger growth condition. Regardless of the notion of stationarity, and even for homogeneous materials, the existence of stationary points is a subtle issue, see [**2**, §§ 2.2, 2.7].

For α, β, M positive constants with $\alpha \leqslant \beta$ we denote by $\mathcal{W}(\alpha,\, \beta,\, M)$ the set of admissible density functions W^h; the precise definition of the class $\mathcal{W}(\alpha,\, \beta,\, M)$ is given by (S1) –(S3) in § 2.2. We can now state the main result of this paper:

THEOREM 1.1. *Let $(W^h) \subset \mathcal{W}(\alpha,\, \beta,\, M)$, $g \in L^2((0,\, L),\, \mathbb{R}^3)$. Let $(y^h) \subset W^{1,2}(\Omega,\, \mathbb{R}^3)$ satisfy $y^h(0,\, x') = (0,\, hx')$ on $\{0\} \times \omega$ for all $h > 0$ in the trace sense. Assume, in addition, that*

$$\limsup_{h\searrow 0} \frac{1}{h^2}\int_\Omega W^h(x, \nabla_h y^h(x))\,\mathrm{d}x < \infty. \tag{1.5}$$

Let $(h_l) \subset (0,\, \infty)$ be a converging sequence with limit 0, such that $\mathcal{E}^{h_l}$, given in (1.1) has the Γ-limit $\mathcal{E}^0$, given in (1.2).

Assume in addition, that for all $l \in \mathbb{N}$ the deformation y^{h_l} satisfies

$$\mathrm{D}\mathcal{E}^{h_l}(y^{h_l})[\psi] = 0 \quad \textit{for all } \psi \in C^\infty_{\mathrm{bdy}}(\overline{\Omega}, \mathbb{R}^3).$$

Then there exists a (not relabelled) subsequence and $(\overline{y},\, \overline{d}_2,\, \overline{d}_3) \in \mathcal{A}$, such that $y^{h_l} \to \overline{y}$ strongly in $W^{1,2}(\Omega,\, \mathbb{R}^3)$ as $l \to \infty$, and

$$\lim_{l\to\infty} \nabla_h y^{h_l} = (\overline{y}' \mid \overline{d}_2 \mid \overline{d}_3) \quad \textit{strongly in } L^2(\Omega, \mathbb{R}^{3\times 3}).$$

Furthermore, $\overline{y}(0) = 0$*,* $\overline{d}_k(0) = e_k$ *for* $k = 2, 3$*, and* $(\overline{y}, \overline{d}_2, \overline{d}_3)$ *is a stationary point of* $\mathcal{E}^0$*.*

REMARK 1.2. The trivial deformation $y^h(x) = (x_1, hx')$ has no stored elastic energy and satisfies the boundary condition $y^h(0, x') = (0, hx')$. Hence an application of Poincaré's inequality shows that (1.5) holds automatically for a minimizing sequence (y^h) of $\mathcal{E}^h$.

REMARK 1.3. The theorem also holds true for more general forces $\widetilde{g} \in L^2(\Omega, \mathbb{R}^3)$. In this case, the forces in the limiting energy must be replaced by the mean of $\widetilde{g}$ on ω, that is, by $\int_\omega g(\cdot, x')\,\mathrm{d}x'$. The more general statement can be proved identically, up to a few additional error terms, but which converge trivially to zero for $h \searrow 0$.

The proof of theorem 1.1 is split into two main parts. For the first one, we follow closely the paper [**16**], where the corresponding result for the homogeneous rod was proved. Their methods for studying the stress can also be applied, with minor modifications, in the more general case considered here. Furthermore, we use additional cancellation effects, which simplifies parts of their proof. To conclude their proof they exploit an explicit, linear relationship between the limiting stress and strain, which allows to easily identify the limit equations. In the inhomogeneous case addressed here, such a relationship is less clear and the identification of the limit equation is more involved. Thus for the second part, we apply results and methods developed in [**5**] to identify the limit equation and conclude the proof.

2. Preliminaries

2.1. Notation

Let $x = (x_1, x') \in \mathbb{R}^3$, and let $\mathfrak{p}(x) = (0, x') \in \mathbb{R}^3$ be the projection of x onto $\{0\} \times \omega$. Let $(e_i)_{i=1}^3$ be the standard basis of $\mathbb{R}^3$. For $A \in \mathbb{R}^{3\times 3}$ let $\mathrm{tr}(A)$ be the trace of A, and for $A, B \in \mathbb{R}^{3\times 3}$ let $A : B := \mathrm{tr}(A^{\mathrm{T}}B)$ be the inner product of $\mathbb{R}^{3\times 3}$. By $v \cdot w$ we denote the inner product for $v, w \in \mathbb{R}^3$. Let the twist function t be given by

$$\mathrm{t}\colon L^1(\Omega) \times L^1(\Omega) \to L^1(0, L), \quad \mathrm{t}(\phi, \psi)(x_1) = \int_\omega x_3\phi(x_1, x') - x_2\psi(x_1, x')\,\mathrm{d}x'.$$

We denote by $\iota\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ the natural inclusion $\iota(v) = v \otimes e_1$, by $\mathrm{axl}\colon \mathbb{R}^{3\times 3}_{\mathrm{skew}} \to \mathbb{R}^3$ the axial vector $\mathrm{axl}(A) = (-A_{23}, A_{13}, -A_{12})$ and by $\mathrm{id}_{3\times 3}$ we denote the 3×3 identity matrix. By $()'$ we denote the derivative with respect to x_1, by $\nabla = (\partial_1, \partial_2, \partial_3)$ the gradient with respect to x and for every $h > 0$ we define the scaled gradient as $\nabla_h = (\partial_1, (1/h)\partial_2, (1/h)\partial_3)$. Finally, we define the function spaces

$$C^\infty_{\mathrm{bdy}}([0, L]) := \{f \in C^\infty([0, L]) \mid f(0) = 0\};$$
$$W^{1,2}_{\mathrm{bdy}}([0, L]) := \{f \in W^{1,2}([0, L]) \mid f(0) = 0\};$$

and

$$C^\infty_{\mathrm{bdy}}(\overline{\Omega}) := \{f \in C^\infty(\overline{\Omega}) \mid f = 0 \text{ on } \{0\} \times \omega\};$$
$$W^{1,2}_{\mathrm{bdy}}(\Omega) := \big\{f \in W^{1,2}(\Omega) \mid f = 0 \text{ on } \{0\} \times \omega \text{ in the trace sense}\big\}.$$

2.2. The nonlinear bending-torsion theory for beams

Let $L > 0$ and let $\omega \subset \mathbb{R}^2$ be an open, bounded, connected Lipschitz domain such that $\mathcal{L}^2(\omega) = 1$, and such that ω is centred, that is, ω satisfies

$$\int_\omega x_2 x_3 \,\mathrm{d}x_2\,\mathrm{d}x_3 = \int_\omega x_2 \,\mathrm{d}x_2\,\mathrm{d}x_3 = \int_\omega x_3 \,\mathrm{d}x_2\,\mathrm{d}x_3 = 0. \tag{2.1}$$

The reference domain $\Omega \subset \mathbb{R}^3$ is given by $\Omega = (0, L) \times \omega$. The assumption on the elastic energy density W are as follows:

Let α, β, M be positive constants with $\alpha \leqslant \beta$. The class $\mathcal{W}(\alpha, \beta, M)$ contains all differentiable functions $W : \mathbb{R}^{3\times 3} \to [0, \infty)$ that satisfy:

(M1) Frame indifference: $W(RF) = W(F)$ for all $F \in \mathbb{R}^{3\times 3}$ and $R \in \mathrm{SO}(3)$.

(M2) Non-degeneracy and continuity:

$$\alpha \operatorname{dist}^2(F, \mathrm{SO}(3)) \leqslant W(F) \leqslant \beta \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3\times 3}.$$

In particular, this implies the minimality at the identity, that is, $W(\mathrm{id}_{3\times 3}) = 0$.

(M3) Linear stress growth: Let $\mathrm{D}W$ be the derivative of W. Then

$$|\mathrm{D}W(F)| \leqslant M(|F| + 1) \text{ for all } F \in \mathbb{R}^{3\times 3}.$$

Remark 2.1. The condition (M3) is already needed for the first term in the first variation of $\mathcal{E}^h$, given in (1.4), to be well-defined. Hence the growth condition appears in a similar form also in [**16**, **17**]. But it is not needed to prove the Γ-convergence, for example, the result in [**14**]. In this case, the upper bound (M2) is also only needed locally, that is,

$$\exists \rho, \beta' > 0 : \; W(F) \leqslant \beta' \operatorname{dist}^2(F, \mathrm{SO}(3)) \;\text{ for all } F \in \mathbb{R}^{3\times 3} \text{ with } \operatorname{dist}(F, \mathrm{SO}(3)) \leqslant \rho.$$

It is, however, easily seen that this local upper bound together with linear stress growth implies the global estimate (M2) for some $\beta > 0$.

Let now α, β, M be as above. A family of energy densities $(W^h)_{h>0}$, $W^h : \Omega \times \mathbb{R}^{3\times 3} \to [0, \infty)$ describes an *admissible composite material of class* $\mathcal{W}(\alpha, \beta, M)$ if for every $h > 0$ it holds:

(S1) W^h is a Borel function on $\Omega \times \mathbb{R}^{3\times 3}$.

(S2) $W^h(x, \cdot) \in \mathcal{W}(\alpha, \beta, M)$ for almost every $x \in \Omega$.

(S3) There exist a monotone function $r : [0, \infty] \to [0, \infty]$ and quadratic forms $Q^h : \Omega \times \mathbb{R}^{3\times 3} \to [0, \infty)$ such that $r(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ and

$$\operatorname*{ess\,sup}_{x\in\Omega} |W^h(x, \mathrm{id}_{3\times 3} + G) - Q^h(x, G)| \leqslant r(|G|)|G|^2 \quad \text{for all } G \in \mathbb{R}^{3\times 3}.$$

Let (Q^h) be the family of corresponding quadratic forms associated with a family $(W^h) \subset \mathcal{W}(\alpha, \beta, L)$. Then for every $h > 0$ we easily obtain, that Q^h is a

Carathéodory function, which for almost every $x \in \Omega$ satisfies

$$\begin{aligned} &\alpha|\mathrm{sym}\, F|^2 \leqslant Q^h(x,F) = Q^h(x, \mathrm{sym}\, F) \leqslant \beta |\mathrm{sym}\, F|^2 \quad \text{for all } F \in \mathbb{R}^{3\times 3},\\ &|Q^h(x,F_1) - Q^h(x,F_2)| \leqslant \beta|\mathrm{sym}\, F_1 - \mathrm{sym}\, F_2| \\ &\qquad\qquad\qquad\qquad \cdot |\mathrm{sym}\, F_1 + \mathrm{sym}\, F_2| \quad \text{for all } F_1, F_2 \in \mathbb{R}^{3\times 3}. \end{aligned} \tag{2.2}$$

Let $\mathbb{A}^h$ denote the linear, symmetric, positive semidefinite operator associated with the quadratic forms Q^h. In particular, $Q^h(F) = (1/2)\mathbb{A}^h F : F$ for all $F \in \mathbb{R}^{3\times 3}$.

