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Self-similar solutions and asymptotic behaviour for a class of degenerate and singular diffusion equations

Chunpeng Wang
Department of Mathematics, Jilin University, Changchun, Jilin 130012, People's Republic of China (wangcp@jlu.edu.cn)

Tong Yang
Department of Mathematics, City University of Hong Kong, Hong Kong (matyang@cityu.edu.hk)

Jingxue Yin
Department of Mathematics, Jilin University, Changchun, Jilin 130012, People's Republic of China (yjx@jlu.edu.cn)

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In this paper, we study the self-similar solutions and the time-asymptotic behaviour of solutions for a class of degenerate and singular diffusion equations in the form

$$u_t = (|(p(u))_x|^{\lambda-2}(p(u))_x)_x, \quad -\infty < x < +\infty, \quad t > 0,$$

where $\lambda > 2$ is a constant. The existence, uniqueness and regularity for the self-similar solutions are obtained. In particular, the behaviour at two end points is discussed. Based on the monotonicity property of the self-similar solutions and the comparison principle, we also investigate the time convergence of the solution for the Cauchy problem to the corresponding self-similar solution when the initial data have some decay in space variable.

1. Introduction

Consider the diffusion equation of the form

$$u_t = (|(p(u))_x|^{\lambda-2}(p(u))_x)_x, \quad -\infty < x < +\infty, \quad t > 0, \tag{1.1}$$

where $\lambda > 2$ and $p(s) \in C([0,+\infty)) \cap C^1((0,+\infty))$. Here we assume that the function $p(s)$ has the property that $\lim_{s\to+\infty} p(s) = +\infty$, $p'(s) > 0$ for $s > 0$ and $p'(s)$ is a monotone function in $(0,+\infty)$. Hence, besides the degeneracy at the points where $u_x = 0$ with $p'(u) < +\infty$, the equation is degenerate if $p'(0) = 0$ and singular if $p'(0) = +\infty$ at $u = 0$. This type of equation has a background in physics and engineering sciences (see [9,16,19,20,22] and the references therein). In recent decades, equations in the form (1.1) have been studied extensively because of the rich phenomena caused by the degeneracy and singularity (see, for example, the important case when $\lambda = 2$ [1,3–8,15,17,21,23,25]). When $\lambda \neq 2$ and $p(s) =$

$s^m (m > 0)$, (1.1) is called the non-Newtonian polytropic filtration equation; it has been thoroughly investigated in [11, 27] and also in [2, 10, 12–14, 18, 24, 26], in which the existence, uniqueness and regularity of solutions, together with the time-asymptotic behaviour of solutions to the Cauchy problem, are established.

It is also interesting to note that (1.1) can be deduced from the compressible Euler equations with damping, and the singularity and degeneracy in (1.1) then correspond to the definition of the pressure function and the order of the nonlinearity of the damping in the system. More precisely, consider the following system of Euler equations for isentropic flow with a damping term:

$$\rho_t + (\rho u)_x = 0, \tag{1.2}$$

$$(\rho u)_t + (\rho u^2 + P(\rho))_x = -\kappa(\rho u)^\alpha, \tag{1.3}$$

where ρ, u and $P(\rho)$ denote density, velocity and pressure, respectively, while $\kappa > 0$ is the damping coefficient and $\alpha > 0$ is the order of the nonlinearity in the damping term. It is known in fluid dynamics that when time $t \to \infty$, the convection term in equation (1.3), i.e. $(\rho u)_t + (\rho u^2)_x$ decays faster than the other terms. Thus, to consider the leading terms in the system, it can be reduced to the following scalar equation:

$$\rho_t = \frac{1}{\kappa}((P(\rho)_x)^{1/\alpha})_x. \tag{1.4}$$

Now it is clear that this is exactly (1.1) when $\lambda = (\alpha + 1)/\alpha$. Hence, the linear damping corresponds to the particular case $\lambda = 2$, while superlinear damping corresponds to the case when $0 < \lambda < 2$ and sublinear damping to the case when $\lambda > 2$. If $0 < \alpha < 1$ and the pressure function satisfies $P'(0) = 0$ or $P'(0) = +\infty$, then it is clear that (1.4) is degenerate or singular, respectively, at $\rho = 0$, i.e. in vacuum states. For the case $\alpha = 1$, it is well known that the parabolic equation (1.4) has a class of self-similar solutions, called Barenblatt solutions, which capture the large-time behaviour of solutions to the Cauchy problem of the Euler equation with linear damping connecting to vacuum, i.e. $P = 0$. Therefore, we believe that our study of the more general case here will be useful for future study of the above system in a more general setting.

In the first part of the paper, we will consider the self-similar solutions to (1.1) of the form

$$u(x,t) = w(\xi), \quad \xi = x(t+1)^{-1/\lambda}, \quad -\infty < x < +\infty, \quad t > 0.$$

Direct calculation shows that $w = w(\xi)$ satisfies

$$-\frac{1}{\lambda}\xi w' = (|(p(w))'|^{\lambda-2}(p(w))')', \quad -\infty < \xi < +\infty. \tag{1.5}$$

For the ordinary differential equation (1.5), we study the infinite two-point boundary-value problem with

$$w(-\infty) = w_-, \qquad w(+\infty) = w_+, \tag{1.6}$$

where $w_\pm \geqslant 0$. Since (1.5) is degenerate at the points where $(p(w))' = 0$ and may be degenerate or singular at the points where $w = 0$, the classical solution may not exist. For this reason, the solutions to (1.5) and the infinite two-point boundary-value problem (1.5), (1.6) are defined as follows.

Definition 1.1. A non-negative function $w(\xi) \in C(-\infty, +\infty)$ is called a solution of (1.5), if $p(w) \in C^1(-\infty, +\infty)$, w and $|(p(w))'|^{\lambda-2}(p(w))'$ are absolutely continuous in $(-\infty, +\infty)$ so that (1.5) holds almost everywhere. Moreover, if

$$\lim_{\xi \to -\infty} w(\xi) = w_-, \qquad \lim_{\xi \to +\infty} w(\xi) = w_+,$$

$w(\xi)$ is called a solution of the infinite two-point boundary-value problem (1.5), (1.6).

Based on the self-similar solutions and the comparison principle, we will investigate further the asymptotic behaviour of solutions to the Cauchy problem (1.1) with the initial data

$$u(x,t) = u_0(x), \quad x \in (-\infty, +\infty), \tag{1.7}$$

where $u_0(x)$ is a monotone, non-negative and bounded function. The solution to the problem (1.1), (1.7) is defined as follows.

Definition 1.2. A function $u(x,t)$ is called a weak solution of the Cauchy problem (1.1), (1.7), if

$$\begin{gathered} p(u) \in C_{\mathrm{loc}}(0, +\infty; L^2_{\mathrm{loc}}(-\infty, +\infty)) \cap L^{\lambda}_{\mathrm{loc}}(0, +\infty; W^{1,\lambda}_{\mathrm{loc}}(-\infty, +\infty)), \\ \varphi \in C_0^\infty((-\infty, +\infty) \times (0, +\infty)), \\ \int_0^{+\infty} \int_{-\infty}^{+\infty} u \varphi_t \, \mathrm{d}x \, \mathrm{d}t = \int_0^{+\infty} \int_{-\infty}^{+\infty} |(p(u))_x|^{\lambda-2} (p(u))_x \varphi_x \, \mathrm{d}x \, \mathrm{d}t \end{gathered}$$

and

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} p(u(x,t)) h(x) \, \mathrm{d}x = \int_{-\infty}^{+\infty} p(u_0(x)) h(x) \, \mathrm{d}x, \quad h \in C_0^\infty(-\infty, +\infty).$$

It will be shown that when the initial data are monotone and decay in the space variable, then the solution to the Cauchy problem decays in the time variable with a rate corresponding to the self-similar solution with two end states at $\pm\infty$.

The paper is organized as follows. In § 2, we state the main results on self-similar solutions and derive some basic formulae, while the proof will be given in § 3. Based on the self-similar solutions, the asymptotic behaviour of solutions to the Cauchy problem (1.1), (1.7) will be investigated in the final section.

2. Self-similar solutions

We first state the main results on the self-similar solutions. For the proof in § 3, we will also derive some basic formulae for the self-similar solutions.

