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# The interaction of skewed vortex pairs: a model for blow-up of the Navier–Stokes equations

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The interaction of two propagating vortex pairs is considered, each pair being initially aligned along the positive principal axis of strain associated with the other. As a preliminary, the action of accelerating strain on a Burgers vortex is considered and the conditions for a finite-time singularity (or 'blow-up') are determined. The asymptotic high Reynolds number behaviour of such a vortex under non-axisymmetric strain, and the corresponding behaviour of a vortex pair, are described. This leads naturally to consideration of the interaction of the two vortex pairs, and identifies a mechanism by which blow-up may occur through self-similar evolution in an interaction zone where scale decreases in proportion to $(t^* - t)^{1/2}$, where $t^*$ is the singularity time. The relevance of Leray scaling in this interaction zone is discussed.

## 1. Introduction

This paper contributes to the continuing debate concerning the possibility of 'blow-up' of solutions of the Euler and/or Navier–Stokes equations, i.e. the development of a singularity of vorticity at finite time, starting from smooth initial conditions. This problem, relating as it does to the regularity of solutions of these equations, is of fundamental importance for fluid dynamics. It assumes particular prominence in the context of turbulence, for if the development of a singularity at finite time is a generic feature of any fully three-dimensional time-dependent flow, then the spatial structure of this singularity may be expected to have an important bearing on, if not to govern, the well-known intermittency of turbulent flow.

The problem of blow-up was first identified by Leray (1934), who recognised the important possibility of self-similar collapse towards a singularity at time $t^*$ say, with all length scales decreasing in proportion to $(t^* - t)^{1/2}$, and velocity in the neighbourhood of the singular point increasing like $(t^* - t)^{-1/2}$. Leray however admitted his inability to make progress in establishing the existence of any solution of the reduced Navier–Stokes equation (or 'Leray' equation) having finite total energy. We shall in this paper encounter the same equation (see § 7), but all the arguments that precede this will suggest that what we should look for is a solution of the Leray equation valid in an inner region and matching in an appropriate way to a non-singular solution of the Navier–Stokes equation in an outer region. It is interesting to note that recent numerical work involving the interaction of vortices in various configurations provides accumulating evidence for the validity of the Leray scaling. This is particularly clear in the recent important paper of Pelz (1997) who

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considers the implosion of six vortex pairs in a configuration with cubic symmetry. The numerical scheme, employing a vortex filament model, is relevant primarily to the Euler equations (as in previous investigations of Pumir & Siggia 1990; Kerr 1993; Boratav & Pelz 1994; and others). Evidence for the Leray scaling appears quite compelling in this study. The associated vorticity field $\boldsymbol{\omega}(\boldsymbol{x}, t)$ scales like $(t^* - t)^{-1}$ near the point of singularity, a behaviour that is compatible with the theorem of Beale, Kato & Majda (1984) which states that if a singularity of the Euler equations occurs, then the time-integral of the maximum of $|\boldsymbol{\omega}|$ diverges as $t \uparrow t^*$. Conservation of circulation implies that any vortex core radius decreases to zero like $(t^* - t)^{1/2}$, a behaviour associated with an axial strain that scales like $(t^* - t)^{-1}$ (like the vorticity).

A singularity of vorticity can be best visualized as the result of vortex stretching in which the applied rate of strain $\gamma$ keeps in step (in some mysterious way) with the maximum vorticity $\omega_m$; in simple terms, we then have

$$\frac{\mathrm{d}\omega_m}{\mathrm{d}t} = \gamma\omega_m, \quad \gamma = A\omega_m, \tag{1.1}$$

where $A$ is a positive constant. This integrates simply to give

$$\omega_m(t) = \frac{1}{A(t^* - t)} \tag{1.2}$$

where

$$t^* = (A\omega_m(0))^{-1}. \tag{1.3}$$

The key question here however is precisely how it is that $\gamma$ can keep in step with $\omega_m$. If we think in terms of a Burgers-type stretched vortex, then the strain rate $\gamma$ is produced by other vortices in the turbulent flow and is presumably determined by their circulations and their geometrical distribution relative to the stretched vortex on which we focus. The circulations are constant in Euler flow (and, if anything, decreasing through viscous reconnection processes in Navier–Stokes flow); so it is to the changing geometrical structure that we must look in order to identify the required mechanism.

We start this investigation by revisiting (in § 2) the Burgers vortex, but allowing the applied strain field $\gamma(t)$ to increase in step with $\omega_m$ in the manner indicated in (1.1). It is perhaps obvious that, if the fluid is inviscid, then the vorticity will become singular (cf. (1.2)) at finite time $t^*$; but, what is perhaps less obvious, we find that viscosity cannot prevent the blow-up process, despite the decreasing length scale (to zero) of the vortex core, provided merely that the initial conditions are such that $\gamma(0)$ exceeds a critical value $\gamma_s$ equal to the strain rate required to maintain a steady Burgers vortex at the initial core radius $\delta_0$.

The treatment of § 2 involves an axisymmetric strain field, for which the solution of the Navier–Stokes equation is exact. The main model of vortex interaction developed in § 6 of the paper requires a modification of this solution in which the strain field is locally two-dimensional rather than axisymmetric, with positive axis of strain aligned along the vortex. An exact solution of the Navier–Stokes equation is no longer available in this situation; however a high Reynolds number asymptotic solution may be obtained by the technique of Moffatt, Kida & Ohkitani (1994, hereafter referred to as MKO'94). This is obtained in § 3, and in § 4 we discuss the remarkable fact that the solution applies equally to a strained vortex pair (and indeed to any other distribution of vortices), under circumstances in which the ratio of core size to vortex separation either tends to zero or remains constant as $t \rightarrow t^*$. In § 5, we describe, as a necessary preliminary to § 6, the structure of the strain field associated with a vortex pair $\pm\Gamma$ located instantaneously at $(0, \pm b_0)$. This is an elementary calculation, which nevertheless presents some interesting features.

The model developed in § 6 builds on the insights developed in the preceding sections. We consider the problem of two vortex pairs, one aligned with the $z$-axis, the other with the $y$-axis, and propagating towards each other in the $\pm x$-directions (see figure 2*a*). Here, 'vortex pair' can mean either a well-separated pair of vortices of strengths $\pm\Gamma$ with diffusing vortex cores; or it may mean any other nearly inviscid two-dimensional vorticity distribution of initial (and subsequently conserved) momentum $2\Gamma b_0$ per unit length. In either case, it is clear that a strong interaction must develop between the vortex pairs as they approach the plane of (skew) symmetry $x = 0$. Each vortex pair is initially aligned along the principal axis of positive rate of strain induced by the other pair; hence conditions are propitious for the mutual intensification of vorticity at a rate which increases because the minimum separation of the vortex pairs decreases. If a self-similar evolution is established in the interaction zone, in which all length scales decrease in proportion to this minimum separation, then the analysis of the 'singularly stretched' vortex pairs developed in the previous sections is applicable, indicating the manner in which a singularity of vorticity may appear at $\boldsymbol{x} = 0$ at finite time $t = t^*$. The key question that remains is: is the assumption of self-similarity in some inner interaction zone valid? This brings us back (in § 7 and § 8) to the Leray (1934) transformation which, applied to vorticity, takes the form

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \frac{1}{t^* - t}\boldsymbol{\Omega}(\boldsymbol{X}), \tag{1.4}$$

where

$$\boldsymbol{X} = \boldsymbol{x}/(\Gamma(t^* - t))^{1/2}, \tag{1.5}$$

and where $\Gamma$ is some measure of (conserved) circulation. This type of solution of the Navier–Stokes equation, if it exists, can be valid only in an inner region where $|\boldsymbol{X}| = O(1)$, and must match to an outer non-singular solution (e.g. that describing the vortex pairs of § 6) as $|\boldsymbol{X}| \to \infty$. The description is entirely compatible with the model of Pelz (1997), and suggests moreover that if a singularity of the above type forms under Euler evolution, then this type of blow-up will not be prevented by the inclusion of weak viscosity.