In [**16**, proposition 4.1] the following compactness result was proved:

Proposition 2.2. *Let $(u^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ be a sequence satisfying*

$$\limsup_{h\searrow 0} \frac{1}{h^2}\int_\Omega \mathrm{dist}^2\big(\nabla_h u^h, \mathrm{SO}(3)\big)\,\mathrm{d}x < \infty. \tag{2.3}$$

Then there exists a constant $C > 0$, depending only on the domain Ω, and a sequence $(R^h) \subset C^\infty([0, L], \mathrm{SO}(3))$, such that

$$\|\nabla_h u^h - R^h\|_{L^2(\Omega)} \leqslant Ch, \tag{2.4}$$

$$\|(R^h)'\|_{L^2(0,L)} + h\|(R^h)'\|_{L^2(0,L)} \leqslant C \tag{2.5}$$

for every $h > 0$. If, in addition, $u^h = h\mathfrak{p}$ on $\{0\}\times\omega$ in the trace sense, then

$$|R^h(0) - \mathrm{id}_{3\times 3}| \leqslant C\sqrt{h}. \tag{2.6}$$

The following observations are standard, and follow the approach taken in [**16**]:

Let (y^h) be the sequence of deformations satisfying the assumptions of Theorem 1.1. The non-degeneracy assumption (M2) implies that (y^h) satisfies (2.3). Thus, by the previous proposition, there exists a sequence (R^h) satisfying (2.4) and (2.5). By using the frame indifference of W^h we have that

$$\begin{aligned} W^h(\cdot, \nabla_h y^h) = W^h(\cdot, (R^h)^{\mathrm{T}}\nabla_h y^h) &= W^h\left(\cdot, \mathrm{id}_{3\times 3} + h\frac{(R^h)^{\mathrm{T}}\nabla_h y^h - \mathrm{id}_{3\times 3}}{h}\right)\\ &= W^h\left(\cdot, \mathrm{id}_{3\times 3} + hG^h\right), \end{aligned}$$

where we introduced

$$G^h = \frac{(R^h)^{\mathrm{T}}\nabla_h y^h - \mathrm{id}_{3\times 3}}{h}. \tag{2.7}$$

Estimate (2.4) implies that (G^h) is uniformly bounded in $L^2(\Omega)$. We define z^h implicitly by introducing the ansatz

$$y^h(x) = \left(\int_0^{x_1} R^h(s)e_1\,\mathrm{d}s\right) + hx_2 R^h(x_1)e_2 + hx_3 R^h(x_1)e_3 + hz^h(x). \tag{2.8}$$

We plug this ansatz into (2.7), and obtain

$$G^h = \frac{(R^h)^{\mathrm{T}}\nabla_h y^h - \mathrm{id}_{3\times 3}}{h} = \iota\big(A^h\mathfrak{p}\big) + (R^h)^{\mathrm{T}}\nabla_h z^h, \tag{2.9}$$

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where we introduced $A^h := (R^h)^{\mathrm{T}}(R^h)'$. Clearly, (A^h) is uniformly bounded in $L^2(I)$, and since (G^h) is uniformly bounded in $L^2(\Omega)$, the sequence $(\nabla_h z^h)$ is uniformly bounded in $L^2(\Omega)$. Furthermore, on $\{0\} \times \omega$ we have the boundary conditions $y^h(x) = hx_2 e_2 + hx_3 e_3$, and thus also we can assume (2.6) holds. With this we obtain $|z^h| \leqslant C\sqrt{h}$ on $\{0\} \times \omega$. Hence Poincaré's inequality implies that (z^h) is uniformly bounded in $W^{1,2}(\Omega)$. Thus, after extracting a subsequence (not relabelled), we have in L^2 the weak convergences

$$G^h \rightharpoonup G, \quad A^h \rightharpoonup A \quad \text{and} \quad (R^h)^{\mathrm{T}} \nabla_h z^h \rightharpoonup R(\partial_1 z \mid q_2 \mid q_3)$$

for some $G \in L^2(\Omega, \mathbb{R}^{3\times 3})$, $z \in W^{1,2}(\Omega, \mathbb{R}^3)$, $A \in L^2((0, L), \mathbb{R}^{3\times 3}_{\text{skew}})$ and $q_2, q_3 \in L^2(\Omega, \mathbb{R}^3)$. The uniform L^2 bound on $((1/h)\partial_2 z^h, (1/h)\partial_3 z^h)$ implies that z does not depend on x_2, x_3. Thus passing to the limit in (2.9) we obtain

$$G(x) = \iota\big(R^{\mathrm{T}}(x_1)\partial_1 z(x_1) + A(x_1)\mathfrak{p}(x)\big) + R^{\mathrm{T}}(x_1)(0 \mid q_2(x) \mid q_3(x)).$$

For brevity, we set $p := R^{\mathrm{T}}\partial_1 z \in L^2((0, L), \mathbb{R}^3)$.

Next, we focus on $\operatorname{sym} G^h$. In [**14**, proof of theorem 2.15] sequences $(v^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$, $(\Psi^h) \subset W^{1,2}((0, L), \mathbb{R}^{3\times 3}_{\text{skew}})$ and $o^h \subset L^2(\Omega, \mathbb{R}^{3\times 3})$ were constructed, such that

$$\operatorname{sym} G^h = \operatorname{sym} \iota\Big(A\mathfrak{p} + p_1 e_1\Big) + \operatorname{sym} \iota\Big((\Psi^h)'\mathfrak{p}\Big) + \operatorname{sym} \nabla_h v^h + o^h,$$

and such that $(\nabla_h v^h)$ is uniformly bounded in $L^2(\Omega)$, and $\Psi^h \rightharpoonup 0$ in $W^{1,2}(0, L)$, $v^h \rightharpoonup 0$ in $W^{1,2}(\Omega)$ and $o^h \to 0$ strongly in $L^2(\Omega)$.

We define the *fixed part* m_d by

$$m_d := A\mathfrak{p} + p_1 e_1 \tag{2.10}$$

and the *corrector sequence* (ψ^h) by

$$\psi^h(x) = \Psi^h \mathfrak{p} - \frac{1}{h}\left(\widehat{\Psi}^h_{12} e_2 + \widehat{\Psi}^h_{13} e_3\right) + v^h, \tag{2.11}$$

where $\widehat{\Psi}^h(x_1) = \int_0^{x_1} \Psi^h(s)\,\mathrm{d}s$. Direct calculation yields

$$\nabla_h \psi^h = \big((\Psi^h)'\mathfrak{p} - \tfrac{1}{h}\Psi^h_{12} e_2 - \tfrac{1}{h}\Psi^h_{13} e_3,\ \tfrac{1}{h}\Psi^h e_2,\ \tfrac{1}{h}\Psi^h e_3\big) + \nabla_h v^h,$$

as well as

$$\operatorname{sym} \nabla_h \psi^h = \operatorname{sym} \iota\Big((\Psi^h)'\mathfrak{p}\Big) + \operatorname{sym} \nabla_h v^h. \tag{2.12}$$

Thus we have that

$$\operatorname{sym} G^h = \operatorname{sym} \iota\big(m_d\big) + \operatorname{sym} \nabla_h \psi^h + o^h, \tag{2.13}$$

with easily verifiable strong convergences

$$(\psi^h_1, h\psi^h_2, h\psi^h_3) \to 0 \ \text{ in } L^2(\Omega, \mathbb{R}^3) \quad \text{and} \quad \mathrm{t}(\psi^h_2, \psi^h_3) \to 0 \text{ in } L^2(0, L).$$

2.3. The Γ-limit

We will briefly introduce the variational approach developed in [**14**], with which the Γ-convergence for the inhomogeneous rod was proved. A similar variational approach for thin elastica was used earlier in [**4**] for the membrane model. The approach was also already adapted and used in [**5**] to show the convergence of stationary points for the inhomogeneous von Kármán rod.

By applying the frame indifference (M1) and Taylor expansion (S3) we obtain

$$\begin{aligned}\frac{1}{h^2}W^h(\cdot,\nabla_h y^h) &= \frac{1}{h^2}W^h\left(\cdot,\mathrm{id}_{3\times 3}+hG^h\right)\\ &\approx \frac{1}{h^2}Q^h(\cdot,hG^h) = Q^h(\cdot,G^h) = Q^h(\cdot,\operatorname{sym}G^h).\end{aligned} \tag{2.14}$$

This motivates, together with the decomposition (2.13), the definitions

$$\mathcal{K}^-_{(h)}(m,O) := \inf\left\{\liminf_{h\searrow 0}\int_{O\times\omega} Q^h(x,\iota(m)+\nabla_h\psi^h)\,\mathrm{d}x\right\},$$

$$\mathcal{K}^+_{(h)}(m,O) := \inf\left\{\limsup_{h\searrow 0}\int_{O\times\omega} Q^h(x,\iota(m)+\nabla_h\psi^h)\,\mathrm{d}x\right\},$$

where we take the infimum over all sequences $(\psi^h)\subset W^{1,2}(O\times\omega,\,\mathbb{R}^3)$, such that

$$(\psi^h_1,h\psi^h_2,h\psi^h_3)\to 0 \ \text{ strongly in } L^2(O\times\omega,\mathbb{R}^3) \quad\text{and}\quad \mathrm{t}(\psi^h_2,\psi^h_3)\to 0 \quad \text{in } L^2(O).$$

It is proved in [**14**, lemma 2.6] that there exists a subsequence, still denoted by (h), such that:

$$\forall m\in L^2(\Omega,\mathbb{R}^3), \forall O\subset[0,L] \text{ open } : \mathcal{K}_{(h)}(m,O) := \mathcal{K}^-_{(h)}(m,O) = \mathcal{K}^+_{(h)}(m,O). \tag{2.15}$$

This can be done by extracting a diagonal sequence such that $\mathcal{K}^-_{(h)}$ and $\mathcal{K}^+_{(h)}$ agree on a dense, countable subset of L^2 and of open subsets of $(0,\,L)$. Utilizing the continuity of the maps $L^2(\Omega,\,\mathbb{R}^3)\to\mathbb{R}$, $m\mapsto\mathcal{K}^-_{(h)}(m,\,O)$, $m\mapsto\mathcal{K}^+_{(h)}(m,\,O)$ for any open set $O\subset(0,\,L)$, proved in [**14**, lemma 2.5], it is then easy to see that (2.15) holds.

We now introduce the relaxation sequence and state its most important properties, which were proved in [**5**, **14**]:

Lemma 2.3. *Let $(h)\subset(0,\,\infty)$ with $h\searrow 0$ be a sequence such that (2.15) holds true. Then there exists a subsequence (not relabelled) such that for every $m\in L^2(\Omega,\,\mathbb{R}^3)$ there exists $(\psi^h_m)\subset W^{1,2}(\Omega,\,\mathbb{R}^3)$, with the property that for every open set $O\subset(0,\,L)$ we have that*

$$\mathcal{K}_{(h)}(m,O) := \lim_{h\searrow 0}\int_{O\times\omega} Q^h(x,\iota(m)+\nabla_h\psi^h_m)\,\mathrm{d}x, \tag{2.16}$$

Moreover, (ψ_m^h) *satisfies the following:*

(a) $(\psi_m^h \cdot e_1,\, h\psi_m^h \cdot e_2,\, h\psi_m^h \cdot e_3) \to 0$ *and* $\mathrm{t}(\psi_m^h \cdot e_2,\, \psi_m^h \cdot e_3) \to 0$ *strongly in* L^2.