Note that if $w(\xi)$ is a solution of the problem (1.5), (1.6), then $\tilde{w}(\xi) = w(-\xi)$ is also a solution of (1.5) with

$$w(-\infty) = w_+, \qquad w(+\infty) = w_-.$$

Thus, in the following discussion, we will assume that $0 \leqslant w_- \leqslant w_+$ without loss of generality.

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The main results on the self-similar solutions, namely the solutions to the infinite two-point boundary-value problem (1.5), (1.6), can be stated as follows.

THEOREM 2.1. *There exists a unique solution of the problem (1.5), (1.6).*

THEOREM 2.2. *Let $w(\xi)$ be the solution of the problem (1.5), (1.6) with $0 < w_- < w_+$. There then exist $-\infty < \xi_* < 0 < \xi^* < +\infty$, such that $w(\xi)$ is strictly increasing in (ξ_*, ξ^*), and*

$$w(\xi) = \begin{cases} w_-, & \xi \in (-\infty, \xi_*], \\ w_+, & \xi \in [\xi^*, +\infty). \end{cases}$$

THEOREM 2.3. *Let $w(\xi)$ be the solution of the problem (1.5), (1.6) with $w_- = 0$ and $w_+ > 0$. There then exists $0 < \xi^* < +\infty$ such that*

$$w(\xi) \begin{cases} < w_+, & \xi \in (-\infty, \xi^*), \\ = w_+, & \xi \in [\xi^*, +\infty). \end{cases}$$

In addition, set

$$\xi_* = \inf\{\xi \in (-\infty, +\infty) : w(\xi) > 0\}.$$

For ξ_*, we reach the following two conclusions.

(i) *If $\int_0^1 p'(s) s^{-1/(\lambda-1)}\, \mathrm{d}s < +\infty$, then $-\infty < \xi_* < 0$ and the solution satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\biggl(\int_0^{w(\xi)} p'(s) s^{-1/(\lambda-1)}\, \mathrm{d}s\biggr)\bigg|_{\xi=(\xi_*)^+} = \biggl(\frac{-\xi_*}{\lambda}\biggr)^{1/(\lambda-1)} > 0.$$

Moreover, $w(\xi)$ is strictly increasing in (ξ_*, ξ^*), while $w(\xi) \equiv 0$ on $(-\infty, \xi_*]$.

(ii) *If $\int_0^1 p'(s) s^{-1/(\lambda-1)}\, \mathrm{d}s = +\infty$, then $\xi_* = -\infty$ and the solution satisfies*

$$\int_{-\infty}^0 w(s)\, \mathrm{d}s < +\infty, \qquad \lim_{\xi \to -\infty} |\xi| w(\xi) = 0.$$

And $w(\xi)$ is strictly increasing in $(-\infty, \xi^*)$.

REMARK 2.4. When $p(s) = s^m$, $m > 0$, the corresponding conclusions on the behaviour of solutions can be stated as follows, with a clear relation between the parameter m and λ.

(i) If $m > 1/(\lambda - 1)$, then $-\infty < \xi_* < 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(w^{m-1/(\lambda-1)}(\xi))\bigg|_{\xi=(\xi_*)^+} = \frac{1}{m}\biggl(m - \frac{1}{\lambda - 1}\biggr)\biggl(\frac{-\xi_*}{\lambda}\biggr)^{1/(\lambda-1)} > 0.$$

(ii) If $0 < m \leqslant 1/(\lambda - 1)$, then $\xi_* = -\infty$.

REMARK 2.5. Consider the Euler equations for a polytropic gas. When $P(\rho) = \sigma^2 \rho^\gamma$ with $\gamma > 0$ the adiabatic constant and σ a constant, the explanation of the vacuum behaviour from theorem 2.3 can be stated as follows. When $\gamma > \alpha$, the gas can connect to vacuum in finite distance from the origin at any time, with

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the physical vacuum boundary condition being $(\rho^{\gamma-\alpha})_x \neq 0$ and bounded at the vacuum interface. This is consistent with the work done on the Euler equations with damping when $\alpha \geqslant 1$. On the other hand, when $0 < \gamma \leqslant \alpha$, the gas canonically does not connect to vacuum in finite distance from the origin for any time, but rather does so at infinity.

To study the large-time behaviour of the solutions to the Cauchy problem (1.1), (1.7), we need the following comparison result for solutions with small perturbation at two end points.

THEOREM 2.6. *Let w_1 and w_2 be the solutions of (1.5) with the boundary value*

$$w_1(-\infty) = w_-, \qquad w_1(+\infty) = w_+$$

and

$$w_2(-\infty) = w_- + \delta, \qquad w_2(+\infty) = w_+ + \delta,$$

respectively, where $0 \leqslant w_- < w_+$, $\delta > 0$ and $w_- + \delta < w_+$.

(i) *If $p'(s)$ is an increasing function, then*

$$0 \leqslant w_2(\xi) - w_1(\xi) \leqslant \left(1 + \frac{p'(w_+ + \delta)}{p'(w_1(0))}\right)\delta, \quad \xi \geqslant 0.$$

(ii) *If $p'(s)$ is a decreasing function, then*

$$0 \leqslant w_2(\xi) - w_1(\xi) \leqslant \left(1 + \frac{p'(w_-)}{p'(w_2(0))}\right)\delta, \quad \xi \leqslant 0.$$

Note that in the above theorem the estimation is on the right state (i.e. $\xi > 0$) in the first case and on the left state (i.e. $\xi < 0$) in the second case.

In the rest of this section, we will derive some basic formulae with basic properties of the solutions to the problem (1.5), (1.6).

Let $v = p(w)$. Then (1.5) is transformed into

$$(|v'|^{\lambda-2}v')' = -\frac{1}{\lambda}\xi(q(v))', \qquad -\infty < \xi < +\infty \tag{2.1}$$

or

$$(|v'|^{\lambda-3}v')' = -\frac{\lambda-2}{\lambda(\lambda-1)}\xi q'(v), \quad -\infty < \xi < +\infty, \tag{2.2}$$

where $q(s) = p^{-1}(s)$ is the inverse function of p, and the boundary data (1.6) become

$$v(-\infty) = v_-, \qquad v(+\infty) = v_+, \tag{2.3}$$

where $v_- = p(w_-)$ and $v_+ = p(w_+)$. From the assumption on the function $p(s)$, we see that $q(s) \in C(R(p)) \cap C^1(R_+(p))$, $\lim_{s\to+\infty} q(s) = +\infty$, $q'(s) > 0$ for $s \in R_+(p)$ and $q'(s) = 1/p'(q(s))$ is a monotone function in $R_+(p)$, where

$$R(p) = \{p(s) : s \in [0, +\infty)\}, \qquad R_+(p) = \{p(s) : s \in (0, +\infty)\}.$$

Following definition 1.1, we obtain the definition of solutions to (2.1) and the corresponding problem (2.1), (2.3).

DEFINITION 2.7. A function $v(\xi) \in C(I)$ is called a solution of (2.1) in an interval I if $v(I) \subset R(p)$, $v \in C^1(I)$, $q(v)$ and $|v'|^{\lambda-2}v'$ are absolutely continuous in I so that (2.1) holds almost everywhere in I. Furthermore, if $I = (-\infty, +\infty)$ and

$$\lim_{\xi \to -\infty} v(\xi) = v_-, \qquad \lim_{\xi \to +\infty} v(\xi) = v_+,$$

$v(\xi)$ is called a solution of the infinite two-point boundary-value problem (2.1), (2.3).

According to the uniqueness theorem on ordinary differential equations, it is straightforward to prove the following proposition.

PROPOSITION 2.8. *Assume that I is an interval and that $v(\xi)$, $\xi \in I$, is a solution of (2.1).*

(i) *If there exists $\xi_1 \in I \cap (-\infty, 0]$ such that $v'(\xi_1) = 0$, then*

$$v'(\xi) = 0, \quad \xi \in I \cap (-\infty, \xi_1].$$

(ii) *If there exists $\xi_2 \in I \cap [0, +\infty)$ such that $v'(\xi_2) = 0$, then*

$$v'(\xi) = 0, \quad \xi \in I \cap [\xi_2, +\infty).$$

As an immediate consequence of proposition 2.8, the problem (2.1), (2.3) has a unique solution when $v_- = v_+$. Therefore, we need to study only the non-trivial solution, namely the case $v_- < v_+$.

