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A sharp trace inequality for functions of bounded variation in the ball

Andrea Cianchi
Dipartimento di Matematica 'U. Dini', Università di Firenze,
Piazza Ghiberti 27, 50122 Firenze, Italy (cianchi@unifi.it)

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The best constant in a mean-value trace inequality for functions of bounded variation on admissible domains $\Omega \subset \mathbb{R}^n$ is shown to agree with an isoperimetric constant associated with Ω. The existence and form of extremals is also discussed. This result is exploited to compute the best constant in the relevant trace inequality when Ω is a ball. The existence and the form of extremals in this special case turn out to depend on the dimension n. In particular, the best constant is not achieved when Ω is a disc in $\mathbb{R}^2$.

1. Introduction and main results

Given an open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, we denote by $\mathrm{BV}(\Omega)$ the space of functions of bounded variation on Ω, namely, those functions $u \in L^1(\Omega)$ whose first-order distributional gradient $\mathrm{D}u$ is a vector-valued Radon measure with finite total variation $\|\mathrm{D}u\|(\Omega)$. The space $\mathrm{BV}(\Omega)$ is a Banach space endowed with the norm given by $\|u\|_{L^1(\Omega)} + \|\mathrm{D}u\|(\Omega)$ for $u \in \mathrm{BV}(\Omega)$. For ease of presentation, we shall assume throughout this paper that Ω is connected.

Traces of functions in $\mathrm{BV}(\Omega)$ on $\partial\Omega$ are well defined if Ω is a Lipschitz domain. More generally, boundary traces of BV functions can be defined if Ω is an admissible domain, namely, a bounded open set such that $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, $\mathcal{H}^{n-1}(\partial\Omega \backslash \partial^{\mathrm{M}}\Omega) = 0$ and

$$\min\{\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \partial\Omega), \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}}E)\} \leqslant C\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega) \tag{1.1}$$

for some positive constant C and every measurable set $E \subset \Omega$ [23, definition 5.10.1]. Here, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, ∂ denotes the topological boundary and ∂^{M} denotes the essential boundary in the sense of geometric measure theory. In this connection, recall that if $E \subset \Omega$ is a measurable set, then its characteristic function $\chi_E \in \mathrm{BV}(\Omega)$ if and only if $\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega) < \infty$; moreover, $\|\mathrm{D}\chi_E\|(\Omega) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega)$ [16, theorem 4.5.11]. The quantity $\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega)$ is called the perimeter of E relative to Ω.

If Ω is an admissible domain, then the trace on $\partial\Omega$ of a function $u \in \mathrm{BV}(\Omega)$ is the function

$$\tilde{u} \colon \partial\Omega \to \mathbb{R}$$

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defined for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$ as

$$\tilde{u}(x) = \lim_{r\to 0} \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x)\cap\Omega} u(y)\,\mathrm{d}y. \tag{1.2}$$

Note that this limit can actually be shown to exist for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$. Here, $B_r(x)$ denotes the ball centred at x with radius r.

Alternative notions of the trace on $\partial\Omega$ of a function $u \in \mathrm{BV}(\Omega)$ are available in the literature. One of them involves the upper and lower approximate limits of the continuation of u by 0 outside Ω [23, definition 5.10.5]. Another makes use of the rough trace [21, § 6.5.1]. Both these definitions agree with $\tilde{u}$, up to subsets of $\partial\Omega$ of $\mathcal{H}^{n-1}$-measure zero.

Boundary traces of functions u from the Sobolev space $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ are more classically defined on a Lipschitz domain Ω as the limit of the standard traces on $\partial\Omega$ of approximating sequences of smooth functions on $\bar{\Omega}$. The trace of a function $u \in W^{1,1}(\Omega)$ obtained via this definition coincides with $\tilde{u}$, up to subsets of $\partial\Omega$ of $\mathcal{H}^{n-1}$-measure zero.

It is well known that $\tilde{u} \in L^1(\partial\Omega)$, the space of integrable functions on $\partial\Omega$ with respect to $\mathcal{H}^{n-1}$, for any function $u \in \mathrm{BV}(\Omega)$, and that $L^1(\partial\Omega)$ is the smallest Lebesgue space to which $\tilde{u}$ belongs for every $u \in \mathrm{BV}(\Omega)$. Furthermore, the linear mapping $\mathrm{BV}(\Omega) \ni u \mapsto \tilde{u} \in L^1(\partial\Omega)$ is bounded. The optimal constant in a Poincaré trace inequality between $\inf_{c\in\mathbb{R}} \|\tilde{u} - c\|_{L^1(\partial B)}$ and $\|\mathrm{D}u\|(\Omega)$ was found in terms of the best constant C in the isoperimetric inequality (1.1) as part of the pioneering work of Maz′ya on the use of isoperimetric inequalities in the characterization of Sobolev-type embeddings [21, theorem 6.5.2].

One purpose of this paper is to show that, for any admissible domain Ω, the optimal constant in a mean-value Poincaré trace inequality in $\mathrm{BV}(\Omega)$ can be expressed via the best constant in a different isoperimetric inequality on Ω. The former constant turns out to be achieved if and only if the latter constant is achieved. Moreover, the characteristic function of any (possible) optimal set in the relevant isoperimetric inequality is an extremal function in the trace inequality.