(b) *The sequence* $(|\mathrm{sym}\,\nabla\psi_m^h|^2)$ *is equi-integrable, and there exist sequences* $B_m^h \subset W^{1,2}((0,\,L),\,\mathbb{R}^{3\times3}_{\mathrm{skew}})$ *and* $(\vartheta_m^h) \subset W^{1,2}(\Omega,\,\mathbb{R}^3)$ *with* $B_m^h \to 0$, $\vartheta_m^h \to 0$ *strongly in their respective* L^2*-norm, and*

$$\mathrm{sym}\,\nabla_h\psi_m^h = \mathrm{sym}\,\iota\big((B_m^h)'\mathfrak{p}\big) + \mathrm{sym}\,\nabla_h\vartheta_m^h.$$

Moreover, for a subsequence $(|(B_m^h)'|^2)$ *and* $(|\nabla_h\vartheta_m^h|^2)$ *are equi-integrable, and the following inequality holds for some* $C > 0$ *independent of* $O \subset (0,\,L)$*:*

$$\limsup_{h\searrow 0}\Big(\|B_m^h\|_{W^{1,2}(O)} + \|\nabla_h\vartheta_m^h\|_{L^2(O\times\omega)}\Big) \leqslant C(\beta\|m\|^2_{L^2(O\times\omega)} + 1).$$

(c) *If* $(\widehat{\psi}^h) \subset W^{1,2}(\Omega,\,\mathbb{R}^3)$ *is any other sequence that satisfies* (a) *and* $(\mathrm{sym}\,\nabla_h\widehat{\psi}^h)$ *is bounded in* $L^2(\Omega,\,\mathbb{R}^{3\times3})$*, then*

$$\lim_{h\searrow 0}\int_\Omega \mathbb{A}^h(\iota(m) + \nabla_h\psi_m^h) : \mathrm{sym}\,\nabla_h\widehat{\psi}^h\,\mathrm{d}x = 0.$$

(d) *If* $(\widehat{\psi}^h) \subset W^{1,2}(\Omega,\,\mathbb{R}^3)$ *is any sequence that satisfies* (2.16) *and* (a) *, then*

$$\|\mathrm{sym}\,\nabla_h\psi_m^h - \mathrm{sym}\,\nabla_h\widehat{\psi}^h\|_{L^2(\Omega)} \to 0,$$

and $(|\mathrm{sym}\,\nabla_h\widehat{\psi}^h|^2)$ *is equi-integrable.*

(e) *The map* $\mathcal{K}_{(h)}(\cdot,\,(0,\,L)) \colon L^2(\Omega,\,\mathbb{R}^3) \to \mathbb{R}$ *is continuously Fréchet-differentiable, and for every* $m,\, n \in L^2(\Omega,\,\mathbb{R}^3)$ *we have that*

$$\frac{\partial\mathcal{K}_{(h)}(m,(0,L))}{\partial m}[n] = \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h(\iota(m) + \mathrm{sym}\,\nabla_h\psi_m^h) : \iota(n)\,\mathrm{d}x. \tag{2.17}$$

The sequence (ψ_m^h) is called the *relaxation sequence* for m. For our purposes, m will always be of the form $m = B\mathfrak{p} + be_1$ for some $B \in L^2((0,\,L),\,\mathbb{R}^{3\times3}_{\mathrm{skew}})$ and $b \in L^2(0,\,L)$. Thus we introduce the linear map

$$m\colon L^2((0,L),\mathbb{R}^{3\times3}_{\mathrm{skew}}) \times L^2((0,L),\mathbb{R}) \to L^2(\Omega,\mathbb{R}^3), \quad m(B,b) := B\mathfrak{p} + be_1. \tag{2.18}$$

By applying the chain rule together with (e) we thus can compute the derivative of $\mathcal{K}_{(h)}(\cdot,\,(0,\,L)) \circ m$. Indeed, for every $B,\, M \in W^{1,2}((0,\,L),\,\mathbb{R}^{3\times3}_{\mathrm{skew}})$, $b,\, \mu \in L^2(0,\,L)$ we have that

$$\begin{aligned}\frac{\partial\mathcal{K}_{(h)}(m(B,b),(0,L))}{\partial B}[M] &= \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h(\iota(m(B,b)) + \mathrm{sym}\,\nabla_h\psi^h_{m(B,b)}) : \iota(M\mathfrak{p})\,\mathrm{d}x,\\ \frac{\partial\mathcal{K}_{(h)}(m(B,b),(0,L))}{\partial b}[\mu] &= \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h(\iota(m(B,b)) + \mathrm{sym}\,\nabla_h\psi^h_{m(B,b)}) : \iota(\mu e_1)\,\mathrm{d}x.\end{aligned} \tag{2.19}$$

To shorten notation we define $\mathcal{K}(m) := \mathcal{K}(m,\,(0,\,L))$ for every $m \in L^2(\Omega,\,\mathbb{R}^3)$.

In [**14**, proposition 2.12] also the following result regarding the existence of a density for $\mathcal{K}_{(h)}$ was proved:

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Proposition 2.4. *Let $(h) \subset (0, \infty)$ with $h \searrow 0$ be a sequence such that (2.15) holds true for every $m \in L^2(\Omega, \mathbb{R}^3)$. Then a measurable function $Q^0_{(h)} \colon [0, L] \times \mathbb{R}^{3\times 3}_{\text{skew}} \times \mathbb{R} \to [0, \infty)$ exists, such that for every $O \subset [0, L]$ open, and every $(B, b) \in L^2((0, L), \mathbb{R}^{3\times 3}_{\text{skew}} \times \mathbb{R})$ we have that*

$$\mathcal{K}_{(h)}(m(B,b),O) = \int_O Q^0_{(h)}(x_1, B(x_1), b(x_1))\,\mathrm{d}x_1.$$

Furthermore, for almost every $x_1 \in [0, L]$ the map $Q^0_{(h)}(x_1, \cdot, \cdot)$ is a quadratic form, and there exists a constant $C > 0$, depending only on the domain ω and the sequence (h), such that for every $\widehat{B} \in \mathbb{R}^{3\times 3}_{\text{skew}}$, $\widehat{b} \in \mathbb{R}$ we have that

$$C^{-1}(|\widehat{B}|^2 + |\widehat{b}|^2) \leqslant Q^0_{(h)}(x_1, \widehat{B}, \widehat{b}) \leqslant C\beta(|\widehat{B}|^2 + |\widehat{b}|^2).$$

Let (h) be as in the hypothesis of proposition 2.4. We define the map

$$\widehat{b}_{(h),\min} \colon [0, L] \times \mathbb{R}^{3\times 3}_{\text{skew}} \to \mathbb{R}, \quad \widehat{b}_{(h),\min}(x_1, \widehat{B}) = \arg\min_{b\in\mathbb{R}} Q^0_{(h)}(x_1, \widehat{B}, b).$$

It is easily seen that $\widehat{b}_{(h),\min}$ is well-defined, linear in $\widehat{B}$, and there exists a constant $C' = C'(\alpha, \beta, \omega, (h)) > 0$ such that for almost every x_1 and all $\widehat{B} \in \mathbb{R}^{3\times 3}_{\text{skew}}$ we have that

$$|\widehat{b}_{(h),\min}(x_1, \widehat{B})| \leqslant C'|\widehat{B}|.$$

We define

$$Q^0_{(h),1} \colon [0, L] \times \mathbb{R}^{3\times 3}_{\text{skew}} \to \mathbb{R}, \quad Q^0_{(h),1}(x_1, \widehat{B}) := Q^0_{(h)}(x_1, \widehat{B}, \widehat{b}_{(h),\min}(x_1, \widehat{B})).$$

Then for almost every $x_1 \in [0, L]$ the map $Q^0_{(h),1}(x_1, \cdot)$ is a quadratic form, and there exists $C' = C''(\alpha, \beta, \omega, (h)) > 0$ such that for all $\widehat{B} \in \mathbb{R}^{3\times 3}_{\text{skew}}$ it holds

$$(C'')^{-1}|\widehat{B}|^2 \leqslant Q^0_{(h),1}(x_1, \widehat{B}) \leqslant C'|\widehat{B}|^2.$$

We now define the limiting bending energy $\mathcal{K}^0_{(h)} \colon L^2((0, L), \mathbb{R}^{3\times 3}_{\text{skew}}) \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{K}^0_{(h)}(B) &:= \int_0^L Q^0_{(h),1}\Big(x_1, B(x_1)\Big)\,\mathrm{d}x_1\\ &= \int_0^L Q^0_{(h)}\Big(x_1, B(x_1), \widehat{b}_{(h),\min}\Big(x_1, B(x_1)\Big)\Big)\,\mathrm{d}x_1.\end{aligned}$$

From the linearity of $\widehat{B} \mapsto \widehat{b}_{(h),\min}(\cdot, \widehat{B})$ and the Fréchet-differentiability of $\mathcal{K}_{(h)}$ we deduce that also $\mathcal{K}^0_{(h)}$ is Fréchet-differentiable. For fixed $\widehat{B} \in \mathbb{R}^{3\times 3}_{\text{skew}}$ and almost every x_1 the function $Q^0_{(h)}(x_1, \widehat{B}, \cdot)$ has quadratic growth, thus $\widehat{b}_{(h),\min}(x_1, \widehat{B})$ is the unique stationary point of $Q^0_{(h)}(x_1,\, \widehat{B},\, \cdot)$. Hence

$$(\partial_b Q^0_{(h)})(x_1, \widehat{B}, b) = 0 \iff b = \widehat{b}_{(h),\min}(x_1, \widehat{B}).$$

Furthermore, the mapping

$$b_{(h),\min}\colon L^2((0,L), \mathbb{R}^{3\times 3}_{\mathrm{skew}}) \to L^2(0,L), \quad b_{(h),\min}(B) = \widehat{b}_{(h),\min}(\cdot, B)$$

is well-defined and linear. Thus for any $B \in L^2((0,\, L),\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ and $b \in L^2(0,\, L)$ we have that

$$\left\{ \frac{\partial \mathcal{K}_{(h)}(m(B,b))}{\partial b}[\mu] = 0 \quad \text{for all } \mu \in L^2(0,L) \right\} \iff b = b_{(h),\min}(B). \tag{2.20}$$

We are now able to compute the variations of $\mathcal{K}^0_{(h)}$. For fixed $B,\, M \in L^2((0,\, L),\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ we calculate by using the chain rule

$$\begin{aligned}\left(\frac{\partial}{\partial B}\mathcal{K}^0_{(h)}\right)(B)[M] &= \frac{\partial}{\partial B}\left(\mathcal{K}_{(h)}\Big(m\big(B, b_{(h),\min}(B)\big)\Big)\right)[M] \\ &= \left(\frac{\partial \mathcal{K}_{(h)}}{\partial m}\Big(m\big(B, b_{(h),\min}(B)\big)\Big)\right)\left[\frac{\partial m(B, b_{(h),\min}(B))}{\partial B}[M]\right].\end{aligned} \tag{2.21}$$

From (2.18) and the linearity of $b_{(h),\min}$ we obtain that

$$\frac{\partial m(B, b_{(h),\min}(B))}{\partial B}[M] = M\mathfrak{p} + ((\partial_B b_{(h),\min})(0) : M)e_1$$

and thus (2.21) can be rewritten as

$$\begin{aligned}\left(\frac{\partial}{\partial B}\mathcal{K}^0_{(h)}\right)(B)[M] = \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h\left(\iota(m(B, b_{(h),\min}(B))) + \operatorname{sym}\nabla_h \psi^h_{m(B,b_{(h),\min}(B))}\right) \\ : \iota(M\mathfrak{p} + ((\partial_B b_{(h),\min})(0) : M)e_1).\end{aligned} \tag{2.22}$$

The function $b_{(h),\min}(B)$ satisfies according to (2.20) the equation

$$\begin{aligned}0 &= \frac{\partial \mathcal{K}_{(h)}}{\partial b}(m(B, b_{(h),\min}(B))[\mu] \\ &= \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h\left(\iota(m(B, b_{(h),\min}(B))) + \operatorname{sym}\nabla_h \psi^h_{m(B,b_{(h),\min}(B))}\right) : \iota(\mu e_1)\end{aligned} \tag{2.23}$$

for all $\mu \in L^2(0,\, L)$. Finally, using $\mu := (\partial_B b_{(h),\min})(0) : M$ in (2.23) allows us to simplify (2.22) to

$$\left(\frac{\partial}{\partial B}\mathcal{K}^0_{(h)}\right)(B)[M] = \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h\left(\iota(m(B, b_{(h),\min}(B))) + \operatorname{sym}\nabla_h \psi^h_{m(B,b_{(h),\min}(B))}\right) : \iota(M\mathfrak{p}). \tag{2.24}$$

2.4. Derivation of the limit Euler-Lagrange equation

Let $\mathcal{A}$ be given by (1.3). We define

$$\mathcal{A}_0 = \{(y, d_2, d_3) \in \mathcal{A} \;:\; y(0) = 0,\; (y' \mid d_2 \mid d_3)(0) = \mathrm{id}_{3\times 3}\}\,.$$

Let $(y,\, d_2,\, d_3) \in \mathcal{A}_0$ and define the associated SO(3)-valued function $R = (y' \mid d_2 \mid d_3)$. The regularity of $y,\, d_2,\, d_3$ implies that $R \in W^{1,2}((0,\, L),\, \mathrm{SO}(3))$. Recall that the limit energy of $(y,\, d_2,\, d_3)$ is given by

$$\begin{aligned}\mathcal{E}^0(y, d_2, d_3) &= \mathcal{K}^0_{(h)}(R^{\mathrm{T}} R') - \int_0^L g \cdot y \,\mathrm{d}x_1 \\ &= \mathcal{K}^0_{(h)}(R^{\mathrm{T}} R') - \int_0^L \widehat{g} \cdot y' \,\mathrm{d}x_1,\end{aligned}$$

where $\widehat{g}(x_1) = \int_{x_1}^L g(s)\,\mathrm{d}s$. We say $(y,\, d_2,\, d_3)$ is a *stationary point* of $\mathcal{E}^0$, if for any C^1-curve $\gamma\colon (-\infty,\, \infty) \to \mathcal{A}_0$ with $\gamma(0) = (y,\, d_2,\, d_3)$ and derivative $\dot{\gamma}$, we have that

$$\left(\partial_\varepsilon \mathcal{E}^0(\gamma(\varepsilon))\right)\big|_{\varepsilon=0} = \mathrm{D}\mathcal{E}^0(y, d_2, d_3)[\dot{\gamma}(0)] = 0,$$

The following lemma gives an alternative characterization by identifying the tangent spaces of $\mathcal{A}_0$ and explicitly computing the derivative $\mathrm{D}\mathcal{E}^0$.