## 2. A singularly stretched vortex

Consider first the following simple modification of the familiar stretched Burger's vortex, illustrating one possible mechanism through which a singularity of vorticity may appear within a finite time. Using cylindrical polar coordinates $(r, \theta, z)$, suppose that the unsteady axisymmetric strain field

$$\boldsymbol{U} = (-\tfrac{1}{2}\gamma(t)r,\ 0,\ \gamma(t)z), \tag{2.1}$$

with $\gamma(t) > 0$, acts upon a vorticity distribution aligned with the $z$-axis,

$$\boldsymbol{\omega} = (0, 0, \omega(r, t)). \tag{2.2}$$

Such a (uniform) strain field of course has infinite energy; we must regard this merely as an idealized model which provides a convenient starting point for subsequent discussion. The additional velocity associated with (2.2) is

$$\boldsymbol{u} = (0, v(r, t), 0) \tag{2.3}$$

where

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) = \omega. \tag{2.4}$$

Noting that $\nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{\omega}) = 0$, the exact vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\boldsymbol{U} \wedge \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}, \tag{2.5}$$

with $z$-component

$$\frac{\partial \omega}{\partial t} = \frac{\gamma(t)}{2r}\frac{\partial}{\partial r}(r^2\omega) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right). \tag{2.6}$$

Let us suppose that at time $t = 0$,

$$\omega(r, 0) = \omega_0 \exp(-r^2/\delta_0^2), \tag{2.7}$$

i.e. the vorticity profile is Gaussian with radial scale $\delta_0$ and circulation

$$\Gamma = \pi \omega_0 \delta_0^2. \tag{2.8}$$

The steady strain rate $\gamma_s$ that would be required to maintain such a vortex against viscous erosion is

$$\gamma_s = 4\nu/\delta_0^2. \tag{2.9}$$

We envisage a situation in which $\omega(r, t)$ becomes singular at some finite time $t = t^* > 0$. To achieve this, it is evident that $\gamma(t)$ must become singular at this time also; we suppose that, through some mechanism, $\gamma(t)$ is prescribed in the form

$$\gamma(t) = c(t^* - t)^{-1}, \quad 0 < t < t^*. \tag{2.10}$$

We may then seek a similarity solution of (2.6) of the form

$$\omega(r, t) = \frac{\Gamma}{\nu(t^* - t)} f(\eta), \quad \eta = \frac{r}{(\nu(t^* - t))^{1/2}}. \tag{2.11}$$

No other power law for $\gamma(t)$ is compatible with a similarity solution, as may be easily checked. The power law (2.10) is distinguished in that the imposed strain becomes singular as $t \to t^*$ in the same manner as the peak vorticity $\omega(0, t)$ (cf. (1.1), (1.2)). (As pointed out by a referee, a wide family of singular solutions may be generated by means of Lundgren's 1982 transformation; however, only one of these, namely the one presented here, has a self-similar power-law form with the above distinguishing property.)

Substitution of (2.11) in (2.6) yields the ordinary differential equation

$$\frac{1}{2\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta^2 f) = \frac{c}{2\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta^2 f) + \frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right); \tag{2.12}$$

and on integration, we find a solution, finite at $r = 0$ (i.e. $\eta = 0$), in the form

$$f(\eta) = \frac{c-1}{4\pi}\exp\left(-\tfrac{1}{4}(c-1)\eta^2\right), \tag{2.13}$$

the coefficient being determined from the requirement that the total circulation remain equal to $\Gamma$. The solution for $\omega(r, t)$ is then from (2.11)

$$\omega(r, t) = \frac{(c-1)\Gamma}{4\pi\nu(t^* - t)}\exp\left(\frac{-(c-1)r^2}{4\nu(t^* - t)}\right). \tag{2.14}$$

The initial condition (2.7) is satisfied provided

$$c - 1 = 4\nu t^*/\delta_0^2. \tag{2.15}$$

Thus a solution of the form (2.11) (with $t^* > 0$) exists only if

$$c > 1, \tag{2.16}$$

and, under this condition, $\omega(r,t)$ becomes singular on the $z$-axis ($r = 0$) as $t \uparrow t^*$. The strain rate is given by

$$\gamma(t) = \frac{c}{t^* - t} = \frac{1 + \gamma_s t^*}{t^* - t}, \tag{2.17}$$

so that

$$\gamma(0) = \gamma_s + (t^*)^{-1}. \tag{2.18}$$

Thus, in order to provoke a singularity at $t = t^*$ with strain rate of the form (2.10), it is necessary that $\gamma(0) > \gamma_s$; and the singularity time is then given by

$$t^* = (\gamma(0) - \gamma_s)^{-1}. \tag{2.19}$$

It is important to observe here that, although a singularity in the strain field is imposed through the assumption (2.10), it is not inevitable that there is a corresponding singularity of vorticity; for this, the additional condition $c > 1$ must be satisfied.

Note that the velocity $v(r,t)\boldsymbol{e}_\theta$ associated with the vortex is given, from (2.10), by

$$v(r,t) = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(\frac{-(c-1)r^2}{4\nu(t^* - t)}\right)\right), \tag{2.20}$$

and that the maximum value of $v(r,t)$ at time $t$ is of order

$$v_m(t) = \frac{\Gamma}{4\pi}\left[\frac{c-1}{\nu(t^* - t)}\right]^{1/2}. \tag{2.21}$$

The ratio of strain rate to maximum vorticity is

$$\frac{\gamma(t)}{\omega(0,t)} = \frac{4\pi c\nu}{(c-1)\Gamma} \tag{2.22}$$

and is constant for $0 < t < t^*$.

Again, we emphasize that this property arises only for the special choice (2.10) of strain-rate time-dependence.

### 2.1. *The inviscid limit*

The similarity assumption (2.11) is clearly inappropriate in the inviscid limit $\nu = 0$. However there is an interesting parallel behaviour which can be obtained directly from (2.6) with $\nu = 0$. If $\gamma(t)$ is given by (2.10), then the solution of (2.6) with initial condition (2.7) is

$$\omega(r,t) = \left(\frac{t^*}{t^* - t}\right)^c \frac{\Gamma}{\pi\delta_0^2} \exp\left\{-\frac{r^2}{\delta_0^2}\left(\frac{t^*}{t^* - t}\right)^c\right\} \tag{2.23}$$

as may be easily verified. In this situation, if $\omega$ is to become singular in the same way as $\gamma$ (i.e. $\sim (t_0 - t)^{-1}$), then necessarily

$$c = 1, \tag{2.24}$$

and so

$$\omega(r,t) = \frac{t^*}{t^*-t}\frac{\Gamma}{\pi\delta_0^2}\exp\left\{-\frac{r^2}{\delta_0^2}\frac{t^*}{t^*-t}\right\}, \tag{2.25}$$

which is formally identical to (2.14) if we use (2.15) to replace $(c-1)/4\nu$ by $t^*/\delta_0^2$. Thus, the inviscid-limit solution is the natural limit of the viscous solution, exhibiting exactly the same type of singularity as $t \uparrow t^*$. The inviscid limit is peculiar however in that (2.25) is a solution of the inviscid equation

$$\frac{\partial\omega}{\partial t} = \frac{1}{2r(t^*-t)}\frac{\partial}{\partial r}(r^2\omega) \tag{2.26}$$

with initial condition (2.7), for arbitrary $t^*$; this is because (2.26) is invariant under the time dilatation $t \to at$, $t^* \to at^*$ for arbitrary $a$.

