Proceedings of the Royal Society of Edinburgh, **138A**, 1345–1362, *2008*

Bernoulli free-boundary problems in strip-like domains and a property of permanent waves on water of finite depth

Eugen Varvaruca*
Department of Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY, UK

(MS received 8 May 2007; accepted 8 November 2007)

We study weak solutions for a class of free-boundary problems which includes as a special case the classical problem of travelling gravity waves on water of finite depth. We show that such problems are equivalent to problems in fixed domains and study the regularity of their solutions. We also prove that in very general situations the free boundary is necessarily the graph of a function.

1. Introduction

One of the classical problems of nonlinear hydrodynamics is that of travelling two-dimensional gravity waves on water of finite depth, which arises from the following physical situation. A wave of permanent form moves with constant speed on the surface of an incompressible irrotational flow, the bottom of the fluid domain being horizontal. With respect to a frame of reference moving with the speed of the wave, the flow is steady and occupies a fixed region Ω in the (X, Y)-plane, which lies between the real axis $\mathcal{B}$ and some *a priori* unknown free surface $\mathcal{S} := \{(u(s), v(s)) : s \in \mathbb{R}\}$. Since the fluid is incompressible and irrotational, the flow can be described by a stream function ψ, which is harmonic in Ω and satisfies the following boundary conditions:

$$\psi = \text{const.} \quad \text{on } \mathcal{B}, \tag{1.1 a}$$

$$\psi = \text{const.} \quad \text{on } \mathcal{S}, \tag{1.1 b}$$

$$|\nabla\psi|^2 + 2gY = \text{const.} \quad \text{on } \mathcal{S}. \tag{1.1 c}$$

The problem consists of determining the curves $\mathcal{S}$ for which a function ψ with these properties exists in Ω.

There are two particular types of waves which have received considerable attention in the literature: periodic waves, where $\mathcal{S}$ is assumed to be periodic in the horizontal direction, and solitary waves, where $\mathcal{S}$ is assumed to be asymptotic to a horizontal line at infinity. Nowadays, the mathematical theory of these problems contains a wealth of results, of which some of the most notable are the existence results of Amick and Toland [3, 4] for both periodic and solitary waves, and the

*Present address: Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ (e.varvaruca@imperial.ac.uk).

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

results on symmetry and monotonicity of solitary waves of Craig and Sternberg [9]. We refer the reader to these papers and their references for historical background and further results. The related problem of travelling gravity waves with vorticity [8] will not be addressed in this paper.

It is always assumed in the literature that the wave profile $\mathcal{S}$ is a graph giving the vertical coordinate as a function of the horizontal coordinate. But it is more natural to assume *a priori* only that $\mathcal{S}$ is a curve in parametric form, and to ask whether there can exist situations in which $\mathcal{S}$ is not such a graph. This is one of the main questions we address in this paper, showing that in very general situations $\mathcal{S}$ is necessarily a graph. The absence until now of any investigation of this problem is surprising, since the related problem for waves of infinite depth [18] was solved a long time ago by Spielvogel [17], in the periodic setting. He derived a differential inequality for a function θ, which gives the angle between the free boundary and the horizontal, and suggested a geometric argument to show that this inequality would prevent the curve from overturning. It is nowadays understood that for a smooth periodic wave the non-overturning property follows easily from this inequality combined with the periodicity of θ (see [19]). Such an approach can easily be adapted to the case of smooth periodic waves on water of finite depth, but it does not work for solitary waves, or for the wave profiles of main interest in this paper, which may have more general geometries and need not be smooth.

It is customary in water-wave theory to assume that $\mathcal{S}$ is smooth enough, for example, that $\mathcal{S}$ is a C^1 curve. However, it is well known that there exist situations in which this smoothness requirement is not satisfied, namely for 'waves of extreme form' (see [5, 14]). In recent years, there has been some interest in weak solutions of the problem of periodic waves of infinite depth, through a series of papers by Shargorodsky and Toland, culminating with the comprehensive treatment in [16]. They noted that the classical theory of Hardy spaces [11, 13] can be used to assign boundary values to harmonic functions in domains whose boundary is a locally rectifiable curve, and required the boundary condition (1.1 c) to be satisfied in such a weak sense. In fact, [16] provides a rich mathematical theory for a more general class of problems, called *Bernoulli free-boundary problems*. Further aspects of this theory, including geometric properties of free boundaries and the nature of their singularities, were examined in [20,21]. (See also [1,10,12] and the references therein for other types of Bernoulli problems.)

In this paper we propose an analogue of the theory in [16] for a class of problems which would naturally generalize the problem of gravity waves of finite depth. Basically, we keep the boundary conditions (1.1 a), (1.1 b), replace (1.1 c) by the more general condition

$$|\nabla\psi| = h(Y) \quad \text{on } \mathcal{S}, \tag{1.2}$$

where h is a given function, and provide a interpretation in a weak sense of these boundary conditions. We impose minimal smoothness requirements on the curve $\mathcal{S}$ and the harmonic function ψ, while the assumptions on the geometry of $\mathcal{S}$ are sufficiently general that, when specialized to the case of water waves, include both the case of periodic waves and that of solitary waves.

In [16] it is assumed that h is a continuous function with values in $[0, \infty]$ (and avoiding at least one of the values 0 and ∞), which is suitably smooth on the open set where it is non-zero and finite. In that situation, the singularities of the curve $\mathcal{S}$ can only occur at *stagnation points*, which are points (X, Y) on $\mathcal{S}$ for which $h(Y) = 0$ or $h(Y) = \infty$. The set of stagnation points, denoted by $\mathcal{S}_{\mathcal{N}}$, is closed and has measure zero on $\mathcal{S}$ (see [16]) and $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$ is a union of smooth open arcs. Aiming for a theory of greater generality, in this paper we consider Bernoulli problems for functions h which are only Borel measurable. (We believe that it should be possible to construct examples of solutions of Bernoulli problems for functions h which are nowhere continuous, and where $\mathcal{S}$ is locally rectifiable but no arc of $\mathcal{S}$ is of class C^1.) We note, however, that the regularity results in [20] extend to the present setting, in that the smoothness of h on some open intervals on which it is non-zero and finite implies the regularity of certain arcs of $\mathcal{S} \setminus \mathcal{S}_{\mathcal{N}}$.

Since one of the main difficulties of Bernoulli problems is the fact that the free boundary $\mathcal{S}$ is unknown *a priori*, it is of interest to see if they can be reduced to problems in fixed domains. This is usually achieved in the hydrodynamical problem by conformal mapping, but in the situation considered here when $\mathcal{S}$ is a curve in parametric form (and not necessarily a graph) this is not a straightforward matter, and we are not aware of any rigorous proof in the literature. We show here that any Bernoulli free-boundary problem is indeed equivalent by means of a conformal mapping to a problem in a fixed domain, namely a strip.

The main result of the paper is that, when h is strictly decreasing and $\log h$ is concave on the interval where h is non-zero, any free boundary $\mathcal{S}$, possibly with many singularities, is necessarily the graph of a function. This result applies in particular to the water-wave problem. In the proof, we first derive a differential inequality generalizing that of Spielvogel [17], and for this purpose the specific Hardy spaces in which the original problem was posed play an essential role. To get to the required conclusion, we then use a geometric argument similar in spirit to that first used in [21] for Bernoulli problems in the setting of [16]. The details here are, however, significantly different from those in [21], since the *a priori* admissible geometries of the curve $\mathcal{S}$ are much more general here.

One may wonder whether any assumptions at all on h are necessary to ensure that $\mathcal{S}$ is a graph. In the setting of [16], examples of free boundaries which are not graphs will be presented in a forthcoming paper. It seems highly likely that such examples exist in the setting considered here too.