The trace inequality in question reads

$$\|\tilde{u} - \tilde{u}_{\partial\Omega}\|_{L^1(\partial\Omega)} \leqslant C(\Omega)\|\mathrm{D}u\|(\Omega) \tag{1.3}$$

for every $u \in \mathrm{BV}(\Omega)$. Here, $\tilde{u}_{\partial\Omega}$ denotes the mean value of $\tilde{u}$ over $\partial\Omega$, given by

$$\tilde{u}_{\partial\Omega} = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} \tilde{u}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x),$$

and $C(\Omega)$ is the best constant in (1.3), namely, the smallest constant which renders (1.3) true. Note that, due to the lack of compactness of the mapping $\mathrm{BV}(\Omega) \ni u \mapsto \tilde{u} \in L^1(\partial\Omega)$, non-constant functions $u \in \mathrm{BV}(\Omega)$ that turn (1.3) into an equality need not exist. If such a function does exist, it will be called an extremal in (1.3). A sequence of functions $\{u_k\}_{k\in\mathbb{N}}$ such that

$$\frac{\|\tilde{u}_k - (\tilde{u}_k)_{\partial\Omega}\|_{L^1(\partial\Omega)}}{\|\mathrm{D}u_k\|(\Omega)} \to C(\Omega)$$

will be called optimizing in (1.3).

The isoperimetric constant coming into play in our discussion is defined as

$$K(\Omega) = \sup_E \frac{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\Omega)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}} E)}{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \Omega)}, \tag{1.4}$$

where the supremum is extended over all measurable sets $E \subset \Omega$ with positive Lebesgue measure. Any set $E \subset \Omega$ at which the supremum in (1.4) is achieved will be called an isoperimetric set in (1.4). The existence of isoperimetric sets in (1.4) is not guaranteed in general, and depends on global geometric properties of Ω. A sequence of sets $\{E_k\}_{k\in\mathbb{N}}$ having the property that

$$\frac{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E_k \cap \partial\Omega)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}} E_k)}{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E_k \cap \Omega)} \to K(\Omega)$$

will be called optimizing in (1.4).

The link between the constants $C(\Omega)$ and $K(\Omega)$ is exhibited by the following result.

THEOREM 1.1. *Let Ω be an admissible domain in $\mathbb{R}^n$, $n \geqslant 2$. Then*

$$C(\Omega) = \frac{2K(\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)}. \tag{1.5}$$

Extremals u exist in (1.3) if and only if isoperimetric sets exist in (1.4). If E is an isoperimetric set in (1.4), then any function of the form $u = a\chi_E + b$ is an extremal in (1.3) for every $a, b \in \mathbb{R}$.

More generally, if $\{E_k\}$ is an optimizing sequence of sets in (1.4), then the sequence $\{u_k\} = \{a_k\chi_{E_k} + b_k\}$ is an optimizing sequence of functions in (1.3) for every $a_k, b_k \in \mathbb{R}$.

Our main result is contained in theorem 1.2. It provides us with the best constant $C(\Omega)$ in the trace inequality (1.3) when Ω is a ball B, and relies upon theorem 1.1. Interestingly, the existence and the form of extremals in the mean-value trace inequality in B turn out to depend on the dimension n. In particular, theorem 1.2 shows that extremals in the trace inequality (1.3) may actually not exist, even for domains with such a simple geometry as the disc in $\mathbb{R}^2$. This should be contrasted with mean-value Poincaré inequalities in $\mathrm{BV}(\Omega)$ inside Ω, where the best constant is always achieved, provided that $\partial\Omega$ is sufficiently smooth (see remark 1.7).

In what follows, we call the (non-empty) intersection of B with a half-space a spherical segment in B.

THEOREM 1.2. *Let B be a ball in $\mathbb{R}^n$, $n \geqslant 2$. Then*

$$\|\tilde{u} - \tilde{u}_{\partial B}\|_{L^1(\partial B)} \leqslant C(n)\|\mathrm{D}u\|(B) \tag{1.6}$$

for every $u \in \mathrm{BV}(B)$, where

$$C(n) = \begin{cases} \dfrac{n\sqrt{\pi}}{2}\dfrac{\Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}(n+2))} & \text{if } n \geqslant 3, \\ 2 & \text{if } n = 2. \end{cases} \tag{1.7}$$

The constant $C(n)$ is the best possible in (1.6).

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If $n \geqslant 4$, the equality holds in (1.6) when u agrees with the characteristic function of a half-ball.

If $n = 3$, the equality holds in (1.6) when u agrees with the characteristic function of any spherical segment.

If $n = 2$, the equality never holds in (1.6), unless u is constant. Any sequence of characteristic functions of spherical segments whose measure converges to 0 is optimizing in (1.6).