## 3. Effect of non-axisymmetric strain

It will be necessary, in what follows, to consider the modification of the above exact solution of the Navier–Stokes equations when, instead of (2.1), the imposed strain field is non-axisymmetric and given (in cylindrical polar coordinates) by

$$\boldsymbol{U} = \gamma(t)(-\tfrac{1}{2}r(1+\lambda\cos 2\theta), \tfrac{1}{2}\lambda r\sin 2\theta, z), \tag{3.1}$$

where $\gamma(t)$ is still given by (2.10), and $\lambda$ is a constant in the range $0 < \lambda < 1$; if $\lambda = 0$, (3.1) reduces to (2.1), while if $\lambda = 1$, (3.1) is a two-dimensional strain field. Note that the Cartesian form of (3.1) is $\boldsymbol{U} = \gamma(t)\left(-\tfrac{1}{2}(1+\lambda)x, -\tfrac{1}{2}(1-\lambda)y, z\right)$.

We shall suppose that the vortex Reynolds number $Re = \Gamma/\nu$ is large, or equivalently that

$$\epsilon \equiv Re^{-1} = \nu/\Gamma \ll 1. \tag{3.2}$$

The asymptotic technique developed by MKO'94 is then applicable with slight modification as follows. With vorticity field now of the form

$$\boldsymbol{\omega} = (0, 0, \omega(r,\theta,t)), \quad \omega = -\nabla^2\psi(r,\theta,t), \tag{3.3}$$

the exact vorticity equation is

$$\frac{\partial\omega}{\partial t} - \frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(r,\theta)} = \frac{\gamma(t)}{2r}\frac{\partial}{\partial r}(r^2\omega) + \tfrac{1}{2}\lambda\gamma(t)\left[\cos 2\theta\, r\frac{\partial}{\partial r} - \sin 2\theta\frac{\partial}{\partial\theta}\right]\omega \tag{3.4}$$

$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2}\right]. \tag{3.5}$$

We introduce dimensionless variables

$$t' = t/t^*, \quad r' = r(\nu t^*)^{-1/2}, \quad \psi' = \psi/\Gamma, \quad \omega' = \omega(\nu t^*/\Gamma). \tag{3.6}$$

Substituting in (3.5) and dropping primes, we obtain

$$\frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(r,\theta)} = -\epsilon L_0\omega - \epsilon\lambda L_1\omega, \tag{3.7}$$

where

$$L_0\omega = -\frac{\partial\omega}{\partial t} + \frac{c}{2(1-t)}\frac{1}{r}\frac{\partial}{\partial r}(r^2\omega) + \nabla^2\omega, \tag{3.8}$$

$$L_1\omega = \frac{c}{2(1-t)}\left(\cos 2\theta\, r\frac{\partial}{\partial r} - \sin 2\theta\frac{\partial}{\partial\theta}\right)\omega. \tag{3.9}$$

Note that, with the scaling (3.5), the term $\partial\omega/\partial t$ appears among the $O(\epsilon)$ terms on the right-hand side of (3.7).

As in MKO'94, we may now seek a vortex-type asymptotic solution of (3.7) in the form

$$\psi = \psi_0(r,t) + \epsilon\psi_1(r,\theta,t) + \epsilon^2\psi_2(r,\theta,t) + \cdots. \tag{3.10}$$

The $O(\epsilon^0)$ term in (3.5) then vanishes, and at $O(\epsilon)$ we have

$$\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial\psi_0}{\partial r}\omega_1 - \frac{1}{r}\frac{\partial\omega_0}{\partial r}\psi_1\right] = -L_0\omega_0 - \frac{\lambda c}{2(1-t)}(\cos 2\theta)r\frac{\partial\omega_0}{\partial r}. \tag{3.11}$$

The solvability condition for this equation, obtained by integrating over $\theta(0 \text{ to } 2\pi)$, is

$$L_0\omega_0 = 0. \tag{3.12}$$

This is simply a scaled version of (2.6), and has solution (2.14) which in the present dimensionless form becomes

$$\omega_0(r,t) = \frac{c-1}{4\pi(1-t)}\exp\left[-\frac{(c-1)r^2}{4(1-t)}\right]. \tag{3.13}$$

The first-order perturbation $\psi_1(r,\theta,t)$ (and hence $\omega_1(r,\theta,t)$) may now be obtained from (3.11) exactly as in MKO'94, $t$ appearing merely as a parameter. The time-dependence of the solution is determined at the leading order (3.13), and the non-axisymmetric part is generated in a quasi-static manner at order $\epsilon$ (and higher orders). We need not pursue the details of these higher-order terms here.

The above procedure clearly requires that $\epsilon > 0$, but $\epsilon$ may be arbitrarily small; it also requires that $c > 1$, as is evident from the leading-order solution (3.13).

## 4. Singular straining of a vortex pair

Consider now a vortex pair $\pm\Gamma$ placed at $t = 0$ at positions $x = 0$, $y = \pm b$ respectively and subjected to the strain field (3.1). We shall suppose below that each vortex core is of scale $\delta \ll b$ and subject to viscous diffusion. However let us first consider the movement of the vortex pair on the assumption that each is a point vortex. The vortex trajectories are evidently $x = X(t)$, $y = \pm Y(t)$ where

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\Gamma}{4\pi Y} - \tfrac{1}{2}\gamma(t)(1+\lambda)X, \tag{4.1}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\tfrac{1}{2}\gamma(t)(1-\lambda)Y, \tag{4.2}$$

the term $\Gamma/4\pi Y$ in (4.1) arising from the vortex interaction. We easily obtain

$$Y(t) = b(1 - t/t^*)^{(1-\lambda)c/2}, \tag{4.3}$$

$$X(t) = \frac{\Gamma t^{*\,1+(1-\lambda)c}}{2\pi b(\lambda c - 1)}\left[\left(1-\frac{t}{t^*}\right)^{1+(1-\lambda)c/2} - \left(1-\frac{t}{t^*}\right)^{(1+\lambda)c/2}\right]. \tag{4.4}$$

If $0 < \lambda < 1$, then evidently

$$Y(t) \to 0, \quad X(t) \to 0 \quad \text{as } t \to t^*. \tag{4.5}$$

The self-induced motion of the vortices first takes them away from the plane $x = 0$, but the $x$-component of the strain field ultimately drives the pair back towards the origin. The important thing to note is the manner in which $Y(t)$ tends to zero as $t \to t^*$. Compare this with the behaviour

$$\delta(t) = \left[\frac{4\nu(t^* - t)}{c-1}\right]^{1/2} \tag{4.6}$$

for the radius of the vortex core given by (2.14) or (3.13): we have

$$\frac{\delta(t)}{Y(t)} \propto (t^* - t)^{(1-c+\lambda c)/2} \tag{4.7}$$

and this tends to zero as $t \to t^*$ provided

$$\lambda > (c-1)/c \ (> 0). \tag{4.8}$$

This is of course satisfied for the particular case of plane strain for which $\lambda = 1$ and $Y = \text{const}$.