2. Bernoulli free-boundary problems

2.1. Some preliminaries

In this paper, by a strip-like domain it is meant a domain Ω whose boundary consists of the real axis $\mathcal{B}$ and a non-self-intersecting curve $\mathcal{S} := \{(u(s), v(s)) : s \in \mathbb{R}\}$ contained in the upper half-plane, such that

$$\mathcal{S} \text{ is locally rectifiable,} \tag{2.1 a}$$

$$\lim_{s \to \pm\infty} u(s) = \pm\infty, \qquad v \text{ is bounded above.} \tag{2.1 b}$$

Let $h : J \to [0, \infty]$ be a given Borel measurable function, where $J \subset (0, \infty)$ is an interval. A Bernoulli problem is one of finding a strip-like domain Ω in which there

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

exists a harmonic function ψ satisfying the following boundary conditions:

$$\psi = 0 \qquad \text{almost everywhere on } \mathcal{B}, \tag{2.2}$$

$$\psi = C \qquad \text{almost everywhere on } \mathcal{S}, \tag{2.3}$$

$$|\nabla\psi| = h(Y) \quad \text{almost everywhere on } \mathcal{S}, \tag{2.4}$$

where C is a positive constant. (Whenever we refer to a relation which holds on a locally rectifiable curve, such as in (2.2)–(2.4), 'almost everywhere' refers to one-dimensional Hausdorff measure, or arc length.) Since the curve $\mathcal{S}$ is not prescribed *a priori*, it is called a *free boundary.*

We now explain the weak sense in which these conditions are to be satisfied. In short, if the following conditions hold:

$$\psi \text{ is bounded in } \Omega, \tag{2.5}$$

$$\text{the subharmonic function } |\nabla\psi| \text{ has a harmonic majorant in } \Omega, \tag{2.6}$$

then the function ψ and its partial derivatives have non-tangential limits almost everywhere on $\mathcal{S}$ and $\mathcal{B}$. In this situation, the required conditions (2.2)–(2.4) would refer to the non-tangential boundary values of ψ and $\nabla\psi$.

To support the above claim, we need a precise definition of a non-tangential limit, and the definition and properties of Hardy spaces in general domains.

2.2. Non-tangential limits

Let Ξ be a bounded open set in the plane whose boundary is a rectifiable Jordan curve $\mathcal{T}$. Let (X_0, Y_0) be a point on $\mathcal{T}$ at which $\mathcal{T}$ has a tangent, and let $\boldsymbol{n}_{\mathrm{i}}$ be the corresponding unit inner normal. We say that a sequence $\{(X_n, Y_n)\}_{n\geqslant 1}$ of points in Ξ *tends to* (X_0, Y_0) *non-tangentially within* Ξ if $(X_n, Y_n) \to (X_0, Y_0)$ as $n \to \infty$ and there exists $\kappa > 0$ such that

$$(X_n - X_0, Y_n - Y_0) \cdot \boldsymbol{n}_{\mathrm{i}} \geqslant \kappa[(X_n - X_0)^2 + (Y_n - Y_0)^2]^{1/2} \quad \text{for all } n \geqslant 1,$$

where the dot denotes the usual inner product in $\mathbb{R}^2$.

Let $F : \Xi \to \mathbb{C}$ be a function, and let $l \in \mathbb{C}$. We say that F *has non-tangential limit* l *at* (X_0, Y_0) *within* Ξ, and write

$$\lim_{(X,Y)\twoheadrightarrow(X_0,Y_0)} F(X, Y) = l,$$

if $\lim_{n\to\infty} F(X_n, Y_n) = l$ for every sequence $\{(X_n, Y_n)\}_{n\geqslant 1}$ which tends to (X_0, Y_0) non-tangentially within Ξ.

Let Ω be a strip-like domain, let $Z_0 := X_0 + \mathrm{i}Y_0$ be a point on $\mathcal{S}$ at which $\mathcal{S}$ has a tangent, and let $F : \Omega \to \mathbb{C}$. Let Z_1 and Z_2 be two points on $\mathcal{S}$ such that Z_0 is between Z_1 and Z_2. Then Z_1 and Z_2 can be joined by a rectifiable arc contained in Ω. This arc, together with the arc $\mathcal{A}$ of $\mathcal{S}$ joining Z_1 and Z_2, determines a rectifiable Jordan curve, which is the boundary of a bounded subdomain Ξ of Ω. Let $l \in \mathbb{C}$. We say that F *has non-tangential limit* l *at* (X_0, Y_0) *within* Ω if F has non-tangential limit l at (X_0, Y_0) within Ξ. It is easy to see that this definition is meaningful, in the sense that it does not depend on the set Ξ used.

2.3. Hardy spaces

Let D be the unit disc in the complex plane. For $p \in [1, \infty)$, the Hardy space $h^p_{\mathbb{C}}(D)$ is usually defined [11, 13, 15] as the class of harmonic functions $F : D \to \mathbb{C}$ with the property that

$$\sup_{r \in (0,1)} \int_{-\pi}^{\pi} |F(r\mathrm{e}^{\mathrm{i}t})|^p \,\mathrm{d}t < +\infty. \tag{2.7}$$

The Hardy space $h^\infty_{\mathbb{C}}(D)$ is the class of bounded harmonic functions in D. For $p \in [1, \infty]$, the Hardy space $\mathcal{H}^p_{\mathbb{C}}(D)$ is the class of holomorphic functions in $h^p_{\mathbb{C}}(D)$. It is well known that any function in $h^p_{\mathbb{C}}(D)$, $p \in [1, \infty]$, has non-tangential limits almost everywhere on the unit circle. The M. Riesz theorem [11, theorem 4.1, p. 54] asserts that, if $U \in h^p_{\mathbb{C}}(D)$ for some $p \in (1, \infty)$, and if V is a harmonic function such that $U + \mathrm{i}V$ is holomorphic in D, then $V \in h^p_{\mathbb{C}}(D)$.

An important fact, which leads to the definition of Hardy spaces in general domains [11, ch. 10], is that, for $p \in [1, \infty)$, a harmonic function F belongs to $h^p_{\mathbb{C}}(D)$ if and only if the subharmonic function $|F|^p$ has a harmonic majorant, i.e. there exists a positive harmonic function u in D such that $|F|^p \leqslant u$ in D. Let Ξ be an open set. For $p \in [1, \infty)$, the space $h^p_{\mathbb{C}}(\Xi)$ is the class of harmonic functions $F : \Xi \to \mathbb{C}$ for which the subharmonic function $|F|^p$ has a harmonic majorant in Ξ. The Hardy space $h^\infty_{\mathbb{C}}(\Xi)$ is the class of bounded harmonic functions in Ξ. The spaces $\mathcal{H}^p_{\mathbb{C}}(\Xi)$ consist of the holomorphic functions in $h^p_{\mathbb{C}}(\Xi)$, for $p \in [1, \infty]$. It is easy to check that the Hardy spaces are conformally invariant: if Ξ_1 and Ξ_2 are two open sets, and $\gamma : \Xi_1 \to \Xi_2$ is a conformal bijection, then $F \in h^p_{\mathbb{C}}(\Xi_2)$ if and only if $F \circ \gamma \in h^p_{\mathbb{C}}(\Xi_1)$, where $p \in [1, \infty]$. Due to this fact, many properties of the Hardy spaces of the disc extend by conformal mapping to Hardy spaces of simply connected domains. If Ξ is a bounded domain whose boundary is a rectifiable Jordan curve, then any function in $h^p_{\mathbb{C}}(\Xi)$, where $1 \leqslant p \leqslant \infty$, has non-tangential boundary values almost everywhere. It is immediate that, if Ω is a strip-like domain, then any function in $h^p_{\mathbb{C}}(\Omega)$, where $1 \leqslant p \leqslant \infty$, has non-tangential boundary values almost everywhere on $\mathcal{S}$ and $\mathcal{B}$.

Finally, we mention that it is possible to define Hardy spaces $\mathcal{H}^p_{\mathbb{C}}(\Xi)$ also for values $p \in (0, 1)$, namely as the class of holomorphic functions $F : \Xi \to \mathbb{C}$ for which the subharmonic function $|F|^p$ has a harmonic majorant in Ξ, an open set. (Note that when F is only harmonic in Ξ and $p \in (0, 1)$, the function $|F|^p$ need not be subharmonic.) For the unit disc D, the class $\mathcal{H}^p_{\mathbb{C}}(D)$, $p \in (0, 1)$, coincides with the class of holomorphic functions for which (2.7) holds. In this paper, only marginal use is made of the spaces $\mathcal{H}^p_{\mathbb{C}}(\Xi)$, $p \in (0, 1)$.

2.4. Further preliminaries

If ψ is harmonic in a strip-like domain Ω and satisfies (2.5) and (2.6), the preceding discussion ensures that ψ and its partial derivatives have non-tangential boundary values almost everywhere on $\mathcal{S}$ and $\mathcal{B}$. In a Bernoulli problem, these boundary values are required to satisfy (2.2)–(2.4).