LEMMA 2.9. *Assume that $v(\xi)$ is a local solution of (2.1) with $v(0) \subset R_+(p)$ and $v'(0) > 0$. There then exists $0 < \xi^* < +\infty$, such that $v'(\xi^*) = 0$ and*

$$v'(\xi) > 0, \quad 0 < \xi < \xi^*.$$

Proof. We prove this by contradiction. Assume the conclusion of the lemma is not true. By (2.1) and the extension theorem, $v(\xi)$ can be extended to $+\infty$ by

$$v'(\xi) > 0, \quad v''(\xi) < 0, \quad \xi > 0.$$

Thus,

$$({v'}^{\lambda-1}(\xi))' = -\frac{1}{\lambda}\xi(q(v(\xi)))', \quad \xi > 0. \tag{2.4}$$

For $\xi > 1$, integrating the above equation from 1 to ξ yields

$$-{v'}^{\lambda-1}(1) < {v'}^{\lambda-1}(\xi) - {v'}^{\lambda-1}(1) = -\frac{1}{\lambda}\int_1^{\xi} s(q(v(s)))' \,\mathrm{d}s \leqslant -\frac{1}{\lambda}(q(v(\xi)) - q(v(1))).$$

Since $v'(\xi) > 0$ for any $\xi > 0$ and $\lim_{s \to +\infty} q(s) = +\infty$, $\lim_{\xi \to +\infty} v(\xi)$ exists and is bounded. Hence,

$$a = v(0) < v(\xi) < \lim_{\xi \to +\infty} v(\xi) = b < +\infty, \quad \xi > 1.$$

By the monotonicity of $q'(s)$,

$$q'(v(\xi)) \geqslant \min\{q'(a), q'(b)\} = \delta > 0, \quad \xi > 1.$$

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Then (2.4) implies that

$$(v'^{\lambda-2}(\xi))' = -\frac{\lambda-2}{\lambda(\lambda-1)}\xi q'(v(\xi)) \leqslant -\frac{\delta(\lambda-2)}{\lambda(\lambda-1)}\xi, \quad \xi > 1,$$

which contradicts the notion that $v'(\xi) > 0$ for any $\xi > 0$, and this completes the proof. □

LEMMA 2.10. *Let $v_1(\xi)$, $v_2(\xi)$ be two strictly increasing solutions of (2.1) in an interval I and let $v_1(I), v_2(I) \subset R_+(p)$. Then we have the following two cases.*

(i) *If $q'(s)$ is monotone decreasing and there exists $\xi_0 \in I$ such that*

$$v_1(\xi_0) \geqslant v_2(\xi_0),$$
$${v'}_1^{\lambda-1}(\xi_0) - {v'}_2^{\lambda-1}(\xi_0) > \max\left\{0, -\frac{1}{\lambda}\xi_0(q(v_1(\xi_0)) - q(v_2(\xi_0)))\right\},$$

then

$$v_1'(\xi) > v_2'(\xi), \quad \xi \in [\xi_0, +\infty) \cap I.$$

In particular, if there exist $\xi_{01}, \xi_{02} \in (-\infty, 0] \cap I$ with $\xi_{01} < \xi_{02}$ such that

$$v_1(\xi_{01}) = v_2(\xi_{02}), \qquad v_1'(\xi_{01}) = v_2'(\xi_{02}), \tag{2.5}$$

then

$$v_1'(\xi) > v_2'(\xi), \quad \xi \in [\xi_{02}, +\infty) \cap I. \tag{2.6}$$

(ii) *If $q'(s)$ is monotone increasing and there exists $\xi_0 \in I$ such that*

$$v_1(\xi_0) \geqslant v_2(\xi_0), \qquad v_1'(\xi_0) > v_2'(\xi_0),$$

then

$$v_1(\xi) > v_2(\xi), \quad \xi \in [\xi_0, +\infty) \cap I.$$

Proof. We prove these two cases by contradiction, as follows.
(i) Assume the conclusion is not true. Let

$$\xi_1 = \inf\{\xi \in [\xi_0, +\infty) \cap I : v_1'(\xi) \leqslant v_2'(\xi)\}.$$

Then $\xi_0 < \xi_1$, $v_1'(\xi_1) = v_2'(\xi_1)$ and $v_1'(\xi) > v_2'(\xi)$ for any $\xi_0 < \xi < \xi_1$. We have the following two subcases.

(a) When $\xi_1 \leqslant 0$, integrating (2.1) from ξ_0 to ξ_1 yields

$$v'^{\lambda-1}\big|_{\xi_0}^{\xi_1} = -\frac{1}{\lambda}\int_{\xi_0}^{\xi_1} \xi(q(v))'\,\mathrm{d}\xi = \frac{1}{\lambda}\int_{\xi_0}^{\xi_1} q(v)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v)\bigg|_{\xi_0}^{\xi_1}.$$

Therefore,

$$\begin{aligned}{v'}_2^{\lambda-1}(\xi_0) - {v'}_1^{\lambda-1}(\xi_0) &= \frac{1}{\lambda}\int_{\xi_0}^{\xi_1}(q(v_1) - q(v_2))\,\mathrm{d}\xi \\ &\quad - \frac{1}{\lambda}\xi_1(q(v_1(\xi_1)) - q(v_2(\xi_1))) + \frac{1}{\lambda}\xi_0(q(v_1(\xi_0)) - q(v_2(\xi_0))).\end{aligned}$$

Because $\xi_1 \leqslant 0$, $q(v_1(\xi_1)) - q(v_2(\xi_1)) > 0$ and $\int_{\xi_0}^{\xi_1}(q(v_1) - q(v_2))\,\mathrm{d}\xi > 0$, we get

$$v_2'^{\lambda-1}(\xi_0) - v_1'^{\lambda-1}(\xi_0) > \frac{1}{\lambda}\xi_0(q(v_1(\xi_0)) - q(v_2(\xi_0))),$$

which contradicts the assumption.

(b) When $\xi_1 > 0$, since $v_1(\xi_1) > v_2(\xi_1)$ and $q'(s)$ is monotone decreasing,

$$q'(v_1(\xi_1)) < q'(v_2(\xi_1)).$$

From (2.2) and $\xi_1 > 0$, we get

$$(v_1'^{\lambda-1})'|_{\xi_1} > (v_2'^{\lambda-1})'|_{\xi_1},$$

which contradicts the notion that $v_1'(\xi_1) = v_2'(\xi_1)$ and $v_1'(\xi) > v_2'(\xi)$ for any $\xi_0 < \xi < \xi_1$.

Now assume that (2.5) holds. Integrating this equation with respect to v_1 from ξ_{01} to ξ_{02} and using (2.5) give

$$\begin{aligned}
v_1'^{\lambda-1}&(\xi_{02}) - v_2'^{\lambda-1}(\xi_{02})\\
&= v_1'^{\lambda-1}|_{\xi_{01}}^{\xi_{02}} = -\frac{1}{\lambda}\int_{\xi_{01}}^{\xi_{02}} \xi(q(v_1))'\,\mathrm{d}\xi\\
&= \frac{1}{\lambda}\int_{\xi_{01}}^{\xi_{02}} q(v_1)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v_1)\bigg|_{\xi_{01}}^{\xi_{02}}\\
&> \frac{1}{\lambda}(\xi_{02} - \xi_{01})q(v_1(\xi_{01})) - \frac{1}{\lambda}\xi_{02}q(v_1(\xi_{02})) + \frac{1}{\lambda}\xi_{01}q(v_1(\xi_{01}))\\
&= -\frac{1}{\lambda}\xi_{02}(q(v_1(\xi_{02})) - q(v_2(\xi_{02}))) > 0.
\end{aligned}$$

Hence, (2.6) follows according to conclusion (i).

(ii) The proof is also divided into two subcases.