REMARK 1.3. The trace inequality of [21] to which we alluded above has the form (1.6), with $\|\tilde{u} - \tilde{u}_{\partial B}\|_{L^1(\partial B)}$ replaced with $\inf_{c\in\mathbb{R}} \|\tilde{u} - c\|_{L^1(\partial B)}$ on the left-hand side [21, theorem 6.5.2 and corollary 6.4.4/3]. When $n \geqslant 3$, the constant in (1.6) agrees with the constant of [21]. Since $\inf_{c\in\mathbb{R}} \|\tilde{u} - c\|_{L^1(\partial B)}$ is attained when c is a median of $\tilde{u}$, which differs from $\tilde{u}_{\partial B}$ in general, we have that

$$\inf_{c\in\mathbb{R}} \|\tilde{u} - c\|_{L^1(\partial B)} \leqslant \|\tilde{u} - \tilde{u}_{\partial B}\|_{L^1(\partial B)},$$

and the inequality is strict for a generic function u. Thus, inequality (1.6) simultaneously improves and recovers the inequality of [21] for $n \geqslant 3$.

REMARK 1.4. Theorem 1.2 yields, in particular, the inequality

$$\|\tilde{u} - \tilde{u}_{\partial B}\|_{L^1(\partial B)} \leqslant C(n)\|\nabla u\|_{L^1(B)} \tag{1.8}$$

for every $u \in W^{1,1}(B)$, where ∇u denotes the (weak) gradient of u, and $C(n)$ is given by (1.7). A standard approximation argument for characteristic functions of spherical segments by Lipschitz functions ensures that the constant $C(n)$ is sharp in (1.8) as well.

REMARK 1.5. Let $q \in (0, 1)$ and let R be the radius of B. An application of the Hölder inequality yields

$$\|\tilde{u} - \tilde{u}_{\partial B}\|_{L^q(\partial B)} \leqslant \mathcal{H}^{n-1}(\partial B)^{(1-q)/q}\|\tilde{u} - \tilde{u}_{\partial B}\|_{L^1(\partial B)}. \tag{1.9}$$

Coupling (1.9) with (1.6) tells us that

$$\|\tilde{u}-\tilde{u}_{\partial B}\|_{L^q(\partial B)} \leqslant \pi^{n(1-q)/(2q)}\Gamma\left(\frac{n+2}{n}\right)^{(q-1)/q} C(n)R^{(1-q)(n-1)/q}\|\mathrm{D}u\|(B) \tag{1.10}$$

for every $u \in \mathrm{BV}(B)$. Moreover, the constant in (1.10) is sharp if $n \geqslant 3$, since the equality holds provided that u agrees with the characteristic function of a half-ball.

REMARK 1.6. Inequality (1.3), and, in particular, (1.6), are a counterpart on bounded domains of a basic inequality for BV functions in the half-space

$$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\colon x_n > 0\},$$

which tells us that

$$\|\tilde{u}\|_{L^1(\partial\mathbb{R}^n_+)} \leqslant \|\mathrm{D}u\|(\mathbb{R}^n_+) \tag{1.11}$$

for every $u \in \mathrm{BV}(\mathbb{R}^n_+)$. Inequality (1.11) is standard and easy to prove via one-dimensional integration along the x_n variable and Fubini's theorem. The constant 1 on the right-hand side of (1.11) is sharp, the sequence

$$\{u_k\} = \{\chi_{\{(x',x_n)\colon |x'|<1,\ 0<x_n<1/k\}}\}$$

being optimizing. Inequality (1.11) continues to hold and to be optimal for functions $u \in W^{1,1}(\mathbb{R}^n_+)$; of course, $\|\mathrm{D}u\|(\mathbb{R}^n_+)$ can be replaced with $\|\nabla u\|_{L^1(\mathbb{R}^n_+)}$ in this case.

The best constant in the trace inequality for functions $u \in W^{1,p}(\mathbb{R}^n_+)$ is also known. It was found in [13] for $p = 2$, and in [22] for $1 < p < n$. A Moser-type trace inequality on arbitrary, sufficiently smooth domains $\Omega \subset \mathbb{R}^n$ for functions in the borderline Sobolev space $W^{1,n}(\Omega)$ is a special case of the results of [9]. Additional references on sharp trace inequalities include [1, 2, 4, 5, 11, 14, 20].

Remark 1.7. Incidentally, let us mention that the optimal constant and the extremal functions in the mean-value Poincaré inequality

$$\|u - u_\Omega\|_{L^{n/(n-1)}(\Omega)} \leqslant C\|\mathrm{D}u\|(\Omega), \tag{1.12}$$

for $u \in \mathrm{BV}(\Omega)$, were found in [8] in the case when $\Omega = B$. Here, u_Ω stands for the mean value of u over Ω. As mentioned above, unlike in (1.3), extremals in (1.12) do always exist for any set $\Omega \subset \mathbb{R}^n$ whose boundary is of class C^2, as recently established in [6]. When $n = 2$, the constant C in (1.12) is the smallest possible among all convex domains when Ω is a disc, as shown in [15]. Existence problems for extremals in mean-value Poincaré inequalities involving $L^p(\Omega)$-norms of the gradient with $p > 1$ have been considered in [12] (see also [19] for an alternate approach in the special case when Ω is a ball). A description of symmetry properties of extremals in the mean-value Poincaré inequality on the ball for the L^2-norm of the gradient is the subject of [17]. One-dimensional Poincaré inequalities are treated in [3, 10]. Related questions are discussed in [7].