Whenever ψ is a harmonic function in a strip-like domain Ω, we consider also a harmonic function φ such that the function $\omega := \varphi + \mathrm{i}\psi$ is holomorphic in Ω. The condition (2.6) can then be reformulated as $\omega' \in \mathcal{H}^1_{\mathbb{C}}(\Omega)$. Here and in what follows,

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

the prime denotes differentiation (it will be clear from the context whether either real or complex differentiation is meant). Let also Ω^{R} denote the reflection of Ω in the line $\mathcal{B}$, let

$$\tilde{\Omega} := \Omega \cup \mathcal{B} \cup \Omega^{\mathrm{R}}, \tag{2.8}$$

and let $L > 0$ be such that $\tilde{\Omega}$ is contained in the strip determined by the lines $Y = -L$ and $Y = L$.

The next result gathers several properties of weak solutions of Bernoulli problems, refining some results in [16]. While in [16] the function ψ was assumed continuous in $\Omega \cup \mathcal{S}$, here we derive the continuity of ω in $\Omega \cup \mathcal{S}$ from (2.5) and (2.3). Also, the claims in [16] that the condition (2.3) ensures that formally the tangential derivative along $\mathcal{S}$ of ψ is zero almost everywhere, and hence that the condition (2.4) is equivalent to a Neumann condition, are put on a rigorous basis here. (The fact proved here that ω has a holomorphic extension to $\tilde{\Omega}$ has no analogue in the setting of [16].)

Proposition 2.1. *Let Ω be a strip-like domain, and let $\omega = \varphi + \mathrm{i}\psi$ be a holomorphic function in Ω such that (2.5) holds.*

(i) *If ψ satisfies (2.2), then ω has a holomorphic extension to $\tilde{\Omega}$ which satisfies $\omega(\bar{Z}) = \overline{\omega(Z)}$ for all $Z \in \tilde{\Omega}$.*

(ii) *If ψ satisfies (2.3), then ω has a continuous extension to $\Omega \cup \mathcal{S}$.*

(iii) *If (2.6) holds and ψ satisfies (2.3), then*

$$\Big(\lim_{(X,Y)\twoheadrightarrow(X_0,Y_0)} \nabla\psi(X,Y)\Big) \cdot \boldsymbol{t}(X_0,Y_0) = 0 \quad \text{for a.e. } (X_0,Y_0) \in \mathcal{S}, \tag{2.9}$$

where $\boldsymbol{t}(X_0,Y_0)$ is a unit tangent to $\mathcal{S}$ at (X_0,Y_0).

(iv) *If (2.6) holds and ψ satisfies (2.3), then ψ satisfies (2.4) if and only if*

$$\Big(\lim_{(X,Y)\twoheadrightarrow(X_0,Y_0)} \nabla\psi(X,Y)\Big) \cdot \boldsymbol{n}_{\mathrm{o}}(X_0,Y_0) = h(Y) \quad \text{for a.e. } (X_0,Y_0) \in \mathcal{S}, \tag{2.10}$$

where $\boldsymbol{n}_{\mathrm{o}}(X_0,Y_0)$ is the unit outer normal to Ω at (X_0,Y_0).

It is necessary for our purposes to strengthen the basic conditions (2.5), (2.6). The formal definition of a Bernoulli problem given below involves a holomorphic function $\omega = \varphi + \mathrm{i}\psi$ in the domain $\tilde{\Omega}$ given by (2.8). The conditions (2.11), (2.12) are motivated by proposition 2.1.

Definition 2.2. A Bernoulli free-boundary problem, or problem (B), is to find a strip-like domain Ω and a function $\omega = \varphi + \mathrm{i}\psi$ in $\tilde{\Omega}$, such that

$$\omega \text{ is holomorphic in } \tilde{\Omega} \text{ and continuous on its closure}, \tag{2.11}$$

$$\omega(\bar{Z}) = \overline{\omega(Z)} \quad \text{for all } Z \in \tilde{\Omega}, \tag{2.12}$$

$$\omega' \in \mathcal{H}^1_{\mathbb{C}}(\tilde{\Omega}), \tag{2.13}$$

$$\omega' \neq 0 \text{ in } \tilde{\Omega} \quad \text{and} \quad \frac{1}{\omega'} \in \mathcal{H}^1_{\mathbb{C}}(\tilde{\Omega}), \tag{2.14}$$

and (2.3), (2.4) hold.

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

REMARK 2.3. It will be seen in the proof of theorem 3.4 that any solution ω of (2.11), (2.12) and (2.3) automatically satisfies the condition $\omega' \neq 0$ in $\tilde{\Omega}$ required in (2.14).

3. The main results

3.1. An equivalent problem in a strip

We now introduce a problem in a strip, to which we then prove that problem (B) is equivalent. Firstly, let us introduce some more notation. For any $a, b \in \mathbb{R}$ with $a < b$, let

$$\Pi_{a,b} := \{x + \mathrm{i}y \in \mathbb{C} : a < y < b\},$$

and for any $c \in \mathbb{R}$, let

$$\mathcal{L}_c := \{x + \mathrm{i}y \in \mathbb{C} : y = c\}.$$

DEFINITION 3.1. We say that a function $W = U + \mathrm{i}V$ in the strip $\Pi_{-C,C}$ is a solution of problem (P) if the following conditions are satisfied:

$$W \text{ is holomorphic in } \Pi_{-C,C} \text{ and continuous on its closure,} \tag{3.1}$$

$$W(\bar{z}) = \overline{W(z)} \quad \text{for all } z \in \Pi_{-C,C}, \tag{3.2}$$

$$V \text{ is bounded in } \Pi_{-C,C}, \qquad V(t, C) > 0 \quad \text{for all } t \in \mathbb{R}, \tag{3.3}$$

$$\text{the mapping } t \mapsto W(t, C) \text{ is injective and } \lim_{t\to\pm\infty} U(t, C) = \pm\infty, \tag{3.4}$$

$$W' \in \mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C}), \tag{3.5}$$

$$W' \neq 0 \text{ in } \Pi_{-C,C} \quad \text{and} \quad \frac{1}{W'} \in \mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C}), \tag{3.6}$$

$$h(V(t, C))|\nabla V(t, C)| = 1 \quad \text{for almost every } t \in \mathbb{R}. \tag{3.7}$$

REMARK 3.2. The condition (3.5) ensures that the partial derivatives of V have non-tangential boundary values almost everywhere. The condition (3.7) refers to these non-tangential boundary values on $\mathcal{L}_C$.

REMARK 3.3. It will be seen in the proof of theorem 3.4 that any solution W of (3.1)–(3.4) necessarily satisfies the condition $W' \neq 0$ in $\Pi_{-C,C}$ required in (3.6).

The main result on equivalence is the following.

THEOREM 3.4. *Let (Ω, ω) be a solution of problem (B). Then ω is a conformal mapping from $\tilde{\Omega}$ onto the strip $\Pi_{-C,C}$ and a homeomorphism from the closure of $\tilde{\Omega}$ onto the closure of $\Pi_{-C,C}$, with Ω being mapped onto $\Pi_{0,C}$, $\mathcal{B}$ being mapped onto $\mathcal{L}_0$ and $\mathcal{S}$ being mapped onto $\mathcal{L}_C$. Let W be the inverse conformal mapping, from $\Pi_{-C,C}$ onto $\tilde{\Omega}$. Then W is a solution of problem (P).*

Conversely, let W be a solution of problem (P). Let $\mathcal{S} := \{W(t, C) : t \in \mathbb{R}\}$, which is a non-self-intersecting curve, and let Ω be the domain whose boundary consists of the curve $\mathcal{S}$ and the real axis $\mathcal{B}$. Then Ω is a strip-like domain. Let $\tilde{\Omega}$ be given by (2.8). Then W is a conformal mapping from $\Pi_{-C,C}$ onto $\tilde{\Omega}$ and a homeomorphism from the closure of $\Pi_{-C,C}$ onto the closure of $\tilde{\Omega}$, with $\Pi_{0,C}$ being mapped onto Ω, $\mathcal{L}_0$ being mapped onto $\mathcal{B}$ and $\mathcal{L}_C$ being mapped onto $\mathcal{S}$. Let ω be the inverse conformal mapping, from $\tilde{\Omega}$ onto $\Pi_{-C,C}$. Then (Ω, ω) is a solution of problem (B).

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

3.2. Local regularity

The following local regularity result is an analogue of [20, theorems 2.1 and 2.3].