Lemma 2.5. *Let $(y,\, d_2,\, d_3) \in \mathcal{A}_0$. Define $R = (y' \mid d_2 \mid d_3)$ and $A = R^{\mathrm{T}} R'$. Then $(y,\, d_2,\, d_3)$ is a stationary point of $\mathcal{E}^0$ if and only if for every $\Phi \in W^{1,2}((0,\, L),\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ with $\Phi(0) = 0$ we have that*

$$\left(\frac{\partial}{\partial B}\mathcal{K}^0_{(h)}\right)(A)[A\Phi - \Phi A + \Phi'] = \int_0^L \widehat{g} \cdot (R\Phi e_1)\,\mathrm{d}x_1. \tag{2.25}$$

The left-hand side of (2.25) can be represented by (2.24).

Proof. Let $(y^\varepsilon,\, d_2^\varepsilon,\, d_3^\varepsilon)_\varepsilon \subset \mathcal{A}_0$ be a C^1-curve with $(y^0,\, d_2^0,\, d_3^0) = (y,\, d_2,\, d_3)$, and define $(R_\varepsilon)_\varepsilon \subset W^{1,2}((0,\, L),\, \mathrm{SO}(3))$ by $R_\varepsilon = ((y^\varepsilon)' \mid d_2^\varepsilon \mid d_3^\varepsilon)$; in particular $R_0 = R$. It is well-known that the tangent space of SO(3) in R is given by $\{R\Phi : \Phi \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}\}$. Thus, denoting the derivative of (R_ε) with respect to ε by $(\dot{R}_\varepsilon)$, we obtain $R_0^{\mathrm{T}} \dot{R}_0 = \Phi$ for some $\Phi \in W^{1,2}((0,\, L),\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$. Moreover, for every $\varepsilon \in \mathbb{R}$ we have that $R_\varepsilon(0) = \mathrm{id}_{3\times 3}$, and thus $\Phi(0) = 0$. Hence the tangent space of $\mathcal{A}_0$ in $(y,\, d_2,\, d_3)$ is given by

$$\begin{aligned}T_{(y,d_2,d_3)}\mathcal{A}_0 = \Big\{&(v_1, v_2, v_3) \in W^{2,2}((0,L),\mathbb{R}^3) \times W^{1,2}((0,L),\mathbb{R}^3) \times W^{1,2}((0,L),\mathbb{R}^3) : \\ &(y' \mid d_2 \mid d_3)^{\mathrm{T}}(v_1' \mid v_2 \mid v_3) \in W^{1,2}((0,L),\mathbb{R}^{3\times 3}_{\mathrm{skew}}),\; (v_1' \mid v_2 \mid v_3)(0) = 0\Big\}.\end{aligned}$$

With the chain rule, we obtain that

$$\begin{aligned}\partial_\varepsilon \mathcal{E}^0(y^\varepsilon, d_2^\varepsilon, d_3^\varepsilon)\big|_{\varepsilon=0} &= \partial_\varepsilon \left(\mathcal{K}^0_{(h)}(R_\varepsilon^{\mathrm{T}} R_\varepsilon') - \int_0^L \widehat{g} \cdot y_\varepsilon' \, \mathrm{d}x_1 \right) \Bigg|_{\varepsilon=0} \\ &= \frac{\partial \mathcal{K}^0_{(h)}}{\partial B}(R^{\mathrm{T}} R')[\partial_\varepsilon (R_\varepsilon^{\mathrm{T}} R_\varepsilon')]\Bigg|_{\varepsilon=0} - \int_0^L \widehat{g} \cdot (R_0 \Phi e_1) \, \mathrm{d}x_1 \\ &= \frac{\partial \mathcal{K}^0_{(h)}}{\partial B}(R^{\mathrm{T}} R')[\dot{R}_0^{\mathrm{T}} R_0' + R_0^{\mathrm{T}} \dot{R}_0'] - \int_0^L \widehat{g} \cdot (R_0 \Phi e_1) \, \mathrm{d}x_1.\end{aligned}$$

By using the relationship $R_0^{\mathrm{T}} \dot{R}_0 = \Phi$ we obtain that

$$\dot{R}_0^{\mathrm{T}} R_0' = -\Phi R^{\mathrm{T}} R' \quad \text{and} \quad R_0^{\mathrm{T}} \dot{R}_0' = R^{\mathrm{T}} (R\Phi)' = R^{\mathrm{T}} R' \Phi + R^{\mathrm{T}} R \Phi',$$

and thus

$$\dot{R}_0^{\mathrm{T}} R_0' + R_0^{\mathrm{T}} \dot{R}_0' = -\Phi R^{\mathrm{T}} R' + R^{\mathrm{T}} R' \Phi + \Phi'. \tag{2.26}$$

Furthermore, we can insert $A = R^{\mathrm{T}} R'$ into (2.26), obtaining

$$\dot{R}_0^{\mathrm{T}} R_0' + R_0^{\mathrm{T}} \dot{R}_0' = -\Phi A + A \Phi + \Phi'.$$

Hence

$$\partial_\varepsilon \mathcal{E}^0(y^\varepsilon, d_2^\varepsilon, d_3^\varepsilon)\big|_{\varepsilon=0} = \frac{\partial \mathcal{K}^0_{(h)}}{\partial B}(A)[A\Phi - \Phi A + \Phi'] - \int_0^L \widehat{g} \cdot (R \Phi e_1).$$

If (y, d_2, d_3) is a stationary point, the left-hand side vanishes and we obtain as claimed

$$\frac{\partial \mathcal{K}^0_{(h)}}{\partial B}(A)[A\Phi - \Phi A + \Phi'] = \int_0^L \widehat{g} \cdot (R \Phi e_1). \tag{2.27}$$

Moreover, if (2.27) holds for every $\Phi \in W^{1,2}((0, L), \mathbb{R}^{3\times 3})$ with $\Phi(0)$, then by the characterization of $T_{(y,d_2,d_3)}\mathcal{A}_0$ we obtain for any C^1-curve γ with $\gamma(0) = (y, d_2, d_3)$ that

$$\left(\partial_\varepsilon \mathcal{E}^0(\gamma(\varepsilon))\right)\big|_{\varepsilon=0} = 0.$$

This concludes the proof. □

3. Proof of the main theorem

We dedicate the whole section to the proof of theorem 1.1. From now on let W^h, y^h, g, (h_k) be as stated in the hypotheses of theorem 1.1. To simplify notation, we will simply write h and $h \searrow 0$, instead of h_k and $k \to \infty$.

From the energy bound (1.5) together with the non-degeneracy hypothesis (M2) on W^h we obtain the inequality

$$\limsup_{h\searrow 0} \frac{1}{h^2}\|\mathrm{dist}(\nabla_h y^h, \mathrm{SO}(3))\|_{L^2}^2 < \infty,$$

and furthermore, by assumption on (y^h) we have that $y^h(0, x_2, x_3) = hx_2e_2 + hx_3e_3$ for all $x' \in \omega$. Thus we may apply proposition 2.2 and deduce that there exists a sequence of rotations $(R^h) \subset C^\infty([0, L], \mathrm{SO}(3))$ with properties (2.4), (2.5) and (2.6).

We introduced once more the linearized strain G^h by

$$G^h = \frac{(R^h)^{\mathrm{T}}\nabla_h y^h - \mathrm{id}_{3\times 3}}{h}.$$

Then (2.4) implies that (G^h) is uniformly bounded in $L^2(\Omega)$. Hence, there exist a subsequence (not relabeled) and $G \in L^2(\Omega, \mathbb{R}^{3\times 3})$ such that $G^h \rightharpoonup G$ in $L^2(\Omega)$. Moreover, the frame indifference of W^h implies that

$$\mathrm{D}W^h(x, F) = R\mathrm{D}W^h(x, R^{\mathrm{T}}F) \quad \text{for all } F \in \mathbb{R}^{3\times 3},\ R \in \mathrm{SO}(3),\ \text{almost every } x \in \Omega.$$

Thus

$$\mathrm{D}W^h(x, \nabla_h y^h) = R^h\mathrm{D}W^h(x, \mathrm{id}_{3\times 3} + hG^h) = hR^hE^h, \tag{3.1}$$

where $E^h := h^{-1}\mathrm{D}W^h(\cdot,\, \mathrm{id}_{3\times 3} + hG^h)$ is the nonlinear stress. On the other hand, a Taylor expansion around the identity yields

$$\mathrm{D}W^h(\cdot, \mathrm{id}_{3\times 3} + hG^h) = h\mathrm{D}^2W^h(\cdot, \mathrm{id}_{3\times 3})G^h + \zeta^h(\cdot, hG^h),$$

where (S3) implies the estimate

$$|\zeta^h(\cdot, F)| \leqslant \widehat{r}(|F|)|F|,$$

for some monotone $\widehat{r}\colon [0, \infty) \to [0, \infty)$ with $\widehat{r}(\varepsilon) \searrow 0$ if $\varepsilon \searrow 0$. Together with $\mathrm{D}^2W^h(\cdot,\, \mathrm{id}_{3\times 3}) = \mathrm{D}^2Q^h(\cdot,\, 0) = \mathbb{A}^h$ we get that

$$E^h = \mathbb{A}^h \mathrm{sym}\, G^h + \frac{1}{h}\zeta^h(\cdot, hG^h). \tag{3.2}$$

The error term $h^{-1}\zeta^h(\cdot,\, hG^h)$ does not necessarily converge strongly to 0 in $L^2(\Omega)$, since G^h might concentrate in $L^2(\Omega)$. We will now show that the error term does not oscillates, and that it weakly converges to zero:

Lemma 3.1. *Let $(\eta^h) \subset L^2(\Omega)$ be such that $(|\eta^h|^2)$ is equi-integrable. Then*

$$\lim_{h\searrow 0} \int_\Omega \eta^h \cdot \left(\frac{1}{h}\zeta^h(\cdot, hG^h)\right)\,\mathrm{d}x = 0.$$

This immediately implies $h^{-1}\zeta^h(\cdot,\, hG^h) \rightharpoonup 0$ in $L^2(\Omega,\, \mathbb{R}^{3\times 3})$, and, in particular, that $(h^{-1}\zeta^h(\cdot,\, hG^h))$ is uniformly bounded in $L^2(\Omega)$.