What this means is that, under the condition (4.8), if the two vortex cores are well-separated at time $t = 0$ (i.e. $\delta_0 \ll b$), then they remain well separated for all $t \in (0, t^*)$. Moreover, under the same condition (4.8), the additional strain ($O(\Gamma/Y^2)$) experienced by each vortex due to the presence of the other remains small (as $t \to t^*$) compared with the 'imposed' strain which is proportional to $(t^* - t)^{-1}$. Hence, provided $\epsilon = \nu/\Gamma \ll 1$, the leading-order solution (3.13) is valid for each vortex: the vortex pair 'collapses' towards the origin $(0, 0)$ as $t \to t^*$, but the circulations $\pm\Gamma$ in the half-planes $y > 0$ and $y < 0$ remain constant for all $t < t^*$ despite the ever-decreasing length-scale of the vorticity distribution.

## 5. The strain field associated with a vortex pair

Consider a vortex pair $\pm\Gamma$ situated instantaneously at $x = 0$, $y = \pm b$, the core of each vortex being diffused on a scale $\delta_0 \ll b$. With the notation

$$E(x, y) = \exp(-(x^2 + y^2)), \tag{5.1}$$

the vorticity distribution is assumed to be

$$\omega(x, y) = \frac{\Gamma}{\pi\delta_0^2}\left[E\left(\frac{x}{\delta_0}, \frac{y-b}{\delta_0}\right) - E\left(\frac{x}{\delta_0}, \frac{y+b}{\delta_0}\right)\right], \tag{5.2}$$

the weak elliptic deformation of the vortex cores that results from mutual interaction being neglected. Of course, this vortex pair will propagate in the $x$-direction with velocity $\Gamma/4\pi b$, and the cores will diffuse under the action of vorticity. For the moment, however, we restrict attention to the instantaneous state described by (5.2), and we investigate the nature of the strain field in the surrounding fluid.

The streamfunction $\psi(x, y)$ outside the vortex cores is given by

$$\psi(x, y) = -\frac{\Gamma}{4\pi}\left(\ln r_1^2 - \ln r_2^2\right), \tag{5.3}$$

where

$$r_1^2 = x^2 + (y-b)^2, \quad r_2^2 = x^2 + (y+b)^2. \tag{5.4}$$

Near any point $(x_0, y_0)$ we may write

$$x = x_0 + \xi, \quad y = y_0 + \eta, \tag{5.5}$$

and expand $\psi$ in the form

$$\psi(x,y) = \psi_0 - \xi v_0 + \eta u_0 + \tfrac{1}{2}(A_0\xi^2 + 2B_0\xi\eta - A_0\eta^2) + \cdots, \tag{5.6}$$

where the suffix zero indicates evaluation at $(x_0, y_0)$; by straightforward calculation, we find

$$u = \psi_y = \frac{\Gamma b}{\pi}\frac{x^2 - y^2 + b^2}{r_1^2 r_2^2}, \tag{5.7}$$

$$v = -\psi_x = \frac{\Gamma b}{\pi}\frac{2xy}{r_1^2 r_2^2}, \tag{5.8}$$

$$A = \psi_{xx} = -\psi_{yy} = \frac{2\Gamma by}{\pi r_1^4 r_2^4}[(x^2+y^2)(3x^2 - y^2 + 2b^2) - b^4], \tag{5.9}$$

$$B = \psi_{xy} = \psi_{yx} = +\frac{2\Gamma bx}{\pi r_1^4 r_2^4}[(x^2+y^2)(3y^2 - x^2 - 2b^2) - b^4]. \tag{5.10}$$

Note that, for $y = 0$,

$$u = \frac{\Gamma b}{\pi}\frac{1}{x^2+b^2}, \quad v = 0, \tag{5.11}$$

$$A = 0, \quad B = -\frac{2\Gamma bx}{\pi(x^2+b^2)^2}. \tag{5.12}$$

The quadratic (strain) contribution to (5.6) is here

$$\psi_s = B_0\xi\eta, \tag{5.13}$$

and, since $B_0 < 0$ (for $x_0 > 0$), the principal axis of positive strain at $(x_0, 0)$ is in the $\eta$ (or $y$) direction.

The principal axes of strain at any point $(x_0, y_0)$ are the directions for which

$$(u_s, v_s) \equiv (B_0\xi - A_0\eta, -A_0\xi - B_0\eta) = \mu(\xi, \eta) \tag{5.14}$$

for some (real) $\mu$. The usual determinantal condition gives

$$\mu^2 = A_0^2 + B_0^2 = \frac{4\Gamma^2 b^2}{\pi^2 r_1^8 r_2^8}P(x_0^2, y_0^2), \tag{5.15}$$

where $P(X, Y)$ is a fifth-order polynomial in $X$ and $Y$. The principal directions of strain at $(x_0, y_0)$ are then given by

$$\frac{\xi}{\eta} = -\frac{B_0 \pm \sqrt{A_0^2 + B_0^2}}{A_0}. \tag{5.16}$$

These principal axes of strain are parallel to $Ox, Oy$ wherever $A = 0$, i.e. on the axis $y = 0$ and also on the quartic curve

$$Q_1 : (x^2+y^2)(3x^2 - y^2 + 2b^2) = b^4. \tag{5.17}$$

The form of this curve is most easily understood by noting that in the plane of the variables

$$X = x^2, \quad Y = y^2, \tag{5.18}$$

it is just the hyperbola $H_1$

$$(X+Y)(3X - Y + 2b^2) = b^4, \tag{5.19}$$

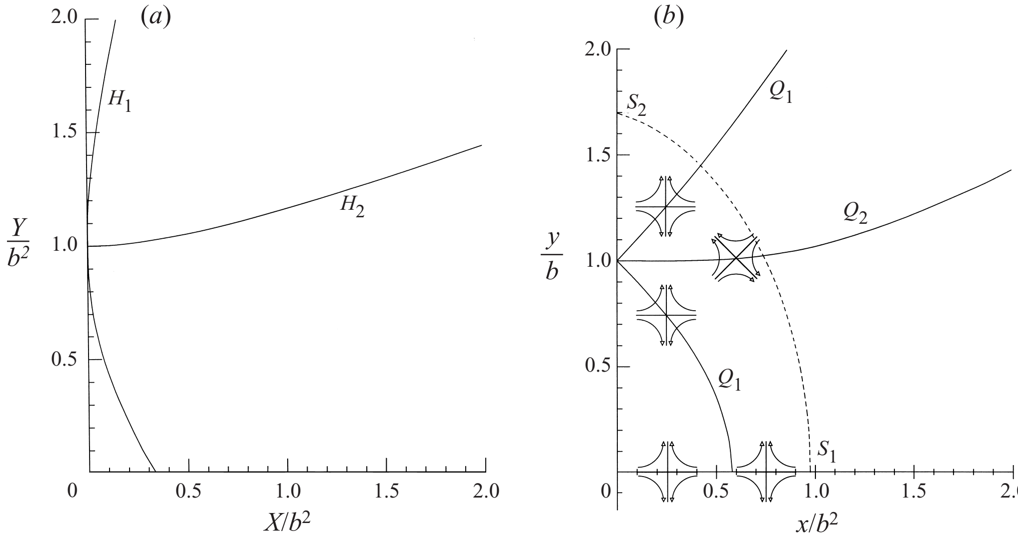


FIGURE 1. Rate-of-strain field associated with a single vortex pair $\pm\Gamma$ placed at $(0, \pm b)$. ($a$) Geometrical divisions in the plane of the variables $X = x^2$, $Y = y^2$; ($b$) variation of strain orientation in $(x, y)$-plane; the curves $Q_1$ and $Q_2$ are symmetric with respect to the $x$- and $y$-axes.

with asymptotes

$$X + Y = 0, \quad 3X - Y + 2b^2 = 0. \tag{5.20}$$

This hyperbola is sketched in figure 1($a$); it is of course only the portion in the quadrant $X > 0$, $Y > 0$ that is relevant here.