Theorem 3.5. *Let $h : (0, \infty) \to [0, \infty]$ be a Borel measurable function. Let $(c, d) \subset (0, \infty)$ be an interval, and suppose that*

$$h \in C^{n,\alpha}_{\mathrm{loc}}((c, d)) \quad \text{where } n \in \mathbb{N} \cup \{0\}, \quad \alpha \in (0, 1),$$
$$h \neq 0 \quad \text{on } (c, d).$$

Let (Ω, ω) be a solution of problem (B). Let $Z_1, Z_2 \in \mathcal{S}$ with $Z_1 \neq Z_2$ be such that the open arc $\mathcal{A}$ of $\mathcal{S}$ joining Z_1 and Z_2 is contained in the strip $\Pi_{c,d}$. Then $\mathcal{A}$ is a curve of class $C^{n+1,\alpha}_{\mathrm{loc}}$ and $\omega \in C^{n+1,\alpha}_{\mathrm{loc}}(\Omega \cup \mathcal{A})$.

3.3. When the free boundary is a graph

The main result of this paper is the following.

Theorem 3.6. *Let $d > 0$ and $h : (0, d] \to [0, \infty)$ be such that*

$$h(d) = 0 \text{ and } h > 0 \text{ on } (0, d), \tag{3.8 a}$$

$$h \in C((0, d]) \cap C^{1,\alpha}_{\mathrm{loc}}((0, d)) \quad \text{for some } \alpha \in (0, 1), \tag{3.8 b}$$

$$h \text{ is strictly decreasing and } \log h \text{ is concave on } (0, d). \tag{3.8 c}$$

Let (Ω, ω) be a solution of problem (B). There then exists a parametrization

$$\{(u(t), v(t)) : t \in \mathbb{R}\}$$

of $\mathcal{S}$, where u, v are locally absolutely continuous and

$$u'(t) > 0 \quad \text{for almost all } t \in \mathbb{R}. \tag{3.9}$$

Hence, there exists a continuous function locally of bounded variation $\eta : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{S} = \{(X, \eta(X)) : X \in \mathbb{R}\}.$$

4. Proofs

4.1. A preliminary result

The following lemma will be useful for some of the proofs of the preceding results. The first part is well known, while the second part is essentially a local version of Privalov's theorem; see [20, lemma 2.2] for the proof of a very similar result.

Lemma 4.1. *Let $x_0 \in \mathbb{R}$ and $r > 0$. For $t > 0$, let $B^+_t(x_0) := \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + y^2 < t^2, y > 0\}$. Let $F \in \mathcal{H}^1_{\mathbb{C}}(B^+_r(x_0))$ be of the form $F = U + \mathrm{i}V$, where U and V are real valued functions, and, for $s \in (x_0 - r, x_0 + r)$, let*

$$u(s) + \mathrm{i}v(s) := \lim_{(x,y) \twoheadrightarrow (s,0)} (U(x, y) + \mathrm{i}V(x, y)).$$

(i) *If u is continuous on $(x_0 - r, x_0 + r)$, then U is continuous on $B^+_r(x_0) \cup \{(x, 0) : x \in (x_0 - r, x_0 + r)\}$.*

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

(ii) *If* $u \in C^{k,\alpha}_{\mathrm{loc}}((-r,r))$ *for some* $k \in \mathbb{N} \cup \{0\}$ *and* $\alpha \in (0,1)$, *then*

$$v \in C^{k,\alpha}_{\mathrm{loc}}((-r,r))$$

and U, V *are of class* $C^{k,\alpha}$ *in the closure of* $B^{+}_{t}(x_0)$ *for every* $t \in (0,r)$.

4.2. Proof of proposition 2.1

(i) Since ψ is bounded in Ω, it is immediate from the first part of lemma 4.1 that ψ has a continuous extension to $\Omega \cup \mathcal{B}$. The conclusion now follows from the classical Schwarz reflection principle (see [15, theorem 11.14, p. 237]).

(ii) Let Z_1 and Z_2 be any two distinct points on $\mathcal{S}$. Then they can be joined by a rectifiable arc contained in Ω. This arc, together with the arc $\mathcal{A}$ of $\mathcal{S}$ joining Z_1 and Z_2, determines a rectifiable Jordan curve, which is the boundary of a bounded subdomain Ξ of Ω. Let $\gamma : D \to \Xi$ be a conformal mapping from D onto Ξ. By Carathéodory's theorem (see [13, ch. II, § C, p. 35]), γ is a homeomorphism from the closure of D onto the closure of Ξ. Let $I \subset \mathbb{R}$ be an interval such that $\gamma(\Gamma_I) = \mathcal{A}$, where $\Gamma_I := \{\mathrm{e}^{\mathrm{i}s} : s \in I\}$.

Let $F := \omega \circ \gamma$. Then $F \in \mathcal{H}^1_{\mathbb{C}}(D)$. Moreover, $F - C$ is real valued on Γ_I, so by the Schwarz reflection principle for $\mathcal{H}^1_{\mathbb{C}}(D)$ functions (see [13, ch. III, § E, p. 63]), F has a holomorphic extension across Γ_I. In particular F extends continuously to $D \cup \Gamma_I$, and therefore ω has a continuous extension to $\Omega \cup \mathcal{A}$. Since Z_1 and Z_2 were arbitrary on $\mathcal{S}$, this proves (ii).

(iii) Suppose now that (2.6) holds. Let Ξ be as in the proof of (ii). Since the boundary of Ξ is a rectifiable Jordan curve, it follows from [11, theorems 3.11 and 3.12, p. 42] that $\gamma' \in \mathcal{H}^1_{\mathbb{C}}(D)$, the mapping $s \mapsto \gamma(\mathrm{e}^{\mathrm{i}s})$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma(\mathrm{e}^{\mathrm{i}s}) = \mathrm{i}\mathrm{e}^{\mathrm{i}s} \lim_{\zeta \twoheadrightarrow \mathrm{e}^{\mathrm{i}s}} \gamma'(\zeta) \quad \text{almost everywhere.} \tag{4.1}$$

Note that the mapping $I \ni s \mapsto \gamma(\mathrm{e}^{\mathrm{i}s})$ is an absolutely continuous parametrization of $\mathcal{A}$ and, almost everywhere on I, $\mathrm{d}(\gamma(\mathrm{e}^{\mathrm{i}s}))/\mathrm{d}s$ gives a tangent vector to $\mathcal{S}$.

Since F is holomorphic across Γ_I and $F - C$ is real valued on Γ_I, it follows that

$$\operatorname{Im}\{\mathrm{i}\mathrm{e}^{\mathrm{i}s} F'(\mathrm{e}^{\mathrm{i}s})\} = 0 \quad \text{for all } s \in I.$$

Since $F = \omega \circ \gamma$ on D, the above means that

$$\operatorname{Im}\{\mathrm{i}\mathrm{e}^{\mathrm{i}s} \lim_{\zeta \twoheadrightarrow \mathrm{e}^{\mathrm{i}s}} (\omega'(\gamma(\zeta))\gamma'(\zeta))\} = 0 \quad \text{almost everywhere on } I.$$

It follows, using (4.1), that

$$\operatorname{Im}\left\{\left(\lim_{X+\mathrm{i}Y \twoheadrightarrow \gamma(\mathrm{e}^{\mathrm{i}s})} \omega'(X+\mathrm{i}Y)\right)\frac{\mathrm{d}}{\mathrm{d}s}\gamma(\mathrm{e}^{\mathrm{i}s})\right\} = 0 \quad \text{almost everywhere on } I,$$

and therefore

$$\left(\lim_{(X,Y) \twoheadrightarrow (X_0,Y_0)} \nabla\psi(X,Y)\right) \cdot \boldsymbol{t}(X_0,Y_0) = 0 \quad \text{for a.e. } (X_0,Y_0) \in \mathcal{A}.$$

Since Z_1 and Z_2 were arbitrary on $\mathcal{S}$, (2.9) follows.

(iv) The maximum principle shows that

$$0<\psi<C \quad \text{everywhere in } \Omega. \tag{4.2}$$

Since (4.2) holds, it follows that

$$\Big(\lim_{(X,Y)\twoheadrightarrow(X_0,Y_0)} \nabla\psi(X,Y)\Big)\cdot \boldsymbol{n}_0(X_0,Y_0)\geqslant 0 \quad \text{for a.e. } (X_0,Y_0)\in\mathcal{S}. \tag{4.3}$$

The required result easily follows from (2.9) and (4.3).

4.3. Hardy spaces of a strip

We now summarize some facts concerning the Hardy spaces of a strip, a thorough treatment of which has recently been given in [6]. However, most (though not all) of the results below follow immediately by means of conformal mapping from the corresponding results in the unit disc, which are well known.