2. Proof of theorem 1.1

In this section we are concerned with the proof of theorem 1.1.

Proof of theorem 1.1. Set $u_+ = \frac{1}{2}(u + |u|)$ and $u_- = \frac{1}{2}(|u| - u)$, the positive and the negative parts of u. Since $\tilde{u} = \tilde{u}_+ - \tilde{u}_-$, we have that

$$\|\tilde{u} - \tilde{u}_{\partial\Omega}\|_{L^1(\partial\Omega)} \leqslant \|\tilde{u}_+ - (\tilde{u}_+)_{\partial\Omega}\|_{L^1(\partial\Omega)} + \|\tilde{u}_- - (\tilde{u}_-)_{\partial\Omega}\|_{L^1(\partial\Omega)}. \tag{2.1}$$

Moreover,

$$\|\mathrm{D}u\|(\Omega) = \|\mathrm{D}(u_+)\|(\Omega) + \|\mathrm{D}(u_-)\|(\Omega). \tag{2.2}$$

Thus, it suffices to prove inequality (1.6) in the case when $u \geqslant 0$. In this case,

$$\tilde{u}(x) = \int_0^\infty \chi_{\{\tilde{u} \geqslant t\}}(x)\,\mathrm{d}t \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega. \tag{2.3}$$

As a consequence,

$$\begin{aligned}\tilde{u}_{\partial\Omega} &= \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} \tilde{u}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} \left(\int_0^\infty \chi_{\{\tilde{u} \geqslant t\}}(x)\,\mathrm{d}t \right) \mathrm{d}\mathcal{H}^{n-1}(x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \biggl(\int_{\partial\Omega} \chi_{\{\tilde u \geqslant t\}}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \biggr)\,\mathrm{d}t \\
&= \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})\,\mathrm{d}t. \qquad (2.4)
\end{aligned}
$$

Owing to (2.3) and (2.4),

$$
\begin{aligned}
&\|\tilde u - \tilde u_{\partial\Omega}\|_{L^1(\partial\Omega)} \\
&\quad = \int_{\partial\Omega} \biggl| \int_0^\infty \chi_{\{\tilde u \geqslant t\}}(x)\,\mathrm{d}t - \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})\,\mathrm{d}t \biggr|\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad \leqslant \int_{\partial\Omega} \int_0^\infty \biggl| \chi_{\{\tilde u \geqslant t\}}(x) - \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \mathcal{H}^{n-1}(\{\tilde u \geqslant t\}) \biggr|\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \int_{\partial\Omega} |\mathcal{H}^{n-1}(\partial\Omega)\chi_{\{\tilde u \geqslant t\}}(x) - \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})|\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\
&\quad = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \biggl(\int_{\{\tilde u \geqslant t\}} (\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\{\tilde u \geqslant t\}))\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&\qquad\qquad + \int_{\partial\Omega \setminus \{\tilde u \geqslant t\}} \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})\,\mathrm{d}\mathcal{H}^{n-1}(x) \biggr)\,\mathrm{d}t \\
&\quad = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty [\mathcal{H}^{n-1}(\{\tilde u \geqslant t\})(\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})) \\
&\qquad\qquad + (\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\{\tilde u \geqslant t\}))\mathcal{H}^{n-1}(\{\tilde u \geqslant t\})]\,\mathrm{d}t \\
&\quad = \frac{2}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})(\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\{\tilde u \geqslant t\}))\,\mathrm{d}t. \qquad (2.5)
\end{aligned}
$$

We have that

$$
\mathcal{H}^{n-1}(\{\tilde u \geqslant t\}) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \partial\Omega) \quad \text{for almost every } t > 0. \qquad (2.6)
$$

Equation (2.6) is a consequence of: the coincidence $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ of $\tilde u$ with the so-called rough trace of u [21, theorem 6.6.2]; the fact that, for almost every $t > 0$, the essential boundary $\partial^{\mathrm{M}}\{u \geqslant t\}$ agrees, up to a set of $\mathcal{H}^{n-1}$ measure zero, with the reduced boundary of $\{u \geqslant t\}$ [23, lemma 5.9.5]; [21, lemma 6.5.1/2], where equation (2.6) is established with $\tilde u$ replaced with the rough trace of u, and $\partial^{\mathrm{M}}\{u \geqslant t\}$ replaced with the reduced boundary of $\{u \geqslant t\}$. Thus,

$$
\begin{aligned}
&\frac{2}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \mathcal{H}^{n-1}(\{\tilde u \geqslant t\})(\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\{\tilde u \geqslant t\}))\,\mathrm{d}t \\
&\quad = \frac{2}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \partial\Omega) \\
&\qquad\qquad \times (\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \partial\Omega))\,\mathrm{d}t \\
&\quad = \frac{2}{\mathcal{H}^{n-1}(\partial\Omega)} \int_0^\infty \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \partial\Omega)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}}\{u \geqslant t\})\,\mathrm{d}t. \qquad (2.7)
\end{aligned}
$$