Similarly, the principal axes of strain are oriented at $\pi/4$ to $Ox, Oy$ wherever $B = 0$, i.e. on $x = 0$ and on the quartic curve

$$Q_2 : (x^2 + y^2)(3y^2 - x^2 - 2b^2) = b^4, \tag{5.21}$$

which corresponds to the hyperbola

$$H_2 : (X + Y)(3Y - X - 2b^2) = b^4 \tag{5.22}$$

in the $(X, Y)$-plane. This is also sketched in figure 1($a$).

Knowledge of these curves allows us to determine the signs of $A$ and $B$ in various regions of the $(x, y)$-plane as shown in figure 1($b$). The corresponding strain orientation can be easily deduced and is as indicated. As might be expected, a material curve like the dashed curve in the figure, which at each point of the segment $S_1S_2$ makes an angle of less than $\pi/4$ with the local principal axis of positive strain, is subject to stretching along the full length of this segment.

## 6. The interaction of two skewed vortex pairs

We are now in a position to consider the model of vortex interaction proposed in the introduction, namely two vortex pairs propagating towards each other, each pair being initially aligned along the principal axis of positive rate of strain associated with the other. Thus we adopt as initial condition the configuration indicated in figure 2($a$): one vortex pair $\pm\Gamma$ (denoted $V_1^{\pm}$) is centred on the lines $x = -a_0$, $y = \pm b_0$ with $b_0/a_0 = O(1)$; and the other pair $\pm\Gamma$ (denoted $V_2^{\pm}$) on $x = a_0$, $z = \pm b_0$. Each of the four constituent vortices is supposed diffused around these lines with Gaussian profile

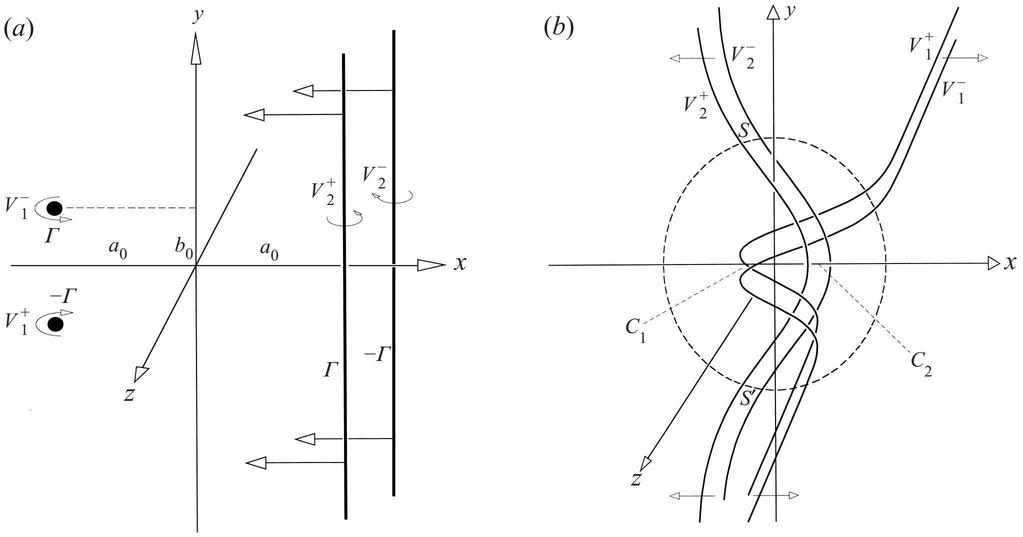


FIGURE 2. Interaction of two vortex pairs, each initially placed parallel to the positive axis of strain associated with the other. (*a*) Initial configuration; (*b*) configuration during the asymptotic interaction phase, the interaction zone being as indicated.

on a scale $\delta_0 \ll a_0, b_0$. Thus, with the notation (5.1), the initial vorticity distribution is assumed to be

$$\boldsymbol{\omega}_0(\boldsymbol{x}) = \frac{\Gamma}{\pi\delta_0^2}\left(0, E\left(\frac{x-a_0}{\delta_0}, \frac{z-b_0}{\delta_0}\right) - E\left(\frac{x-a_0}{\delta_0}, \frac{z+b_0}{\delta_0}\right),\right.$$
$$\left. E\left(\frac{x+a_0}{\delta_0}, \frac{y-b_0}{\delta_0}\right) - E\left(\frac{x+a_0}{\delta_0}, \frac{y+b_0}{\delta_0}\right)\right). \qquad (6.1)$$

This vorticity field is invariant under rotation ($x \to -x$, $z \to -z$) through $\pi$ about the $y$-axis followed by rotation ($y \to z$, $z \to -y$) through $\pi/2$ about the $x$-axis. This skew symmetry is preserved under the subsequent evolution, and may (for $t > 0$) be expressed in the form

$$(\omega_1(\boldsymbol{x},t), \omega_2(\boldsymbol{x},t), \omega_3(\boldsymbol{x},t)) = \left(-\omega_1(\boldsymbol{x}^*,t), \omega_3(\boldsymbol{x}^*,t), \omega_2(\boldsymbol{x}^*,t)\right) \qquad (6.2)$$

where $\boldsymbol{x}^* = (-x, z, y)$. This simply means that we may focus attention on the evolution of one vortex pair, the evolution of the other being governed by this symmetry.

Consider now, in physical terms, what happens for $t > 0$. The vortex pairs propagate towards each other and a ‘zone of interaction’ of the vortex pairs $|\boldsymbol{x}| = O(b_0)$ may be identified; far outside this zone, the vortex pairs $V_1^{\pm}$, $V_2^{\pm}$ propagate with undisturbed velocities $\pm\Gamma/4\pi b_0$ respectively. After a time of order $4\pi a_0 b_0/\Gamma$ a situation like that depicted in figure 2(*b*) must presumably develop, all the interesting effects taking place within the interaction zone. Clearly the minimum $x$-wise separation of the vortex pairs $2a(t)$ must continue to decrease until either viscous effects cause reconnection of vortex lines (the conventional view), or a singularity of the vorticity distribution appears at $\boldsymbol{x} = 0$ at some finite time $t^* > 0$.

In this scenario, if a singularity appears, then it does so at the single point $\boldsymbol{x} = 0$ at time $t = t^*$, consistent with the fact that the space–time Hausdorff dimension of any singularity of the Navier–Stokes equations is not greater than 1/2 (Cafarelli, Kohn & Nirenberg 1982). The singularity must also clearly involve a singularity of the direction of the vorticity field at $\boldsymbol{x} = 0, t = t^*$, consistent with the result of Constantin & Fefferman (1993) (see also Constantin, Fefferman & Majda 1996).