Let $q:\mathbb{R}\to\mathbb{R}$ be given by $q(t)=\mathrm{e}^{-(\pi/2C)|t|}$ for all $t\in\mathbb{R}$. For any $p\in[1,\infty)$, let

$$L^p(\mathbb{R},q(t)\,\mathrm{d}t):=\bigg\{F:\mathbb{R}\to\mathbb{C}:\int_{-\infty}^{\infty}|F(t)|^p q(t)\,\mathrm{d}t<+\infty\bigg\}. \tag{4.4}$$

If F is in the Hardy space $h^p_{\mathbb{C}}(\Pi_{-C,C})$, where $p\in[1,\infty]$, then F has non-tangential boundary values almost everywhere, which, when $p\in[1,\infty)$, satisfy $|F(\cdot,\pm C)|\in L^p(\mathbb{R},q(t)\,\mathrm{d}t)$. For $p\in(1,\infty]$, but not necessarily when $p=1$, F can be expressed as a Poisson integral of its boundary values. Here, for any two functions $F_C, F_{-C}\in L^1(\mathbb{R},q(t)\,\mathrm{d}t)$, the Poisson integral [22] associated to F_C and F_{-C} is given, for all $(x,y)\in\Pi_{-C,C}$, by

$$\begin{aligned}\mathcal{P}[F_C,F_{-C}](x,y):=\frac{1}{4C}\int_{-\infty}^{\infty}\frac{\cos((\pi/2C)y)}{\cosh((\pi/2C)(x-s))-\sin((\pi/2C)y)}F_C(s)\,\mathrm{d}s\\ +\frac{1}{4C}\int_{-\infty}^{\infty}\frac{\cos((\pi/2C)y)}{\cosh((\pi/2C)(x-s))+\sin((\pi/2C)y)}F_{-C}(s)\,\mathrm{d}s.\end{aligned}$$

However, if $F\in\mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C})$, then, by the F. and M. Riesz theorem [13, ch. 2], F can be expressed as the Poisson integral of its boundary values.

For any $F\in L^1(\mathbb{R},q(t)\,\mathrm{d}t)$, we denote by $\mathcal{P}_{\mathrm{e}}[F]$ and $P_{\mathrm{o}}[F]$ the functions in $\Pi_{-C,C}$ given by

$$\mathcal{P}_{\mathrm{e}}[F]:=\mathcal{P}[F,F] \quad\text{and}\quad \mathcal{P}_{\mathrm{o}}[F]:=\mathcal{P}[F,-F]. \tag{4.5}$$

The next result is slightly more general than needed for our purposes, but we think it might be of interest in itself. It is an analogue of [11, theorem 3.11, p. 42].

Lemma 4.2. *Let the holomorphic function $F:\Pi_{-C,C}\to\mathbb{C}$ be such that $F'\in\mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C})$. Then F is continuous on the closure of $\Pi_{-C,C}$, the mappings $\mathbb{R}\ni t\mapsto F(t\pm \mathrm{i}C)$ are locally absolutely continuous, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t\pm\mathrm{i}C)=\lim_{(x,y)\twoheadrightarrow(t,\pm C)}F'(x+\mathrm{i}y) \quad \textit{for a.e. } t\in\mathbb{R}.$$

Proof of lemma 4.2. For any $A > 0$, let

$$\mathcal{R}_A := \{(x,y) \in \Pi_{-C,C} : -A < x < A\}.$$

Let $\gamma : D \to \mathcal{R}_A$ be a conformal mapping. Then γ is a homeomorphism from the closure of D onto the closure of $\mathcal{R}_A$ and, since the boundary of $\mathcal{R}_A$ is a rectifiable Jordan curve, it follows from [11, theorem 3.12, p. 44] that $\gamma' \in \mathcal{H}^1_{\mathbb{C}}(D)$. Let $G := F \circ \gamma$. Then, since $G' = F'(\gamma)\gamma'$, we deduce that $G' \in \mathcal{H}^{1/2}_{\mathbb{C}}(D)$. By a classical result of Hardy and Littlewood (see [11, theorem 5.12, p. 88]), it follows that $G \in \mathcal{H}^1_{\mathbb{C}}(D)$, and therefore $F \in \mathcal{H}^1_{\mathbb{C}}(\mathcal{R}_A)$. Since this is true for all $A > 0$, it follows that F has non-tangential boundary values almost everywhere, which we denote $F(\cdot \pm \mathrm{i}C)$. Let us also define

$$f(t \pm \mathrm{i}C) := \lim_{(x,y)\to(t,\pm C)} F'(x + \mathrm{i}y) \quad \text{for a.e. } t \in \mathbb{R}.$$

Let us now focus our attention on the behaviour of F near the line $\mathcal{L}_C$. It is a consequence of [6, corollary 2.1] that, for every $A > 0$,

$$\|F'(\cdot + \mathrm{i}y) - f(\cdot + \mathrm{i}C)\|_{L^1(-A,A)} \to 0 \quad \text{as } y \nearrow C. \tag{4.6}$$

Let $x_0 \in \mathbb{R}$ be such that there exists $\lim_{y\nearrow C} F(x_0 + \mathrm{i}y) =: F(x_0 + \mathrm{i}C)$. For every $x \in \mathbb{R}$ and $y \in (0, C)$,

$$F(x + \mathrm{i}y) - F(x_0 + \mathrm{i}y) = \int_{x_0}^{x} F'(s + \mathrm{i}y)\,\mathrm{d}s. \tag{4.7}$$

We deduce from (4.7), upon passing to the limit as $y \nearrow C$ and taking into account (4.6), that for every x such that there exists $\lim_{y\nearrow C} F(x + \mathrm{i}y) =: F(x + \mathrm{i}C)$, the following holds:

$$F(x + \mathrm{i}C) - F(x_0 + \mathrm{i}C) = \int_{x_0}^{x} f(s + \mathrm{i}C)\,\mathrm{d}s.$$

Hence, $\mathbb{R} \ni x \mapsto F(x + \mathrm{i}C)$ coincides almost everywhere with a locally absolutely continuous function. By the first part of lemma 4.1, F is continuous on $\Pi_{-C,C} \cup \mathcal{L}_C$. Hence, the mapping $\mathbb{R} \ni x \mapsto F(x + \mathrm{i}C)$ is locally absolutely continuous with

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t + \mathrm{i}C) = f(t + \mathrm{i}C) \quad \text{for a.e. } t \in \mathbb{R}.$$

A similar argument can be used to deal with the behaviour of F near the line $\mathcal{L}_{-C}$. This completes the proof of lemma 4.2. □

4.4. Proof of theorem 3.4

We start by proving the result claimed in remark 2.3. Let ω satisfy (2.11), (2.12) and (2.3). We first prove the existence of a conformal mapping W_0 from $\Pi_{-C,C}$ onto $\tilde{\Omega}$, such that $W_0(\bar{z}) = \overline{W_0(z)}$ for all $z \in \Pi_{-C,C}$, and which has an extension as a homeomorphism from the closure of $\Pi_{-C,C}$ onto the closure of $\tilde{\Omega}$, with $\mathcal{L}_0$ being mapped onto $\mathcal{B}$ and $\mathcal{L}_C$ being mapped onto $\mathcal{S}$. Indeed, let β be the conformal mapping from D onto $\Pi_{-C,C}$ given by

$$\beta(\xi) = \frac{2C}{\pi} \log \frac{1 + \xi}{1 - \xi} \quad \text{for all } \xi \in D. \tag{4.8}$$

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

Let α be a conformal map from D onto the strip containing $\tilde{\Omega}$ which is determined by the lines $Y = -L$ and $Y = L$, given by

$$\alpha(\zeta) = \frac{2L}{\pi} \log \frac{1+\zeta}{1-\zeta} \quad \text{for all } \zeta \in D. \tag{4.9}$$

Let $\Xi := \alpha^{-1}(\tilde{\Omega})$. Then Ξ is a subdomain of D, whose boundary is a Jordan curve consisting of two arcs which are symmetric about the real axis, contained in D and joining the points -1 and 1. An immediate application of Carathéodory's theorem ensures the existence of a conformal mapping δ from D onto Ξ which has an extension as a homeomorphism between the closures of these domains, and is such that $\delta(\pm 1) = \pm 1$ and δ maps the segment $[-1, 1]$ of the real line onto itself. Defining $W_0 := \alpha \circ \delta \circ \beta^{-1}$, this mapping W_0 has all the required properties. It follows that $\psi \circ W_0$ is bounded in $\Pi_{-C,C}$ and satisfies $\psi \circ W_0 = C$ on $\mathcal{L}_C$ and $\psi \circ W_0 = -C$ on $\mathcal{L}_{-C}$. It follows from the maximum principle that $\psi(W_0(x, y)) = y$ in $\Pi_{-C,C}$, and therefore $\varphi(W_0(x, y)) = x + c_0$ in $\Pi_{-C,C}$, for some $c_0 \in \mathbb{R}$. Hence, $\omega - c_0$ is the inverse of W_0, and therefore $\omega' \neq 0$ in $\tilde{\Omega}$. This proves the result claimed in remark 2.3.