By (1.4),

$$\int_0^\infty \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \partial\Omega)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}}\{u \geqslant t\})\,\mathrm{d}t \leqslant K(\Omega)\int_0^\infty \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \Omega)\,\mathrm{d}t. \tag{2.8}$$

The coarea formula for BV functions [23, theorem 5.4.4] tells us that

$$\int_0^\infty \mathcal{H}^{n-1}(\partial^{\mathrm{M}}\{u \geqslant t\} \cap \Omega)\,\mathrm{d}t = \|\mathrm{D}u\|(\Omega). \tag{2.9}$$

From (2.1), (2.2) and (2.5)–(2.9), we deduce that $C(\Omega) \leqslant 2K(\Omega)/\mathcal{H}^{n-1}(\partial\Omega)$.

To prove the reverse inequality, given any measurable set E such that

$$\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega) < \infty,$$

one can make use of the function $\chi_E \in \mathrm{BV}(\Omega)$ as a trial function in (1.3). Doing so, we easily obtain that

$$\begin{aligned} C(\Omega) &\geqslant \frac{\|\tilde{\chi}_E - (\tilde{\chi}_E)_{\partial\Omega}\|_{L^1(\partial\Omega)}}{\|\mathrm{D}\chi_E\|(\Omega)} \\ &= \frac{2\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \partial\Omega)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}}E)}{\mathcal{H}^{n-1}(\partial\Omega)\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega)}. \end{aligned} \tag{2.10}$$

Note that, in (2.10), we have made use of (2.6), applied with $u = \chi_E$, and of the fact that $\|\mathrm{D}\chi_E\|(\Omega) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega)$. Inequality (2.10) implies that

$$C(\Omega) \geqslant \frac{2K(\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)}.$$

Equation (1.5) is fully proved.

Next, if E is an isoperimetric set in (1.4), then, by (1.5) and (2.10),

$$\begin{aligned} \frac{2K(\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)} &= C(\Omega) \\ &\geqslant \frac{\|\tilde{\chi}_E - (\tilde{\chi}_E)_{\partial\Omega}\|_{L^1(\partial\Omega)}}{\|\mathrm{D}\chi_E\|(\Omega)} \\ &= \frac{2\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \partial\Omega)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^{\mathrm{M}}E)}{\mathcal{H}^{n-1}(\partial\Omega)\mathcal{H}^{n-1}(\partial^{\mathrm{M}}E \cap \Omega)} \\ &= \frac{2K(\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)}. \end{aligned} \tag{2.11}$$

Hence, the equality holds in the inequality in (2.11). Thus, χ_E, and hence any function of the form $a\chi_E + b$ with $a, b \in \mathbb{R}$, is an extremal in (1.3). An analogous argument proves the assertion concerning optimizing sequences in (1.3) and (1.4).

Conversely, assume that there exists an extremal u in (1.3). An inspection of the above proof tells us that if the equality holds in (1.3), then, in particular, the equality must hold in (2.8), with u replaced with u_+ and u_-, for almost

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every $t \in [0, \operatorname{ess\,sup} u_+)$ and $t \in [0, \operatorname{ess\,sup} u_-)$, respectively. Equality in (2.8) entails that the level sets $\{u_\pm \geqslant t\}$ are isoperimetric in (1.4) for almost every $t \in (\operatorname{ess\,inf} u_\pm, \operatorname{ess\,sup} u_\pm)$. □

3. Proof of theorem 1.2

Let $\mathbb{B}^n$ be the ball in $\mathbb{R}^n$, which is centred at 0 and has radius 1. For each $\vartheta \in [0, \pi]$, we denote by $T(\vartheta)$ the spherical segment in $\mathbb{B}^n$ given by

$$T(\vartheta) = \mathbb{B}^n \cap \{(x_1, \dots, x_n) \colon x_1 \geqslant \cos\vartheta\}. \tag{3.1}$$

Let us call ω_n the n-dimensional Lebesgue measure of $\mathbb{B}^n$, namely,

$$\omega_n = \pi^{n/2}/\Gamma(1 + \tfrac{1}{2}n).$$

Define the function $\Phi\colon [0, \pi] \to [0, \omega_{n-1}]$ as

$$\Phi(\vartheta) = \omega_{n-1} \sin^{n-1}\vartheta \quad \text{for } \vartheta \in [0, \pi], \tag{3.2}$$

and the function $\Psi\colon [0, \pi] \to [0, n\omega_n]$ as

$$\Psi(\vartheta) = (n-1)\omega_{n-1} \int_0^\vartheta \sin^{n-2}\eta \, \mathrm{d}\eta \quad \text{for } \vartheta \in [0, \pi]. \tag{3.3}$$

Elementary geometric considerations show that

$$\Phi(\vartheta) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}} T(\vartheta) \cap \mathbb{B}^n) \tag{3.4}$$

and

$$\Psi(\vartheta) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}} T(\vartheta) \cap \partial\mathbb{B}^n) \tag{3.5}$$

for every $\vartheta \in [0, \pi]$.