If the behaviour envisaged does indeed occur, then the development of the singularity should presumably be describable in terms of a local similarity solution of the Navier–Stokes equations representing a characteristic structure whose scale decreases in proportion to $a(t)$. Thus, for example, if $2b(t)$ is the minimum separation of $V_1^+$ and $V_1^-$, then geometrical self-similarity implies that

$$b(t)/a(t) = s \tag{6.3}$$

where $s$ (the 'separation' parameter) is a constant of order unity; and if $\delta(t)$ is the scale of the core radius where this is minimal, then

$$\delta(t)/a(t) = \sigma \tag{6.4}$$

where $\sigma$ is a constant which may depend on the vortex Reynolds number $Re = \Gamma/\nu$ (since the core size is influenced by viscosity). This means that $\delta(t)/a(t)$ remains small, if small initially, so that the circulation $\pm\Gamma$ in each of the four vortices remains effectively constant.

We may now estimate $\mathrm{d}a/\mathrm{d}t$, which is determined by the instantaneous configuration of the vortices of strength $\pm\Gamma$ and by the current scale $a(t)$ in the interaction zone; on dimensional grounds

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{k\Gamma}{2a} \tag{6.5}$$

where $k$ is a positive constant of order unity (positive because we know on physical grounds that $a(t)$ is decreasing). Hence, integrating,

$$a^2 = k\Gamma(t^* - t) \tag{6.6}$$

where $t^*$ is a constant. This result holds only once the self-similar behaviour is established. If, during this self-similar stage, $a = a_1$ when $t = t_1$, then from (6.6) the 'singularity time' $t^*$ is given by

$$t^* - t_1 = a_1^2/k\Gamma. \tag{6.7}$$

Note further that the rate of strain acting on the pair $V_2^\pm$ at the point $C_2$ (figure 2($b$) (due mainly to the influence of the segment of $V_1^\pm$ in the interaction zone) is also determined instantaneously by $\Gamma$ and by the scale $a(t)$, and is therefore given (cf. (2.10)) by

$$\gamma(t) = k'\Gamma/a^2 = \frac{c}{t^* - t} \quad (t_1 < t < t^*), \tag{6.8}$$

where $k'$ and $c = k'/k$ are constants of order unity. This strain field is two-dimensional with positive strain aligned parallel to $V_2^\pm$ at $C_2$. The relevance of the model developed in §§ 2–4 should now be apparent! The implication is that, in some neighbourhood of the points $C_1^\pm$, $C_2^\pm$ on the four vortices, the solution (2.14) (with $\pm$ now inserted in front of $\Gamma$) is valid. Note that the scale $\delta(t)$ of the vortex cores is given by (4.6), so that now

$$\frac{\delta(t)}{a(t)} = \left(\frac{4}{k(c-1)}\right)^{1/2}\left(\frac{\nu}{\Gamma}\right)^{1/2} = \left(\frac{4}{k(c-1)}\right)^{1/2} Re^{-1/2}, \tag{6.9}$$

a constant, consistent with (6.4). As $t \uparrow t^*$, the whole configuration collapses onto $\boldsymbol{x} = 0$, where a singularity of vorticity appears.

A necessary condition for the validity of this description is that $c > 1$ (see (2.15) and (2.16)). It therefore becomes necessary to estimate more accurately the values of $k$ and $k'$. Consider first the values of these constants if the vortices are treated as straight (as in figure 2*a*). In this configuration

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{\Gamma}{4\pi b} + \frac{\Gamma}{\pi}\frac{b}{4a^2 + b^2}, \tag{6.10}$$

the first term coming from the self-induced velocity of either pair, and the second coming from the interaction (equation (5.7) with $y = 0$, $x = 2a$). With $b = sa$, (6.10) gives (6.5) with

$$k = \frac{1}{2\pi}\left[\frac{4 - 3s^2}{s(4 + s^2)}\right]. \tag{6.11}$$

Since $k$ must be positive to give $\mathrm{d}a/\mathrm{d}t < 0$, we must have $s^2 < 4/3$, i.e.

$$s < 1.15. \tag{6.12}$$

Similarly, the strain acting on either vortex pair at the point of minimum separation is given by (5.12) with $x = 2a$, $b = sa$, i.e.

$$\gamma = \frac{k'\Gamma}{a^2} \quad \text{with} \quad k' = \frac{4s}{\pi(4 + s^2)^2}. \tag{6.13}$$

Hence, from (6.11) and (6.13), we have

$$c = k'/k = \frac{8s^2}{(4 + s^2)(4 - 3s^2)} \tag{6.14}$$

and the condition $c > 1$ is satisfied provided

$$s^2 > -\tfrac{8}{3} + \tfrac{4}{3}\sqrt{7} = 0.86, \quad \text{i.e. } s > 0.92. \tag{6.15}$$

The conditions (6.12), (6.15) evidently place tight constraints on $s$ for self-consistency of the description.

Now of course in the self-similar phase in the interaction zone, the vortex pairs are necessarily curved as in figure 2(*b*), and so the above values of $k$ and $k'$ ((6.11) and (6.13)) are an overestimate. However, the curvature effect may be expected to reduce $k$ and $k'$ by the same factor, so that the result (6.14) for the ratio remains accurate.

The curvature of the vortices in the interaction zone is actually a vitally important ingredient of the model. The portions of $V_2^{\pm}$ in the neighbourhoods of the inflection points $S$, $S'$ (figure 2*b*) contribute the $y$-component of velocity at $V_1^{\pm}$ which decreases $b(t)$ in step with $a(t)$ according to (6.3). This inward-directed component must be sufficient to overcome the curvature-induced velocity of $V_1^+$ and $V_1^-$ which is outwards in the direction of the binormal.

It seems likely that the relative magnitude of these effects will depend critically on the ratio of core radius to vortex separation. A numerical study by Boratav (1991) has indicated that the curvature effect associated with local induction may, for some initial conditions, dominate, in which case a singularity of the type envisaged will not occur (R. B. Pelz 1999, private communication). The diverse effects are evidently all quite delicately balanced, and it is clearly desirable to identify more precisely the nature of the similarity solution that is postulated in the interaction zone. A preliminary exploration is described in the following section.

## 7. Local similarity solution of the Navier–Stokes equations

The problem of determining both a local similarity solution valid in an inner region, and a global solution in the outer region to which it can be matched, is very difficult. Nevertheless, we seek to clarify in this section the necessary condition that the inner solution must satisfy for a self-consistent matching to be possible.