Suppose now that (Ω, ω) is a solution of problem (B). The preceding considerations show that ω is a conformal mapping from $\tilde{\Omega}$ onto the strip $\Pi_{-C,C}$ and a homeomorphism from the closure of $\tilde{\Omega}$ onto the closure of $\Pi_{-C,C}$. It is immediate that the inverse conformal mapping W satisfies (3.1)–(3.6). Also, an argument similar to that in [11, § 3.5, p. 43] shows that, for any Borel set $\mathcal{A}$ of $\mathbb{R}$, $\mathcal{A}$ has measure zero if and only if $W(\mathcal{A})$ has one-dimensional Hausdorff measure zero on $\mathcal{S}$. Moreover, almost every $t_0 \in \mathbb{R}$ has the following property: a sequence $\{(x_n, y_n)\}_{n \geqslant 1}$ tends to (t_0, C) non-tangentially within $\Pi_{0,C}$ if and only if the sequence $\{W(x_n, y_n)\}_{n \geqslant 1}$ tends to $W(t_0, C)$ non-tangentially within Ω. In view of these facts, (3.7) follows from (2.4). This completes the proof of the fact that W is a solution of problem (P).

We now prove the result claimed in remark 3.3. Let W satisfy (3.1)–(3.4). Let $\mathcal{S} := \{W(t, C) : t \in \mathbb{R}\}$, which by the first part of (3.4) is a non-self-intersecting curve. Let Ω be the domain whose boundary consists of the curve $\mathcal{S}$ and the real axis $\mathcal{B}$, and let $\tilde{\Omega}$ be given by (2.8). The first part of (3.3) ensures, by means of the M. Riesz theorem, that U can be recovered as the Poisson integral of its boundary values in the strip $\Pi_{-C,C}$. We deduce from this that

$$\lim_{x \to \pm\infty} U(x, y) = \pm\infty \quad \text{uniformly in } y \in [-C, C]. \tag{4.10}$$

Consider again the mappings β and α given by (4.8) and (4.9). Let $\sigma := \alpha^{-1} \circ W \circ \beta$. Then σ is holomorphic in D, and the continuity of W in the closure of $\Pi_{-C,C}$ and its behaviour expressed by (4.10) ensure that σ has a continuous extension to the closure of D. Moreover, (3.4), (3.2) and (3.3) show that σ is injective on the boundary of D. It follows from the classical Darboux–Picard theorem (see [7, corollary 9.16, p. 310]) that σ is a conformal bijection from D onto Ξ, where $\Xi := \alpha^{-1}(\tilde{\Omega})$, a subdomain of D whose boundary is a Jordan curve consisting of two arcs which are symmetric about the real axis, contained in D and joining the points -1 and 1. It now follows that W is a conformal mapping from $\Pi_{-C,C}$ onto $\tilde{\Omega}$, and therefore $W' \neq 0$ on $\Pi_{-C,C}$, as required.

Suppose now that W is a solution of problem (P). It follows from (3.5) upon invoking lemma 4.2 that the mapping $\mathbb{R} \ni t \mapsto W(t, C)$ is locally absolutely continuous, and hence $\mathcal{S}$ is a locally rectifiable curve. Therefore, Ω is a strip-like domain. We have already seen that W is a conformal mapping from $\Pi_{-C,C}$ onto $\tilde{\Omega}$ and a homeomorphism from the closure of $\Pi_{-C,C}$ onto the closure of $\tilde{\Omega}$. It is immediate that, if ω is the inverse conformal mapping, then (2.11)–(2.14) and (2.3) hold. The same argument used at the end of the first part of the proof shows that (2.4) follows from (3.7). This completes the proof of the fact that (Ω, ω) is a solution of problem (B).

4.5. Sketch of the proof of theorem 3.5

The proof is based on arguments which are very similar to some used in [20]. Because of theorem 3.4, one can concentrate on proving the regularity of the corresponding solution of problem (P).

We start by deriving some further properties of solutions W of problem (P). Since $W' \neq 0$ in $\Pi_{-C,C}$ and is real valued on the real axis, one can write

$$\log W' = \log |W'| + \mathrm{i}\Theta, \tag{4.11}$$

where Θ is a harmonic function in $\Pi_{-C,C}$, with $\Theta(x, 0) = 0$ for all $x \in \mathbb{R}$. Then

$$\frac{\partial U}{\partial x} = |W'| \cos \Theta, \qquad \frac{\partial V}{\partial x} = |W'| \sin \Theta. \tag{4.12}$$

Since $W' \in \mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C})$ and $1/W' \in \mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C})$, it follows that

$$\log |W'| \in h^p_{\mathbb{C}}(\Pi_{-C,C}) \quad \text{for all } p \in (1, \infty)$$

and hence, by the M. Riesz theorem, $\Theta \in h^p_{\mathbb{C}}(\Pi_{-C,C})$ for all $p \in (1, \infty)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be given, for almost every $t \in \mathbb{R}$, by

$$\theta(t) := \lim_{(x,y) \twoheadrightarrow (t,C)} \Theta(x, y). \tag{4.13}$$

Let us also define, for all $t \in \mathbb{R}$,

$$w(t) = u(t) + \mathrm{i}v(t) := U(t, C) + \mathrm{i}V(t, C). \tag{4.14}$$

Since $W' \in \mathcal{H}^1_{\mathbb{C}}(\Pi_{-C,C})$, it follows from lemma 4.2 that w is locally absolutely continuous. Moreover, taking into account (4.12), we deduce that

$$u' = |w'| \cos \theta \quad \text{and} \quad v' = |w'| \sin \theta \quad \text{almost everywhere.} \tag{4.15}$$

In particular, θ gives the angle between the tangent to the free boundary and the horizontal. The condition (3.7) means that

$$|w'| h(v) = 1 \quad \text{almost everywhere.} \tag{4.16}$$

It follows from (4.15) and (4.16) that

$$u' = \frac{\cos \theta}{h(v)} \quad \text{and} \quad v' = \frac{\sin \theta}{h(v)} \quad \text{almost everywhere.} \tag{4.17}$$

Suppose now that (Ω, ω) is a solution of problem (B) as in theorem 3.5, and let W be the corresponding solution of problem (P). Let $t_1, t_2 \in \mathbb{R}$ be such that

$\omega(Z_j) = t_j + \mathrm{i}C$ for $j = 1, 2$. It follows that $v(t) \in (c, d)$ for all $t \in (t_1, t_2)$. As in [20, proofs of theorems 2.1 and 2.3], a simple bootstrap argument based on the second part of lemma 4.1 and making use of (4.11), (4.12), (4.13), (4.14) and (4.17) yields that $W \in C^{n+1,\alpha}_{\mathrm{loc}}(\Pi_{0,C} \cup \{(t, C) : t \in (t_1, t_2)\})$. The required result is now immediate.

4.6. Proof of theorem 3.6

We make use of the notation and results in the proof of theorem 3.5 concerning the corresponding solution W of problem (P). In particular, let u, v be given by (4.14). Then clearly $\mathcal{S} = \{(u(t), v(t)) : t \in \mathbb{R}\}$. Moreover, as we have seen, u and v are locally absolutely continuous. It remains to prove that (3.9) holds.

Let $\mathcal{N} \subset \mathbb{R}$ be given by

$$\mathcal{N} := \{t \in \mathbb{R} : h(v(t)) = 0\} = \{t \in \mathbb{R} : v(t) = d\}.$$

In the terminology of [16, 20], $\mathcal{N}$ is the set of stagnation points. Obviously, $\mathcal{N}$ is a closed set and, in view of (4.16), has measure zero. By the proof of theorem 3.5, $W \in C^{2,\alpha}_{\mathrm{loc}}(\Pi_{0,C} \cup \{(t, C) : t \in \mathbb{R} \setminus \mathcal{N}\})$.