(a) When $\xi_0 \geqslant 0$, if the conclusion of the lemma is not true, then we can let

$$\xi_2 = \inf\{\xi \in (\xi_0, +\infty) \cap I : v_1(\xi) \leqslant v_2(\xi)\}.$$

Thus, $0 \leqslant \xi_0 < \xi_2$, $v_1(\xi_2) = v_2(\xi_2)$ and $v_1(\xi) > v_2(\xi)$ for any $\xi_0 < \xi < \xi_2$. Integrating (2.1) from ξ_0 to ξ_2 yields

$$v'^{\lambda-1}|_{\xi_0}^{\xi_2} = -\frac{1}{\lambda}\int_{\xi_0}^{\xi_2} \xi(q(v))'\,\mathrm{d}\xi = \frac{1}{\lambda}\int_{\xi_0}^{\xi_2} q(v)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v)\bigg|_{\xi_0}^{\xi_2}.$$

Hence,

$$\begin{aligned}
(v_1'^{\lambda-1}(\xi_2) - v_2'^{\lambda-1}(\xi_2)) &+ (v_2'^{\lambda-1}(\xi_0) - v_1'^{\lambda-1}(\xi_0))\\
&= \frac{1}{\lambda}\int_{\xi_0}^{\xi_2}(q(v_1) - q(v_2))\,\mathrm{d}\xi + \frac{1}{\lambda}\xi_0(q(v_1(\xi_0)) - q(v_2(\xi_0))).
\end{aligned}$$

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By $\int_{\xi_0}^{\xi_2}(q(v_1)-q(v_2))\,\mathrm{d}\xi>0$, $\xi_0\geqslant 0$, $v_1(\xi_0)\geqslant v_2(\xi_0)$ and $v_1'(\xi_0)>v_2'(\xi_0)$, we get

$$v_1'^{\lambda-1}(\xi_2)-v_2'^{\lambda-1}(\xi_2)>0,$$

which contradicts the notion that $v_1(\xi_2)=v_2(\xi_2)$ and $v_1(\xi)>v_2(\xi)$ for any $\xi_0<\xi<\xi_2$.

(b) When $\xi_0<0$, we will show that

$$v_1'(\xi)>v_2'(\xi),\quad \xi\in[\xi_0,0)\cap I. \tag{2.7}$$

If this condition does not apply, let

$$\xi_3=\inf\{x\in[\xi_0,0]\cap I: v_1'(\xi)\leqslant v_2'(\xi)\}.$$

Then $\xi_0<\xi_3<0$, $v_1'(\xi_3)=v_2'(\xi_3)$ and $v_1'(\xi)>v_2'(\xi)$ for any $\xi_0<\xi<\xi_3$. Since $v_1(\xi_0)\geqslant v_2(\xi_0)$ and $q'(s)$ is monotone increasing, $q'(v_1(\xi_3))>q'(v_2(\xi_3))$. By (2.2) and $\xi_3<0$, we get

$$(v_1'^{\lambda-1})'|_{\xi_3}>(v_2'^{\lambda-1})'|_{\xi_3},$$

which contradicts the notion that $v_1'(\xi_3)=v_2'(\xi_3)$ and $v_1'(\xi)>v_2'(\xi)$ for any $\xi_0<\xi<\xi_3$. Therefore, (2.7) holds and this completes the proof of the lemma. □

Lemma 2.11. *Assume $v_-<v_+$ and v is a solution of the problem (2.1), (2.3). Set*

$$\xi_*=\inf\{\xi\in(-\infty,+\infty): v(\xi)>v_-\}.$$

Then v is monotone increasing in $(-\infty,+\infty)$ and $-\infty\leqslant\xi_*<0$. Moreover, we have the following two conclusions on ξ_*.

(i) *If $\xi_*>-\infty$, then*

$$\int_{v_-}^{v_-+1}(q(s)-q(v_-))^{-1/(\lambda-1)}\,\mathrm{d}s<+\infty.$$

Furthermore, if in addition, $v_-=p(0)$, then

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\biggl(\int_{v_-}^{v(\xi)}q^{-1/(\lambda-1)}(s)\,\mathrm{d}s\biggr)\bigg|_{\xi=(\xi_*)^+}=\biggl(\frac{-\xi_*}{\lambda}\biggr)^{1/(\lambda-1)}>0. \tag{2.8}$$

(ii) *If $\xi_*=-\infty$, then*

$$\int_{v_-}^{v_-+1}(q(s)-q(v_-))^{-1/(\lambda-1)}\,\mathrm{d}s=+\infty.$$

Proof. From (2.2) and proposition 2.8, v is monotone increasing and $v'(\xi)\leqslant v'(0)$ for all $\xi\in(-\infty,+\infty)$. Hence, $\xi_*<0$. The two cases on ξ_* can be discussed as follows.

Case 1 ($\xi_*>-\infty$). For any $\xi_*<\xi<0$, integrating (2.1) from ξ_* to ξ yields

$$v'^{\lambda-1}\big|_{\xi_*}^{\xi}=-\frac{1}{\lambda}\int_{\xi_*}^{\xi}s(q(v(s)))'\,\mathrm{d}s.$$

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Then,

$$v'^{\lambda-1}(\xi) = -\frac{1}{\lambda}\int_{\xi_*}^{\xi} s(q(v(s)))'\,\mathrm{d}s$$
$$\leqslant \frac{|\xi_*|}{\lambda}\int_{\xi_*}^{\xi} |(q(v(s)))'|\,\mathrm{d}s = \frac{|\xi_*|}{\lambda}\int_{\xi_*}^{\xi}(q(v(s)))'\,\mathrm{d}s = \frac{|\xi_*|}{\lambda}(q(v(\xi)) - q(v_-)).$$

Thus,

$$(q(v(\xi)) - q(v_-))^{-1/(\lambda-1)}v'(\xi) \leqslant \left(\frac{|\xi_*|}{\lambda}\right)^{1/(\lambda-1)}, \quad \xi \in (\xi_*, 0).$$

Integrating the above inequality from ξ_* to 0, we get

$$\int_{v_-}^{v(0)}(q(s) - q(v_-))^{-1/(\lambda-1)}\,\mathrm{d}s = \int_{\xi_*}^{0}(q(v(\xi)) - q(v_-))^{-1/(\lambda-1)}v'(\xi)\,\mathrm{d}\xi$$
$$\leqslant \left(\frac{|\xi_*|}{\lambda}\right)^{1/(\lambda-1)}|\xi_*|,$$

which implies that

$$\int_{v_-}^{v_-+1}(q(s) - q(v_-))^{-1/(\lambda-1)}\,\mathrm{d}s < +\infty.$$

For (2.8), we integrate (2.1) from ξ_* to $\xi > \xi_*$ to obtain

$$v'^{\lambda-1}\Big|_{\xi_*}^{\xi} = -\frac{1}{\lambda}\int_{\xi_*}^{\xi} s(q(v(s)))'\,\mathrm{d}s = \frac{1}{\lambda}\int_{\xi_*}^{\xi} q(v(s))\,\mathrm{d}s - \frac{1}{\lambda}sq(v(s))\Big|_{\xi_*}^{\xi}.$$

Owing to $v'(\xi_*) = 0$ and $q(v(\xi_*)) = q(v_-) = 0$,

$$v'^{\lambda-1}(\xi) = \frac{1}{\lambda}\int_{\xi_*}^{\xi} q(v(s))\,\mathrm{d}s - \frac{1}{\lambda}\xi q(v(\xi)), \quad \xi > \xi_*.$$

Hence,

$$\frac{v'(\xi)}{q^{1/(\lambda-1)}(v(\xi))} = \left(\frac{1}{\lambda}\int_{\xi_*}^{\xi}\frac{q(v(s))}{q(v(\xi))}\,\mathrm{d}s - \frac{1}{\lambda}\xi\right)^{1/(\lambda-1)}, \quad \xi > \xi_*.$$

By letting $\xi \to (\xi_*)^+$ and noticing that $q(v(s)) < q(v(\xi))$ for all $\xi_* < s < \xi$, we achieve (2.8).