Proof. Let $0 < \alpha < 1$. We define the sets $S_h^\alpha = \{x \in \Omega : h|G^h(x)| \leqslant h^\alpha\}$, and the truncated function $\widehat{G}^h := G^h \chi_{S_h^\alpha}$. Obviously $h\widehat{G}^h \to 0$ in L^∞, $G^h = \widehat{G}^h$ on S_h^α, and by Chebyshev inequality we have that $\mathcal{L}^2(\Omega \setminus S_h^\alpha) \to 0$ for $h \searrow 0$. We can now compute

$$\begin{aligned}\left\|\frac{1}{h}\zeta^h(\cdot, h\widehat{G}^h)\right\|_{L^2}^2 &= \frac{1}{h^2}\int_\Omega |\zeta^h(x, h\widehat{G}^h)|^2 \, \mathrm{d}x \\ &\leqslant \frac{1}{h^2}\int_\Omega \widehat{r}^2(\|h\widehat{G}^h\|_\infty)|h\widehat{G}^h|^2 \, \mathrm{d}x \\ &\leqslant \widehat{r}^2(\|h\widehat{G}^h\|_\infty)\|\widehat{G}^h\|_{L^2}^2 \leqslant \widehat{r}(\|h\widehat{G}^h\|_\infty)^2\|G^h\|_{L^2}^2 \to 0,\end{aligned}$$

by the uniform bound of G^h in the L^2-norm. Finally, applying Hölder's inequality yields

$$\left|\int_\Omega \eta^h \cdot \left(\frac{1}{h}\Big(\zeta^h(x, hG^h) - \zeta^h(x, h\widehat{G}^h)\Big)\right) \mathrm{d}x\right| \leqslant C\int_{S_h} |\eta^h|^2 \, \mathrm{d}x \to 0,$$

which implies the claim. □

With this result, we can deduce the limit PDE in terms of the stress. The part follows closely the corresponding proof in [**16**], and thus we skip some details.

Compactness

From the properties (2.4)–(2.6) for the sequence (R^h), we deduce that there exist a subsequence (not relabelled) and limit $R \subset W^{1,2}((0, L), \mathrm{SO}(3))$ such that $R(0) = \mathrm{id}_{3\times 3}$ and $R^h \rightharpoonup R$ in $W^{1,2}((0, L), \mathrm{SO}(3))$. Defining $\overline{y}(x_1) = \int_0^{x_1} R(s)e_1 \, \mathrm{d}s$, $\overline{d}_k = Re_k$ for $k = 2, 3$ we obtain $\overline{y} \in W^{2,2}_{\mathrm{bdy}}([0, L], \mathbb{R}^3)$, $y^h \to \overline{y}$ strongly in $W^{1,2}(\Omega, \mathbb{R}^3)$, $\nabla_h y^h \to (\overline{y}' \mid \overline{d}_2 \mid \overline{d}_3)$ strongly in $L^2(\Omega, \mathbb{R}^{3\times 3})$, $\overline{d}_k(0) = e_k$ for $k = 2, 3$. Hence $(\overline{y}, \overline{d}_2, \overline{d}_3) \in \mathcal{A}_0$.

Properties of E^h

We start by using only the uniform energy bound of the deformations (y^h), that is, stationarity is not yet needed. Recall the decomposition (3.2), that is,

$$E^h = \mathbb{A}^h(x)\mathrm{sym}\, G^h + \frac{1}{h}\zeta^h(x, hG^h).$$

The uniform bound on $|\mathbb{A}^h| \leqslant C\beta$ given by (2.2), the uniform L^2 bound on G^h and the uniform L^2 bound on the sequence $(h^{-1}\zeta^h(\cdot, hG^h))_{h>0}$, following from lemma 3.1, imply a uniform L^2 bound on the sequence E^h. Thus (E^h) weakly subconverges to some $E \in L^2(\Omega, \mathbb{R}^{3\times 3})$. The frame indifference (M1) implies that $\mathrm{D}W^h(\cdot, F)F^{\mathrm{T}}$ is symmetric for every $F \in \mathbb{R}^{3\times 3}$ almost everywhere on Ω. For $F = \mathrm{id}_{3\times 3} + hG^h$ the statement $\mathrm{skew}\,(\mathrm{D}W^h(\cdot, F)F^{\mathrm{T}}) = 0$ can be rewritten as

$$\mathrm{skew}\,(E^h) = h\mathrm{skew}\,(G^h(E^h)^{\mathrm{T}}). \tag{3.3}$$

From the uniform $L^2(\Omega)$ bound on both (E^h) and (G^h), we deduce that $(h^{-1}\mathrm{skew}\,(E^h))$ is uniformly bounded in $L^1(\Omega)$.

Deriving Euler-Lagrange equations
As (y^h) are stationary points of $\mathcal{E}^h$, for any $\psi \in C^\infty_{\mathrm{bdy}}(\overline{\Omega}, \mathbb{R}^3)$ we have that

$$\int_\Omega \big(\mathrm{D}W^h(x, \nabla_h y^h(x)) : \nabla_h \psi(x) - h^2 g(x_1)\cdot\psi(x)\big)\, \mathrm{d}x = 0.$$

By density, the equation also holds for arbitrary $\psi \in W^{1,2}_{\mathrm{bdy}}(\Omega, \mathbb{R}^3)$. Using relationship (3.1) we rewrite this equation as

$$\int_\Omega \big(R^h E^h : \nabla_h\psi - hg\cdot\psi\big)\, \mathrm{d}x = 0. \tag{3.4}$$

For $\psi(x) = \varphi(x_1)$ with $\varphi \in C^\infty_{\mathrm{bdy}}([0, L], \mathbb{R}^3)$ the equation (3.4) reduces to

$$\int_0^L \int_\omega \big(R^h E^h e_1\cdot\varphi' - hg\cdot\varphi\big)\, \mathrm{d}x'\, \mathrm{d}x_1 = \int_0^L \Big(R^h \overline{E}^h e_1\cdot\varphi' - hg\cdot\varphi\Big)\, \mathrm{d}x_1 = 0, \tag{3.5}$$

where $\overline{E}^h$ is the zero-th moment of E^h, that is,

$$\overline{E}^h \colon (0, L) \to \mathbb{R}^{3\times3}, \quad \overline{E}^h(x_1) = \int_\omega E^h(x_1, x')\, \mathrm{d}x'.$$

Furthermore, we denote the first moments with respect to x_2 and x_3 of E^h by $\widetilde{E}^h$, $\widehat{E}^h \in L^2((0, L), \mathbb{R}^{3\times3})$ respectively; more precisely let

$$\widetilde{E}^h(x_1) = \int_\omega x_2 E^h(x_1, x')\, \mathrm{d}x'; \quad \widehat{E}^h(x_1) = \int_\omega x_3 E^h(x_1, x')\, \mathrm{d}x'.$$

Let $\phi \in C^\infty_{\mathrm{bdy}}([0, L])$ and define $\psi^h(x) = x_2\phi(x_1)R^h(x_1)e_1$. Then $\psi^h \in W^{1,2}_{\mathrm{bdy}}(\Omega, \mathbb{R}^3)$ and

$$\nabla_h\psi^h(x) = \Big(x_2\phi'(x_1)R^h(x_1)e_1 + x_2\phi(x_1)(R^h)'(x_1)e_1 \,\Big|\, \tfrac{1}{h}\phi(x_1)R^h(x_1)e_1 \,\Big|\, 0\Big).$$

Plugging ψ^h into (3.4) yields

$$\begin{aligned} 0 &= \int_\Omega \big(R^h E^h : \nabla_h\psi^h - hg\cdot\psi^h\big)\, \mathrm{d}x \\ &= \int_0^L \Big(R^h\widetilde{E}^h e_1\cdot\phi' R^h e_1 + R^h\widetilde{E}^h e_1\cdot\phi(R^h)' e_1 + \frac{1}{h}R^h\overline{E}^h e_2\cdot\phi R^h e_1\Big)\, \mathrm{d}x_1. \end{aligned}$$

Introducing $A^h := (R^h)^{\mathrm{T}}(R^h)'$ this simplifies to

$$\int_0^L \Big(\widetilde{E}^h_{11}\cdot\phi' + \phi\widetilde{E}^h e_1\cdot A^h e_1 + \phi\frac{1}{h}\overline{E}^h_{12}\Big)\, \mathrm{d}x_1 = 0. \tag{3.6}$$

Analogously, for $\psi(x) = x_3\phi(x_1)R^h(x_1)e_1$ we get

$$\int_0^L \Big(\widehat{E}^h_{11}\cdot\phi' + \phi\widehat{E}^h e_1\cdot A^h e_1 + \phi\frac{1}{h}\overline{E}^h_{13}\Big)\, \mathrm{d}x_1 = 0, \tag{3.7}$$

and finally, $\psi(x) = x_3\phi(x_1)R^h(x_1)e_2 - x_2\phi(x_1)R^h(x_1)e_3$ yields

$$\int_0^L \left(\phi'(\widehat{E}^h_{21} - \widetilde{E}^h_{31}) + \phi(\widehat{E}^h e_1 \cdot A^h e_2 - \widetilde{E}^h e_1 \cdot A^h e_3) + \phi\frac{1}{h}(\overline{E}^h_{23} - \overline{E}^h_{32})\right) \mathrm{d}x_1 = 0. \tag{3.8}$$

Consequences of the Euler-Lagrange equations

Now, by stationarity of (y^h), the equation (3.5) holds for arbitrary $\varphi \in C_c^\infty((0,\, L),\, \mathbb{R}^3)$, and thus

$$\overline{E}^h e_1 = -h(R^h)^{\mathrm{T}}\widehat{g} \quad \text{almost everywhere in } (0, L). \tag{3.9}$$

Passing to the limit we also obtain

$$\overline{E} e_1 = 0 \quad \text{almost everywhere in } (0, L). \tag{3.10}$$

Furthermore, the equations (3.6), (3.7) and (3.8) imply that $\widetilde{E}^h_{11}$, $\widehat{E}^h_{11}$ and $(\widehat{E}^h_{21} - \widetilde{E}^h_{31})$ are weakly differentiable. The respective derivatives are in L^1, as seen by combining (3.3), (3.9) together with the uniform L^2 bound on A^h, which was just $(R^h)^{\mathrm{T}}(R^h)'$. By Sobolev's Embedding Theorem we thus obtain that

$$(\widetilde{E}^h_{11}), (\widehat{E}^h_{11}), (\widehat{E}^h_{21} - \widetilde{E}^h_{31}) \text{ converge strongly in } L^2(0, L). \tag{3.11}$$

From this we immediately obtain the following:

Let $(M^h) \subset L^2((0,\, L),\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ with $M^h \rightharpoonup M \in L^2((0,\, L),\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$. Then by direct calculation we obtain

$$\int_\Omega E^h : \iota(M^h\mathfrak{p})\, \mathrm{d}x = \int_0^L (\widetilde{E}^h_{31} - \widehat{E}^h_{21}, \widehat{E}^h_{11}, \widetilde{E}^h_{11}) \cdot \mathrm{axl} M^h\, \mathrm{d}x_1$$

and thus by applying (3.11) we get

$$\lim_{h\searrow 0} \int_\Omega E^h : \iota(M^h\mathfrak{p})\, \mathrm{d}x = \lim_{h\searrow 0} \int_\Omega E^h : \iota(M\mathfrak{p})\, \mathrm{d}x. \tag{3.12}$$

The limit of the PDE in terms of the stress

Fix some $\Phi \in C^\infty_{\mathrm{bdy}}([0,\, L],\, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ and let $\phi_1,\, \phi_2,\, \phi_3$ be given by $\mathrm{axl}(\Phi) = (\phi_1,\, \phi_2,\, \phi_3)$. We then define the test functions

$$\psi^h(x_1, x_2, x_3) = R^h(x_1)\Phi(x_1)\mathfrak{p}(x).$$

Then $\psi^h \in W^{1,2}_{\mathrm{bdy}}(\Omega,\, \mathbb{R}^3)$ and

$$\nabla_h \psi^h = \left(R^h\Phi'\mathfrak{p} + (R^h)'\Phi\mathfrak{p} \;\middle|\; \tfrac{1}{h}R^h\Phi e_2 \;\middle|\; \tfrac{1}{h}R^h\Phi e_3\right).$$

Plugging ψ^h into (3.4) yields that

$$\begin{aligned} &\int_\Omega \left(R^h E^h : \nabla_h \psi^h - hg \cdot \psi^h\right) \mathrm{d}x \\ &= \int_\Omega \left(E^h e_1 \cdot \Phi'\mathfrak{p} + E^h e_1 \cdot A^h \Phi\mathfrak{p} + \frac{1}{h}E^h e_2 \cdot \Phi e_2 + \frac{1}{h}E^h e_3 \cdot \Phi e_3\right) \mathrm{d}x. \end{aligned} \tag{3.13}$$

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Using the skew-symmetry of Φ we obtain that

$$E^h e_2 \cdot \Phi e_2 + E^h e_3 \cdot \Phi e_3 = E^h : \Phi - E^h e_1 \cdot \Phi e_1$$
$$= (\Phi : \operatorname{skew} E^h) - E^h e_1 \cdot \Phi e_1.$$

The preceding calculations imply that

$$\int_\Omega \left(R^h E^h : \nabla_h \psi^h - hg \cdot \psi^h\right) \,\mathrm{d}x = \mathrm{I}^h + \mathrm{II}^h + \mathrm{III}^h, \tag{3.14}$$

where

$$\mathrm{I}^h := \int_\Omega \left(E^h e_1 \cdot \Phi' \mathfrak{p}\right) \,\mathrm{d}x = \int_\Omega E^h : \iota(\Phi' \mathfrak{p}) \,\mathrm{d}x, \tag{3.15}$$

$$\mathrm{II}^h := \int_\Omega \left(\frac{1}{h} E^h e_1 \cdot \Phi e_1\right) \,\mathrm{d}x,$$

$$\mathrm{III}^h := \int_\Omega \left(E^h e_1 \cdot A^h \Phi \mathfrak{p} + \frac{1}{h} \Phi : \operatorname{skew} E^h\right) \,\mathrm{d}x. \tag{3.16}$$

The third one will be the most difficult to handle.