Note now that (4.16) can be rewritten as

$$-\log|w'| = \log h(v) \quad \text{almost everywhere.} \tag{4.18}$$

Let $Q : \Pi_{0,C} \to \mathbb{R}$ be given by

$$Q(x, y) = -\log|W'(x + \mathrm{i}y)| - \log h(V(x, y)) \quad \text{for all } (x, y) \in \Pi_{0,C}. \tag{4.19}$$

The concavity of $\log h$ ensures that Q is a subharmonic function in $\Pi_{0,C}$. Also, it follows from (3.2) and (4.18) that, in the notation of (4.5),

$$\begin{aligned} Q &= \mathcal{P}_{\mathrm{e}}[-\log|w'|] - \log h(\mathcal{P}_{\mathrm{o}}[v]) \\ &= \mathcal{P}_{\mathrm{e}}[\log h(v)] - \log h(\mathcal{P}_{\mathrm{o}}[v]). \end{aligned} \tag{4.20}$$

But, since $v > 0$, it follows that

$$\mathcal{P}_{\mathrm{e}}[v] > \mathcal{P}_{\mathrm{o}}[v] > 0.$$

We deduce from this and (3.8 c) by an obvious application of Jensen's inequality [15, theorem 3.3, p. 62] that

$$-\log h(\mathcal{P}_{\mathrm{o}}[v]) < -\log h(\mathcal{P}_{\mathrm{e}}[v]) \leqslant \mathcal{P}_{\mathrm{e}}[-\log h(v)].$$

Now (4.20) shows that $Q < 0$ in $\Pi_{0,C}$. Since Q is subharmonic in $\Pi_{0,C}$, $Q \in C^1(\Pi_{0,C} \cup \{(x, C) : x \in \mathbb{R} \setminus \mathcal{N}\})$ and $Q = 0$ in $\{(x, C) : x \in \mathbb{R} \setminus \mathcal{N}\}$, it follows from the Hopf boundary-point lemma that

$$\frac{\partial Q}{\partial y}(x, C) > 0 \quad \text{for all } x \in \mathbb{R} \setminus \mathcal{N}.$$

This means, upon using (4.19) and the Cauchy–Riemann equations, that, for all $t \in \mathbb{R} \setminus \mathcal{N}$,

$$\theta'(t) - \frac{h'(v(t))}{h^2(v(t))} \cos\theta(t) > 0. \tag{4.21}$$

We aim to prove that

$$\cos\theta(t) > 0 \quad \text{for all } t \in \mathbb{R} \setminus \mathcal{N}, \tag{4.22}$$

as this would yield (3.9).

We argue by contradiction and assume that there exists $f \in \mathbb{R} \setminus \mathcal{N}$ such that $\cos\theta(f) \leqslant 0$. We distinguish the following four cases:

(i) $\mathcal{N} = \emptyset$;

(ii) $\mathcal{N} \neq \emptyset$ and $f \in (a, b) \subset \mathbb{R} \setminus \mathcal{N}$, where $a, b \in \mathcal{N}$;

(iii) $\mathcal{N} \neq \emptyset$ and $f \in (a, \infty) \subset \mathbb{R} \setminus \mathcal{N}$, where $a \in \mathcal{N}$;

(iv) $\mathcal{N} \neq \emptyset$ and $f \in (-\infty, b) \subset \mathbb{R} \setminus \mathcal{N}$, where $b \in \mathcal{N}$.

We aim to show that a contradiction is reached in each of these cases. We give a full treatment only of case (i), and point out the necessary modifications of the argument to deal with the remaining cases.

As in [21], the proof in all four cases here ultimately rests on an application of the following classical theorem in the global differential geometry of plane curves (see, for example, [2, theorem 24.15, p. 340] and the references therein) to a suitably devised Jordan curve.

THEOREM 4.3. *Let $\sigma : [a, b] \to \mathbb{C}$ be a parametrization of a Jordan curve, where σ is a function of class C^1, with $\sigma'(a) = \sigma'(b)$ and $|\sigma'| > 0$ on $[a, b]$. Let $\phi : [a, b] \to \mathbb{R}$ be a continuous function such that*

$$\sigma'(t) = |\sigma'(t)| \exp\{\mathrm{i}\phi(t)\} \quad \textit{for all } t \in [a, b].$$

Then $\phi(b) - \phi(a)$ equals either 2π or -2π.

We also use the following lemmas [16, proofs of lemmas 4.16 and 4.17].

LEMMA 4.4. *Let $(a, b) \subset \mathbb{R} \backslash \mathcal{N}$ and suppose that (4.21) holds on (a, b). Let $e \in (a, b)$. If $\ell_1, \ell_2 \in \mathbb{Z}$ are such that $\ell_1\pi + \frac{1}{2}\pi \leqslant \theta(e) \leqslant \ell_2\pi + \frac{1}{2}\pi$, then*

$$\theta(t) > \ell_1\pi + \tfrac{1}{2}\pi \quad \textit{for all } t \in (e, b),$$
$$\theta(t) < \ell_2\pi + \tfrac{1}{2}\pi \quad \textit{for all } t \in (a, e).$$

LEMMA 4.5. *Let $(a, b) \subset \mathbb{R} \setminus \mathcal{N}$, where $a \in \mathcal{N} \cup \{-\infty\}$ and $b \in \mathcal{N} \cup \{+\infty\}$, and suppose that (4.21) holds on (a, b). Let $f \in (a, b)$ be such that $\cos\theta(f) \leqslant 0$. If $m \in \mathbb{Z}$ is such that*

$$2m\pi + \tfrac{1}{2}\pi \leqslant \theta(f) \leqslant 2m\pi + \tfrac{3}{2}\pi,$$

then there exists $g_1, g_2 \in (a, b)$ with $g_1 \leqslant f \leqslant g_2$ such that θ is strictly increasing on $[g_1, g_2]$ and

$$\theta(g_1) = 2m\pi + \tfrac{1}{2}\pi \quad \textit{and} \quad \theta(g_2) = 2m\pi + \tfrac{3}{2}\pi.$$

Proof of lemma 4.5. We prove only the existence of g_2 with the required properties, since the proof for g_1 is entirely similar.

When $b \in \mathcal{N}$, the existence of g_2 follows by the argument in [16, proof of lemma 4.17].

Suppose now that $b = +\infty$. Note from (4.21) that θ is strictly increasing on any interval on which $\cos\theta \leqslant 0$. If such g_2 does not exist, then obviously $\cos\theta < 0$ on (f, ∞), and this implies that $u' < 0$ on (f, ∞), which contradicts the fact that $\lim_{t\to\infty} u(t) = +\infty$. This proves the existence of g_2. □

In what follows we deal with case (i), thus assuming that $\mathcal{N} = \emptyset$ and that there exists $f \in \mathbb{R}$ such that $\cos\theta(f) \leqslant 0$. It follows that

$$\theta(f) \in [2m\pi + \tfrac{1}{2}\pi, 2m\pi + \tfrac{3}{2}\pi] \quad \text{for some } m \in \mathbb{Z}. \tag{4.23}$$

Let g_1, g_2 be given by lemma 4.5, so that

$$\theta(g_1) = 2m\pi + \tfrac{1}{2}\pi, \qquad \theta(g_2) = 2m\pi + \tfrac{3}{2}\pi. \tag{4.24}$$

Let $M_1, M_2 \in \mathbb{R}$ be such that

$$M_1 < \min\{u(s) : s \in [g_1, g_2]\}, \qquad M_2 > \max\{u(s) : s \in [g_1, g_2]\}.$$

Let p_1, p_2 be such that $p_1 < g_1$, $p_2 > g_2$ and

$$u(p_1) = M_1, \quad u(p_2) = M_2, \quad M_1 < u(s) < M_2 \quad \text{for all } s \in (p_1, p_2).$$

Then

$$u'(p_1) \geqslant 0, \qquad u'(p_2) \geqslant 0,$$

so there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$\theta(p_1) \in [2m_1\pi - \tfrac{1}{2}\pi, 2m_1\pi + \tfrac{1}{2}\pi], \qquad \theta(p_2) \in [2m_2\pi - \tfrac{1}{2}\pi, 2m_2\pi + \tfrac{1}{2}\pi]. \tag{4.25}$$