Case 2 ($\xi_* = -\infty$). For any $\xi_1 < \xi < 0$, integrating (2.1) from ξ_1 to ξ gives

$$v'^{\lambda-1}\Big|_{\xi_1}^{\xi} = -\frac{1}{\lambda}\int_{\xi_1}^{\xi} s(q(v(s)))'\,\mathrm{d}s.$$

By letting $\xi_1 \to -\infty$, we get

$$v'^{\lambda-1}(\xi) = -\frac{1}{\lambda}\int_{-\infty}^{\xi} s(q(v(s)))'\,\mathrm{d}s = -\frac{1}{\lambda}\int_{-\infty}^{\xi} s|(q(v(s)))'|\,\mathrm{d}s$$
$$\geqslant \frac{|\xi|}{\lambda}\int_{-\infty}^{\xi}(q(v(s)))'\,\mathrm{d}s = \frac{|\xi|}{\lambda}(q(v(\xi)) - q(v_-)).$$

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Hence,

$$(q(v(\xi)) - q(v_-))^{-1/(\lambda-1)} v'(\xi) \geqslant \left(\frac{|\xi|}{\lambda}\right)^{1/(\lambda-1)}, \quad \xi < 0.$$

Integrating the above inequality from $-\infty$ to 0 yields

$$\begin{aligned}\int_{v_-}^{v(0)} (q(s) - q(v_-))^{-1/(\lambda-1)} \,\mathrm{d}s &= \int_{-\infty}^{0} (q(v(\xi)) - q(v_-))^{-1/(\lambda-1)} v'(\xi) \,\mathrm{d}\xi \\ &\geqslant \int_{-\infty}^{0} \left(\frac{|\xi|}{\lambda}\right)^{1/(\lambda-1)} \mathrm{d}\xi \\ &= +\infty.\end{aligned}$$

Therefore,

$$\int_{v_-}^{v_-+1} (q(s) - q(v_-))^{-1/(\lambda-1)} \,\mathrm{d}s = +\infty.$$

The proof of the lemma is complete. □

Corollary 2.12. *Under the assumption of lemma 2.11, if $v_- > 0$ additionally, then $-\infty < \xi_* < 0$.*

Proof. From the monotonicity property of $q'(s)$ and $v_- > 0$, we have

$$(q(s) - q(v_-)) \geqslant \delta(s - v_-), \quad v_- < s < v_- + 1,$$

where

$$\delta = \min\{q'(v_-), q'(v_- + 1)\} > 0.$$

Thus,

$$(q(s) - q(v_-))^{-1/(\lambda-1)} \leqslant \delta^{-1/(\lambda-1)} (s - v_-)^{-1/(\lambda-1)}, \quad v_- < s < v_- + 1.$$

As $\lambda > 2$,

$$\int_{v_-}^{v_-+1} (q(s) - q(v_-))^{-1/(\lambda-1)} \,\mathrm{d}s < +\infty.$$

By lemma 2.11, $-\infty < \xi_* < 0$, which completes the proof. □

3. Proofs of theorems on self-similar solutions

This section is devoted to the proofs of theorems 2.1–2.3 and 2.6, which are based on the following lemmas and propositions. We first consider the case without degeneracy, namely $v_\pm \in R_+(p)$ and $v_- < v_+$. Noting that $\lambda > 2$ in (2.2), by the extension theorem and uniqueness theorem for the initial-value problem in ordinary differential equations, lemma 2.9 implies the following lemma.

Lemma 3.1. *Assume that $\xi_0 < 0$ and $v_- \in R_+(p)$. Then (2.1) with the initial conditions*

$$v(\xi_0) = v_-, \qquad v'(\xi_0) = 0 \tag{3.1}$$

admits a unique solution in $(0, +\infty)$. Moreover, there exists $0 < \xi^* < +\infty$, such that $v'(\xi) > 0$ in (ξ_0, ξ^*) and $v'(\xi) = 0$ on $[\xi^*, +\infty)$.

PROPOSITION 3.2. *Assume that $v_\pm \in R_+(p)$ and $v_- < v_+$. There then exists a unique solution of the problem (2.1), (2.3).*

Proof. For $\xi_0 < 0$, we denote $v(\xi;\xi_0)$ the solution of (2.1) with the initial data (3.1). Set

$$\xi^*(\xi_0) = \sup\{\xi > \xi_0 : v'(\xi;\xi_0) > 0\}.$$

For any fixed $\xi_{01} < \xi_{02} < 0$, lemma 2.10 implies that

$$v(\xi;\xi_{01}) > v(\xi;\xi_{02}), \quad \xi > \xi_{02}.$$

Thus,

$$v(\xi^*(\xi_{01});\xi_{01}) > v(\xi^*(\xi_{02});\xi_{02}),$$

namely, $v(\xi^*(\xi_0);\xi_0)$ is strictly decreasing in $\xi_0 \in (-\infty, 0)$. Therefore,

$$\lim_{\xi_0\to 0^-} v(\xi^*(\xi_0);\xi_0) = 0, \qquad \lim_{\xi_0\to-\infty} v(\xi^*(\xi_0);\xi_0) = +\infty.$$

By the continuous dependence of the solutions on the initial data, we obtain the existence. The uniqueness follows from the strictly monotonicity property of $v(\xi^*(\xi_0);\xi_0)$ and corollary 2.12. □

Now we consider the case that may contain degeneracy, that is, $v_- = p(0)$ and $v_+ \in R_+(p)$.

PROPOSITION 3.3. *Assume that $v_- = p(0)$ and $v_+ \in R_+(p)$. There then exists at least one solution of the problem (2.1), (2.3).*

Proof. We denote by $v_n(\xi)$ the solution of (2.1) with the boundary value

$$v(-\infty) = v_- + \frac{1}{n}, \qquad v(+\infty) = v_+,$$

where n is a positive integer. By proposition 3.2 and lemma 2.10, v_n exists and

$$0 < \xi_n^* \leqslant \xi_{n+1}^*, \quad v_- \leqslant v_{n+1}(\xi) \leqslant v_n(\xi) \leqslant v_+, \quad \xi \in (-\infty,+\infty), \quad n = 1,2,\dots,$$

where $\xi_n^* = \sup\{\xi \in (-\infty,+\infty) : v_n'(\xi) > 0\}$. Let

$$\xi^* = \lim_{n\to\infty} \xi_n^*, \quad v(\xi) = \lim_{n\to\infty} v_n(\xi), \quad \xi \in (-\infty,+\infty).$$

Integrating the equation for v_n from 0 to ξ_n^* gives

$$v_n'^{\,\lambda-1}\big|_0^{\xi_n^*} = -\frac{1}{\lambda}\int_0^{\xi_n^*} \xi(q(v_n))'\,\mathrm{d}\xi = \frac{1}{\lambda}\int_0^{\xi_n^*} q(v_n)\,\mathrm{d}\xi - \frac{1}{\lambda}\xi q(v_n)\bigg|_0^{\xi_n^*}.$$

Thus,

$$v_n'^{\,\lambda-1}(0) + \frac{1}{\lambda}\int_0^{\xi_n^*} q(v_n)\,\mathrm{d}\xi = \frac{1}{\lambda}\xi_n^* q(v_+) \geqslant \frac{1}{\lambda}\xi_1^* q(v_+) > 0, \quad n = 1,2,\dots,$$

which implies that $v \not\equiv v_-$ in $(-\infty,+\infty)$. It is standard to show that v is a solution of the problem (2.1), (2.3) and this completes the proof. □

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Proposition 3.4. *Assume that*

$$\int_{v_-}^{v_-+1} q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$$

and that v is a solution to the problem (2.1), (2.3) with $v_- = p(0)$ and $v_+ \in R_+(p)$. Then

$$\int_{-\infty}^{0} q(v(s))\,\mathrm{d}s < +\infty, \qquad \lim_{\xi\to-\infty} \xi q(v(\xi)) = 0.$$

Proof. From lemma 2.11, we have $\xi_* = -\infty$, where

$$\xi_* = \inf\{\xi \in (-\infty,+\infty) : v(\xi) > v_-\}.$$

Integrating (2.1) and (2.2) from $-\infty$ to 0 yields

$$v'^{\lambda-1}(0) = -\frac{1}{\lambda}\int_{-\infty}^{0} s(q(v(s)))'\,\mathrm{d}s, v'^{\lambda-2}(0) = -\frac{\lambda-2}{\lambda(\lambda-1)}\int_{-\infty}^{0} sq'(v(s))\,\mathrm{d}s. \quad (3.2)$$

Due to

$$\int_{v_-}^{v_-+1} q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$$

and the monotonicity property of $q'(s)$, $q(v_-) = 0$ and $q'(s)$ is increasing. Moreover, for any $s \in (-\infty,-1]$,

$$|s|q(v(s)) = |s|(q(v(s)) - q(v_-)) \leqslant |s|q'(v(s))(v(s) - v_-) \leqslant -sq'(v(s))(v_+ - v_-).$$

From (3.2), we get

$$\int_{-\infty}^{0} |s|q(v(s))\,\mathrm{d}s < +\infty, \qquad \int_{-\infty}^{0} q(v(s))\,\mathrm{d}s < +\infty$$

and

$$v'^{\lambda-1}(0) = \frac{1}{\lambda}\lim_{\xi\to-\infty} \xi q(v(\xi)) + \frac{1}{\lambda}\int_{-\infty}^{0} q(v(s))\,\mathrm{d}s.$$