The estimates of § 6 suggests that we should seek a local similarity solution of the Navier–Stokes equation of the form

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{\Gamma^{1/2}}{\sqrt{(t^*-t)}}\boldsymbol{U}(\boldsymbol{X}),\qquad \boldsymbol{X}=\frac{\boldsymbol{x}}{(\Gamma(t^*-t))^{1/2}},\tag{7.1}$$

where $\Gamma$ is the conserved vortex strength. This is just the Leray (1934) transformation discussed in the introduction. We then have

$$\boldsymbol{\omega}=\nabla\wedge\boldsymbol{u}=\frac{1}{(t^*-t)}\boldsymbol{\Omega}(\boldsymbol{X}),\qquad \boldsymbol{\Omega}=\nabla_X\wedge\boldsymbol{U},\tag{7.2}$$

which may be compared with (2.11). Noting that

$$\frac{\partial\boldsymbol{X}}{\partial t}=\frac{\boldsymbol{X}}{2(t^*-t)},\qquad \frac{\partial X_i}{\partial x_j}=\frac{1}{(\Gamma(t^*-t))^{1/2}}\delta_{ij},\tag{7.3}$$

we have further

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\frac{-1}{(t^*-t)^2}\left(\boldsymbol{\Omega}+\tfrac{1}{2}\boldsymbol{X}\cdot\nabla_X\wedge\boldsymbol{\Omega}\right)=-\frac{1}{2(t^*-t)^2}\nabla_X\wedge(\boldsymbol{X}\wedge\boldsymbol{\Omega}),\tag{7.4}$$

$$\nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{\omega})=\frac{1}{(t^*-t)^2}\nabla_X\wedge(\boldsymbol{U}\wedge\boldsymbol{\Omega}),\tag{7.5}$$

and

$$\nu\nabla^2\boldsymbol{\omega}=\frac{\nu}{\Gamma(t^*-t)^2}\nabla_X^2\boldsymbol{\Omega}.\tag{7.6}$$

Hence, the (Navier–Stokes) vorticity equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{\omega})+\nu\nabla^2\boldsymbol{\omega}\tag{7.7}$$

yields the quasi-steady equation

$$-\nabla\wedge(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{X})\wedge\boldsymbol{\Omega}=\epsilon\nabla^2\boldsymbol{\Omega}\tag{7.8}$$

where $\nabla$ now represents $\partial/\partial\boldsymbol{X}$. Note that, for the particular problem considered in § 2, (7.8) is equivalent to (2.12), allowing for slight change of notation.

It must here be emphasized that we cannot reasonably expect to find any solution of (7.8) for which $\boldsymbol{\Omega}(\boldsymbol{X})$ is localized in $\boldsymbol{X}$ (i.e. $O(|\boldsymbol{X}|^{-2-\rho})$ for $\rho>0$ as $|\boldsymbol{X}|\to\infty$, for this would yield a solution $\boldsymbol{\omega}(\boldsymbol{x},t)$ of (7.7) similarly localized in $\boldsymbol{x}$, and with the property that

$$\boldsymbol{\omega}\to 0\quad\text{as}\quad t\to t^*\quad\text{for all}\quad \boldsymbol{x}\neq 0,\tag{7.9}$$

a behaviour that cannot be reconciled with the fact that, under evolution governed by (7.7), vortex lines are convected and diffused by the fluid motion. This statement is compatible with the theorem of Nečas, Růžička & Šverák (1997) that no smooth solution of (7.8) exists for which $|\boldsymbol{U}|\in L^3(R^3)$.

What we must rather look for is a solution of (7.8) which matches, as $|\boldsymbol{X}|\to\infty$, to an 'outer solution' of (7.7), this outer solution being non-singular as $t\to t^*$. The appropriate matching condition is

$$\lim_{|\boldsymbol{X}|\to\infty} \boldsymbol{\Omega}(\boldsymbol{X}) = \lim_{|\boldsymbol{x}|\to 0} \boldsymbol{\omega}(\boldsymbol{x}, t). \tag{7.10}$$

This can be achieved provided

$$\boldsymbol{\Omega}(\boldsymbol{X}) \sim |\boldsymbol{X}|^{-2} \quad \text{as} \quad |\boldsymbol{X}| \to \infty, \tag{7.11}$$

$$\boldsymbol{\omega}(\boldsymbol{x}, t) \sim \Gamma\, |\boldsymbol{x}|^{-2} \quad \text{as} \quad |\boldsymbol{x}| \to 0. \tag{7.12}$$

More precisely, for the problem of two vortex pairs considered in §6, we require a solution of (7.8) satisfying

$$\boldsymbol{\Omega} \cdot \hat{\boldsymbol{e}}_r \sim f(\theta, \varphi)/|\boldsymbol{X}|^2 \quad \text{as} \quad |\boldsymbol{X}| \to \infty, \tag{7.13}$$

where $f(\theta, \varphi)$ is a function of the spherical polar angles $(\theta, \varphi)$ integrating to zero over the sphere $|\boldsymbol{X}| = \text{const.}$ and to $\pm 1$ over the (small) surface areas on which the outgoing (incoming) vortices are located. The corresponding behaviour for the outer field is then

$$\boldsymbol{\omega} \cdot \hat{\boldsymbol{e}}_r \sim \Gamma f(\theta, \varphi)/|\boldsymbol{x}|^2 \quad \text{as} \quad |\boldsymbol{x}| \to 0 \tag{7.14}$$

each vortex thus expanding conically in passing from the inner to the outer region (figure 3). We note here that precisely this sort of structure is evident in the solution computed by Pelz (1997).

The problem (7.8), (7.13) is still a difficult one, which will require careful numerical analysis. The great advantage of this formulation however is that, if a smooth solution is found by numerical or analytical means, an objective that we must defer to a subsequent investigation, then this immediately implies a finite-time singularity of the Navier–Stokes equations via the transformation (7.1), (7.2).

Here, we simply infer certain properties of the singularity, assuming such a solution can indeed be found. First, we may easily estimate the contributions to momentum $\boldsymbol{P}$, angular momentum $\boldsymbol{M}$, kinetic energy $K$, helicity $H$ and enstrophy $\bar{\Omega}$ from the interaction zone $V$, which we may take to be $|X| < R$, for some fixed $R$. We find

$$\boldsymbol{P} = \frac{1}{2}\int_V \boldsymbol{x} \wedge \boldsymbol{\omega}\, \mathrm{d}^3\boldsymbol{x} = \tfrac{1}{2}\Gamma^2(t^* - t)\int_0^R \boldsymbol{X} \wedge \boldsymbol{\Omega}(\boldsymbol{X})\, \mathrm{d}^3\boldsymbol{X}, \tag{7.15}$$

$$\boldsymbol{M} = \frac{1}{3}\int_V \boldsymbol{x} \wedge (\boldsymbol{x} \wedge \boldsymbol{\omega})\, \mathrm{d}^3\boldsymbol{x} = \tfrac{1}{3}\Gamma^{5/2}(t^* - t)^{3/2}\int_0^R \boldsymbol{X} \wedge (\boldsymbol{X} \wedge \boldsymbol{\Omega})\, \mathrm{d}^3\boldsymbol{X}, \tag{7.16}$$

$$K = \frac{1}{2}\int_V \boldsymbol{u}^2 \mathrm{d}^3\boldsymbol{x} = \tfrac{1}{2}\Gamma^{5/2}(t^* - t)^{1/2}\int_0^R \boldsymbol{U}^2 \mathrm{d}^3\boldsymbol{X}, \tag{7.17}$$

$$H = \int_V \boldsymbol{u} \cdot \boldsymbol{\omega}\, \mathrm{d}^3\boldsymbol{x} = \Gamma^2\int_0^R \boldsymbol{U} \cdot \boldsymbol{\Omega}\, \mathrm{d}^3\boldsymbol{X}, \tag{7.18}$$

$$\bar{\Omega} = \int \boldsymbol{\omega}^2 \mathrm{d}^3\boldsymbol{x} = \frac{\Gamma^{3/2}}{(t^* - t)^{1/2}}\int_0^R \boldsymbol{\Omega}^2 \mathrm{d}^3\boldsymbol{X}. \tag{7.19}$$

Thus the contributions to $\boldsymbol{P}$, $\boldsymbol{M}$ and $K$ from the interaction zone $V$ tend to zero as $t \to t^*$, whereas the contribution to $H$ is constant. The contribution to enstrophy, and hence to rate of dissipation of energy,† from $V$ tends to infinity like $(t^* - t)^{-1/2}$, a

† The dissipation integral $\int_V (\partial u_i/\partial x_j)^2\, \mathrm{d}V$ obviously has the same $(t^* - t)^{-1/2}$ time-dependence.