It follows from (4.24), (4.25) and lemma 4.4 that

$$m_1 \leqslant m, \qquad m_2 \geqslant m + 1. \tag{4.26}$$

Let $\tilde{w} : [p_1, p_2] \to \mathbb{C}$ be the restriction of w to $[p_1, p_2]$. It is obvious that, for some $q_1, q_2 \in \mathbb{R}$ with $q_1 < p_1$, $p_2 < q_2$, one can construct a function $\hat{w} : [q_1, q_2] \to \mathbb{C}$, where

$$\hat{w}(q) := \hat{u}(q) + \mathrm{i}\hat{v}(q) \quad \text{for all } q \in [q_1, q_2],$$

such that $\hat{w}$ is an extension of $\tilde{w}$, and it has the following additional properties:

$$\hat{u}, \hat{v} : [q_1, q_2] \to \mathbb{R} \quad \text{are of class } C^1; \tag{4.27 a}$$

$$\hat{u}'(q)^2 + \hat{v}'(q)^2 > 0 \quad \text{for all } q \in [q_1, q_2]; \tag{4.27 b}$$

$$\hat{u}' \geqslant 0 \quad \text{on } [q_1, p_1] \cup [p_2, q_2]; \tag{4.27 c}$$

$$\hat{u}'(q_1) = \hat{u}'(q_2) = 1; \tag{4.27 d}$$

$$\hat{v}'(q_1) = \hat{v}'(q_2) = 0. \tag{4.27 e}$$

Let E be such that

$$E > \max\{\hat{v}(q) : q \in [q_1, q_2]\}.$$

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

Let $\mathcal{A}_1$ be the semicircle having as a diameter the segment joining the points $(\hat{u}(q_1), E)$ and $(\hat{u}(q_1), \hat{v}(q_1))$, and situated to the left of this segment. Let $\mathcal{A}_2$ be the semicircle having as a diameter the segment joining the points $(\hat{u}(q_2), \hat{v}(q_2))$ and $(\hat{u}(q_2), E)$, and situated to the right of this segment. Let us choose some $r_1, r_2 \in \mathbb{R}$ with $r_1 < q_1$, $r_2 > q_2$ and let us consider a C^1 function $w_* : [r_1, r_2] \to \mathbb{C}$ which is an extension of $\hat{w}$, such that

$$w_*(r_1) = \hat{u}(q_1) + \mathrm{i}E, \qquad w_*(r_2) = \hat{u}(q_2) + \mathrm{i}E, \tag{4.28 a}$$

$$w_*'(r_1) = w_*'(r_2) = -1, \qquad |w_*'(r)| > 0 \quad \text{for all } r \in [r_1, r_2], \tag{4.28 b}$$

$$w_*|_{[r_1,q_1]} \quad \text{is an injective parametrization of } \mathcal{A}_1, \tag{4.28 c}$$

$$w_*|_{[q_2,r_2]} \quad \text{is an injective parametrization of } \mathcal{A}_2. \tag{4.28 d}$$

Let us define $\tilde{r}_2 := r_2$ and let us choose $\tilde{r}_1$ with $\tilde{r}_1 < r_1$ so that w_* has an extension to $[\tilde{r}_1, \tilde{r}_2]$ as a C^1 function such that

$$w_*(\tilde{r}_1) = \hat{u}(q_2) + \mathrm{i}E,$$
$$w_*'(r) = -1 \quad \text{for all } r \in [\tilde{r}_1, r_1].$$

It is very easy to prove that $w_* : [\tilde{r}_1, \tilde{r}_2] \to \mathbb{C}$, constructed above, provides a parametrization of a Jordan curve with a continuously varying tangent. Let us write

$$w_*'(r) = |w_*'(r)| \exp\{\mathrm{i}\theta_*(r)\} \quad \text{for all } r \in [\tilde{r}_1, \tilde{r}_2],$$

where $\theta_* : [\tilde{r}_1, \tilde{r}_2] \to \mathbb{R}$ is a continuous function which extends $\theta : [p_1, p_2] \to \mathbb{R}$. It follows from (4.25) and (4.27) that

$$\theta_*(q_1) = 2m_1\pi, \qquad \theta_*(q_2) = 2m_2\pi.$$

Using (4.28) we deduce that

$$\theta_*(\tilde{r}_1) = 2m_1\pi - \pi, \qquad \theta_*(\tilde{r}_2) = 2m_2\pi + \pi.$$

Therefore,

$$\theta_*(\tilde{r}_2) - \theta_*(\tilde{r}_1) = 2(m_2 - m_1)\pi + 2\pi, \tag{4.29}$$

where, by (4.26),

$$m_2 - m_1 \geqslant 1. \tag{4.30}$$

But the validity of (4.29) with (4.30) contradicts theorem 4.3. This proves the required result (4.22) in case (i).

Case (ii) can be dealt with by an argument which is entirely similar to that in [21, proof of theorem 3.3]. Cases (iii) and (iv) can be treated using a geometric construction which combines elements of those used for cases (i) and (ii). This completes the proof of theorem 3.6.

Acknowledgments

The author is very grateful to Professor J. F. Toland for many valuable discussions on the theory of water waves. This work was supported by EPSRC Grant no. EP/D505402/1.

https://doi.org/10.1017/S0308210507000522 Published online by Cambridge University Press

References

1 H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), 105–144.

2 H. Amann. *Ordinary differential equations* (Berlin: Walter de Gruyter, 1990).

3 C. J. Amick and J. F. Toland. On solitary water waves of finite amplitude. *Arch. Ration. Mech. Analysis* **76** (1981), 9–95.

4 C. J. Amick and J. F. Toland. On periodic water waves and their convergence to solitary waves in the long-wave limit. *Phil. Trans. R. Soc. Lond.* A **303** (1981), 633–669.

5 C. J. Amick, L. E. Fraenkel and J. F. Toland. On the Stokes conjecture for the wave of extreme form. *Acta Math.* **148** (1982), 193–214.

6 A. Bakan and S. Kaijser. Hardy spaces for the strip. *J. Math. Analysis Applic.* **333** (2007), 347–364.

7 R. B. Burckel. *An introduction to classical complex analysis*, vol. 1 (Academic, 1979).

8 A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Commun. Pure Appl. Math.* **57** (2004), 481–527.

9 W. Craig and P. Sternberg. Symmetry of solitary waves. *Commun. PDEs* **13** (1988), 603–633.

10 I. I. Daniljuk. On integral functionals with a variable domain of integration. In *Proceedings of the Steklov Institute of Mathematics*, vol. 118 (Moscow: Nauka, 1972). (In Russian.) (English translation Providence, RI: American Mathematical Society, 1976.)

11 P. L. Duren. *Theory of H^p spaces* (Mineola, NY: Dover, 2000).

12 M. Flucher and M. Rumpf. Bernoulli's free-boundary problem, qualitative theory and numerical approximation. *J. Reine Angew. Math.* **486** (1997), 165–204.

13 P. Koosis. *Introduction to H_p spaces*, 2nd edn (Cambridge University Press, 1999).

14 P. I. Plotnikov. Proof of the Stokes conjecture in the theory of surface waves. *Dinamika Splosh. Sredy* **57** (1982), 41–76. (In Russian.) (English translation *Stud. Appl. Math.* **108** (2002), 217–244.)

15 W. Rudin. *Real and complex analysis*, 3rd edn (McGraw-Hill, 1986).

16 E. Shargorodsky and J. F. Toland. *Bernoulli free-boundary problems*, Memoirs of the American Mathematical Society, vol. 196 (Providence, RI: American Mathematical Society, 2008).

17 E. R. Spielvogel. A variational principle for waves of infinite depth. *Arch. Ration. Mech. Analysis* **39** (1970), 189–205.

18 J. F. Toland. Stokes waves. *Topolog. Meth. Nonlin. Analysis* **7** (1996), 1–48. (Errata. *Topolog. Meth. Nonlin. Analysis* **8** (1997), 412–414.)

19 J. F. Toland. On a pseudo-differential equation for Stokes waves. *Arch. Ration. Mech. Analysis* **162** (2002), 179–189.

20 E. Varvaruca. Singularities of Bernoulli free boundaries. *Commun. PDEs* **31** (2006), 1451–1477.

21 E. Varvaruca. Some geometric and analytic properties of solutions of Bernoulli free-boundary problems. *Interfaces Free Boundaries* **9** (2007), 367–381.

22 D. V. Widder. Functions harmonic in a strip. *Proc. Am. Math. Soc.* **12** (1961), 67–72.

(*Issued 5 December 2008*)