Thus, $\lim_{\xi\to-\infty} \xi q(v(\xi))$ exists and the limit is zero by

$$\int_{-\infty}^{0} |s|q(v(s))\,\mathrm{d}s < +\infty.$$

□

Proposition 3.5. *The problem (2.1), (2.3) admits at most one solution.*

Proof. Assume that v_1 and v_2 are two solutions to the problem (2.1), (2.3). From lemmas 2.9 and 2.10, we may assume that

$$v_1(\xi) \geqslant v_2(\xi), \quad \xi \in (-\infty,+\infty). \quad (3.3)$$

For any $\xi_1 < 0 < \xi_2$, integrating (2.1) from ξ_1 to ξ_2 gives

$$v_i'^{\lambda-1}|_{\xi_1}^{\xi_2} = -\frac{1}{\lambda}\int_{\xi_1}^{\xi_2} \xi(q(v_i))'\,\mathrm{d}\xi = -\frac{1}{\lambda}\xi q(v_i)\bigg|_{\xi_1}^{\xi_2} + \frac{1}{\lambda}\int_{\xi_1}^{\xi_2} q(v_i)\,\mathrm{d}\xi, \quad i = 1,2. \quad (3.4)$$

Based on (3.4), we have the following two cases.

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(i) If $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s < +\infty$, then from lemmas 2.11, 2.9 and 2.10, there exist $-\infty < \xi_* < 0 < \xi^* < +\infty$ such that

$$v_1(\xi_*) = v_2(\xi_*) = v_-, \qquad v_1(\xi^*) = v_2(\xi^*) = v_+$$

and

$$v_1'(\xi_*) = v_2'(\xi_*) = v_1'(\xi^*) = v_2'(\xi^*) = 0.$$

Choosing $\xi_1 = \xi_*$ and $\xi_2 = \xi^*$ in (3.4) yields

$$\int_{\xi_*}^{\xi^*} q(v_1)\,\mathrm{d}\xi = \int_{\xi_*}^{\xi^*} q(v_2)\,\mathrm{d}\xi.$$

Due to the monotonicity of $q(s)$, (3.3) leads to $v_1 \equiv v_2$.

(ii) If $\int_0^1 q^{-1/(\lambda-1)}(s)\,\mathrm{d}s = +\infty$, then from lemmas 2.11, 2.9 and 2.10 and proposition 3.4, there exists $0 < \xi^* < +\infty$ such that

$$v_1(\xi^*) = v_2(\xi^*) = v_+, \qquad v_1'(\xi^*) = v_2'(\xi^*) = 0,$$

and

$$\lim_{\xi\to-\infty} \xi q(v_1(\xi)) = \lim_{\xi\to-\infty} \xi q(v_2(\xi)) = 0.$$

By choosing $\xi_2 = \xi^*$ in (3.4) and letting $\xi_1 \to -\infty$, we get

$$\int_{-\infty}^{\xi^*} q(v_1)\,\mathrm{d}\xi = \int_{-\infty}^{\xi^*} q(v_2)\,\mathrm{d}\xi.$$

Due to the monotonicity of $q(s)$, (3.3) and proposition 3.4 imply that $v_1 \equiv v_2$. It then completes the proof. □

Since the problem (1.5), (1.6) is equivalent to the problem (2.1), (2.3), the case when $0 < w_- < w_+$ in theorems 2.1 and 2.2 follows from proposition 3.2, lemmas 2.9, 2.11 and corollary 2.12 directly. The case when $0 = w_- < w_+$ in theorems 2.1 and 2.3 then follows from propositions 3.3–3.5 and lemmas 2.9 and 2.11 directly. Therefore, we have completed the proofs for theorems 2.1–2.3.

Finally, we prove theorem 2.6, which can be restated as follows.

Proposition 3.6. *Let v_1 and v_2 be the solutions of (2.1) with the boundary value*

$$v_1(-\infty) = v_-, \qquad v_1(+\infty) = v_+,$$

and let

$$q(v_2(-\infty)) = q(v_-) + \delta, \qquad q(v_2(+\infty)) = q(v_+) + \delta,$$

respectively, where $v_\pm \in R(p)$ and $q(v_-) < q(v_-) + \delta < q(v_+)$. Then we have the following two cases.

(i) *If $q'(s)$ is a decreasing function, then*

$$0 \leqslant q(v_2(\xi)) - q(v_1(\xi)) \leqslant (1 + q'(v_1(0))p'(q(v_+) + \delta))\delta, \quad \xi \geqslant 0. \tag{3.5}$$

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(ii) *If $q'(s)$ is an increasing function, then*

$$0 \leqslant q(v_2(\xi)) - q(v_1(\xi)) \leqslant (1 + q'(v_2(0))p'(q(v_-)))\delta, \quad \xi \leqslant 0. \tag{3.6}$$

Proof. Denote by v_0 the solution of (2.1) with the boundary value

$$q(v_0(-\infty)) = q(v_-) + \delta, \quad v_0(+\infty) = v_+.$$

Then, by lemma 2.10, we have $v_1(0) < v_0(0) < v_2(0)$ and

$$v_1(\xi) \leqslant v_0(\xi) \leqslant v_2(\xi), \quad \xi \in (-\infty, +\infty).$$

Now we discuss the two cases separately.

(i) If $q'(s)$ is a decreasing function, then we will show that

$$q(v_0(\xi)) - q(v_1(\xi)) \leqslant \delta, \quad \xi \geqslant 0 \tag{3.7}$$

and

$$v_2'(\xi) - v_0'(\xi) \geqslant 0, \quad \xi \in (-\infty, +\infty). \tag{3.8}$$

In fact, from (3.8), we see that, for any $\xi \geqslant 0$,

$$\begin{aligned} q(v_2(\xi)) - q(v_0(\xi)) &\leqslant q'(v_0(\xi))(v_2(\xi) - v_0(\xi)) \\ &\leqslant q'(v_0(\xi))(p(q(v_+) + \delta) - v_+) \\ &\leqslant q'(v_1(0))p'(q(v_+) + \delta)\delta. \end{aligned}$$

This, together with (3.7), implies (3.5).

By lemma 2.10, (3.8) holds and

$$v_1'(\xi) - v_0'(\xi) \geqslant 0, \quad \xi \geqslant 0. \tag{3.9}$$

Define

$$h_1(\xi) = q(v_0(\xi)) - q(v_1(\xi)), \quad \xi \in (-\infty, +\infty).$$

Since $q'(s)$ is a decreasing function, from (3.9), we have

$$h_1(\xi) \leqslant h_1(0), \quad \xi \geqslant 0.$$

Now if (3.7) is not true, then

$$h_1(0) > \delta = h_1(-\infty).$$

Let

$$\xi_1 = \sup\{\xi < 0 : h_1(\xi) \leqslant h_1(0)\}.$$

Due to $h_1'(0) < 0$ and $h_1(0) > h_1(-\infty)$, ξ_1 exists with $\xi_1 < 0$. Moreover, $h_1'(\xi_1) \geqslant 0$, $h_1(\xi_1) = h_1(0)$ and

$$h_1(\xi) > h_1(0), \quad \xi \in (\xi_1, 0).$$

Integrating (2.1) from ξ_1 to 0 gives

$$v_i'^{\lambda-1}\big|_{\xi_1}^0 = -\frac{1}{\lambda}\int_{\xi_1}^0 \xi(q(v_i))' \,\mathrm{d}\xi = -\frac{1}{\lambda}\xi q(v_i)\bigg|_{\xi_1}^0 + \frac{1}{\lambda}\int_{\xi_1}^0 q(v_i)\,\mathrm{d}\xi, \quad i = 0, 1.$$

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Thus,

$$v_1'^{\lambda-1}(\xi_1) - v_0'^{\lambda-1}(\xi_1) = v_1'^{\lambda-1}(0) - v_0'^{\lambda-1}(0) + \frac{1}{\lambda}\xi_1 h_1(\xi_1) + \frac{1}{\lambda}\int_{\xi_1}^{0} h_1(\xi)\,\mathrm{d}\xi > 0.$$

This implies that $v_1'(\xi_1) > v_0'(\xi_1)$, which leads to $h_1'(\xi_1) < 0$. Hence, this contradicts $h_1'(\xi_1) \geqslant 0$ so that (3.7) holds.