Given a measurable set $E \subset \mathbb{B}^n$, we denote by E^{s} the spherical symmetral of E about the half-axis $H = \{(x_1, \dots, x_n) \colon x_1 \geqslant 0,\ x_2 = \dots = x_n = 0\}$. The set E^{s} is defined as the subset of $\mathbb{B}^n$ such that the intersection of E^{s} with any sphere S centred at 0 is a spherical cap, centred at $S \cap H$, such that

$$\mathcal{H}^{n-1}(E^{\mathrm{s}} \cap S) = \mathcal{H}^{n-1}(E \cap S).$$

In particular, E^{s} is symmetric about the x_1-axis.

The next result is an inequality between $\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n)$ and $\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n)$ for any measurable set $E \subset \mathbb{B}^n$. It will be exploited to show that, when $\Omega = \mathbb{B}^n$, the supremum in (1.4) agrees with the supremum of the same functional restricted to the class of spherical segments in $\mathbb{B}^n$. The relevant inequality is essentially contained in [21, lemma 6.4.4]. We reproduce a proof here for completeness.

Proposition 3.1. *Spherical segments in $\mathbb{B}^n$ minimize $\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n)$ among all measurable sets $E \subset \mathbb{B}^n$ with prescribed $\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n)$. In formulae,*

$$\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n) \geqslant \Phi(\Psi^{-1}(\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n))) \tag{3.6}$$

for every measurable set $E \subset \mathbb{B}^n$.

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Proof. Fix $s \in (0, n\omega_n)$. Let $E \subset \mathbb{B}^n$ be a measurable set such that

$$\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n) = s.$$

We may obviously assume that $\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n) < \infty$, and hence that

$$\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E) < \infty.$$

By [21, lemma 6.4.1/1], there exists a sequence of polyhedra $\{P_k\}$ enjoying the following properties. Define $Q_k = P_k \cap \mathbb{B}^n$ for $k \in \mathbb{N}$. Then

$$\lim_{k\to\infty} \chi_{Q_k} = \chi_E \quad \text{in } L^1(\mathbb{B}^n),$$
$$\lim_{k\to\infty} \mathcal{H}^{n-1}(\partial Q_k \cap \mathbb{B}^n) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n), \tag{3.7}$$

and

$$\lim_{k\to\infty} \mathcal{H}^{n-1}(\partial Q_k \cap \partial\mathbb{B}^n) = \mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n). \tag{3.8}$$

Fix any $k \in \mathbb{N}$. By the very definition of spherical symmetrization,

$$\mathcal{H}^{n-1}(\partial Q_k^{\mathrm{s}} \cap \partial\mathbb{B}^n) = \mathcal{H}^{n-1}(\partial Q_k \cap \partial\mathbb{B}^n). \tag{3.9}$$

Moreover, since spherical symmetrization does not increase the perimeter relative to B (see, for example, [18]),

$$\mathcal{H}^{n-1}(\partial Q_k^{\mathrm{s}} \cap \mathbb{B}^n) \leqslant \mathcal{H}^{n-1}(\partial Q_k \cap \mathbb{B}^n). \tag{3.10}$$

Now, let T_k be the spherical segment in $\mathbb{B}^n$ such that

$$\partial T_k \cap \partial\mathbb{B}^n = \partial Q_k^{\mathrm{s}} \cap \partial\mathbb{B}^n. \tag{3.11}$$

Obviously,

$$\mathcal{H}^{n-1}(\partial T_k \cap \mathbb{B}^n) \leqslant \mathcal{H}^{n-1}(\partial Q_k^{\mathrm{s}} \cap \mathbb{B}^n). \tag{3.12}$$

By (3.11),

$$\mathcal{H}^{n-1}(\partial T_k \cap \partial\mathbb{B}^n) = \mathcal{H}^{n-1}(\partial Q_k \cap \partial\mathbb{B}^n), \tag{3.13}$$

and, by (3.12) and (3.10),

$$\mathcal{H}^{n-1}(\partial T_k \cap \mathbb{B}^n) \leqslant \mathcal{H}^{n-1}(\partial Q_k \cap \mathbb{B}^n). \tag{3.14}$$

Owing to (3.4) and (3.5), the equality holds in (3.6) if $E = T_k$, namely,

$$\mathcal{H}^{n-1}(\partial T_k \cap \mathbb{B}^n) = \Phi(\Psi^{-1}(\mathcal{H}^{n-1}(\partial T_k \cap \partial\mathbb{B}^n))). \tag{3.15}$$

Inequality (3.6) follows from (3.13)–(3.15) and (3.7) and (3.8). □

Let us now define the function $f\colon (0,\pi) \to [0,\infty)$ as

$$f(\vartheta) = \frac{1}{\sin^{n-1}\vartheta}\biggl(\int_0^{\vartheta} \sin^{n-2}\eta \,\mathrm{d}\eta\biggr)\biggl(\int_{\vartheta}^{\pi} \sin^{n-2}\eta \,\mathrm{d}\eta\biggr) \quad \text{for } \vartheta \in (0,\pi). \tag{3.16}$$

Note that $f(\vartheta)$ is related to the functional on the right-hand side of (1.4) evaluated for $E = T(\vartheta)$ by the equality

$$\frac{\mathcal{H}^{n-1}(\partial T(\vartheta) \cap \partial\mathbb{B}^n)\mathcal{H}^{n-1}(\partial\mathbb{B}^n \setminus \partial T(\vartheta))}{\mathcal{H}^{n-1}(\partial T(\vartheta) \cap \mathbb{B}^n)} = (n-1)^2\omega_{n-1} f(\vartheta) \quad \text{for } \vartheta \in (0, \pi). \tag{3.17}$$

In the next lemma, monotonicity properties of f are established in order to determine the supremum of the left-hand side of (3.17).