Regarding II^h, from (3.9) we obtain $\overline{E}^h e_1 = -h(R^h)^{\mathrm{T}} \widehat{g}$ and thus

$$\mathrm{II}^h = \int_0^L \left(\frac{1}{h} \overline{E}^h e_1 \cdot \Phi e_1\right) \,\mathrm{d}x_1 = -\int_0^L \widehat{g} \cdot (R^h \Phi e_1) \,\mathrm{d}x_1. \tag{3.17}$$

Regarding III^h, we claim that we have that

$$\lim_{h \searrow 0} \mathrm{III}^h = \lim_{h \searrow 0} \int_\Omega \left(E^h e_1 \cdot (A\Phi - \Phi A)\mathfrak{p}\right) \,\mathrm{d}x. \tag{3.18}$$

Indeed, recall that from (2.7) we have that

$$G^h = A^h \mathfrak{p} \otimes e_1 + (R^h)^{\mathrm{T}} \nabla_h z^h,$$

where z^h was defined by (2.8). By making use of (3.3) we obtain

$$\frac{1}{h} \operatorname{skew}(E^h) = \operatorname{skew}\left(G^h (E^h)^{\mathrm{T}}\right)$$
$$= \operatorname{skew}\left(\left((R^h)^{\mathrm{T}} \nabla_h z^h + A^h \mathfrak{p} \otimes e_1\right)(E^h)^{\mathrm{T}}\right)$$
$$= \operatorname{skew}\left((R^h)^{\mathrm{T}} \nabla_h z^h (E^h)^{\mathrm{T}}\right) + \operatorname{skew}\left((A^h \mathfrak{p} \otimes e_1)(E^h)^{\mathrm{T}}\right).$$

Furthermore, by the skew-symmetry of Φ we thus obtain

$$\frac{1}{h} \Phi : \operatorname{skew}(E^h) = \Phi : \left((R^h)^{\mathrm{T}} \nabla_h z^h (E^h)^{\mathrm{T}}\right) + \Phi : \left(A^h \mathfrak{p} \otimes E^h e_1\right). \tag{3.19}$$

For any $M \in \mathbb{R}^{n \times n}$ and $v, w \in \mathbb{R}^n$ we have the algebraic identity

$$M : (v \otimes w) = \operatorname{tr}(M^{\mathrm{T}}(v \otimes w)) = \operatorname{tr}((M^{\mathrm{T}} v) \otimes w) = (M^{\mathrm{T}} v) \cdot w,$$

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which applied to $M = \Phi$, $v = A^h\mathfrak{p}$, $w = E^h e_1$ yields for the second term in (3.19) the equality

$$\Phi : (A^h\mathfrak{p} \otimes E^h e_1) = -E^h e_1 \cdot (\Phi A^h \mathfrak{p}).$$

With this, we can simplify III^h, given by (3.16), to

$$\begin{aligned} \mathrm{III}^h &= \int_\Omega \left(E^h e_1 \cdot A^h \Phi \mathfrak{p} + \frac{1}{h}\Phi : \operatorname{skew} E^h \right) \mathrm{d}x \\ &= \int_\Omega \left(E^h e_1 \cdot (A^h\Phi - \Phi A^h)\mathfrak{p} + \Phi : ((R^h)^{\mathrm{T}} \nabla_h z^h (E^h)^{\mathrm{T}}) \right) \mathrm{d}x. \end{aligned} \tag{3.20}$$

We start by proving that,

$$\lim_{h\searrow 0} \int_\Omega \left(E^h e_1 \cdot (A^h\Phi - \Phi A^h)\mathfrak{p} \right) \mathrm{d}x = \lim_{h\searrow 0} \int_\Omega \left(E^h e_1 \cdot (A\Phi - \Phi A)\mathfrak{p} \right) \mathrm{d}x.$$

This, however, follows immediately from (3.12) by setting $M^h := A^h\Phi - \Phi A^h$ and $M := A\Phi - \Phi A$. We now show that the second term on the right-hand side of (3.20) converges to 0, thus proving the claim (3.18).

The skew-symmetry of Φ implies that almost everywhere on Ω we have that

$$\begin{aligned} \Phi : \left((R^h)^{\mathrm{T}} \nabla_h z^h (E^h)^{\mathrm{T}} \right) &= (\Phi E^h) : \left((R^h)^{\mathrm{T}} \nabla_h z^h \right) \\ &= -E^h : \left(\Phi (R^h)^{\mathrm{T}} \nabla_h z^h \right) \\ &= -R^h E^h : \left(R^h \Phi (R^h)^{\mathrm{T}} (\nabla_h z^h) \right). \end{aligned}$$

Integrating over Ω then yields

$$\int_\Omega \Phi : \left((R^h)^{\mathrm{T}} \nabla_h z^h (E^h)^{\mathrm{T}} \right) \mathrm{d}x = -\int_\Omega R^h E^h : \left(R^h \Phi (R^h)^{\mathrm{T}} (\nabla_h z^h) \right) \mathrm{d}x. \tag{3.21}$$

Let $h_0 > 0$ and $h \in (0, h_0)$. Then

$$\begin{aligned} R^h \Phi (R^h)^{\mathrm{T}} &= R^{h_0} \Phi (R^h)^{\mathrm{T}} + (R^h - R^{h_0}) \Phi (R^h)^{\mathrm{T}} \\ &= R^{h_0} \Phi (R^{h_0})^{\mathrm{T}} + (R^h - R^{h_0}) \Phi (R^h)^{\mathrm{T}} + R^{h_0} \Phi (R^h - R^{h_0})^{\mathrm{T}}, \end{aligned}$$

and hence

$$\begin{aligned} \int_\Omega R^h E^h : \left(R^h \Phi (R^h)^{\mathrm{T}} (\nabla_h z^h) \right) \mathrm{d}x &= \int_\Omega R^h E^h : \left(R^{h_0} \Phi (R^{h_0})^{\mathrm{T}} (\nabla_h z^h) \right) \mathrm{d}x \\ &\quad + \int_\Omega R^h E^h : \left((R^h - R^{h_0}) \Phi (R^h)^{\mathrm{T}} (\nabla_h z^h) \right) \mathrm{d}x \\ &\quad + \int_\Omega R^h E^h : \left(R^{h_0} \Phi (R^h - R^{h_0})^{\mathrm{T}} (\nabla_h z^h) \right) \mathrm{d}x. \end{aligned} \tag{3.22}$$

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We estimate the last two terms in (3.22) by

$$\begin{aligned}\bigg|\int_\Omega & R^h E^h : \big((R^h - R^{h_0})\Phi(R^h)^{\mathrm{T}}(\nabla_h z^h)\big)\,\mathrm{d}x \\ &+ \int_\Omega R^h E^h : \big(R^{h_0}\Phi(R^h - R^{h_0})^{\mathrm{T}}(\nabla_h z^h)\big)\,\mathrm{d}x\bigg| \\ &\leqslant 6\sup_{h>0}\Big(\|E^h\|_{L^2(\Omega)}\|\Phi\|_{L^\infty(0,L)}\|\nabla_h z^h\|_{L^2(\Omega)}\Big)\cdot\|R^h - R^{h_0}\|_{L^\infty(0,L)}.\end{aligned} \tag{3.23}$$

As $R^h \to R$ strongly in $L^\infty(0, L)$, passing first with h to the limit 0, and then with h_0 to 0, the right-hand side of (3.23) converges to 0. It remains to show that the first term on the right-hand side of (3.22) converges to 0. We rewrite this term as

$$\begin{aligned}\int_\Omega R^h E^h : \big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}}(\nabla_h z^h)\big)\,\mathrm{d}x &= \int_\Omega R^h E^h : \Big[\nabla_h\Big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}} z^h\Big)\Big]\,\mathrm{d}x \\ &\quad - \int_\Omega R^h E^h e_1 \cdot \Big[\big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}}\big)' z^h\Big]\,\mathrm{d}x.\end{aligned} \tag{3.24}$$

As R^{h_0} is a smooth function on the compact set $[0, L]$, we have that $R^{h_0} \in W^{1,\infty}(0, L)$. Together with $\Phi \in C^\infty_{\mathrm{bdy}}([0, L], \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ and $z^h \in W^{1,2}(\Omega, \mathbb{R}^3)$ this implies that $R^{h_0}\Phi(R^{h_0})^{\mathrm{T}} z^h \in W^{1,2}_{\mathrm{bdy}}(\Omega, \mathbb{R}^3)$. Plugging $R^{h_0}\Phi(R^{h_0})^{\mathrm{T}} z^h$ into (3.4) yields

$$\begin{aligned}\int_\Omega R^h E^h : \nabla_h\Big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}} z^h\Big)\,\mathrm{d}x &= h\int_\Omega f\cdot\Big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}} z^h\Big)\,\mathrm{d}x \\ &\leqslant 3\,h\|f\|_{L^2(0,L)}\|\Phi\|_{L^\infty(0,L)}\|z^h\|_{L^2(\Omega)}.\end{aligned} \tag{3.25}$$

By the uniform L^2 bound on (z^h), the right-hand side of (3.25) converges to 0 as $h \searrow 0$, independently of h_0. Moreover, the weak convergence of (E^h) in $L^2(\Omega)$, the strong convergence of (z^h) in $L^2(\Omega)$ and the strong convergence of (R^h) in $L^\infty(0, L)$ allows us to pass to the limit $h \searrow 0$ in the second term on the right-hand side of (3.24). We obtain that

$$\lim_{h\searrow 0}\int_\Omega R^h E^h e_1\cdot\Big[\big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}}\big)' z^h\Big]\,\mathrm{d}x = \int_\Omega R E e_1\cdot\Big[\big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}}\big)' z\Big]\,\mathrm{d}x.$$

As R, R^{h_0}, Φ and z are independent of x_2, x_3, we obtain that

$$\int_\Omega R E e_1\cdot\Big[\big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}}\big)' z\Big]\,\mathrm{d}x = \int_0^L\Big(R\overline{E}e_1\cdot\Big[\big(R^{h_0}\Phi(R^{h_0})^{\mathrm{T}}\big)' z\Big]\Big)\,\mathrm{d}x_1. \tag{3.26}$$

By (3.10), we have $\overline{E}e_1 = 0$ almost everywhere on $(0, L)$ and thus (3.26) is identically 0 for all h_0. Combining (3.21)–(3.26) yields

$$\lim_{h\searrow 0}\int_\Omega \Phi : \big(\nabla_h z^h (R^h E^h)^{\mathrm{T}}\big)\,\mathrm{d}x = 0.$$

This concludes the proof of the claim (3.18).