(ii) Similarly, if $q'(s)$ is increasing, we will then show that

$$v_1'(\xi) - v_0'(\xi) \geqslant 0, \quad \xi \in (-\infty, +\infty) \tag{3.10}$$

and

$$q(v_2(\xi)) - q(v_0(\xi)) \leqslant \delta, \quad \xi \leqslant 0. \tag{3.11}$$

In fact, from (3.10), we see that, for any $\xi \leqslant 0$,

$$\begin{aligned} q(v_0(\xi)) - q(v_1(\xi)) &\leqslant q'(v_0(\xi))(v_0(\xi) - v_1(\xi)) \\ &\leqslant q'(v_0(\xi))(p(q(v_-) + \delta) - v_-) \\ &\leqslant q'(v_2(0))p'(q(v_-))\delta. \end{aligned}$$

This, together with (3.11), implies (3.6).

By lemmas 2.10, 2.9, and (2.7), we achieve (3.10) and

$$v_2'(\xi) - v_0'(\xi) \geqslant 0, \quad \xi \leqslant 0. \tag{3.12}$$

Define

$$h_2(\xi) = q(v_2(\xi)) - q(v_0(\xi)), \quad \xi \in (-\infty, +\infty).$$

Since $q'(s)$ is increasing, (3.12) leads to

$$h_2(\xi) \leqslant h_2(0), \quad \xi \leqslant 0.$$

Again, assume (3.11) is not true. Then

$$h_2(0) > \delta = h_2(+\infty).$$

Let

$$\xi_2 = \inf\{\xi > 0 : h_2(\xi) \geqslant h_2(0)\}.$$

As $h_2'(0) > 0$ and $h_2(0) > h_2(+\infty)$, we have $\xi_2 > 0$ satisfying $h_2'(\xi_2) \leqslant 0$, $h_2(\xi_2) = h_2(0)$ and

$$h_2(\xi) > h_2(0), \quad \xi \in (0, \xi_2).$$

Integrating (2.1) from 0 to ξ_2 yields

$$v_i'^{\lambda-1}\big|_0^{\xi_2} = -\frac{1}{\lambda}\int_0^{\xi_2} \xi(q(v_i))'\,\mathrm{d}\xi = -\frac{1}{\lambda}\xi q(v_i)\bigg|_0^{\xi_2} + \frac{1}{\lambda}\int_0^{\xi_2} q(v_i)\,\mathrm{d}\xi, \quad i = 0, 2.$$

Hence,

$$v_2'^{\lambda-1}(\xi_2) - v_0'^{\lambda-1}(\xi_2) = v_2'^{\lambda-1}(0) - v_0'^{\lambda-1}(0) - \frac{1}{\lambda}\xi_2 h_2(\xi_2) + \frac{1}{\lambda}\int_{\xi_2}^{0} h_2(\xi)\,\mathrm{d}\xi > 0.$$

This implies that $v_2'(\xi_2) > v_0'(\xi_2)$, which leads to $h_2'(\xi_2) > 0$. Thus, it contradicts $h_2'(\xi_2) \leqslant 0$ so that (3.11) holds. The proof of the proposition is then complete. □

4. Asymptotic behaviour of solutions

Finally, in this section, we will investigate the asymptotic behaviour of solutions to the Cauchy problem (1.1), (1.7) based on the properties of the self-similar solutions proved in the previous sections.

Similar to the non-Newtonian filtration equation or non-Newtonian polytropic filtration equation (see, for example, [11, 27]), the following existence theorem and the comparison principle hold. We state them here without proof for brevity.

THEOREM 4.1 (existence theorem). *Assume that* $0 \leqslant u_0(x) \in L^\infty(-\infty, +\infty)$, *and that* $u_0(x)$ *is a monotone function. Then the Cauchy problem (1.1), (1.7) admits a unique weak solution.*

THEOREM 4.2 (comparison principle). *Assume that* u_1 *and* u_2 *are two weak solutions to (1.1) satisfying*

$$0 \leqslant u_1(x,0) \leqslant u_2(x,0), \quad x \in (-\infty, +\infty),$$

with $u_2(x,0) \in L^\infty(-\infty, +\infty)$, *and* $u_{01}(x)$ *and* $u_{02}(x)$ *being monotone. Then*

$$u_1(x,t) \leqslant u_2(x,t), \quad x \in (-\infty, +\infty), \quad t > 0.$$

The asymptotic behaviour of solutions to the Cauchy problem (1.1), (1.7) is given by the following two theorems.

THEOREM 4.3. *Assume that* $0 \leqslant u_0(x) \in L^\infty(-\infty, +\infty)$, *and that* $u_0(x)$ *is a monotone function. Let* $u(x,t)$ *be the solution of the Cauchy problem (1.1), (1.7). Then, for any* $l > 0$,

$$\lim_{t \to +\infty} \sup_{-l<x<l} |u(x,t) - w(xt^{-1/\lambda})| = 0, \tag{4.1}$$

where w *is the solution to the infinite two-point boundary-value problem (1.5), (1.6) with*

$$w_- = \lim_{x \to -\infty} u_0(x), \qquad w_+ = \lim_{x \to +\infty} u_0(x).$$

Proof. Without loss of generality, we may assume that u_0 is increasing and that $0 \leqslant w_- < w_+$. For any $0 < \varepsilon < w_+ - w_-$, there exists $L > l$ such that

$$u_0(-L) < w_- + \varepsilon, \qquad u_0(L) > w_+ - \varepsilon.$$

Let w_1 and w_2 be the solutions of (1.5) with the boundary conditions

$$w_1(-\infty) = w_-, \qquad w_1(+\infty) = w_+ - \varepsilon,$$

and

$$w_2(-\infty) = w_- + \varepsilon, \quad w(+\infty) = w_+.$$

Then, from lemma 2.10,

$$w_1(\xi) \leqslant w(\xi) \leqslant w_2(\xi), \quad \xi \in (-\infty, +\infty). \tag{4.2}$$

Define

$$u_1(x,t) = w_1((x-L)t^{-1/\lambda}), \quad u_2(x,t) = w_2((x+L)t^{-1/\lambda}), \quad x \in (-\infty, +\infty),\ t > 0.$$

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Then, u_1 and u_2 are the two solutions of (1.1) with the following initial data, respectively,

$$u_1(x,0)=\begin{cases} w_-, & x<L,\\ w_+-\varepsilon, & x>L,\end{cases} \qquad u_2(x,0)=\begin{cases} w_-+\varepsilon, & x<-L,\\ w_+, & x>-L.\end{cases}$$

By the comparison principle,

$$u_1(x,t)\leqslant u(x,t)\leqslant u_2(x,t), \quad x\in(-\infty,+\infty), \quad t>0. \tag{4.3}$$

We then need to discuss the following two cases.

(i) If $p'(s)$ is an increasing function, then

$$\begin{aligned} u_2(x,t)-u_1(x,t) &= w_2((x+L)t^{-1/\lambda})-w_1((x-L)t^{-1/\lambda})\\ &\leqslant |w_2((x+L)t^{-1/\lambda})-w_1((x+L)t^{-1/\lambda})|\\ &\qquad + |w_1((x+L)t^{-1/\lambda})-w_1((x-L)t^{-1/\lambda})|. \end{aligned}$$

By theorem 2.6,

$$|w_2((x+L)t^{-1/\lambda})-w_1((x+L)t^{-1/\lambda})|\leqslant C_1\varepsilon, \quad -l<x<l, \quad t>0,$$

and, by the mean value theorem,

$$|w_1((x+L)t^{-1/\lambda})-w_1((x-L)t^{-1/\lambda})|\leqslant 2w_1'(0)Lt^{-1/\lambda}, \quad x\in(-\infty,+\infty), \quad t>0.$$

(ii) If $p'(s)$ is a decreasing function, then

$$\begin{aligned} u_2(x,t)-u_1(x,t) &= w_2((x+L)t^{-1/\lambda})-w_1((x-L)t^{-1/\lambda})\\ &\leqslant |w_2((x+L)t^{-1/\lambda})-w_2((x-L)t^{-1/\lambda})|\\ &\qquad + |w_2((x-L)t^{-1/\lambda})-w_1((x-L)t^{-1/\lambda})|. \end{aligned}$$