LEMMA 3.2. *Let $n \geqslant 2$, and let f be the function defined as in (3.16). Then, $f(\vartheta) = f(\pi - \vartheta)$ for $\vartheta \in (0, \pi)$. Moreover, we have the following.*

(i) *If $n = 2$, then f is strictly decreasing in $(0, \frac{1}{2}\pi]$. Hence,*

$$\sup_{\vartheta \in (0,\pi)} f(\vartheta) = \lim_{\vartheta \to 0^+} f(\vartheta) = \lim_{\vartheta \to \pi^-} f(\vartheta),$$

and $\sup_{\vartheta \in (0,\pi)} f(\vartheta)$ is not achieved.

(ii) *If $n = 3$, then f is constant in $(0, \pi)$.*

(iii) *If $n \geqslant 4$, then f is strictly increasing in $(0, \frac{1}{2}\pi]$. Hence,*

$$\max_{\vartheta \in (0,\pi)} f(\vartheta) = f(\tfrac{1}{2}\pi).$$

Proof. (i) If $n = 2$, then

$$f(\vartheta) = \frac{\vartheta(\pi - \vartheta)}{\sin\vartheta} \quad \text{for } \vartheta \in (0, \tfrac{1}{2}\pi].$$

Thus,

$$f'(\vartheta) = \frac{g(\vartheta)}{\sin^2\vartheta} \qquad \text{for } \vartheta \in (0, \tfrac{1}{2}\pi],$$

where we have set

$$g(\vartheta) = (\pi - 2\vartheta)\sin\vartheta - \vartheta(\pi - \vartheta)\cos\vartheta \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi].$$

Note that $g(0) = g(\frac{1}{2}\pi) = 0$, and

$$g'(\vartheta) = (-\vartheta^2 + \pi\vartheta - 2)\sin\vartheta \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi].$$

Thus,

$$g' < 0 \text{ in } [0, \tfrac{1}{2}(\pi - \sqrt{\pi^2 - 8})) \quad \text{and} \quad g' > 0 \text{ in } (\tfrac{1}{2}(\pi - \sqrt{\pi^2 - 8}), \tfrac{1}{2}\pi].$$

Hence, $g < 0$ in $(0, \frac{1}{2}\pi)$, and f is strictly decreasing in $(0, \frac{1}{2}\pi]$.

(ii) If $n = 3$, then

$$f(\vartheta) = \frac{(1 - \cos\vartheta)(1 + \cos\vartheta)}{\sin^2\vartheta} = 1 \quad \text{for } \vartheta \in (0, \pi).$$

Hence, f is constant in $(0, \pi)$.

(iii) If $n \geqslant 4$, then

$$f'(\vartheta) = \frac{h(\vartheta)}{\sin^n \vartheta} \quad \text{for } \vartheta \in (0, \tfrac{1}{2}\pi],$$

where we have set

$$h(\vartheta) = \sin^{n-1}\vartheta\biggl(\int_\vartheta^\pi \sin^{n-2}\eta \,\mathrm{d}\eta - \int_0^\vartheta \sin^{n-2}\eta \,\mathrm{d}\eta\biggr) - (n-1)\cos\vartheta\biggl(\int_\vartheta^\pi \sin^{n-2}\eta \,\mathrm{d}\eta\biggr)\biggl(\int_0^\vartheta \sin^{n-2}\eta \,\mathrm{d}\eta\biggr)$$

for $\vartheta \in [0, \frac{1}{2}\pi]$. Observe that $h(0) = h(\frac{1}{2}\pi) = 0$, and

$$h'(\vartheta) = m(\vartheta)\sin\vartheta \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi],$$

where

$$m(\vartheta) = -2\sin^{2n-4}\vartheta + (n-1)\biggl(\int_\vartheta^\pi \sin^{n-2}\eta \,\mathrm{d}\eta\biggr)\biggl(\int_0^\vartheta \sin^{n-2}\eta \,\mathrm{d}\eta\biggr) \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi].$$

Next, note that $m(0) = 0$, and

$$m'(\vartheta) = z(\vartheta)\sin^{n-2}\vartheta \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi]$$

where

$$z(\vartheta) = -4(n-2)\sin^{n-3}\vartheta\cos\vartheta + (n-1)\biggl(\int_\vartheta^\pi \sin^{n-2}\eta \,\mathrm{d}\eta - \int_0^\vartheta \sin^{n-2}\eta \,\mathrm{d}\eta\biggr) \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi].$$