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Inserting (3.15), (3.17) and (3.18) into (3.14) we obtain

$$\lim_{h\searrow 0}\int_\Omega \left(R^hE^h : \nabla_h\psi - hg\cdot\psi\right)\,\mathrm{d}x = \lim_{h\searrow 0}\int_\Omega E^h : \iota((A\Phi-\Phi A)\mathfrak{p}+\Phi'\mathfrak{p})\,\mathrm{d}x + \lim_{h\searrow 0}\int_0^L \widehat{g}\cdot(R^h\Phi e_1)\,\mathrm{d}x_1. \tag{3.27}$$

From the strong convergence $R^h \to R$ in L^∞ we obtain for the second term

$$\lim_{h\searrow 0}\int_0^L \widehat{g}\cdot(R^h\Phi e_1)\,\mathrm{d}x_1 = \int_0^L \widehat{g}\cdot(R\Phi e_1)\,\mathrm{d}x_1, \tag{3.28}$$

while for the first term we will show that

$$\lim_{h\searrow 0}\int_\Omega E^h : \iota((A\Phi-\Phi A)\mathfrak{p}+\Phi'\mathfrak{p})\,\mathrm{d}x = \frac{\partial\mathcal{K}_{(h)}}{\partial m}(m_d)[(A\Phi-\Phi A+\Phi')\mathfrak{p}]. \tag{3.29}$$

Identification of the limit

To show (3.29) we will first prove the analogue to [**5**, lemma 3.1], whose approach we will follow from now on.

Lemma 3.2. *Let* $(u^h)\subset W^{1,2}(\Omega,\mathbb{R}^3)$ *be such that* $\mathrm{t}(u_2^h, u_3^h)\to 0$ *strongly in* $L^2(0, L)$, $\left(|\mathrm{sym}\,\nabla_h u^h|^2\right)$ *is equi-integrable and* $(u_1^h, hu_2^h, hu_3^h)\to 0$ *strongly in* $L^2(\Omega)$. *Then for all* $\phi\in C^\infty_{\mathrm{bdy}}([0, L])$ *we have that*

$$\lim_{h\searrow 0}\int_\Omega \left(\phi\mathbb{A}^hG^h : \nabla_h u^h\right)\mathrm{d}x = 0. \tag{3.30}$$

Proof. Let (u^h) be as in the hypothesis, and fix some $\phi\in C^\infty_{\mathrm{bdy}}([0, L])$. By proposition A.1 there exists a constant $C_\omega>0$, depending only on ω, and sequences $(B^h)\subset W^{1,2}((0, L), \mathbb{R}^{3\times 3}_{\mathrm{skew}})$, $(\vartheta^h)\subset W^{1,2}(\Omega,\mathbb{R}^3)$ and $(o^h)\subset L^2(\Omega,\mathbb{R}^{3\times 3})$ with

$$\mathrm{sym}\,\nabla_h u^h = \mathrm{sym}\,\iota((B^h)'\mathfrak{p}) + \mathrm{sym}\,\nabla_h\vartheta^h + o^h, \tag{3.31}$$

that, in addition, satisfy the bounds

$$\|B^h\|_{W^{1,2}(0,L)} + \|\vartheta^h\|_{L^2(\Omega)} + \|\nabla_h\vartheta^h\|_{L^2(\Omega)} \leqslant C_\omega\|\mathrm{sym}\,\nabla_h u^h\|_{L^2(\Omega)}.$$

Furthermore, B^h, ϑ^h, $o^h\to 0$ strongly in $L^2(\Omega)$, and $(|(B^h)'|^2)$, $(|\nabla_h\vartheta^h|^2)$ are both equi-integrable. Using (3.2) we can write (3.30) as

$$\int_\Omega\left(\phi\mathbb{A}^h\mathrm{sym}\,G^h : \mathrm{sym}\,(\nabla_h u^h)\right)\mathrm{d}x = \int_\Omega\left(\phi E^h : \mathrm{sym}\,(\nabla_h u^h)\right)\mathrm{d}x - \frac{1}{h}\int_\Omega\left(\phi\zeta^h(x, hG^h) : \mathrm{sym}\,(\nabla_h u^h)\right)\mathrm{d}x. \tag{3.32}$$

The first term on the right-hand side can be decomposed with (3.31) to

$$\int_\Omega\left(\phi E^h : \mathrm{sym}\,(\nabla_h u^h)\right)\,\mathrm{d}x = \int_\Omega\left(\phi E^h : \mathrm{sym}\left(\iota((B^h)'\mathfrak{p}) + \nabla_h\vartheta^h + o^h\right)\right)\mathrm{d}x. \tag{3.33}$$

Clearly, the term containing o^h converges to 0 as $h \searrow 0$. By symmetry of $\mathbb{A}^h$ we have that skew $E^h = \frac{1}{h}\mathrm{skew}\,\zeta^h(\cdot,\, hG^h)$ and thus

$$\begin{aligned}
&\int_\Omega \Big(\phi E^h : \mathrm{sym}\,\big(\iota((B^h)'\mathfrak{p}) + \nabla_h \vartheta^h\big)\Big)\,\mathrm{d}x \\
&= \int_\Omega \Big(\phi E^h : \big(\iota((B^h)'\mathfrak{p}) + \nabla_h \vartheta^h\big)\Big)\,\mathrm{d}x \\
&\quad - \int_\Omega \Big(\phi \frac{1}{h}\zeta^h(x, hG^h) : \mathrm{skew}\,\big(\iota((B^h)'\mathfrak{p}) + \nabla_h \vartheta^h\big)\Big)\,\mathrm{d}x.
\end{aligned} \tag{3.34}$$

Combining (3.33) with (3.34) yields

$$\begin{aligned}
\lim_{h\searrow 0}\int_\Omega \Big(\phi \mathbb{A}^h G^h : \mathrm{sym}\,(\nabla_h u^h)\Big)\,\mathrm{d}x &= \lim_{h\searrow 0}\int_\Omega \Big(\phi E^h : \big(\iota((B^h)'\mathfrak{p}) + \nabla_h \vartheta^h\big)\Big)\,\mathrm{d}x \\
&\quad - \lim_{h\searrow 0}\int_\Omega \phi \frac{1}{h}\zeta^h(x, hG^h) : \Big(\iota((B^h)'\mathfrak{p}) + \nabla_h \vartheta^h\Big)\,\mathrm{d}x.
\end{aligned} \tag{3.35}$$

By applying (3.12) with $M^h = (B^h)'$ and $M = 0$, the first term on the right-hand side of (3.35) is 0. For the second term on the right-hand side of (3.35) we write

$$\begin{aligned}
\int_\Omega \Big(\phi E^h : \nabla_h \vartheta^h\Big)\,\mathrm{d}x &= \int_\Omega \Big(\phi R^h E^h : R^h \nabla_h \vartheta^h\Big)\,\mathrm{d}x \\
&= \int_\Omega \Big(R^h E^h : \nabla_h (R^h \phi \vartheta^h)\Big)\,\mathrm{d}x - \int_\Omega \Big(R^h E^h : \iota((R^h\phi)'\vartheta^h)\Big)\,\mathrm{d}x.
\end{aligned} \tag{3.36}$$

For the first term on the right-hand side of (3.36) we use the Euler-Lagrange equation and obtain

$$\int_\Omega E^h : \nabla_h (R^h \phi \vartheta^h)\,\mathrm{d}x = h\int_\Omega g \cdot (R^h \phi \vartheta^h) \to 0,$$

while we split once more the second term on the right-hand side of (3.36) into

$$\begin{aligned}
\int_\Omega \Big(R^h E^h : \iota((R^h\phi)'\vartheta^h)\Big)\,\mathrm{d}x &= \int_\Omega \Big(R^h E^h e_1 \cdot (R^h\phi)'(\vartheta^h - \overline{\vartheta}^h)\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega \Big(R^h E^h e_1 \cdot (R^h\phi)'\overline{\vartheta}^h\Big)\,\mathrm{d}x,
\end{aligned} \tag{3.37}$$

where $\overline{\vartheta}^h(x_1) = \int_\omega \vartheta(x_1,\, x')\,\mathrm{d}x$. By the uniform bound of $h(R^h)''$ in $L^2(0,\, L)$, stated in (2.5), we obtain the uniform bound of $h(R^h)'$ in $W^{1,2}(0,\, L)$. From the compact Sobolev embedding we obtain that $(h(R^h)')$ is strongly compact in $L^\infty(0,\, L)$. Since $(R^h)'$ is bounded in $L^2(0,\, L)$, we have that $h(R^h)' \to 0$ strongly in $L^2(0,\, L)$. By uniqueness of the limit, we have that $h(R^h)' \to 0$ strongly in $L^\infty(0,\, L)$. We apply

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Poincaré's inequality and obtain

$$\|\vartheta^h - \overline{\vartheta}^h\|_{L^2(\Omega)} \leqslant C\|\partial_2\vartheta^h\|_{L^2(\Omega)} \leqslant Ch\|\nabla_h\vartheta^h\|_{L^2(\Omega)} \leqslant Ch.$$

This bound, together with $h(R^h)' \to 0$ strongly in $L^\infty(0,\, L)$, implies that

$$h(R^h\phi)'\frac{(\vartheta^h - \overline{\vartheta}^h)}{h} \to 0 \quad \text{strongly in } L^2(\Omega).$$

For the second term in (3.37) we use Sobolev embedding to obtain $\overline{\vartheta}^h \to 0$ strongly in $L^\infty(0,\, L)$. Combining both we conclude the vanishing of (3.37). Finally, for the last remaining term in (3.32), namely

$$\lim_{h\searrow 0}\int_\Omega \phi(x_1)\frac{1}{h}\zeta^h(x, hG^h) : \Big(\iota((B^h)'\mathfrak{p}) + \nabla_h\vartheta^h\Big)\,\mathrm{d}x,$$

we use that $(|(B^h)'|^2)$ and $(|\nabla_h\vartheta^h|^2)$ are equi-integrable. Hence lemma 3.1 implies that this term vanishes as well. This finishes the proof of the lemma. □

We finally prove (3.29). For this, we decompose E^h into $E^h = \mathbb{A}^h G^h + (1/h)\zeta^h(\cdot,\, hG^h)$ and apply lemma 3.1 to obtain

$$\begin{aligned}\lim_{h\searrow 0}\int_\Omega E^h : \iota((A\Phi - \Phi A)\mathfrak{p} + \Phi'\mathfrak{p})\,\mathrm{d}x \\ = \lim_{h\searrow 0}\int_\Omega \mathbb{A}^h G^h : \iota((A\Phi - \Phi A)\mathfrak{p} + \Phi'\mathfrak{p})\,\mathrm{d}x.\end{aligned} \tag{3.38}$$

From the decomposition (2.13) we get that

$$\operatorname{sym} G^h = \operatorname{sym}\iota(m_d) + \operatorname{sym}\nabla_h\psi^h + o^h,$$

where the fixed part $m_d \in L^2(\Omega,\, \mathbb{R}^3)$ and the corrector sequence ψ^h were introduced in (2.10) and (2.11) respectively, and o^h converges strongly to zero in $L^2(\Omega)$.