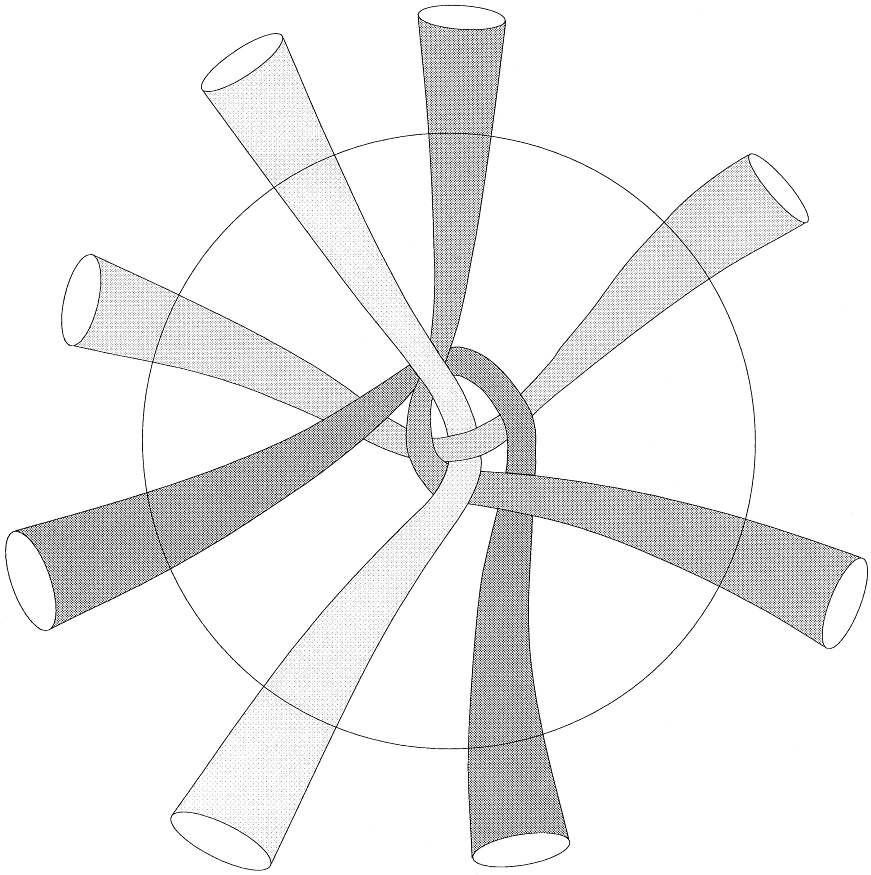

FIGURE 3. A hypothetical configuration showing conical expansion of vortex tubes as they leave the interaction zone.

result that may have an important bearing on the phenomenon of intermittency of energy dissipation in turbulent flow.

The decreasing contribution to $K$ in the interaction zone is particularly noteworthy. If conservation of energy were imposed as an additional requirement, this would require a scale decreasing like $(t^* - t)^{2/5}$ rather than $(t^* - t)^{1/2}$ (Constantin 1994); there is however no need to impose such a condition, because a flux of energy from the inner to the outer region cannot be excluded.

## 8. Inviscid limit

In the inviscid limit $\epsilon = 0$, (7.8) becomes simply

$$\nabla \wedge [(\boldsymbol{U} + \tfrac{1}{2}\boldsymbol{X}) \wedge \boldsymbol{\Omega}] = 0. \tag{8.1}$$

We note here a peculiarity of this situation. If, instead of (7.1) we seek an 'expanding' similarity solution of the Euler equations of the form

$$\boldsymbol{u}(\boldsymbol{x}, t) = \left(\frac{\Gamma}{t^* + t}\right)^{1/2} \boldsymbol{U}(\boldsymbol{X}_1), \qquad \boldsymbol{X}_1 = \frac{\boldsymbol{x}}{(\Gamma(t^* + t))^{1/2}} \tag{8.2}$$

for $t > -t^*$, then, instead of (8.1) we obtain

$$\nabla \wedge [(\boldsymbol{U} - \tfrac{1}{2}\boldsymbol{X}_1) \wedge \boldsymbol{\Omega}] = 0. \tag{8.3}$$

Evidently if $\boldsymbol{U} = \boldsymbol{U}_1(\boldsymbol{X}_1)$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_1(\boldsymbol{X}_1)$ is a solution of (8.3), then $\boldsymbol{U} = -\boldsymbol{U}_1(\boldsymbol{X})$, $\boldsymbol{\Omega} = -\boldsymbol{\Omega}_1(\boldsymbol{X})$ is a solution of (8.1); thus if there exists an expanding similarity solution of the form (8.2) matching to an external flow via a condition of the form (7.10), then there exists a dual 'imploding' solution obtained by replacing $\boldsymbol{U}_1(\boldsymbol{X}_1)$ by $-\boldsymbol{U}(\boldsymbol{X})$. This duality, which is a consequence of the time-reversibility of the Euler equations, was noted by Pelz (1997), who in fact suggested that an imploding solution may be replaced by its dual as $t$ passes through the critical time $t^*$.

Equation (8.1) is satisfied provided

$$(\boldsymbol{U} + \tfrac{1}{2}\boldsymbol{X}) \wedge \boldsymbol{\Omega} = \nabla h \tag{8.4}$$

for some scalar field $h$. We then have

$$\boldsymbol{\Omega} \cdot \nabla h = 0 \tag{8.5}$$

and

$$\boldsymbol{U} \cdot \nabla h = -\tfrac{1}{2}\boldsymbol{X} \cdot \nabla h = -\tfrac{1}{2}R\frac{\partial h}{\partial R} \tag{8.6}$$

where $R = |\boldsymbol{X}|$. From (8.5), the vortex lines lie on surfaces $h =$ const. in the interaction zone. From (8.6), the component of the flow $\boldsymbol{U}$ across these surfaces must exactly compensate the term $\frac{1}{2}\boldsymbol{X} \cdot \nabla h$ which is associated with the inclination of the vorticity $\boldsymbol{\Omega}$ to the radius vector $\boldsymbol{X}$.

Reference back to the exact solution (2.25) of the Euler equation shows that the same behaviour is evident there: the condition $c = 1$ gives a balance of the first two terms of (2.12), one coming from $\partial\omega/\partial t$, the other from the action of the imposed strain; in the similarity variables, two radial velocities (perpendicular to the vortex tube) exactly compensate.

It is an honour and a pleasure to dedicate this paper to Philip Saffman, whose contributions to fluid dynamics, and in particular to vortex dynamics, have provoked admiration and provided inspiration over more than four decades. I thank K. Ohkitani for drawing my attention to the seminal work of Leray (1934) concerning the possibility of a finite-time singularity; and R. B. Pelz for providing me with relevant preprints and reprints and for his illuminating comments.

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