The function z has the following properties:

$$z(0) = (n-1)\int_0^\pi \sin^{n-2}\eta \,\mathrm{d}\eta > 0, \qquad z(\tfrac{1}{2}\pi) = 0,$$

and

$$z'(\vartheta) = 2(n-3)\sin^{n-4}\vartheta((2n-3)\sin^2\vartheta - 2n + 4) \quad \text{for } \vartheta \in [0, \tfrac{1}{2}\pi].$$

Thus,

$$z' < 0 \text{ in } \biggl(0, \sqrt{\arcsin\frac{2n-4}{2n-3}}\biggr) \quad \text{and} \quad z' > 0 \text{ in } \biggl(\sqrt{\arcsin\frac{2n-4}{2n-3}}, \frac{\pi}{2}\biggr).$$

As a consequence, there exists $\vartheta_1 \in (0, \sqrt{\arcsin(2n-4)/(2n-3)})$ such that

$$z > 0 \text{ in } (0, \vartheta_1) \quad \text{and} \quad z < 0 \text{ in } (\vartheta_1, \tfrac{1}{2}\pi).$$

Therefore,

$$m' > 0 \text{ in } (0, \vartheta_1) \quad \text{and} \quad m' < 0 \text{ in } (\vartheta_1, \tfrac{1}{2}\pi);$$

hence, there exists $\vartheta_2 \in (0, \frac{1}{2}\pi)$ such that

$$m > 0 \text{ in } (0, \vartheta_2) \quad \text{and} \quad m < 0 \text{ in } (\vartheta_2, \tfrac{1}{2}\pi).$$

Note that m necessarily has to be negative in a neighbourhood of $\frac{1}{2}\pi$, since $h(0) = h(\frac{1}{2}\pi)(= 0)$. Consequently,

$$h' > 0 \text{ in } (0, \vartheta_2) \quad \text{and} \quad h' < 0 \text{ in } (\vartheta_2, \tfrac{1}{2}\pi),$$

whence $h > 0$ in $(0, \frac{1}{2}\pi)$. Thus, $f' > 0$ in $(0, \frac{1}{2}\pi)$, hence, f strictly increases in $(0, \frac{1}{2}\pi]$. □

Proposition 3.3. *Let $n \geqslant 2$, and let $K(\mathbb{B}^n)$ be defined as in (1.4), with $\Omega = \mathbb{B}^n$, namely,*

$$K(\mathbb{B}^n) = \sup_E \frac{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n)\mathcal{H}^{n-1}(\partial\mathbb{B}^n \setminus \partial^{\mathrm{M}} E)}{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n)}. \tag{3.18}$$

Then

$$K(\mathbb{B}^n) = \begin{cases} \dfrac{(n\omega_n)^2}{4\omega_{n-1}} & \text{if } n \geqslant 3, \\ 2\pi & \text{if } n = 2. \end{cases} \tag{3.19}$$

If $n \geqslant 4$, the supremum in (3.18) is attained if E is a half-ball. If $n = 3$, the supremum in (3.18) is attained if E is any spherical segment. If $n = 2$, the supremum in (3.18) is not attained; any sequence of spherical segments of the form $T(\vartheta_k)$ is optimizing, provided that the sequence $\{\vartheta_k\}$ converges either to 0^+ or to π^-.

Proof. For every measurable set $E \subset \mathbb{B}^n$, define the spherical segment T_E as

$$T_E = T(\Psi^{-1}(\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n))),$$

where T and Ψ are defined as in (3.1) and (3.3), respectively. Therefore,

$$\begin{aligned} &\frac{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n)\mathcal{H}^{n-1}(\partial\mathbb{B}^n \setminus \partial^{\mathrm{M}} E)}{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \mathbb{B}^n)} \\ &\qquad \leqslant \frac{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} T_E \cap \partial\mathbb{B}^n)\mathcal{H}^{n-1}(\partial\mathbb{B}^n \setminus \partial^{\mathrm{M}} T_E)}{\mathcal{H}^{n-1}(\partial^{\mathrm{M}} T_E \cap \mathbb{B}^n)} \\ &\qquad = (n-1)^2\omega_{n-1} f(\Psi^{-1}(\mathcal{H}^{n-1}(\partial^{\mathrm{M}} E \cap \partial\mathbb{B}^n))) \\ &\qquad \leqslant (n-1)^2\omega_{n-1} \sup_{\vartheta\in(0,\pi)} f(\vartheta) \\ &\qquad = K(\mathbb{B}^n), \end{aligned} \tag{3.20}$$

where f is defined by (3.16). Note that the first inequality holds by proposition 3.1, and the last equality holds by lemma 3.2. Furthermore, the equality holds in the first inequality whenever E is a spherical segment. Thus, the conclusion follows via lemma 3.2 again. □

Proof of theorem 1.2. An dilation scaling and translation argument shows that the constant $C(n)$ in (1.6) is independent of the radius and of the centre of B. The conclusion is thus a consequence of theorem 1.1 and proposition 3.3. □

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