By theorem 2.6,

$$|w_2((x-L)t^{-1/\lambda})-w_1((x-L)t^{-1/\lambda})|\leqslant C_2\varepsilon, \quad -l<x<l, \quad t>0.$$

On the other hand, by the mean value theorem,

$$|w_2((x+L)t^{-1/\lambda})-w_2((x-L)t^{-1/\lambda})|\leqslant 2w_2'(0)Lt^{-1/\lambda}, \quad x\in(-\infty,+\infty), \quad t>0.$$

In summary, we have

$$u_2(x,t)-u_1(x,t)\leqslant C(\varepsilon+Lt^{-1/\lambda}), \quad -l<x<l, \quad t>0,$$

where $C>0$ is a constant independent of ε and L. Therefore, (4.2) and (4.3) imply that

$$|u(x,t)-w(xt^{-1/\lambda})|\leqslant u_2(x,t)-u_1(x,t)\leqslant C(\varepsilon+Lt^{-1/\lambda}), \quad -l<x<l, \quad t>0,$$

which leads to (4.1) and completes the proof. □

The following theorem shows that if we know the spatial decay rate of the initial data, then we will have the convergence rate of the solution to the corresponding self-similar solution.

THEOREM 4.4. *Assume that* $0 \leqslant u_0(x) \in L^\infty(-\infty, +\infty)$, *that* $u_0(x)$ *is a monotone function and that*

$$\limsup_{x\to-\infty} |x|^\beta |u_0(x) - w_-| < +\infty, \qquad \limsup_{x\to+\infty} |x|^\beta |u_0(x) - w_+| < +\infty, \tag{4.4}$$

with $\beta > 0$ *being a constant. Here,*

$$w_- = \lim_{x\to-\infty} u_0(x), \qquad w_+ = \lim_{x\to+\infty} u_0(x).$$

Let $u(x,t)$ *be the solution of the Cauchy problem (1.1), (1.7). Then, for any* $l > 0$,

$$\sup_{-L(t)<x<L(t)} |u(x,t) - w(xt^{-1/\lambda})| \leqslant C(l + l^{-\beta}) t^{-\beta/((1+\beta)\lambda)}, \quad t > 0, \tag{4.5}$$

where

$$L(t) = l t^{1/((1+\beta)\lambda)}, \quad t > 0,$$

and w *is the solution of the infinite two-point boundary-value problem (1.5), (1.6), while* $0 < C < +\infty$ *is a constant independent of* t *and* l.

Proof. Without loss of generality, we may assume that u_0 is increasing and that $0 \leqslant w_- < w_+$. From (4.4), there exist $L_0 > 0$ and $C_0 > 0$ such that $2C_0 L_0^{-\beta} < w_+ - w_-$ and

$$\begin{aligned} w_- \leqslant u_0(x) &\leqslant w_- + C_0|x|^{-\beta}, & x &\leqslant -L_0, \\ w_+ - C_0|x|^{-\beta} \leqslant u_0(x) &\leqslant w_+, & x &\geqslant L_0. \end{aligned}$$

Let $s_0 = (L_0/l)^{(1+\beta)\lambda}$. Then, for any fixed $s \geqslant s_0$, we have

$$\begin{aligned} w_- \leqslant u_0(x) &\leqslant w_- + C_0 L^{-\beta}(s), & x &\leqslant -L(s), \\ w_+ - C_0 L^{-\beta}(s) \leqslant u_0(x) &\leqslant w_+, & x &\geqslant L(s). \end{aligned}$$

Let w_1 and w_2 be the solutions of (1.5) with the boundary data

$$w_1(-\infty) = w_-, \quad w_1(+\infty) = w_+ - L^{-\beta}(s)$$

and

$$w_2(-\infty) = w_- + L^{-\beta}(s), \quad w(+\infty) = w_+,$$

respectively. Then, from lemma 2.10,

$$w_1(\xi) \leqslant w(\xi) \leqslant w_2(\xi), \quad \xi \in (-\infty, +\infty). \tag{4.6}$$

Define

$$\begin{aligned} u_1(x,t) &= w_1((x - L(s))t^{-1/\lambda}), \quad x \in (-\infty, +\infty), \quad t > 0, \\ u_2(x,t) &= w_2((x + L(s))t^{-1/\lambda}), \quad x \in (-\infty, +\infty), \quad t > 0. \end{aligned}$$

Then, u_1 and u_2 are the solutions of (1.1) with the initial values

$$u_1(x,0) = \begin{cases} w_-, & x < L(s), \\ w_+ - L^{-\beta}(s), & x > L(s), \end{cases} \quad u_2(x,0) = \begin{cases} w_- + L^{-\beta}(s), & x < -L(s), \\ w_+, & x > -L(s), \end{cases}$$

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respectively. By the comparison principle,

$$u_1(x,t) \leqslant u(x,t) \leqslant u_2(x,t), \quad x \in (-\infty,+\infty), \quad t > 0. \tag{4.7}$$

Again, we need to discuss the following two cases.

(i) If p' is increasing, then

$$\begin{aligned} u_2(x,s) - u_1(x,s) &= w_2((x+L(s))s^{-1/\lambda}) - w_1((x-L(s))s^{-1/\lambda}) \\ &\leqslant |w_2((x+L(s))s^{-1/\lambda}) - w_1((x+L(s))s^{-1/\lambda})| \\ &\quad + |w_1((x+L(s))s^{-1/\lambda}) - w_1((x-L(s))s^{-1/\lambda})|. \end{aligned}$$

Theorem 2.6 implies that

$$\begin{aligned} &|w_2((x+L(s))s^{-1/\lambda}) - w_1((x+L(s))s^{-1/\lambda})| \\ &\qquad \leqslant C_1 L^{-\beta}(s) = C_1 l^{-\beta} s^{-\beta/((1+\beta)\lambda)}, \quad -L(s) < x < L(s). \end{aligned}$$

On the other hand, the mean value theorem gives

$$\begin{aligned} &|w_1((x+L(s))s^{-1/\lambda}) - w_1((x-L(s))s^{-1/\lambda})| \\ &\qquad \leqslant 2w_1'(0)L(s)s^{-1/\lambda} = 2w_1'(0) l s^{-\beta/((1+\beta)\lambda)}, \quad x \in (-\infty,+\infty). \end{aligned}$$

(ii) If p' is decreasing, then

$$\begin{aligned} u_2(x,s) - u_1(x,s) &= w_2((x+L(s))s^{-1/\lambda}) - w_1((x-L(s))s^{-1/\lambda}) \\ &\leqslant |w_2((x+L(s))s^{-1/\lambda}) - w_2((x-L(s))s^{-1/\lambda})| \\ &\quad + |w_2((x-L(s))s^{-1/\lambda}) - w_1((x-L(s))s^{-1/\lambda})|. \end{aligned}$$

Theorem 2.6 yields

$$\begin{aligned} &|w_2((x-L(s))s^{-1/\lambda}) - w_1((x-L(s))s^{-1/\lambda})| \\ &\qquad \leqslant C_2 L^{-\beta}(s) = C_2 l^{-\beta} s^{-\beta/((1+\beta)\lambda)}, \quad -L(s) < x < L(s), \end{aligned}$$

and the mean value theorem leads to

$$\begin{aligned} &|w_2((x+L(s))s^{-1/\lambda}) - w_2((x-L(s))s^{-1/\lambda})| \\ &\qquad \leqslant 2w_2'(0)L(s)s^{-1/\lambda} = 2w_2'(0) l s^{-\beta/((1+\beta)\lambda)}, \quad x \in (-\infty,+\infty). \end{aligned}$$

We combine these two cases to obtain

$$u_2(x,s) - u_1(x,s) \leqslant C(l + l^{-\beta}) s^{-\beta/((1+\beta)\lambda)}, \quad -L(s) < x < L(s),$$

where $C > 0$ is a constant independent of s and l. By (4.6) and (4.7), we then have

$$\begin{aligned} |u(x,s) - w(xs^{-1/\lambda})| &\leqslant u_2(x,s) - u_1(x,s) \\ &\leqslant C(l + l^{-\beta}) s^{-\beta/((1+\beta)\lambda)}, \quad -L(s) < x < L(s),\ s > s_0, \end{aligned}$$

which implies (4.5) and completes the proof. □

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