We show that $\operatorname{sym}\nabla_h\psi^h$ and $\operatorname{sym}\nabla_h\psi^h_{m_d}$ are, up to L^2-concentration, close in $L^2(\Omega)$, where $(\psi^h_{m_d})$ is the relaxation sequence given by lemma 2.3. Indeed, we first use identity (2.12) to obtain

$$\operatorname{sym}\nabla_h\psi^h = \operatorname{sym}\iota((\Psi^h)'\mathfrak{p}) + \operatorname{sym}\nabla_h v^h,$$

where Ψ^h, v^h are defined prior to (2.12). By applying [**14**, lemma 2.17] to (Ψ^h) and (v^h), we obtain a subsequence (h) (not relabelled), a sequence of measurable sets O^h with $\lim_{h\searrow 0}\mathcal{L}^3(\Omega\setminus O^h) = 0$ and a sequences $(\widetilde{\Psi}^h)$, $(\widetilde{v}^h)$ such that

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$(|(\widetilde{\Psi}^h)'|^2, |(\nabla_h \widetilde{v}^h)|^2$ are equi-integrable and such that

$$\|(\Psi^h - \widetilde{\Psi}^h)'\|_{L^2(O^h)} + \|\nabla_h (v^h - \widetilde{v}^h)\|_{L^2(O^h)} \to 0.$$

By proposition A.2 there exists $(\widetilde{\psi}^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ such that

$$\operatorname{sym} \nabla_h \widetilde{\psi}^h = \operatorname{sym} \iota((\widetilde{\Psi}^h)'\mathfrak{p}) + \operatorname{sym} \nabla_h \widetilde{v}^h.$$

By construction, we have that

$$(|\operatorname{sym} \nabla_h \widetilde{\psi}^h|^2) \text{ is equi-integrable, and } \lim_{h \searrow 0} \|\operatorname{sym} (\nabla_h \psi^h - \nabla_h \widetilde{\psi}^h)\|_{L^2(O^h)} = 0. \tag{3.39}$$

Let $a \in (0, L)$. Then

$$\begin{aligned} &\|\operatorname{sym} \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d})\|^2_{L^2(\Omega)} \\ &= \|\operatorname{sym} \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d})\|^2_{L^2((0,a)\times\omega)} + \|\operatorname{sym} \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d})\|^2_{L^2((a,L)\times\omega)}. \end{aligned} \tag{3.40}$$

For the second term on the right-hand side we use the coercivity of Q^h to obtain

$$\alpha \|\operatorname{sym} \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d})\|^2_{L^2((a,L)\times\omega)} \leqslant \frac{1}{2} \int_{(a,L)\times\omega} \mathbb{A}^h \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}).$$

Let $\rho \in C^\infty([0, L])$ be a cut-off function such that $\rho \geqslant 0$, $\rho = 0$ on $[0, a/2]$ and $\rho = 1$ on $[a, L]$. With this we calculate

$$\begin{aligned} \alpha \|\operatorname{sym} \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d})\|^2_{L^2((a,L)\times\omega)} &\leqslant \frac{1}{2} \int_\Omega \rho \mathbb{A}^h \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}) \\ &= \frac{1}{2} \int_\Omega \rho \mathbb{A}^h (\iota(m_d) + \nabla_h \widetilde{\psi}^h) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}) \\ &\quad - \frac{1}{2} \int_\Omega \rho \mathbb{A}^h (\iota(m_d) + \nabla_h \psi^h_{m_d}) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}). \end{aligned}$$

The second term vanishes by virtue of lemma 2.3, while for the second one we use the decomposition (2.13), i.e.,

$$\operatorname{sym} G^h = \operatorname{sym} \iota(m_d) + \operatorname{sym} \nabla_h \psi^h + o^h,$$

to write

$$\begin{aligned} \int_\Omega \rho \mathbb{A}^h (\iota(m_d) + \nabla_h \widetilde{\psi}^h) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}) &= \int_\Omega \rho \mathbb{A}^h (G^h + o^h) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}) \\ &\quad + \int_\Omega \rho \mathbb{A}^h (\nabla_h (\widetilde{\psi}^h - \psi^h)) : \nabla_h (\widetilde{\psi}^h - \psi^h_{m_d}). \end{aligned} \tag{3.41}$$

The sequence (o^h) converges to 0 strongly in $L^2(\Omega)$, and thus the term containing it vanishes in the limit. By lemma 2.3 the sequence $|\operatorname{sym} (\nabla_h \psi^h_{m_d})|^2$, and by construction the sequence $|\operatorname{sym} (\nabla_h \widetilde{\psi}^h)|^2$, are both equi-integrable. Thus by applying lemma 3.2 the first term on the right-hand side of (3.41) vanishes. For the second term we decompose $\Omega = O^h \cup (\Omega \setminus O^h)$ and estimate with Hölder's inequality

$$\begin{aligned} &\left|\int_\Omega \rho \mathbb{A}^h \nabla_h(\widetilde{\psi}^h - \psi^h) : \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\right| \\ &\leqslant \beta \left|\int_\Omega \operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h) : \operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\right| \\ &\leqslant \beta \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h)\|_{L^2(O^h)} \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\|_{L^2(\Omega)} \\ &\quad + \beta \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h)\|_{L^2(\Omega)} \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\|_{L^2(\Omega \setminus O^h)}. \end{aligned} \tag{3.42}$$

First note that

$$\|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\|_{L^2(\Omega)}, \quad \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h)\|_{L^2(\Omega)}$$

are uniformly bounded in h. Furthermore, utilizing (3.39) we obtain that

$$\lim_{h \searrow 0} \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h)\|_{L^2(O^h)} = 0,$$

and thus the first term on the right-hand side of (3.42) converges to 0. For the second term on the right-hand side of (3.42) we use the equi-integrability of $(|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})|^2)$ together with $\mathcal{L}^3(\Omega \setminus O^h) \to 0$ as $h \searrow 0$, and obtain that the sequence converges to 0 as well.

Returning to (3.40), we take a sequence $a = a(h)$ with $a(h) \searrow 0$ as $h \searrow 0$ such that

$$\lim_{h \searrow 0} \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\|_{L^2((a(h),L)\times\omega)} = 0.$$

By equi-integrability we also obtain

$$\lim_{h \searrow 0} \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\|_{L^2((0,a(h))\times\omega)} = 0,$$

and thus

$$\lim_{h \searrow 0} \|\operatorname{sym} \nabla_h(\widetilde{\psi}^h - \psi^h_{m_d})\|_{L^2(\Omega)} = 0.$$

Returning to (3.38), we first approximate $\operatorname{sym} \nabla_h \psi^h$ by $\operatorname{sym} \nabla_h \widetilde{\psi}^h$, and then the latter by $\operatorname{sym} \nabla_h \psi^h_{m_d}$, thus obtaining

$$\begin{aligned} &\lim_{h \searrow 0} \int_\Omega E^h : \iota((A\Phi - \Phi A)\mathfrak{p} + \Phi'\mathfrak{p})\, \mathrm{d}x \\ &= \lim_{h \searrow 0} \int_\Omega \mathbb{A}^h G^h : \iota((A\Phi - \Phi A)\mathfrak{p} + \Phi'\mathfrak{p})\, \mathrm{d}x \\ &= \lim_{h \searrow 0} \int_\Omega \mathbb{A}^h(\iota(m_d) + \nabla_h \psi^h_{m_d}) : \iota((A\Phi - \Phi A)\mathfrak{p} + \Phi'\mathfrak{p})\, \mathrm{d}x \\ &= \frac{\partial \mathcal{K}_{(h)}}{\partial m}(m_d)[(A\Phi - \Phi A + \Phi')\mathfrak{p}]. \end{aligned} \tag{3.43}$$

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We combine (3.27), (3.28) and (3.43), obtaining

$$0 = \lim_{h\searrow 0}\int_\Omega \left(R^h E^h : \nabla_h\psi - h\widehat{g}\cdot\partial_1\psi\right)\,\mathrm{d}x = \frac{\partial\mathcal{K}_{(h)}}{\partial m}(m_d)[(A\Phi - \Phi A + \Phi')\mathfrak{p}] - \int_0^L \widehat{g}\cdot(R\Phi e_1)\,\mathrm{d}x_1. \tag{3.44}$$

If

$$\frac{\partial\mathcal{K}_{(h)}}{\partial m}(m_d)[(A\Phi - \Phi A + \Phi')\mathfrak{p}] = \left(\frac{\partial}{\partial B}\mathcal{K}^0_{(h)}\right)(A)[A\Phi - \Phi A + \Phi'] \tag{3.45}$$

holds, then (3.44) reads

$$\left(\frac{\partial}{\partial B}\mathcal{K}^0_{(h)}\right)(A)[A\Phi - \Phi A + \Phi'] = \int_0^L \widehat{g}\cdot(R\Phi e_1)\,\mathrm{d}x_1,$$

and by lemma 2.5 this is equivalent to $(\overline{y}, \overline{d}_2, \overline{d}_3)$ being a stationary point of $\mathcal{E}^0$.

After replacing both sides in (3.45) by the more explicit representations (2.17) and (2.24), we see that it suffices to show that the fixed part m_d is given by $m(A, b_{(h),\min}(A))$. By definition of m_d in (2.10) we have that $m_d = m(A, p_1)$ where p_1 is some L^2 function. By the characterization given in (2.20) the equality $p_1 = b_{(h),\min}(A)$ follows, if

$$\frac{\partial\mathcal{K}_{(h)}(m(A,\cdot))}{\partial b}(p_1)[\mu] = 0 \quad \text{for all } \mu\in L^2(0,L).$$

Using (2.19) we see that this is equivalent to

$$\lim_{h\searrow 0}\int_\Omega \mathbb{A}^h(\iota(m_d)) + \operatorname{sym}\nabla_h\psi^h_{m_d}) : \iota(\mu e_1)\,\mathrm{d}x = 0 \quad \text{for all } \mu\in L^2(0,L). \tag{3.46}$$

Similar to before we can replace $\operatorname{sym}\nabla_h\psi^h_{m_d}$ by $\operatorname{sym}\nabla_h\psi^h$. Then we can approximate $\mathbb{A}^h G^h$ by E^h with lemma 3.1, and the statement (3.46) is then seen to be equivalent to

$$\lim_{h\searrow 0}\int_0^L \overline{E}^h_{11}\mu\,\mathrm{d}x = 0 \quad \text{for all } \mu\in L^2(0,L),$$

which now easily follows from (3.10).

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The result stated in theorem 1.1 is part of author's Ph.D. thesis [**19**].

Appendix A.

For convenience of the reader we recall a type of decomposition introduced in [**10**, **11**]. More precisely the variant proved in [**14**, corollary 2.3, lemma 2.4]

Proposition A.1. *Let $L > 0$ and $\Omega = (0, L) \times \omega$, where ω is an open, connected bounded Lipschitz-domain, which is centered at the origin in the sense of (2.1). Let $(u^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ with* $\mathrm{t}(u_2^h, u_3^h) \to 0$ *in $L^2(0, L)$,*

$$\sup_{h>0} \left\| \operatorname{sym} \nabla_h u^h \right\|_{L^2} < \infty \quad \text{and} \quad (u_1^h, hu_2^h, hu_3^h) \to 0 \text{ strongly in } L^2(\Omega, \mathbb{R}^3).$$

Then there exists a constant $C_\omega > 0$, depending only on ω, and sequences $(B^h) \subset W^{1,2}((0, L), \mathbb{R}^{3\times3}_{\text{skew}})$, $(\vartheta^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ and $(o^h) \subset L^2(\Omega, \mathbb{R}^{3\times3})$ with

$$\operatorname{sym} \nabla_h u^h = \operatorname{sym} \iota((B^h)'\mathfrak{p}) + \operatorname{sym} \nabla_h \vartheta^h + o^h,$$

and satisfying the bounds

$$\|B^h\|_{W^{1,2}} + \|\vartheta^h\|_{L^2} + \|\nabla_h \vartheta^h\|_{L^2} \leqslant C_\omega \|\operatorname{sym} \nabla_h u^h\|_{L^2}.$$

Furthermore, B^h, o^h, $\vartheta^h \to 0$ strongly in L^2. If, in addition, $(|\operatorname{sym} \nabla_h u^h|^2)$ is equi-integrable, then so are $(|(B^h)'|^2)$ and $(|\nabla_h \vartheta^h|^2)$.

The reverse holds true as well:

Proposition A.2. *Let $L > 0$ and $\Omega = (0, L) \times \omega$, where ω is an open, connected bounded Lipschitz-domain, which is centered at the origin in the sense of (2.1). Let $(B^h) \subset W^{1,2}((0, L), \mathbb{R}^{3\times3}_{\text{skew}})$, $(\vartheta^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ be sequences with $B^h \to 0$ strongly in $L^2((0, L), \mathbb{R}^{3\times3})$ and $\vartheta^h \to 0$ strongly in $L^2(\Omega, \mathbb{R}^3)$. Then there exists $(u^h) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ with* $\mathrm{t}(u_2^h, u_3^h) \to 0$ *in $L^2(0, L)$ and*

$$(u_1^h, hu_2^h, hu_3^h) \to 0 \text{ strongly in } L^2(\Omega, \mathbb{R}^3)$$

such that

$$\operatorname{sym} \nabla_h u^h = \operatorname{sym} \iota((B^h)'\mathfrak{p}) + \operatorname{sym} \nabla_h \vartheta^h.$$

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